Schubert Varieties

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1

Standard Monomial Theory for Graßmann varieties

Standard Monomial theory (abbreviated as SMT) is the central theme of this book. We think that the classical case of Schubert varieties in the Graßmannian deserves special attention because the philosophy and the strategy are the same as in the general case, but the technical requirements used in the constructions and in the proofs are elementary.

We shall see how one can develop SMT in this classical case by using just the Plücker embedding and the associated Plücker coordinates. It should be supportive to have this elementary case ready as a guideline for the later chapters.

SMT consists in constructing explicit bases for the homogeneous coordinate rings of the Graßmann variety and its Schubert varieties. After introducing the Graßmannian and its Schubert varieties, we present the Standard Monomial Theory for Schubert varieties in the Graßmannian in Section 1.5. As an application we present a proof of the "vanishing theorems" for these Schubert varieties, where we also deduce their normality.

Let us fix some notation that we shall use throughout this chapter. Let k be an algebraically closed field of arbitrary characteristic and set $k^* = k \setminus \{0\}$. Let $V = k^n$ and let $e_1, \ldots, e_n$ be the standard basis of V. The group $SL_n(k)$ acts naturally on V. Let T be the subgroup of of $SL_n(k)$ consisting of diagonal matrices, and B the subgroup consisting of upper triangular matrices. We have $B = TU = UT$ where U is the subgroup of B consisting of upper triangular unipotent matrices. Given a positive integer d, we denote by $\mathfrak{S}_d$ the symmetric group on d letters.

1.1 The Plücker embedding

1.1.1. Graßmann variety. Let us start with the most simple example of a Graßmann variety, the projective space $\mathbb{P}^{n-1}$. Recall that the projective space $\mathbb{P}^{n-1}$ is defined as the set of all lines in V. Another way to formulate the definition is to say that the projective space is the quotient $(V\setminus\{0\})/\sim$, where the equivalence relation is defined by: $v \sim v'$ if there exists an element $t \in k^*$ such that $tv = v'$.

The definition of a Graßmann variety is a straight forward generalization of the above, only one has to replace lines, i.e. 1-dimensional subspaces, by d-dimensional subspaces.

Definition 1.1.2. Let $1 \leq d < n$. The *Graßmann variety* $\mathrm{Gr}_{d,n}$ is defined as the set of all d-dimensional subspaces in V.

In particular, $\mathrm{Gr}_{1,n} = \mathbb{P}^{n-1}$. To get a description of $\mathrm{Gr}_{d,n}$ as a quotient similar to the description of the projective space above, let $U \in \mathrm{Gr}_{d,n}$ be a d-dimensional subspace of k^n. Fix a basis $\{v_1, \dots, v_d\}$ of U, then we can associate to U an $n \times d$ matrix $A = (a_{i,j})$ of rank d such that the j-th column consists of the coefficients of v_j with respect to the standard basis $\{e_1, \dots, e_n\}$ of V, *i.e.* $v_j = \sum_{i=1}^n a_{ij} e_i$.

Vice versa, to an $n \times d$ matrix $A \in M_{n,d}(k)$ of rank d one associates naturally the d-dimensional subspace U of V obtained as the span of the column vectors. In this language we can give a description of $\mathrm{Gr}_{d,n}$ similar to that of the projective space above: let Z be the set of $n \times d$ matrices of rank strictly less than d, then $\mathrm{Gr}_{d,n} = (M_{n,d}(k) \setminus Z)/\sim$, where the equivalence relation is defined by: $A \sim A'$ if the column vectors span the same subspace of V.

Above we defined the relation "$\sim$" on $V \setminus \{0\}$ in terms of the group action of k^* on V. Here we can do the same by using the fact that $GL_d(k)$ acts transitively on the set of bases of a d-dimensional subspace:

$$\mathrm{Gr}_{d,n} = (M_{n,d}(k) \setminus Z)/\sim, \text{ where } A \sim A' \Leftrightarrow \begin{array}{c} A' = AC \text{ for some} \\ C \in GL_d(k) \end{array}$$

For $d = 1$, this is exactly the description of the projective space $\mathbb{P}^{n-1} = \mathrm{Gr}_{1,n}$ given above.

1.1.3. $\mathrm{Gr}_{d,n}$ **as homogeneous space.** Another very useful description of the Graßmann variety is that of $\mathrm{Gr}_{d,n}$ as a homogeneous space. If $U \subset V$ is a d-dimensional subspace and $g \in SL_n(k)$, then $gU = \{gu \mid u \in U\}$ is again a d-dimensional subspace. In fact, given $U, U' \in \mathrm{Gr}_{d,n}$, there exists always a $g \in SL_n(k)$ such that $gU = U'$.

Denote by $F_j \subset V$ the j-dimensional subspace $F_j = \langle e_1, e_2 \dots, e_j \rangle$ spanned by the first j elements of the standard basis. Then we can identify $\mathrm{Gr}_{d,n}$ with the coset space $SL_n(k)/P_d$, where P_d is the isotropy group of the d-dimensional subspace F_d. Now $g \in SL_n(k)$ is an element of P_d if and only if $ge_j \in F_d$ for $1 \leq j \leq d$, and hence:

$$\mathrm{Gr}_{d,n} = SL_n(k)/P_d \,, \text{ where } P_d = \left\{ A \in SL_n(k) \;\middle|\; A = \begin{pmatrix} * & * \\ 0_{(n-d)\times d} & * \end{pmatrix} \right\}.$$

Note that the isotropy group P_d contains B.

1.1.4. Plücker coordinates. To endow the Graßmann variety with the structure of an algebraic variety, we will identify $\mathrm{Gr}_{d,n}$ with a subset of the projective space $\mathbb{P}(\Lambda^d V)$. A first step in this direction is the introduction of Plücker coordinates, which can be viewed as linear functions on $\Lambda^d V$ as well as multilinear alternating functions on $M_{n,d}(k)$.

The d-fold wedge product is alternating, the ordered products of elements in the canonical basis of V: $e_{i_1} \wedge \cdots \wedge e_{i_d}$, $1 \leq i_1 < \cdots < i_d \leq n$, form a basis of $\Lambda^d V$, called the canonical basis of $\Lambda^d V$.

Definition 1.1.5. Let $I_{d,n} := \{\underline{i} = (i_1, \ldots, i_d) | 1 \leq i_1 < \cdots < i_d \leq n\}$ be the set of all strictly increasing sequences of length d between 1 and n. For $\underline{i} = (i_1, \ldots, i_d) \in I_{d,n}$ we write $e_{\underline{i}} = e_{i_1} \wedge \cdots \wedge e_{i_d}$. We define a partial order "$\geq$" on $I_{d,n}$ as follows: $\underline{i} \geq \underline{j} \Leftrightarrow i_t \geq j_t$ for all $t = 1, \ldots, d$.

So the canonical basis of $\Lambda^d V$ can be written as $\{e_{\underline{i}} \mid \underline{i} \in I_{d,n}\}$. Denote by $\{p_{\underline{i}} \mid \underline{i} \in I_{d,n}\}$ the dual basis of $(\Lambda^d V)^*$, i.e., $p_{\underline{i}}(e_{\underline{j}}) = \delta_{\underline{i},\underline{j}}$.

Definition 1.1.6. The linear functions $p_{\underline{i}}$, $\underline{i} \in I_{d,n}$, on $\Lambda^d V$ are called *Plücker coordinates.*

By the definition of the d-fold wedge product the space of linear functions on $\Lambda^d V$ can be naturally identified with the space of multilinear alternating functions on d-copies of V, i.e., on $M_{d,n}(k) = \underbrace{V \times \ldots \times V}_{d \text{ times}}$.

Remark 1.1.7. We use the same name *Plücker coordinates* and the same symbol $p_{\underline{i}}$ for the *linear functions on* $\Lambda^d V$ as well as the corresponding *multilinear alternating function on the space* $M_{n,d}(k)$.

To make this relationship more explicit, recall that we have a natural map, the exterior product map:

$$\begin{array}{rcc} \pi_d : M_{n,d}(k) & \to & \Lambda^d V \\ A = (v_1, \ldots, v_d) & \mapsto & v_1 \wedge \cdots \wedge v_d \end{array}. \tag{1.1}$$

Here $v_1, \ldots, v_d$ are the column vectors of the matrix A. If we express the product $v_1 \wedge \cdots \wedge v_d$ as a linear combination of the elements of the canonical basis, then, by the definition of the dual basis, we have

$$v_1 \wedge \cdots \wedge v_d = \sum_{\underline{i} \in I_{d,n}} p_{\underline{i}}(\pi_d(A)) e_{\underline{i}}.$$

The alternating multilinear function on $M_{n,d}(k)$ associated to $p_{\underline{i}}$ is just the $\underline{i}$-th coordinate of the linear combination above, i.e., it is the composition $p_{\underline{i}} \circ \pi_d$. So by abuse of notation we write just $p_{\underline{i}}(A)$ instead of $p_{\underline{i}}(\pi_d(A))$.

Denote by $A_{\underline{i}}$ the $d \times d$ submatrix of A consisting of the i_1-th, i_2-th, ... and the i_d-th row of A. It follows that:

Lemma 1.1.8. *$p_{\underline{i}}(A)$ is the determinant $\det A_{\underline{i}}$ of the submatrix $A_{\underline{i}}$ of A.*

1.1.9. Plücker embedding. Our next step is to identify the Graßmann variety with a subset of the projective space $\mathbb{P}(\Lambda^d V)$.

For $A \in M_{n,d}(k)$ of rang d let $v_1, \dots, v_d \in k^n$ be the column vectors, let $U \subset V$ be the span of these column vectors and let $u_1, \dots, u_d \in U$. Denote by $C = (c_{i,j})$ the $d \times d$-matrix expressing the u_j as linear combinations of the v_i. i.e., $u_j = \sum_{i=1}^d c_{i,j} v_i$. The exterior product is alternating, so we get $v_1 \wedge \ldots \wedge v_d = (\det C) u_1 \wedge \ldots \wedge u_d$. As a consequence we see that the exterior product map induces a well defined map:

$$\pi : \mathrm{Gr}_{d,n} = ((M_{n,d}(k) \setminus Z)/\sim) \longrightarrow \mathbb{P}(\Lambda^d V)$$

called the *Plücker embedding*. We have a left action of $SL_n(k)$ on $M_{n,d}(k)$ defined by $g(v_1, \dots, v_d) = (gv_1, \dots, gv_d)$, and we have a natural action of $SL_n(k)$ on $\Lambda^d V$ given by $g(v_1 \wedge \cdots \wedge v_d) = (gv_1) \wedge \cdots \wedge (gv_d)$. It follows that the exterior product map $\pi_d : M_{n,d}(k) \to \Lambda^d V$ is equivariant with respect to these $SL_n(k)$-actions, and hence so is the Plücker embedding. The term *embedding* is justified because:

Proposition 1.1.10. *The Plücker map $\pi : \mathrm{Gr}_{d,n} \to \mathbb{P}(\Lambda^d V)$ is injective.*

Proof. Let F_d be the d-dimensional subspace of V spanned by $e_1, \dots, e_d$. By the homogeneity of the $SL_n(k)$-action on $\mathrm{Gr}_{d,n}$, it is sufficient to show if $\pi(U) = \pi(F_d)$, then $U = F_d$.

So suppose $\pi(U) = \pi(F_d)$ and let $\{v_1, \dots, v_d\}$ be a basis of U. Denote by $A \in M_{n,d}(k)$ the corresponding matrix. Since $[\pi_d(A)] = [e_1 \wedge \ldots \wedge e_d]$, we can choose the basis such that $\pi_d(A) = e_1 \wedge \ldots \wedge e_d$. It follows that the submatrix $A_{1,\dots,d}$ consisting of the first d-rows of A has determinant one, so by replacing A by $A \cdot A_{1,\dots,d}^{-1}$ if necessary we can (and will) assume that the submatrix of A consisting of the first d rows is the $d \times d$ identity matrix.

Now all $d \times d$ minors except $p_{1,2,\dots,d}(A)$ vanish. In particular, for $i > d$ we have $\pm a_{i,j} = p_{1,\dots,j-1,j+1,\dots,d,i}(A) = 0$ and hence $U = F_d$. □

1.1.11. Again Plücker coordinates. In section 1.1.4 we introduced the name Plücker coordinate for the dual basis $p_{\underline{i}}$ of the canonical basis of $\Lambda^d V$. To simplify the notation we use $p_{\underline{i}}$ in the following for arbitrary d-tuples and not only for elements $\underline{i} \in I_{d,n}$.

We give a description of the functions as alternating multilinear functions on the columns of $M_{n,d}(k)$ (instead of describing them as linear functions on $\Lambda^d V$).

For $1 \leq i_1, \ldots, i_d \leq n$ (not necessarily distinct nor in increasing order) set $\underline{i} = (i_1, \ldots, i_d)$. For an $n \times d$ matrix A let $A_{\underline{i}}$ be the $d \times d$ matrix having as first row the i_1-th row of A, as second row the i_2-th row of A and so on. We set $p_{\underline{i}}(A) = \det A_{\underline{i}}$.

Clearly, $p_{\underline{i}} = 0$ if the i_j's are not distinct, and if they are all distinct, then

$$p_{i_1,\ldots,i_d} = \mathrm{sgn}(\sigma) p_{\sigma(i_1),\ldots,\sigma(i_d)} \tag{1.2}$$

where $\sigma \in \mathfrak{S}_d$ is such that $(\sigma(i_1), \ldots, \sigma(i_d)) \in I_{d,n}$.

1.1.12. Alternating functions. In view of Proposition 1.1.10, we can identify $\mathrm{Gr}_{d,n}$ with $\mathrm{Im}\,\pi$. In general the image will not be all of $\mathbb{P}(\Lambda^d V)$, so the Plücker coordinates restricted to $\mathrm{Gr}_{d,n}$ must satisfy some relations.

By definition, the Plücker coordinates are *i)* linear functions on $\Lambda^d V$ as well as *ii)* multilinear alternating functions on the columns of $M_{n,d}(k)$ (the latter being identified with d-copies of V).

These functions are defined as determinants of maximal submatrices, so they have a third property: *iii)* the Plücker coordinate $p_{\underline{i}}$ is a *multilinear and alternating function in the i_1-th, i_2-th etc. row of $M_{n,d}(k)$.*

Suppose now $\underline{i} \cap \underline{j} = \emptyset$, then the product $p_{\underline{i}} p_{\underline{j}}$ is a quadratic function on $\Lambda^d V$ which is definitely not anymore multilinear in the columns of $M_{n,d}(k)$. But this function is still multilinear in the i_1-th, i_2-th, …, j_1-th, j_2-th etc. row of $M_{n,d}(k)$, and alternating separately in the i_k and j_ℓ. So if we alternate this function so that it becomes alternating in the rows say $i_1, \ldots, i_d, j_1$, then we have an alternating function on $d+1$-copies of a d-dimensional vector space (the space of row vectors of $M_{n,d}(k)$). Hence this function is zero on $M_{n,d}(k)$. Or, in other words, viewed as a quadratic function on $\Lambda^d V$, we have a function such that the restriction to $\mathrm{Im}\,\pi_d$ vanishes.

Example 1.1.13. Before starting with the formal approach consider the example $\mathrm{Gr}_{2,4}$ and the product of Plücker coordinates $p_{1,2}p_{3,4} \in k[\Lambda^2 k^4]$. The composition with $\pi_2 : M_{4,2} \to \Lambda^2 k^4$ gives a function which is of course not anymore multilinear in the columns of $M_{4,2}(k)$, but which is still multilinear in the rows of this space of matrices. We will "formally alternate" this function. For example (we will see below why this is the alternated function)

$$p_{1,2}p_{3,4} + p_{2,3}p_{1,4} - p_{2,4}p_{1,3} \tag{1.3}$$

is a quadratic polynomial on $\Lambda^2 k^4$. The restriction to $\mathrm{Im}\,\pi_2$ is a multilinear function on $M_{4,2}(k)$ which is alternating in the first, the third and the fourth row of $M_{4,2}(k)$. The only function with this property (i.e. being alternating on 3 copies of a 2-dimensional space) is the zero function, so the function above vanishes identically on $\mathrm{Im}\,\pi_2$. But this means that the restriction of the quadratic polynomial in (1.3) to $\mathrm{Gr}_{2,4}$ is identically zero, and hence the Plücker coordinates satisfy on $\mathrm{Gr}_{2,4}$ a quadratic relation.

To formalize this idea, let us start with some generalities. We work inside the ring $k[x_{i,j}]$ of polynomial functions on $M_{n,d}(k)$ and we write just $x_1, \ldots, x_n$ for the vector variables corresponding to the **rows** of $M_{n,d}(k)$. Let $f(x_1, \ldots, x_n)$ be a multilinear function, then we can alternate it by setting:

$$Alt(f) := \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma)({}^{\sigma} f)(x_1, \ldots, x_n),$$

where ${}^{\sigma} f(x_1, \ldots, x_n) = f(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})$.

Suppose $n \geq d+1$. Instead of assuming that the function is multilinear in all vector variables, fix a subset $M = \{k_1, \ldots, k_{d+1}\}$, $1 \leq k_1 \leq \ldots \leq k_{d+1} \leq n$, of pairwise different indices, and assume the function is multilinear in the rows corresponding to the indices $k_1, \ldots, k_{d+1}$. The function

$$Alt_M(f) := \sum_{\sigma \in \mathfrak{S}_{d+1}} \mathrm{sgn}(\sigma) f(\ldots, x_{k_{\sigma^{-1}(1)}}, \ldots, x_{k_{\sigma^{-1}(d+1)}}, \ldots)$$

(i.e. all vector variables different from $x_{k_1}, \ldots, x_{k_{d+1}}$ are not changed) is alternating and multilinear in $x_{k_1}, \ldots, x_{k_{d+1}}$.

For $1 \leq t < d+1$ let $M = M_1 \cup M_2$ be a disjoint decomposition such that $\sharp M_1 = t$. If f is alternating separately in the variables $\{x_k \mid k \in M_1\}$ and $\{x_\ell \mid \ell \in M_2\}$, then

$$\begin{aligned} \mathrm{sgn}(\sigma) f(\ldots, x_{k_{\sigma^{-1}(1)}}, & \ldots, x_{k_{\sigma^{-1}(d+1)}}, \ldots) \\ &= \mathrm{sgn}(\sigma') f(\ldots, x_{k_{\sigma'^{-1}(1)}}, \ldots, x_{k_{\sigma'^{-1}(d+1)}}, \ldots) \end{aligned}$$

whenever σ and σ' are in the same coset in $\mathfrak{S}_{d+1}/\mathfrak{S}_t \times \mathfrak{S}_{d+1-t}$. Here we identify the subgroup $\mathfrak{S}_t \times \mathfrak{S}_{d+1-t}$ with the subgroup of permutations in $\mathfrak{S}_{d+1}$ which separately permute only the elements in M_1 and M_2 among themselves.

So to get an alternating function one has to take the sum

$$Alt_{M_1,M_2}(f) := \sum_{\sigma \in \mathfrak{S}_{d+1}/\mathfrak{S}_t \times \mathfrak{S}_{d+1-t}} \mathrm{sgn}(\sigma) f(\ldots, x_{k_{\sigma^{-1}(1)}}, \ldots, x_{k_{\sigma^{-1}(d+1)}}, \ldots)$$

only over a system of representatives of the cosets.

Example 1.1.14. Suppose $n = 4$ and $d = 2$. Let $f(x_1, x_2, x_3, x_4) = p_{1,2}p_{3,4}$ be the product of these two Plücker coordinates, then f is a multilinear function on $M_{4,2}(k)$, alternating separately in the 1st and 2nd and the 3rd and 4th row. Set $M_1 = \{1\}$, $M_2 = \{3,4\}$ and $M = M_1 \cup M_2$, and denote by $\mathfrak{S}_M$ respectively $\mathfrak{S}_{M_i}$ the permutation groups of the sets. Then

$$id = \begin{pmatrix} 134 \\ 134 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 134 \\ 314 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 134 \\ 341 \end{pmatrix},$$

is a set of representatives of $\mathfrak{S}_M/\mathfrak{S}_{M_1} \times \mathfrak{S}_{M_2}$ (see section 1.1.20 for a procedure to get the representatives) and

$$\begin{aligned} Alt_{M_1,M_2}(f) &= f + \operatorname{sgn}(\sigma_1)({}^{\sigma_1}f) + \operatorname{sgn}(\sigma_2)({}^{\sigma_2}f) \\ &= f(x_1,x_2,x_3,x_4) - f(x_3,x_2,x_1,x_4) + f(x_4,x_2,x_1,x_3) \\ &= p_{1,2}p_{3,4} + p_{2,3}p_{1,4} - p_{2,4}p_{1,3} \end{aligned}$$

is the function on $M_{4,2}(k)$ in equation 1.3, which is alternating in the 1st, 3rd and 4th row.

1.1.15. Quadratic relations. A product $f = p_{\underline{i}}p_{\underline{j}}$ of Plücker coordinates is a quadratic polynomial on $\Lambda^d V$. Suppose now all indices i_k, j_ℓ are different. The product is a function on $M_{n,d}(k)$ which is multilinear with respect to the rows of this space of matrices. Fix $1 \le t < d$, then f is, by construction, alternating separately in the (row) vector variables $x_{i_1}, \dots, x_{i_t}$ and $x_{j_t}, \dots, x_{j_d}$.

Given $\sigma \in \mathfrak{S}_{d+1}$, note that σ shuffles the indicees $i_1, \dots, i_t$ and $j_t, \dots, j_d$. Denote by $\underline{i}^\sigma$ and $\underline{j}^\sigma$ the d-tuples $(\sigma^{-1}(i_1), \dots, \sigma^{-1}(i_t), i_{t+1}, \dots, i_d)$ and $(j_1, \dots, j_{t-1}, \sigma^{-1}(j_t), \dots, \sigma^{-1}(j_d))$. Recall that the function $\operatorname{sgn}(\sigma)({}^\sigma f)$, $\sigma \in \mathfrak{S}_{d+1}/\mathfrak{S}_t \times \mathfrak{S}_{d+1-t}$, is independent of the choice of a representative for σ. The function we get by alternating $f = p_{\underline{i}}p_{\underline{j}}$ is:

$$Alt_{\{i_1,\dots,i_t\},\{j_t,\dots,j_d\}}(p_{\underline{i}}p_{\underline{j}}) = \sum_{\sigma \in \mathfrak{S}_{d+1}/\mathfrak{S}_t \times \mathfrak{S}_{d+1-t}} \operatorname{sgn}(\sigma) p_{\underline{i}^\sigma} p_{\underline{j}^\sigma}.$$

Lemma 1.1.16. *Suppose $n \ge 2d$. Let $\underline{i}, \underline{j}$ be two d-tuples, $1 \le i_k, j_l \le n$, such that the entries are all distinct. Fix $1 \le t < d$, the homogeneous polynomial $Alt_{\{i_1,\dots,i_t\},\{j_t,\dots,j_d\}}(p_{\underline{i}}p_{\underline{j}}) \in k[\Lambda^d V]$ vanishes on $\mathrm{Gr}_{d,n} \subset \mathbb{P}[\Lambda^d V]$.*

Proof. By composing the function with the exterior product map, we see that the quadratic polynomial vanishes on $\mathrm{Gr}_{d,n}$ if and only if, viewed as a sum of products of minors, the function vanishes on $M_{n,d}(k)$. But this function is multilinear and alternating in the $d+1$ row vector variables $x_{i_1}, \dots, x_{i_t}$, $x_{j_t}, \dots, x_{j_d}$. The space of the row vectors is of dimension d, so this function vanishes on $M_{n,d}(k)$. □

To weaken the condition that all indices have to be different, consider two arbitrary d-tuples $\underline{i}$ and $\underline{j}$, $1 \le i_k, j_l \le n$. We will now define a new pair $\underline{i}', \underline{j}'$ such that all entries are different. Set

$$\begin{aligned} i'_k &= i_k + mn \quad \text{where} \quad m = \sharp\{\ell \mid \ell < k, i_k = i_\ell\} \\ j'_k &= j_k + mn \quad \text{where} \quad m = \sharp\{\ell \mid j_k = i_\ell\} + \sharp\{\ell \mid \ell < k, j_k = j_\ell\} \end{aligned}$$

For example, suppose i_1, i_2, j_1, j_2 are pairwise different, then this procedure applied to the pair

$$\begin{gathered} \underline{i} = (i_1, i_2, i_1, i_1, i_2) \;\; \underline{j} = (j_1, i_2, i_1, j_1, j_2) \\ \downarrow \\ \underline{i}' = (i_1, i_2, i_1 + n, i_1 + 2n, i_2 + n) \;\; \underline{j}' = (j_1, i_2 + 2n, i_1 + 3n, j_1 + n, j_2) \end{gathered}$$

provides a new pair $(\underline{i}', \underline{j}')$ such that all entries are different. So we can formally define the quadratic polynomial (in a larger ring with more vector variables)

$$Alt_{\{i'_1,\dots,i'_t\},\{j'_t,\dots,j'_d\}}(p_{\underline{i}'}p_{\underline{j}'}). \tag{1.4}$$

We define the polynomial (which is either zero or a quadratic polynomial)

$$Alt_{(i_1,\dots,i_t),(j_t,\dots,j_d)}(p_{\underline{i}}p_{\underline{j}})$$

now as the function obtained from (1.4) by replacing in the Plücker coordinates all indices i'_k, j'_ℓ by the original indices, i.e. all indices $i'_k, j'_\ell > n$ are replaced by $i'_k \pmod n$ respectively $j'_\ell \pmod n$.

Example 1.1.17. *Suppose $n = 5$, $d = 3$ and $\underline{i} = (2, 1, 5)$ and $\underline{j} = (1, 3, 4)$. Then $\underline{i}' = \underline{i}$ and $\underline{j}' = (6, 3, 4)$. For $t = 1$ we have $M_1 = \{2\}$, $M_2 = \{6, 3, 4\}$ and $M = M_1 \cup M_2$. For the permutation groups we have $\mathfrak{S}_{M_1} \simeq \mathfrak{S}_1$, $\mathfrak{S}_{M_2} \simeq \mathfrak{S}_3$, $\mathfrak{S}_M \simeq \mathfrak{S}_4$ and $\mathfrak{S}_M/(\mathfrak{S}_{M_1} \times \mathfrak{S}_{M_2}) \simeq \mathfrak{S}_4/(\mathfrak{S}_1 \times \mathfrak{S}_3)$. Denote by s_1, s_2, s_3 the simple transpositions of $\mathfrak{S}_4$. The elements $id, s_1, s_2s_1, s_3s_2s_1$ form a system of representatives for the cosets in $\mathfrak{S}_4/\mathfrak{S}_1 \times \mathfrak{S}_3$ and we get*

$$Alt_{\{2\},\{6,3,4\}}(p_{\underline{i}'}p_{\underline{j}'}) = p_{2,1,5}p_{6,3,4} - p_{6,1,5}p_{2,3,4} + p_{3,1,5}p_{2,6,4} - p_{4,1,5}p_{2,6,3}$$

After specializing (i.e. replacing 6 back by 1) we get

$$\begin{aligned} Alt_{(2),(1,3,4)}(p_{\underline{i}}p_{\underline{j}}) &= p_{2,1,5}p_{1,3,4} - p_{1,1,5}p_{2,3,4} + p_{3,1,5}p_{2,1,4} - p_{4,1,5}p_{2,1,3} \\ &= -p_{1,2,5}p_{1,3,4} + p_{1,3,5}p_{1,2,4} - p_{1,4,5}p_{1,2,3} \end{aligned}$$

Theorem 1.1.18. *Let $\underline{i}$ and $\underline{j}$, $1 \le i_k, j_l \le n$, be two arbitrary d-tuples. For all $1 \le t < d$, the polynomial $Alt_{(i_1,\dots,i_t),(j_t\dots,j_d)}(p_{\underline{i}}p_{\underline{j}})$ vanishes on the Graßmann variety $\mathrm{Gr}_{d,n}$.*

Proof. Suppose the polynomial is different from zero. As above, by composing the function with the exterior product map, one sees that this quadratic polynomial vanishes on $\mathrm{Gr}_{d,n}$ if and only if, viewed as a sum of products of minors, the function vanishes on $M_{n,d}(k)$. If the entries in $\underline{i}$ and $\underline{j}$ are all different, then this is Lemma 1.1.16. Otherwise consider first the multilinear function $Alt_{(i'_1,\dots,i'_t),(j'_t\dots,j'_d)}(p_{\underline{i}'}p_{\underline{j}'})$ defined in (1.4), this function is defined on the space $M_{2dn,d}(k)$ of $2dn \times d$ matrices, and vanishes identically since it is multilinear and alternating in $d+1$ of the vector variables.

The original space $M_{n,d}(k)$ can be seen as a subspace of $M_{2dn,d}(k)$ by identifying a $n \times d$-matrix A with the $2dn \times d$-matrix obtained by putting $2d$ copies of A on the top of each other. By construction we have then

$$Alt_{(i_1,\dots,i_t),(j_t\dots,j_d)}(p_{\underline{i}}p_{\underline{j}}) = Alt_{(i'_1,\dots,i'_t),(j'_t\dots,j'_d)}(p_{\underline{i}'}p_{\underline{j}'})|_{M_{n,d}(k)} \equiv 0.$$

□

Definition 1.1.19. If $Alt_{(i_1,\dots,i_t),(j_t\dots,j_d)}(p_{\underline{i}}p_{\underline{j}})$ is not the zero polynomial in $k[\Lambda^d V]$, then this quadratic polynomial is called a *shuffle relation* or a *Plücker relation.*

1.1.20. Shuffles. We will describe how to obtain shuffles or coset representatives. Fix $1 \leq t \leq d$, we want to describe a special set of coset representatives of $\mathfrak{S}_{d+1}/\mathfrak{S}_t \times \mathfrak{S}_{d+1-t}$. Let $\mathfrak{S}_{d+1}$ act on the set $\{1,\dots,d+1\}$. Then a coset $\overline{\sigma} \in \mathfrak{S}_{d+1}/\mathfrak{S}_t \times \mathfrak{S}_{d+1-t}$ is identified by the relative position of the first t and the second $d+1-t$ elements. Expressed in a pictorial way: suppose we are given a configuration of t-balls and $d+1-t$ triangles:

$$\bigcirc \bigcirc \triangle \bigcirc \triangle\triangle\triangle \bigcirc \dots$$

If we fill the balls with any permutation of $\{1,\dots,t\}$ and the triangles with any permutation of $\{t+1,\dots,d+1\}$, we always get a permutation which is an element of the same coset.

A canonical representative of such a coset is hence obtained by putting $1,2,\dots,t$ in order in the balls and $t+1,\dots,d+1$ in order in the triangles. Such a representative is called a t*-shuffle.*

Example 1.1.21. *To determine the set of all* 2*-shuffles in* $\mathfrak{S}_4$ *consider first the set of all configuration of* 2*-balls and* 2 *triangles:*

$$\bigcirc\bigcirc\triangle\triangle,\ \bigcirc\triangle\bigcirc\triangle,\ \triangle\bigcirc\bigcirc\triangle,\ \bigcirc\triangle\triangle\bigcirc,\ \triangle\bigcirc\triangle\bigcirc,\ \triangle\triangle\bigcirc\bigcirc.$$

The 2*-shuffles and the decomposition of the inverse are given by:*

$$\begin{array}{cccccccc} \sigma & = & \begin{pmatrix}1234\\1234\end{pmatrix}, & \begin{pmatrix}1234\\1324\end{pmatrix}, & \begin{pmatrix}1234\\3124\end{pmatrix}, & \begin{pmatrix}1234\\1342\end{pmatrix}, & \begin{pmatrix}1234\\3142\end{pmatrix}, & \begin{pmatrix}1234\\3412\end{pmatrix} \\ \sigma^{-1} & = & id, & s_2, & s_2s_1, & s_2s_3, & s_2s_1s_3, & s_2s_1s_3s_2 \end{array}$$

Example 1.1.22. *Let* $n=5$, $d=3$, $\underline{i}=(2,3,4), \underline{j}=(1,4,5)$, $t=2$. *By the example above we have*

$$\begin{aligned} Alt_{(2,3)(4,5)}p_{\underline{i}}p_{\underline{j}} &= p_{2,3,4}p_{1,4,5} - p_{2,4,4}p_{1,3,5} + p_{3,4,4}p_{1,2,5} \\ &\qquad + p_{2,5,4}p_{1,3,4} - p_{3,5,4}p_{1,2,4} + p_{4,5,4}p_{1,2,3} \\ &= p_{2,3,4}p_{1,4,5} - p_{2,4,5}p_{1,3,4} + p_{3,4,5}p_{1,2,4} \end{aligned}$$

1.1.23. Closed embedding. Next we will see that one can identify $\mathrm{Gr}_{d,n}$ with is a closed subset of $\mathbb{P}(\Lambda^d V)$, i.e. the Graßmann variety is naturally endowed with the structure of a projective variety.

Theorem 1.1.24. *The Graßmann variety* $\mathrm{Gr}_{d,n} \subset \mathbb{P}(\Lambda^d V)$ *is the zero set of the homogeneous ideal generated by the following polynomials:*

$$\sum_{l=1}^{d+1} (-1)^l p_{i_1,\dots,\widehat{i_l},\dots,i_{d+1}} p_{j_1,\dots,j_{d-1},i_l}, \tag{1.5}$$

where $i_1,\dots,i_{d+1}$ *and* $j_1,\dots,j_{d-1}$ *are any numbers between* 1 *and* n.

Proof. The relation in (1.5) is a special case of the shuffle relations (see Theorem 1.1.18, $t = d$), so $\mathrm{Gr}_{d,n}$ is contained in the zero set of the homogeneous ideal generated by these equations.

Conversely, let $y = [\sum_{\underline{i}\in I_{d,n}} y_{\underline{i}} e_{\underline{i}}]$ satisfy the equations in (1.5). Suppose $y_{l_1,\dots,l_d} \neq 0$ for some $\underline{\ell} = (l_1,\dots,l_d) \in I_{d,n}$, without loss of generality we may (and will) assume $y_{l_1,\dots,l_d} = 1$. For $1 \le i \le n$, $1 \le j \le d$, set

$$a_{ij} = y_{l_1,\dots,l_{j-1},i,l_{j+1},\dots,l_d}.$$

We apply the usual rules as in (1.2): $y_{l_1,\dots,l_{j-1},i,l_{j+1},\dots,l_d}$ is zero if two indices are equal etc. Let A be the $n \times d$ matrix $A = (a_{ij})$. By construction $A_{l_1,\dots,l_d} = \mathrm{I}_d$ because $a_{l_j,j} = y_{l_1,\dots,l_d} = 1$ for $j = 1,\dots,d$ and for $i \neq j$ we have $a_{l_j,i} = y_{l_1,\dots,l_{i-1},l_j,l_{i+1},\dots,l_d} = 0$. Clearly $\operatorname{rank} A = d$, let U be the d-dimensional subspace spanned by the columns of A. We have to show that $\pi(U) = [\sum_{\underline{i}\in I_{d,n}} p_{\underline{i}}(A) e_{\underline{i}}] = [\sum_{\underline{i}\in I_{d,n}} y_{\underline{i}} e_{\underline{i}}] = y$ and hence $y \in \mathrm{Gr}_{d,n}$.

For two d-tuples $\underline{k}, \underline{k}'$ denote by $\sharp\{\underline{k}\cap\underline{k}'\}$ the number of common entries. We will show $p_{\underline{j}}(A) = y_{\underline{j}}$ by decreasing induction on $\sharp\{\underline{\ell}\cap\underline{j}\}$. We know already that $p_{\underline{\ell}}(A) = 1 = y_{\underline{\ell}}$. For $\underline{j} = (l_1,\dots,l_{j-1},i,l_{j+1},\dots,l_d)$ we have $p_{\underline{j}}(A) = a_{i,j} = y_{\underline{j}}$ by the definition of A, so this proves the claim if $\sharp\{\underline{\ell}\cap\underline{j}\} \ge d-1$.

Let $\underline{j}$ be arbitrary such that $\sharp\{\underline{\ell}\cap\underline{j}\} < d-1$. There exists an entry in $\underline{j}$ which is not an entry in $\underline{\ell}$. Without loss of generality (i.e., after permuting the entries if necessary) we assume that j_d has this property. Now y satisfies all the relations in (1.5), so the coordinates $y_{\underline{\ell}}$ and $y_{\underline{j}}$ satisfy a relation of the form above: $y_{\underline{\ell}} y_{\underline{j}} + \sum \pm y_{\underline{\ell}'} y_{\underline{j}'} = 0$, where $\underline{\ell}'$ differs from $\underline{\ell}$ in just one place. Further, if $y_{\underline{\ell}'} y_{\underline{j}'} \neq 0$, then $\sharp\{\underline{j}'\cap\underline{\ell}\} > \sharp\{\underline{j}\cap\underline{\ell}\}$ since j_d has been replaced by an element in $\underline{\ell}$. Thus we know by induction $y_{\underline{\ell}'} = p_{\underline{\ell}'}(A)$, $y_{\underline{j}'} = p_{\underline{j}'}(A)$.

By Theorem 1.1.18, the d-minors of A satisfy the relations in (1.5), so $p_{\underline{\ell}}(A) p_{\underline{j}}(A) + \sum \pm p_{\underline{\ell}'}(A) p_{\underline{j}'}(A) = 0$. Now $p_{\underline{\ell}}(A) = y_{\underline{\ell}} = 1$, so $p_{\underline{j}}(A) = -\sum \pm p_{\underline{\ell}'}(A) p_{\underline{j}'}(A) = -\sum \pm y_{\underline{\ell}'} y_{\underline{j}'} = y_{\underline{j}}$. □

1.2 Monomials and tableaux

Let $I(\mathrm{Gr}_{d,n}) = \{f \in k[\Lambda^d V] \mid f|_{\mathrm{Gr}_{d,n}} \equiv 0\}$ be the homogeneous vanishing ideal of the Grassmann variety and denote by

$$k[\mathrm{Gr}_{d,n}] = k[\Lambda^d V]/I(\mathrm{Gr}_{d,n})$$

the homogeneous coordinate ring of this projective variety. The homogeneous coordinate ring of the projective space $\mathbb{P}(\Lambda^d V)$ is the polynomial ring $k[\Lambda^d V] = k[p_{\underline{i}} \mid \underline{i} \in I_{d,n}]$. This ring has a k-basis the monomials in the Plücker coordinates. Our aim is to find a subset of special monomials, the "standard monomials", such that the classes of these monomials form a k-basis of $k[\mathrm{Gr}_{d,n}]$. We prepare in this section the necessary combinatorial background.

To have a "normal form" for the monomials in $k[\Lambda^d V]$, we fix a total order $\succeq$ on the set $I_{d,n}$:

$$\underline{i} \succ \underline{j} \text{ if and only if } \exists\, 1 \leq t \leq d : i_1 = j_1, \ldots, i_{t-1} = j_{t-1}, i_t > j_t.$$

It follows that the ordered or *weakly standard* monomials

$$p_{\underline{i}} p_{\underline{j}} \cdots p_{\underline{k}} \text{ such that } \underline{i} \succeq \underline{j} \succeq \ldots \succeq \underline{k}$$

form a k-basis for the ring $k[\Lambda^d V]$. Note that $\underline{i} \geq \underline{j}$ (see Definition 1.1.5) implies $\underline{i} \succeq \underline{j}$.

Definition 1.2.1. Let $p_{\underline{i}} \cdots p_{\underline{k}}$ be a weakly standard monomial (i.e. $\underline{i} \succeq \ldots \succeq \underline{k}$) in the Plücker coordinates. The monomial is called a *standard monomial* if $\underline{i} \geq \ldots \geq \underline{k}$.

It is often convenient to use the language of Young tableaux for tuples of elements in $I_{d,n}$. Expressed in a pictorial way, a tableau is a filling of a Young diagram.

Definition 1.2.2. A *Young diagram* of *shape* $m^d = (m, \ldots, m)$ is a sequence of d left adjusted rows of boxes, each row of length m, and a Young tableau is a filling of the boxes with the numbers $\{1, \ldots, n\}$. Such a filling is called *column standard* if the entries in the columns are strictly increasing (top to bottom).

Unless stated otherwise, all tableaux we consider are column standard. So by abuse of notation we write in the following just Young tableau for a column standard Young tableau.

Let $\mathcal{T}$ be a Young tableau of shape m^d. The entries in each column, read from top to the bottom, define an element in $I_{d,n}$. So we can view a tableau $\mathcal{T}$ of shape m^d also as an m-tuple of elements in $I_{d,n}$. Since the world of combinatorics and the world of commutative algebra do not always live in harmony, note a slight slip in the notation. Tuples are to be read from the left to the right, filling of diagrams are to be read from the right to the left. So the tuple associated to the Young tableau of shape 6^2:

$$\mathcal{T} = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 3 & 3 & 4 \\ \hline 2 & 5 & 4 & 4 & 5 & 5 \\ \hline \end{array} \qquad (1.6)$$

is $((4,5),(3,5),(3,4),(2,4),(1,5),(1,2))$. The monomial $p_{\mathcal{T}}$ associated to a tableau is the product of the Plücker coordinates corresponding to the columns of the tableau. For the example above we have

$$p_{\mathcal{T}} = p_{4,5} p_{3,5} p_{3,4} p_{2,4} p_{1,5} p_{1,2}.$$

Definition 1.2.3. Let $\mathcal{T}$ be a Young tableau of *shape* m^d and let $(\underline{i}, \underline{j}, \dots, \underline{k})$ be the corresponding m-tuple of elements in $I_{d,n}$. The tableau is called *weakly standard* if $\underline{i} \succeq \underline{j} \succeq \dots \succeq \underline{k}$, and the tableau is called *semi-standard* if $\underline{i} \geq \underline{j} \geq \dots \geq \underline{k}$. In terms of the entries of the diagram, this means the tableau is semi-standard if and only if the entries in the boxes are strictly increasing in the columns and weakly increasing (left to right) in the rows.

In this language we see that

$$\{p_{\mathcal{T}} \mid \mathcal{T} \text{ Young tableau, shape } m^d, \text{ weakly standard}\}$$

is the set of all weakly standard monomials of degree m, and

$$\{p_{\mathcal{T}} \mid \mathcal{T} \text{ Young tableau, shape } m^d, \text{ semi-standard}\}$$

is the set of all semi-standard monomials of degree m.

Remark 1.2.4. The odd looking correspondence of names *semi-standard tableaux* $\leftrightarrow$ *standard monomials* comes from the fact that the name standard tableau is already reserved for a class of Young tableau occuring in the representation theory of the symmetric group.

We denote by $\succ_l$ the induced homogeneous lexicographic ordering on the weakly standard tableaux, i.e. if $\mathcal{T} = (\underline{i}^1, \underline{i}^2, \dots, \underline{i}^r)$ and $\mathcal{T}' = (\underline{j}^1, \underline{j}^2, \dots, \underline{j}^s)$ are weakly standard tableaux, then we say $\mathcal{T} \succ_l \mathcal{T}'$ if $r > s$ or if $r = s$, and $\underline{i}^k \succ \underline{j}^k$ for the first index k such that $\underline{i}^k \neq \underline{j}^k$.

We define an induced ordering on the weakly standard monomials by $p_{\mathcal{T}} \succ_l p_{\mathcal{T}'}$ if $\mathcal{T} \succ_l \mathcal{T}'$. Note that $\succ_l$ defines a *monomial ordering* on $k[\Lambda^d k^n]$, i.e. if $p_{\mathcal{T}} \succ_l p_{\mathcal{T}'}$, then we have for weakly standard tableaux $\mathcal{T}, \mathcal{T}', \mathcal{T}''$:

$$p_{\mathcal{T}''} p_{\mathcal{T}} \succ_l p_{\mathcal{T}''} p_{\mathcal{T}'} \succ_l p_{\mathcal{T}'}.$$

1.3 Straightening relation

Our aim is to show that the standard monomials form a basis of $k[\mathrm{Gr}_{d,n}] = k[\Lambda^d V]/I(\mathrm{Gr}_{d,n})$. As a first step we show that the Plücker relations imply straightening relations, i.e. provide an algorithm to "straighten out" non-standard monomials as a sum of standard monomials.

Proposition 1.3.1. *The images of the standard monomials in* $k[\mathrm{Gr}_{d,n}]$ *span the homogeneous coordinate ring.*

The proof uses the Plücker relations described in Theorem 1.1.18. We start with some preliminaries. Let $\underline{i}, \underline{j} \in I_{d,n}$ be such that $\underline{i} \succ \underline{j}$ but $\underline{i} \not\geq \underline{j}$, then there exists $1 \leq t \leq d$ such that $i_r \geq j_r,\ 1 \leq r \leq t-1$, and $i_t < j_t$. Consider the shuffle relation

$$Alt_{(i_1,\dots,i_t),(j_t\dots,j_d)}(p_{\underline{i}} p_{\underline{j}}) = \sum_{\sigma \in \mathfrak{S}_{d+1}/\mathfrak{S}_t \times \mathfrak{S}_{d+1-t}} \mathrm{sgn}(\sigma) p_{\underline{i}^\sigma} p_{\underline{j}^\sigma}. \tag{1.7}$$

Denote by $\underline{i}^\sigma\!\uparrow$ the tuple obtained from $\underline{i}^\sigma$ by writing the entries in ascending order.

Lemma 1.3.2. *If* $\sigma \neq \mathrm{id}$*, then* $\underline{i}^\sigma\!\uparrow \succ \underline{i}, \underline{j} \succ \underline{j}^\sigma\!\uparrow$.

Proof. If $\sigma \neq \mathrm{id}$, then the shuffle replaces some elements in $\{i_1, \ldots, i_t\}$ by some elements in $\{j_t, \ldots, j_d\}$. Now by the choice of t we know $j_t, \ldots, j_d > i_1, \ldots, i_t$. So after reordering the elements we get for $\underline{i}^\sigma\!\uparrow = (i'_1, \ldots, i'_d)$: $i'_1 \geq i_1, \ldots, i'_t \geq i_t$, and for at least one $1 \leq \ell \leq t$ we have $i'_\ell > i_\ell$. It follows: $\underline{i}^\sigma\!\uparrow \succ \underline{i} \succ \underline{j}$. The proof for $\underline{j}^\sigma\!\uparrow$ is similar. □

Proposition 1.3.3 (Straightening relation)**.** *If $p_{\underline{i}} p_{\underline{j}}$ is weakly standard but not standard, then*

$$p_{\underline{i}} p_{\underline{j}} \equiv \sum_{\substack{\underline{i}' > \underline{j}' \\ \underline{i}' \succ \underline{i}, \underline{j} \succ \underline{j}'}} a_{\underline{j}', \underline{i}'} p_{\underline{i}'} p_{\underline{j}'} \mod I(\mathrm{Gr}_{d,n}) \tag{1.8}$$

In other words: the image of $p_{\underline{i}} p_{\underline{j}}$ in $k[\mathrm{Gr}_{d,n}]$ is a linear combination of the images of standard monomials.

A relation as in (1.8) is called a *straightening relation*.

Proof. If the product is weakly standard but not standard, then let $1 \leq t \leq d$ be as above. The polynomial $Alt_{(i_1,\ldots,i_t),(j_t\ldots,j_d)}(p_{\underline{i}} p_{\underline{j}})$ vanishes on $\mathrm{Gr}_{d,n}$, so, modulo $I(\mathrm{Gr}_{d,n})$, by (1.7) we can write $p_{\underline{i}} p_{\underline{j}}$ as a linear combination of products $p_{\underline{i}^\sigma\uparrow} p_{\underline{j}^\sigma\uparrow}$, $\sigma \neq \mathrm{id}$. These products are again weakly standard by Lemma 1.3.2, and $\mathcal{T}' = (\underline{i}^\sigma\!\uparrow, \underline{j}^\sigma\!\uparrow) \succ_l \mathcal{T} = (\underline{i}, \underline{j})$.

If one of the $\mathcal{T}'$ is not semi-standard, then we can repeat the *straightening procedure* and express $p_{\mathcal{T}'}$ (modulo $I(\mathrm{Gr}_{d,n})$) as a linear combination of $p_{\mathcal{T}''}$, where $\mathcal{T}'' \succ_l \mathcal{T}' \succ_l \mathcal{T}$. The number of weakly standard tableaux of shape 2^d is finite, so this process has to end after a finite number of steps. As a consequence we see that we can express $p_{\mathcal{T}} = p_{\underline{i}} p_{\underline{j}}$ as linear combination of standard monomials $p_{\mathcal{T}'}$ (modulo $I(\mathrm{Gr}_{d,n})$) such that $\underline{i}' \succ \underline{i}, \underline{j} \succ \underline{j}'$ for $\mathcal{T}' = (\underline{i}', \underline{j}')$. □

Proof of Proposition 1.3.1. The homogeneous coordinate ring is spanned by the classes of the weakly standard monomials. If a monomial $p_{\mathcal{T}}$ is weakly standard but not standard, then we can find a pair of columns in the corresponding tableau satisfying the conditions of Proposition 1.3.3. Since we have a monomial order, it follows that, modulo $I(\mathrm{Gr}_{d,n})$, we can rewrite the monomial as a linear combination of monomials $p_{\mathcal{T}'}$ where $\mathcal{T}'$ and $\mathcal{T}$ are of the same shape and $\mathcal{T}' \succ \mathcal{T}$. If one of the $\mathcal{T}'$ is not standard, then we can repeat the procedure. But the number of weakly standard Young tableaux of a fixed shape is finite, so the algorithm has to end and expresses $p_{\mathcal{T}}$, modulo $I(\mathrm{Gr}_{d,n})$, as a linear combination of standard monomials, which finishes the proof. □

1.4 Schubert Varieties in $\mathrm{Gr}_{d,n}$

For $1 \leq t \leq n$, let F_t be the standard t-dimensional subspace of V spanned by $\{e_1, \dots, e_t\}$. We denote by $\mathcal{F}$ the complete flag of subspaces

$$\mathcal{F}: \ F_0 = 0 \subset F_1 \subset F_2 \subset \ldots \subset F_{n-1} \subset F_n = V.$$

For a d-dimensional subspace $W \in \mathrm{Gr}_{d,n}$ of V consider the intersections of the subspace with the flag:

$$0 \subseteq (F_1 \cap W) \subseteq (F_2 \cap W) \subseteq \ldots \subseteq (F_{n-1} \cap W) \subseteq W = W \cap F_n.$$

The tuple $(i_1, \dots, i_d)$ given by the indices where the dimension jumps (i.e. $\dim(F_{i_j} \cap W) = j = \dim(F_{i_j - 1} \cap W) + 1)$ is an element in $I_{d,n}$. For $\underline{i} \in I_{d,n}$ let $C_{\underline{i}}$ be defined by

$$C_{\underline{i}} = \left\{ W \in \mathrm{Gr}_{d,n} \mid \forall\, 1 \leq t \leq d : \begin{array}{l} \dim(W \cap F_{i_t}) = t \text{ and} \\ \dim(W \cap F_\ell) < t \text{ for all } \ell < i_t \end{array} \right\}.$$

Definition 1.4.1. The subset $C_{\underline{i}} \subset \mathrm{Gr}_{d,n}$ is called the *Schubert cell* $C_{\underline{i}}$ associated to $\underline{i}$.

The Graßmann variety is the disjoint union of the Schubert cells. The *Schubert variety* associated to $\underline{i}$ is defined to be

$$X_{\underline{i}} = \{ W \in \mathrm{Gr}_{d,n} \mid \dim(W \cap F_{i_t}) \geq t \ , \ 1 \leq t \leq d \}.$$

We have obviously $C_{\underline{i}} \subset X_{\underline{i}}$.

As in the case of the Graßmannian, there are other descriptions of Schubert varieties and Schubert cells. Let $M_{\underline{i}} \subset M_{n,d}(k)$ be the set of matrices $A = (a_{i,i})$ such that $a_{i,j} = 0$ for $i > i_j$ and $a_{i_k,\ell} = \delta_{k,\ell}$ (so the submatrix $A_{\underline{i}}$ is the $d \times d$ identity matrix).

Example 1.4.2. For $n = 8$, $d = 4$ and $\underline{i} = (2, 3, 5, 7)$ the matrices in $M_{\underline{i}} \subset M_{n,d}(k)$ are of the form

$$\begin{pmatrix} * & * & * & * \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Lemma 1.4.3. *The exterior product map* $\pi_d : (v_1, \dots, v_d) \mapsto [v_1 \wedge \ldots \wedge v_d]$ *induces a bijection* $\pi_d : M_{\underline{i}} \to C_{\underline{i}}$.

Proof. Let $W \in C_{\underline{i}}$. By choosing a basis $\{v_1\} \subset W \cap F_{i_1}$, $\{v_1, v_2\} \subset W \cap F_{i_2}$ etc., we can fix a basis $\{v_1, \ldots, v_d\}$ of W such that $v_k = e_{i_k} + \sum_{\ell < i_k} a_{\ell,k} e_\ell$, denote $A' = (a'_{p,q})$ the corresponding matrix. The submatrix $A_{\underline{i}}$ is upper triangular and unipotent. By replacing A' by $A = A'(A_{\underline{i}})^{-1}$ we get a matrix in $M_{\underline{i}}$ such that the columns span W. On the other hand, a subspace spanned by the column vectors of a matrix in $M_{\underline{i}}$ satisfies the intersection conditions defining $C_{\underline{i}}$, so the map $\pi_d : M_{\underline{i}} \to C_{\underline{i}}$, $(v_1, \ldots, v_d) \mapsto [v_1 \wedge \ldots \wedge v_d]$, is well defined and surjective. The coefficients $a_{i,j}$ of the matrix A can be obtained (see also proof of Theorem 1.1.24) back from $W = [v_1 \wedge \ldots \wedge v_d]$: $a_{i,j} = p_{\underline{i}'}(W)/p_{\underline{i}}(W)$, where $\underline{i}' = (i_1, \ldots, i_{j-1}, i, i_{j+1}, \ldots, i_d)$, so the map is bijective. □

The subspaces F_t are stable under the subgroups B and U, and hence $X_{\underline{i}}$ as well as $C_{\underline{i}}$ are stable under the induced action of U, T and $B = TU = UT$. The points $[e_{\underline{i}}]$ can be characterized as follows: the T-fixed points in $\mathbb{P}(\Lambda^d V)$ correspond to the T-stable lines in $\Lambda^d V$. Now as T-module we have $\Lambda^d V \simeq \bigoplus_{\underline{i} \in I_{d,n}} k e_{\underline{i}}$, and every one-dimensional T-submodule is of the form $k e_{\underline{i}}$ for some $\underline{i}$. So the only T-fixed points in $\mathbb{P}(\Lambda^d V)$ are the $[e_{\underline{i}}] \in \mathrm{Gr}_{d,n}$.

Lemma 1.4.4. $C_{\underline{i}} = B \cdot [e_{\underline{i}}] = U \cdot [e_{\underline{i}}]$, *and* $\mathrm{Gr}_{d,n} = \dot{\bigcup}_{\underline{i} \in I_{d,n}} B \cdot [e_{\underline{i}}]$.

Proof. For $W \in C_{\underline{i}}$ let $A \in M_{\underline{i}}$ be the corresponding matrix. By standard linear algebra arguments, we can find an upper triangular unipotent matrix $u \in U$ such that $uA = E_{\underline{i}} = (e_{p,q})$ is the matrix with entries $e_{i_k,k} = 1$ for $k = 1, \ldots, d$, and all other entries equal to zero. Since the exterior product map is equivariant with respect to the $SL_n(k)$-action, this means $u.W = [e_{\underline{i}}]$. It follows that $C_{\underline{i}} = U \cdot [e_{\underline{i}}]$, and, since $[e_{\underline{i}}]$ is a T-fixed point, $C_{\underline{i}} = B \cdot [e_{\underline{i}}]$. The rest is immediate. □

A Schubert variety is in fact a projective variety, for example it can be seen as the intersection of $\mathrm{Gr}_{d,n}$ with a finite number of hyperplanes in $\mathbb{P}(\Lambda^d V)$:

Lemma 1.4.5. $X_{\underline{i}} = \{W \in \mathrm{Gr}_{d,n} \mid p_{\underline{j}}(W) = 0 \ \forall \underline{j} \in I_{d,n}$ *such that* $\underline{j} \not\leq \underline{i}\}$, $C_{\underline{j}} \subseteq X_{\underline{i}}$ *if and only if* $\underline{j} \leq \underline{i}$, *and* $C_{\underline{j}} \cap X_{\underline{i}} = \emptyset$ *if and only if* $\underline{j} \not\leq \underline{i}$.

Proof. Let $W \in C_{\underline{j}}$, by definition $\dim(W \cap F_{j_t}) = t$ and $\dim(W \cap F_\ell) < t$ for $\ell < j_t$. By the definition of $X_{\underline{i}}$ it follows that $W \in X_{\underline{i}}$ if and only if $j_1 \leq i_1$, $j_2 \leq i_2$, etc., or, in other words, $\underline{i} \geq \underline{j}$. Hence: $C_{\underline{j}} \subseteq X_{\underline{i}}$ if and only if $\underline{j} \leq \underline{i}$, and the intersection $C_{\underline{j}} \cap X_{\underline{i}}$ is empty otherwise.

Suppose now $W \in C_{\underline{j}} \subseteq X_{\underline{i}}$, let $A \in M_{\underline{j}}$ be the corresponding matrix and let $v_1, \ldots, v_d$ be the column vectors. Since $v_k = e_{j_k} + \sum_{\ell < j_k} a_{\ell,k} e_\ell$, we have in $\mathbb{P}(\Lambda^d V)$

$$W = [v_1 \wedge \ldots \wedge v_d] = [e_{\underline{j}} + \sum_{\underline{j}' < \underline{j}} a_{\underline{j}'} e_{\underline{j}'}]$$

In particular, $p_{\underline{k}}(W) = 0$ for all $\underline{k} \not\leq \underline{j}$. Next suppose $W' \in \mathrm{Gr}_{d,n}$ is such that $p_{\underline{k}}(W') = 0$ for all $\underline{k} \not\leq \underline{i}$. Now $W' \in C_{\underline{j}}$ for some $\underline{j} \in I_{d,n}$. Since $p_{\underline{j}}(e_{\underline{j}}) \neq 0$ by the above, we have $\underline{j} \leq \underline{i}$ and hence $W' \in C_{\underline{j}} \subset X_{\underline{i}}$. □

Proposition 1.4.6. *The Schubert variety $X_{\underline{i}}$ is the Zariski closure of the Schubert cell $C_{\underline{i}}$, and $X_{\underline{j}} \subseteq X_{\underline{i}}$ if and only if $\underline{j} \leq \underline{i}$.*

Proof. The second part follows from the first and Lemma 1.4.5. Suppose first that $\underline{j} < \underline{i}$ is such that $j_k = i_k - 1$ for some k and all other entries are equal. Let $E_{i,j}$ be the matrix having 1 as entry in the i-th row and j-th column, and having entry zero everywhere else. Let I be the identity matrix, then

$$(I + tE_{i_k-1,i_k}).[e_{\underline{i}}] = [e_{\underline{i}} + te_{\underline{j}}] = [t^{-1}e_{\underline{i}} + e_{\underline{j}}] \in C_{\underline{i}}$$

for all $t \in k^*$. It follows: $\lim_{t\to\infty}[t^{-1}e_{\underline{i}} + e_{\underline{j}}] = [e_{\underline{j}}]$ and hence $[e_{\underline{j}}] \in \overline{C_{\underline{i}}}$. The cell is B-stable and hence so is the Zariski closure. It follows $\overline{C}_{\underline{j}} \subset \overline{C_{\underline{i}}}$.

Let now $\underline{i} > \underline{j}$ be arbitrary and let k be minimal such that $i_k > j_k$. Let $\underline{i}' \in I_{d,n}$ be obtained from $\underline{i}$ by replacing the entry i_k by $i_k - 1$, then $\underline{i} > \underline{i}' \geq \underline{j}$. Since $\overline{C}_{\underline{i}'} \subset \overline{C_{\underline{i}}}$, to prove $C_{\underline{j}} \subset \overline{C_{\underline{i}}}$ it suffices to prove $C_{\underline{j}} \subset \overline{C_{\underline{i}'}}$. But this follows now by decreasing induction (repeating the procedure above).

It follows $\bigcup_{\underline{j}\leq\underline{i}} C_{\underline{j}} \subseteq \overline{C_{\underline{i}}} \subseteq X_{\underline{i}}$, and hence $\overline{C_{\underline{i}}} = X_{\underline{i}}$ by Lemma 1.4.5. □

Proposition 1.4.7. *$C_{\underline{i}}$ is an affine cell of dimension $d(\underline{i}) = \sum_{1\leq t\leq d} i_t - t$ and $X_{\underline{i}}$ is an irreducible projective variety of dimension $d(\underline{i})$.*

Proof. The Schubert variety is the Zariski closure of the Schubert cell, so the first part implies the second. Let $\mathbb{P}(\Lambda^d V)_{p_{\underline{i}}}$ be the affine open set

$$\mathbb{P}(\Lambda^d V)_{p_{\underline{i}}} = \{[w] \in \mathbb{P}(\Lambda^d V) \mid p_{\underline{i}}([w]) \neq 0\},$$

then $C_{\underline{i}}$ is the affine variety $X_{\underline{i}} \cap \mathbb{P}(\Lambda^d V)_{p_{\underline{i}}}$. The map $\pi_d : M_{\underline{i}} \to C_{\underline{i}}$ and its inverse defined in the proof of Lemma 1.4.3 are morphisms of affine varieties, and hence define an isomorphism between $C_{\underline{i}}$ and the affine space $M_{\underline{i}}$ of dimension $d(\underline{i}) = \sum_{1\leq t\leq d} i_t - t$. □

Lemma 1.4.8. *Let $X_{\underline{i}}$, $X_{\underline{j}}$ be two Schubert varieties in $\mathrm{Gr}_{d,n}$. Then $X_{\underline{i}} \cap X_{\underline{j}}$ is irreducible, i.e. $X_{\underline{i}} \cap X_{\underline{j}}$ is a Schubert variety (set-theoretically).*

Proof. The intersection is closed and B-stable, so it has to be a union of Schubert varieties. Set $k_\ell = \min\{i_\ell, j_\ell\}$, $1 \leq \ell \leq d$, and $\underline{k} = (k_1, \ldots, k_d)$. Then $\underline{k} \in I_{d,n}$, and if $X_{\underline{k}'} \subset X_{\underline{i}} \cap X_{\underline{j}}$, then $\underline{k}' \leq \underline{i}, \underline{j}$ and hence $\underline{k}' \leq \underline{k}$. It follows that $X_{\underline{i}} \cap X_{\underline{j}} = X_{\underline{k}}$. □

Lemma 1.4.9. *Let $X_{\underline{i}}$, $X_{\underline{j}}$ be two distinct Schubert divisors in a Schubert variety $X_{\underline{k}}$ in $\mathrm{Gr}_{d,n}$, i.e. $X_{\underline{i}}$, $X_{\underline{j}}$ are Schubert varieties of codimension 1 in $X_{\underline{k}}$. Then the intersection is a Schubert variety of codimension 2 in $X_{\underline{k}}$.*

Proof. By Lemma 1.4.8 we know that the intersection is the Schubert variety $X_{\underline{k}'}$, where $k'_\ell = \min\{i_\ell, j_\ell\}$. Since $\underline{k} > \underline{i}$, by the dimension fomula in Proposition 1.4.7 we know that $\underline{k}$ and $\underline{i}$ are the same but for one place where the entry in $\underline{i}$ is one less than the corresponding entry in $\underline{k}$, and the same holds for $\underline{j}$. It follows that $\underline{k}$ and $\underline{k}'$ are the same but for two places where the entries in $\underline{k}'$ are one less than the corresponding entries in $\underline{k}$, and hence $X_{\underline{k}'}$ is of codimension 2 in $X_{\underline{k}}$. □

1.5 SMT for Schubert varieties in the Graßmannian

Let $R = k[\mathrm{Gr}_{d,n}] = k[\Lambda^d V]/I(\mathrm{Gr}_{d,n})$ be the homogeneous coordinate ring of $\mathrm{Gr}_{d,n}$ for the Plücker embedding with the induced grading $R = \bigoplus_{m\in\mathbb{N}} R_m$. More generally, for $\underline{i} \in I_{d,n}$, let $R(\underline{i}) = k[X_{\underline{i}}]$ be the homogeneous coordinate ring of the Schubert variety $X_{\underline{i}}$ in $\mathrm{Gr}_{d,n}$. In this section, we present a Standard Monomial Theory for $X_{\underline{i}}$ and as a consequence, we obtain a basis for the graded components $R(\underline{i})_m$, $m \in \mathbb{Z}^+$. Recall that $\mathrm{Gr}_{d,n} = X_{[n-d+1,\ldots,n]}$, so this includes the case of the Graßmann variety.

Definition 1.5.1. A monomial $p_{\mathcal{T}} = p_{\underline{i}^1} \cdots p_{\underline{i}^m}$ is said to be *standard* on a Schubert variety $X_{\underline{i}}$, if the monomial is standard (or, equivalently, the tableau $\mathcal{T}$ is semi-standard) and, in addition, $\underline{i} \geq \underline{i}^1$ (so $\underline{i} \geq \underline{i}^1 \geq \ldots \geq \underline{i}^m$).

Remark 1.5.2. By Lemma 1.4.5 we know that $p_{\underline{j}}$ vanishes identically on $X_{\underline{i}}$ if and only if $\underline{j} \not\leq \underline{i}$. So for a standard monomial $p_{\mathcal{T}}$ (on $\mathrm{Gr}_{d,n}$) we conclude

$$p_{\mathcal{T}} \text{ is standard on } X_{\underline{i}} \Leftrightarrow p_{\mathcal{T}}|_{X_{\underline{i}}} \text{ is not identically zero}$$

Theorem 1.5.3. *The standard monomials on $X_{\underline{i}}$ form a basis of the homogeneous coordinate ring $R(\underline{i})$.*

Proof. By Proposition 1.3.1, we know that the standard monomials span R. Since the standard monomials, not standard on $X_{\underline{i}}$, vanish identically on $X_{\underline{i}}$, it follows that the standard monomials, standard on $X_{\underline{i}}$, span the homogeneous coordinate ring $R(\underline{i})$. It remains to prove the linear independence. We proceed by induction on $\dim X_{\underline{i}}$, the case $\underline{i} = (1, 2, \ldots, d)$ (i.e. $\dim X_{\underline{i}} = 0$) being obvious because $k[X_{\underline{i}}] = k[p_{(1,2,\ldots,d)}]$ is isomorphic to the polynomial ring in one variable. Suppose now $\dim X_{\underline{i}} > 0$ and let

$$\sum_{\ell=1}^{r} c_{\mathcal{T}_\ell} p_{\mathcal{T}_\ell} \equiv 0,\ c_{\mathcal{T}_\ell} \in k\,, \tag{1.9}$$

be a linear relation of standard monomials $p_{\mathcal{T}_\ell}$ of degree m. If $m = 1$, then let $\mathcal{T}_1 = (\underline{j}^1), \ldots, \mathcal{T}_r = (\underline{j}^r)$. Note that $\underline{j}^s \leq \underline{i}$ for all $s = 1, \ldots, r$. Without loss of generality let $\underline{j}^1$ be a minimal element such that $c_{\underline{j}^1} \neq 0$. But then $p_{\underline{j}^t}([e_{\underline{j}^1}]) = 0$ for all $t \geq 2$ and hence the sum in (1.9) does not vanish in $[e_{\underline{j}^1}]$, which is a contradiction.

Suppose now $m > 1$ and assume there exists an ℓ such that $\mathcal{T}_\ell = (\underline{j}_1, \ldots, \underline{j}_d)$ is standard and $\underline{j}_1 \neq \underline{i}$. Now note: *a)* the restriction of the standard monomials $p_{\mathcal{T}_k}$ occuring in this sum which are not standard on $X_{\underline{j}_1}$, they vanish identically on $X_{\underline{j}_1}$, and *b)* at least $p_{\mathcal{T}_\ell}$ is standard on $X_{\underline{j}_1}$. So the restriction of (1.9) to $X_{\underline{j}_1}$ is a non-trivial linear dependence relation between standard monomials, standard on $X_{\underline{j}_1}$. By induction on the dimension, this is impossible.

So for all $\ell = 1, \ldots, r$, $p_{\mathcal{T}_\ell}$ is of the form $p_{\underline{i}} p_{\mathcal{T}'_\ell}$, where $p_{\mathcal{T}'_\ell}$ is a standard monomial, standard on $X_{\underline{i}}$. Now $X_{\underline{i}}$ is an irreducible variety, so

$$\sum_{\ell=1}^{r} c_{\mathcal{T}_\ell} p_{\mathcal{T}_\ell} \equiv 0 \Leftrightarrow p_{\underline{i}}\Big(\sum_{\ell=1}^{r} c_{\mathcal{T}_\ell} p_{\mathcal{T}'_\ell}\Big) \equiv 0 \Leftrightarrow \sum_{\ell=1}^{r} c_{\mathcal{T}_\ell} p_{\mathcal{T}'_\ell} \equiv 0$$

Now the latter is a homogeneous linear dependence relation of lesser degree, and hence by induction $c_{\mathcal{T}_\ell} = 0$ for all ℓ. □

As a consequence of Theorem 1.5.3, we have a qualitative description of a typical quadratic relation on a Schubert variety $X_{\underline{i}}$, which is a stronger version of Proposition 1.3.3:

Proposition 1.5.4. *Let $\underline{k}, \underline{i}, \underline{j} \in I_{d,n}$, $\underline{k} > \underline{i}, \underline{j}$. We assume that $\underline{i} \succeq \underline{j}$ but $\underline{i} \not\geq \underline{j}$, so $p_{\underline{i}} p_{\underline{j}}$ is a monomial of degree 2 on $X_{\underline{k}}$ but not standard. Let*

$$p_{\underline{i}} p_{\underline{j}} = \sum_{\substack{\underline{a},\underline{b} \in I_{d,n} \\ \underline{k} \geq \underline{a} \geq \underline{b}}} c_{\underline{a},\underline{b}} p_{\underline{a}} p_{\underline{b}}, \;\; c_{\underline{a},\underline{b}} \in k^* \tag{1.10}$$

be the expression for $p_{\underline{i}} p_{\underline{j}}$ as a sum of standard monomials on $X_{\underline{k}}$. Then for every $\underline{a}, \underline{b}$ on the right hand side, we have $\underline{a} > \underline{i}, \underline{j} > \underline{b}$.

Proof. Among all the $\underline{a}$'s occuring in (1.10) choose a minimal one, call it $\underline{a}_0$. Restricting (1.10) to $X_{\underline{a}_0}$, the restriction of the right hand side is a non-zero sum of standard monomials on $X_{\underline{a}_0}$. Note that the restriction of $p_{\underline{a}} p_{\underline{b}}$ to $X_{\underline{a}_0}$ is (by the minimality of $\underline{a}_0$) non-zero if and only if $\underline{a} = \underline{a}_0$; and there is at least one term, namely $p_{\underline{a}_0} p_{\underline{b}}$ whose restriction to $X_{\underline{a}_0}$ is non-zero. Hence in view of the linear independence of standard monomials on $X_{\underline{a}_0}$, we obtain that the restriction of the left hand side to $X_{\underline{a}_0}$ is non-zero. From this we conclude that $\underline{a}_0 \geq$ both $\underline{i}$ and $\underline{j}$.

In fact, $\underline{a}_0 >$ both $\underline{i}$ and $\underline{j}$ for if $\underline{a}_0$ equals one of them, say $\underline{a}_0 = \underline{i}$, then $p_{\underline{i}} p_{\underline{j}}$ would vanish on $X_{\underline{a}_0}$ because $\underline{j}$ and $\underline{i} = \underline{a}_0$ are not comparable. But this is not possible.

Among all the $\underline{b}$'s choose a maximal one, call it $\underline{b}_0$. For $\underline{c} = (c_1, \ldots, c_d) \in I_{d,n}$ set $\underline{c}^\perp = (n+1-c_d, \ldots, n+1-c_1)$, then $\underline{c}^\perp \in I_{d,n}$ and $^\perp$ induces an order reversing involution on $I_{d,n}$. Let w_0 be the permutation exchanging i and $n+1-i$ for all $i = 1, \ldots, n$. Denote by n_{w_0} the corresponding permutation matrix, then $n_{w_0}(p_{\underline{c}}) = cp_{\underline{c}^\perp}$ for some $c \in k^*$. So by applying n_{w_0} to (1.10), we conclude from the above that $\underline{b}_0^\perp >$ both $\underline{i}^\perp$ and $\underline{j}^\perp$, *i.e.* $\underline{b}_0 <$ both $\underline{i}$ and $\underline{j}$. □

1.5.5. Equations defining Schubert varieties in the Graßmannian. As a first consequence of standard monomial theory we give a description of the vanishing ideals of the Graßmann variety and its Schubert varieties.

Proposition 1.5.6. *1. The homogeneous vanishing ideal $I(\mathrm{Gr}_{d,n}) \subseteq k[\Lambda^d(V)]$ of the embedded Graßmann variety $\mathrm{Gr}_{d,n} \subseteq \mathbb{P}(\Lambda^d V)$ is the homogeneous ideal generated by the Plücker relations (see Definition 1.1.19).*

2. For $\underline{i} \in I_{d,n}$, the vanishing ideal $I_{\underline{i}} \subseteq k[\mathrm{Gr}_{d,n}]$ of the Schubert variety $X_{\underline{i}} \subseteq \mathrm{Gr}_{d,n}$ is the ideal generated by the Plücker coordinates $p_{\underline{j}}$ such that $\underline{j} \not\leq \underline{i}$.

3. The standard monomials $p_{\mathcal{T}}$, $\mathcal{T} = (\underline{i}^1, \ldots, \underline{i}^m)$, such that $\underline{i}^1 \not\leq \underline{i}$, form a basis of the kernel of the restriction map $k[\mathrm{Gr}_{d,n}] \to k[X_{\underline{i}}]$.

Proof. Let I be the ideal generated by the Plücker relations. By Proposition 1.3.3, we can write a homogeneous function f as a linear combination of standard monomials of the same degree plus an element of I. The linear independence of the standard monomials (Theorem 1.5.3) implies: f vanishes on $\mathrm{Gr}_{d,n}$ if and only $f \in I$.

For $\underline{i} \in I_{d,n}$ let $B_{\underline{i}}$ the set of standard monomials, standard on $X_{\underline{i}}$ and let $K_{\underline{i}}$ be the set of standard monomials, not standard on $X_{\underline{i}}$. Let $f \in k[\mathrm{Gr}_{d,n}]$, then we can write f as a linear combination of standard monomials. We can break up this sum into two parts $f = k_{\underline{i}} + b_{\underline{i}}$, where $k_{\underline{i}}$ is a linear combination of the elements in $K_{\underline{i}}$ and $b_{\underline{i}}$ is a linear combination of elements in $B_{\underline{i}}$. By Theorem 1.5.3, f is in the kernel of the restriction map $k[\mathrm{Gr}_{d,n}] \to k[X_{\underline{i}}]$ if and only if $b_{\underline{i}} = 0$, which proves part *3)*. Part *2)* of the proposition is an immediate consequence of this. □

Remark 1.5.7. Part *2)* of the proposition above implies that $X_{\underline{i}}$ is scheme-theoretically (even at the cone level) the intersection of $\mathrm{Gr}_{d,n}$ with all hyperplanes in $\mathbb{P}(\Lambda^d V)$ containing $X_{\underline{i}}$. Further, as a closed subvariety of $\mathrm{Gr}_{d,n}$, $X_{\underline{i}}$ is defined scheme-theoretically by the vanishing of the $p_{\underline{j}}$'s, $\underline{j} \not\leq \underline{i}$.

We now extend the results above to unions of Schubert varieties. Let $\underline{i}^1, \ldots, \underline{i}^s \in I_{d,n}$ and let $X = \bigcup_{\ell=1,\ldots,s} X_{\underline{i}^\ell}$ be a union of Schubert varieties.

Definition 1.5.8. A monomial $p_{\mathcal{T}}$ in the Plücker coordinates is called *standard* on the union X of Schubert varieties if it is standard on at least one of the Schubert varieties $X_{\underline{i}^1}, \ldots, X_{\underline{i}^s}$.

Theorem 1.5.9. *Let R_X be the homogeneous coordinate ring of the union of Schubert varieties $X \subset \mathrm{Gr}_{d,n}$. Then the (images of the) set of standard monomials, standard on X, forms a basis for R_X.*

Proof. Let B_X be the set of standard monomials, standard on X and let K_X be the set of standard monomials, not standard on X. Let $f \in k[\mathrm{Gr}_{d,n}]$, then we can write f as a linear combination of standard monomials. We can break up this sum into two parts $f = k_X + b_X$, where k_X is a linear combination of the elements in K_X and b_X is a linear combination of elements in B_X. The elements in K_X vanish identically on X, so we see that R_X is spanned by the (images of the) elements in B_X.

To prove the linear independence, suppose one has a linear dependence relation between the standard monomials, standard on X. The restriction of this relation to one of the Schubert varieties $X_{\underline{i}^\ell}$ gives a linear dependence relation for those summands in this linear dependence relation which are standard on $X_{\underline{i}^\ell}$. But by Theorem 1.5.3, this is impossible, so non of the summands is standard on $X_{\underline{i}^\ell}$. This holds for all $\ell = 1, \ldots, s$, so the standard monomials, standard on X, are linearly independent. □

1.6 Consequences

Theorem 1.6.1. *Let X_1, X_2 be unions of Schubert varieties in $\mathrm{Gr}_{d,n}$. Then the scheme-theoretic union $X_1 \cup X_2$ and the scheme-theoretic intersection $X_1 \cap X_2$ are reduced.*

Proof. For a closed subscheme Y in $\mathrm{Gr}_{d,n}$, let $I(Y)$ denote the ideal defining Y in $\mathrm{Gr}_{d,n}$. We have $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$, and hence $I(X_1 \cup X_2)$ is reduced since $I(X_1), I(X_2)$ are reduced.

Let X be the set theoretic intersection of X_1 and X_2, then X is a union of Schubert varieties. By Theorem 1.5.3 we know that the vanishing ideal $I \subset k[\mathrm{Gr}_{d,n}]$ of X has as basis all standard monomials not standard on X. Similar, let I_j, $j = 1, 2$ be the vanishing ideal of X_j, then I_j has as basis all standard monomials not standard on X_j. But then $I_1 + I_2$ has as basis all standard monomials which are not standard on either X_1 or X_2, in other words, which are not standard on X. But this implies $I = I_1 + I_2$, and hence X is also the scheme theoretic intersection of the two. □

By a *Schubert divisor* in $X_{\underline{i}}$ we mean a Schubert variety $X_{\underline{j}} \subset X_{\underline{i}}$ of codimension one. The union $\partial X_{\underline{i}} = \bigcup_{\ell=1}^{r} X_{\underline{j}^\ell}$ of all the Schubert divisors in $X_{\underline{i}}$ is called the *border* of $X_{\underline{i}}$.

Theorem 1.6.2. (Pieri's formula) *We have the following scheme-theoretic equality*

$$X_{\underline{i}} \cap \{p_{\underline{i}} = 0\} = \partial X_{\underline{i}}.$$

Proof. Let R_∂ be the homogeneous coordinate ring of $\partial X_{\underline{i}}$ and denote by $I_\partial \subset k[X_{\underline{i}}]$ the vanishing ideal of $\partial X_{\underline{i}}$, so $R_\partial = k[X_{\underline{i}}]/I_\partial$. Clearly, the principle ideal $(p_{\underline{i}}) \subseteq R_{X_{\underline{i}}}$ generated by $p_{\underline{i}}$ is contained in I_∂. Let $f \in I_\partial$, then we can write f as

$$f = \sum a_k p_{\mathcal{T}_k} + \sum b_j p_{\mathcal{T}'_j}$$

where $a_k, b_j \in k$, and in the first sum we have standard monomials $p_{\mathcal{T}_k}$ in $k[X_{\underline{i}}]$ starting with $p_{\underline{i}}$, while in the second sum we have standard monomials $p_{\mathcal{T}'_j}$ in $k[X_{\underline{i}}]$ starting with $p_{\underline{j}}$, where $\underline{j} < \underline{i}$. Obviously, the first sum is an element in $(p_{\underline{i}}) \subseteq I_\partial$, and hence also the second sum $\sum b_j p_{\mathcal{T}'_j}$ vanishes identically on $\partial X_{\underline{i}}$. But the $p_{\mathcal{T}'_j}$ are all standard on $\partial X_{\underline{i}}$ by construction, so $b_j = 0$ for all j and $f \in (p_{\underline{i}})$. □

1.6.3. Vanishing theorem and basis for cohomology. Denote by $\mathcal{L}$ the line bundle on $\mathrm{Gr}_{d,n}$ obtained as the restriction of $\mathcal{O}_{\mathbb{P}(\Lambda^d V)}$ to $\mathrm{Gr}_{d,n} \hookrightarrow \mathbb{P}(\Lambda^d V)$.

Denote by $S(Y, m)$ the set of standard monomials of degree m, standard on a union of Schubert varieties Y, and let $s(Y, m)$ denote the cardinality of this set. The restrictions of the standard monomials to Y form a linearly independent subset of the space of global sections $H^0(Y, \mathcal{L}^m)$ of the line bundle $\mathcal{L}^m$ restricted to Y.

The aim of this section is to show that SMT actually provides a basis for $H^0(Y, \mathcal{L}^m)$ and implies the vanishing of higher cohomology.

Theorem 1.6.4. (Vanishing Theorems) *Let Y be a union of Schubert varieties in $\mathrm{Gr}_{d,n}$. Then*

(a) $H^i(Y, \mathcal{L}^m) = 0$ for $i \geq 1$, $m \geq 0$.

(b) The set $S(Y, m)$ is a basis for $H^0(Y, \mathcal{L}^m)$.

(c) $H^i(X_{\underline{i}}, \mathcal{L}^m) = 0$ for $0 \leq i < \dim X_{\underline{i}}$, $m < 0$ for all $\underline{i} \in I_{d,n}$.

We start with some preliminaries. In a first step we will reduce the proof of the theorem to case where Y is just a Schubert variety. If $Y = X_{\underline{i}}$ for some $\underline{i}$, then $S(Y, m)$ and $s(Y, m)$ will also be denoted by $S(\underline{i}, m)$, respectively $s(\underline{i}, m)$.

Lemma 1.6.5. *1. Let $Y = Y_1 \cup Y_2$ where Y_1, Y_2 are unions of Schubert varieties in $\mathrm{Gr}_{d,n}$. Then*

$$s(Y, m) = s(Y_1, m) + s(Y_2, m) - s(Y_1 \cap Y_2, m).$$

2. $s(\underline{i}, m) = s(\underline{i}, m-1) + s(\partial X_{\underline{i}}, m)$

Proof. Both (1) and (2) are easy consequences of Theorem 1.5.9, Theorem 1.6.1 and Theorem 1.6.2. □

Proposition 1.6.6. *Let r be an integer $\leq d(n-d)$. Suppose that all Schubert varieties X in $\mathrm{Gr}_{d,n}$ of dimension at most r satisfy the following two properties:*

1. $H^i(X, \mathcal{L}^m) = 0$*, for* $i \geq 1$, $m \geq 0$.
2. *The set* $S(X,m)$ *is a basis for* $H^0(X, \mathcal{L}^m)$, $m \geq 0$.

Then all unions and intersections of Schubert varieties of dimension at most r satisfy (1) *and* (2).

Proof. We will prove the result by induction on r and induction on the number of irreducible components. Let S_r denote the set of Schubert varieties X in $\mathrm{Gr}_{d,n}$ of dimension at most r. Let Y be a union of Schubert varieties, say $Y = \bigcup_{j=1}^{t} X_{\underline{i}^j}$ is a decomposition of Y as a union of its irreducible components, where $X_{\underline{i}^j} \in S_r$. Set $Y_1 = \bigcup_{j=1}^{t-1} X_{\underline{i}^j}$ and set $Y_2 = X_{\underline{i}^t}$. Consider the exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{O}_{Y_1} \oplus \mathcal{O}_{Y_2} \to \mathcal{O}_{Y_1 \cap Y_2} \to 0.$$

Tensoring with $\mathcal{L}^m$, we obtain the long exact sequence,

$$\begin{aligned} \cdots \to H^{i-1}(Y_1 \cap Y_2, \mathcal{L}^m) \to H^i(Y, \mathcal{L}^m) \to \\ H^i(Y_1, \mathcal{L}^m) \oplus H^i(Y_2, \mathcal{L}^m) \to H^i(Y_1 \cap Y_2, \mathcal{L}^m) \to \cdots . \end{aligned}$$

Now by Theorem 1.6.1, $Y_1 \cap Y_2$ is reduced and $Y_1 \cap Y_2$ is a union of Schubert varieties in S_{r-1}. Hence by the induction hypothesis, (1) and (2) hold for $Y_1 \cap Y_2$. In particular, if $m \geq 0$, then (2) implies that the map $H^0(Y_1, \mathcal{L}^m) \oplus H^0(Y_2, \mathcal{L}^m) \to H^0(Y_1 \cap Y_2, \mathcal{L}^m)$ is surjective. Hence we obtain that the sequence

$$0 \to H^0(Y, \mathcal{L}^m) \to H^0(Y_1, \mathcal{L}^m) \oplus H^0(Y_2, \mathcal{L}^m) \to H^0(Y_1 \cap Y_2, \mathcal{L}^m) \to 0$$

is exact. This implies that $H^0(Y_1 \cap Y_2, \mathcal{L}^m) \to H^1(Y, \mathcal{L}^m)$ is the zero map. Also, $H^1(Y, \mathcal{L}^m) \to H^1(Y_1, \mathcal{L}^m) \oplus H^1(Y_2, \mathcal{L}^m)$ is the zero map since by induction $H^1(Y_1, \mathcal{L}^m) = 0 = H^1(Y_2, \mathcal{L}^m)$. Hence we obtain $H^1(Y, \mathcal{L}^m) = 0$, for $m \geq 0$. For $i \geq 2$, the assertion that $H^i(Y, \mathcal{L}^m) = 0$, for $m \geq 0$ follows from the long exact cohomology sequence above and the induction hypothesis. This proves assertion (1) for Y.

To prove assertion (2) for Y, we observe that

$$\begin{aligned} \dim H^0(Y, \mathcal{L}^m) &= \dim H^0(Y_1, \mathcal{L}^m) + \dim H^0(Y_2, \mathcal{L}^m) \\ &\quad - \dim H^0(Y_1 \cap Y_2, \mathcal{L}^m) \\ &= s(Y_1, m) + s(Y_2, m) - s(Y_1 \cap Y_2, m). \end{aligned}$$

Hence Lemma 1.6.5 and the induction hypothesis imply that

$$\dim H^0(Y, \mathcal{L}^m) = s(Y, m).$$

This together with the linear independence of standard monomials on Y proves assertion (2) for Y. $\square$

Proposition 1.6.7. *Let* $Y = \bigcup_{j=1}^{t} X_{\underline{i}^j}$ *be a union of Schubert divisors in a Schubert variety* $X_{\underline{i}}$ *in* $\mathrm{Gr}_{d,n}$*. Suppose that* $H^i(X_{\underline{i}^j}, \mathcal{L}^m) = 0$*, for* $m < 0$*,* $0 \le i \le \dim Y - 1 = \dim X_{\underline{i}} - 2$*. Then* $H^i(Y, \mathcal{L}^m) = 0$*, for* $m < 0$*,* $0 \le i \le \dim Y - 1$*.*

Proof. We will prove the result by induction on t and $\dim Y = \dim X_{\underline{i}} - 1$. As in the proof of Proposition 1.6.6, we set $Y_1 = \bigcup_{j=1}^{t-1} X_{\underline{i}^j}$ and $Y_2 = X_{\underline{i}^t}$, and consider the long exact cohomology sequence

$$\cdots \to H^{i-1}(Y_1 \cap Y_2, \mathcal{L}^m) \to H^i(Y, \mathcal{L}^m) \to$$
$$H^i(Y_1, \mathcal{L}^m) \oplus H^i(Y_2, \mathcal{L}^m) \to H^i(Y_1 \cap Y_2, \mathcal{L}^m) \to \cdots$$

We have, $Y_1 \cap Y_2 = \bigcup_{j=1}^{t-1} X_{\underline{i}^j} \cap X_{\underline{i}^t}$. By Lemma 1.4.9, $X_{\underline{i}^j} \cap X_{\underline{i}^t}$, $1 \le j \le t-1$ is irreducible of codimension 2 in $X_{\underline{i}}$. Hence by induction on $\dim Y$, we have $H^i(Y_1 \cap Y_2, \mathcal{L}^m) = 0$, for $m < 0$, $0 \le i \le \dim Y_1 - 1 = \dim Y - 2$. Further, by induction on t, $H^i(Y_1, \mathcal{L}^m) = 0$ for $m < 0$, $0 \le i \le \dim Y_1$, and by hypothesis, $H^i(Y_2, \mathcal{L}^m) = 0$, for $m < 0$, $0 \le i \le \dim Y_1$. The required result now follows. $\square$

Theorem 1.6.4 is now a consequence of the following proposition and Proposition 1.6.6:

Proposition 1.6.8. *Let* $X_{\underline{i}}$ *be a Schubert variety in* $\mathrm{Gr}_{d,n}$*. Then*

(a) $H^i(X_{\underline{i}}, \mathcal{L}^m) = 0$ *for* $i \ge 1$*,* $m \ge 0$*.*

(b) $H^i(X_{\underline{i}}, \mathcal{L}^m) = 0$ *for* $0 \le i < \dim X_{\underline{i}}$*,* $m < 0$*.*

(c) The set $S(\underline{i}, m)$ *is a basis for* $H^0(X_{\underline{i}}, \mathcal{L}^m)$*.*

Proof. We prove the result by induction on m and $\dim X_{\underline{i}}$.

If $\dim X_{\underline{i}} = 0$, X is just a point, and the result is obvious. Assume now that $\dim X_{\underline{i}} \ge 1$. Let $X = X_{\underline{i}}$. Let $X_1, \ldots, X_s$ be all the Schubert divisors in X, and set $\partial X = \bigcup_{i=1}^{s} X_i$. Then by Pieri's formula (cf. Theorem 1.6.2), we have,

$$\partial X = X \cap \{p_{\underline{i}} = 0\} \quad \text{(scheme-theoretically)}.$$

Hence the sequence

$$0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to \mathcal{O}_{\partial X} \to 0$$

is exact. Tensoring it with $\mathcal{L}^m$, we obtain the long exact cohomology sequence

$$\cdots \to H^{i-1}(\partial X, \mathcal{L}^m) \to H^i(X, \mathcal{L}^{m-1}) \to H^i(X, \mathcal{L}^m) \to H^i(\partial X, \mathcal{L}^m) \to \cdots .$$

Let $m \geq 0$, $i \geq 2$. Then the induction hypothesis on $\dim X$ implies (in view of Proposition 1.6.6) that $H^i(\partial X, \mathcal{L}^m) = 0$, $i \geq 1$. Hence we obtain that the sequence $0 \to H^i(X, \mathcal{L}^{m-1}) \to H^i(X, \mathcal{L}^m)$, $i \geq 2$, is exact. If $i = 1$, again the induction hypothesis implies the surjectivity of $H^0(X, \mathcal{L}^m) \to H^0(\partial X, \mathcal{L}^m)$. This in turn implies that the map $H^0(\partial X, \mathcal{L}^m) \to H^1(X, \mathcal{L}^{m-1})$ is the zero map, and hence we obtain that the sequence $0 \to H^1(X, \mathcal{L}^{m-1}) \to H^1(X, \mathcal{L}^m)$ is exact. Thus we obtain that $0 \to H^i(X, \mathcal{L}^{m-1}) \to H^i(X, \mathcal{L}^m)$, $m \geq 0$, $i \geq 1$ is exact. But $H^i(X, \mathcal{L}^m) = 0$, $m \gg 0$, $i \geq 1$ (cf. [244]). Hence we obtain

$$H^i(X, \mathcal{L}^m) = 0 \text{ for } i \geq 1,\ m \geq 0, \tag{1}$$

and

$$h^0(X, \mathcal{L}^m) = h^0(X, \mathcal{L}^{m-1}) + h^0(\partial X, \mathcal{L}^m) \tag{2}$$

where $h^0(X, \mathcal{L}^m) = \dim H^0(X, \mathcal{L}^m)$. In particular, assertion (a) follows from (1). The induction hypothesis on m implies that $h^0(X, \mathcal{L}^{m-1}) = s(X, m-1)$. On the other hand, the induction hypothesis on $\dim X$ implies (in view of Proposition 1.6.6) that $h^0(\partial X, \mathcal{L}^m) = s(\partial X, m)$. Hence we obtain

$$h^0(X, \mathcal{L}^m) = s(X, m-1) + s(\partial X, m). \tag{3}$$

Now (3) together with part 2 of Lemma 1.6.5 imply $h^0(X, \mathcal{L}^m) = s(X, m)$. Hence (c) follows in view of the linear independence of standard monomials on $X_{\underline{i}}$ (cf. Theorem 1.5.3).

To prove (b), consider the long exact cohomology sequence

$$\cdots \to H^{i-1}(\partial X, \mathcal{L}^m) \to H^i(X, \mathcal{L}^{m-1}) \to H^i(X, \mathcal{L}^m) \to \cdots$$

We have, $H^i(\partial X, \mathcal{L}^m) = 0,\ m \in \mathbb{Z},\ 1 \leq i \leq \dim \partial X - 1$ (by induction hypothesis and Propositions 1.6.6 and 1.6.7). Hence we obtain the exact sequence

$$0 \to H^i(X, \mathcal{L}^{m-1}) \to H^i(X, \mathcal{L}^m),\ 2 \leq i < \dim X.$$

But $H^i(X, \mathcal{L}^m) = 0$, $m \gg 0$ (cf. [244]). Hence we obtain, $H^i(X, \mathcal{L}^m) = 0$, $2 \leq i < \dim X$, $m \in \mathbb{Z}$ (in particular for $m < 0$).

It remains to prove (b) for $i = 0, 1$. Let then $m < 0$. Induction hypothesis together with Proposition 1.6.7 implies that $H^0(\partial X, \mathcal{L}^m) = 0$. Hence we obtain the exact sequence

$$0 \to H^1(X, \mathcal{L}^{m-1}) \to H^1(X, \mathcal{L}^m),\ m < 0.$$

Thus we obtain inclusions

$$H^1(X, \mathcal{L}^{m-1}) \subset H^1(X, \mathcal{L}^m) \subset \cdots \subset H^1(X, \mathcal{L}^{-1}).$$

But now the isomorphism $H^0(X, \mathcal{O}_X) \cong H^0(\partial X, \mathcal{O}_{\partial X})$ together with the fact that $H^1(X, \mathcal{O}_X) = 0$ implies that $H^1(X, \mathcal{L}^{-1}) = 0$. This now implies (in view of the above inclusions) that $H^1(X, \mathcal{L}^m) = 0$, $m < 0$. Now

$H^0(\partial X, \mathcal{L}^m) = 0$ implies

$$H^0(X, \mathcal{L}^{m-1}) \cong H^0(X, \mathcal{L}^m), \ m < 0. \tag{4}$$

The isomorphism $H^0(X, \mathcal{O}_X) \cong H^0(\partial X, \mathcal{O}_{\partial X})$ implies that $H^0(X, \mathcal{L}^{-1}) = 0$, and it follows that $H^0(X, \mathcal{L}^m) = 0$, $m < 0$ (note that in view of (4), we have, $H^0(X, \mathcal{L}^m) \cong H^0(X, \mathcal{L}^{-1}) = 0$). □

Corollary 1.6.9. *For the homogeneous coordinate ring $R = k[\mathrm{Gr}_{d,n}]$ of the Graßmann variety and the homogeneous coordinate ring $R(\underline{i}) = k[X_{\underline{i}}]$, $\underline{i} \in I_{d,n}$, of a Schubert variety we have*

1. $R = \bigoplus_{m \in \mathbb{Z}^+} H^0(X, \mathcal{L}^m)$.
2. $R(\underline{i}) = \bigoplus_{m \in \mathbb{Z}^+} H^0(X_{\underline{i}}, \mathcal{L}^m)$.

Proof. The assertions follow immediately from Theorems 1.5.3 and 1.6.4. □

Proposition 1.6.10. *The Schubert variety $X_{\underline{i}}$ in $\mathrm{Gr}_{d,n}$ is arithmetically normal, i.e. the affine cone $\widehat{X_{\underline{i}}}$ over $X_{\underline{i}}$ is normal.*

Proof. Let $Y = \mathbb{P}(\Lambda^d V)$. The canonical homomorphism

$$\phi_m : H^0(Y, \mathcal{L}^m) \to H^0(X_{\underline{i}}, \mathcal{L}^m)$$

is surjective for $m \gg 0$. Let $S_m = H^0(X_{\underline{i}}, \mathcal{L}^m)$. We have (see [107] for example) $\mathrm{Im}\, \phi_m = R(\underline{i})_m$, and $S = \bigoplus_{m \in \mathbb{Z}^+} S_m$ is the integral closure of $R(\underline{i})$ in its quotient field. Now by Corollary 1.6.9, we have $R(\underline{i})_m = S_m$ for all m, and hence $R(\underline{i})$ is normal. This implies: $X_{\underline{i}}$ is arithmetically normal. □

2
SMT for flag varieties

There are two immediate directions to generalize SMT: one is to consider a generalized Graßmannian $G/P \subset \mathbb{P}(V(\omega))$ for $P \subset G$ a maximal parabolic subgroup of a simple algebraic group and ω the corresponding fundamental weight, the other is to replace the maximal parabolic subgroup P by a parabolic subgroup $Q \subset SL_n(k)$, and to consider the corresponding multicone over $SL_n(k)/Q$.

We will consider both points of view later in a much more general setting (Chapter **??**). But before attacking the general case, we consider again $G = SL_n(k)$ because it will be the guiding example for the corresponding constructions (combinatorial as well as geometric) in the sequel. But differently from the rather elementary methods used in the previous chapter, for the proof that the standard monomials span the homogenous coordinate ring we need some non-trivial algebraic-geometric facts. The approach in Chapter **??** is in some sense more in the style of the elementary previous chapter, the non-trivial algebraic geometry part being hidden in the representation theoretic version of Frobenius splitting.

We use the same notation as in Chapter 1 for $V = k^n$, the standard basis $\{e_1, \ldots, e_n\}$, $B = TU$, the Weyl group W is the symmetric group $\mathfrak{S}_n$ etc. For the combinatorial and geometric language (Bruhat decomposition, minimal lifts etc.) used in this section we refer to Chapter **??**, where we recall some standard facts on Weyl groups and the combinatorics of Schubert varieties in the general setting of semisimple algebraic groups.

2.1 SMT for Schubert varieties in the flag variety

The standard monomial theory for the Graßmann variety can also be used to develop a standard monomial theory for (partial) flag varieties and their Schubert varieties. This construction is an important example for the theory of standard monomials for a multicone.

2.1.1. Partial flag varieties. Let $\underline{d} = (d_1, \ldots, d_r)$, $1 \leq d_1 < \ldots < d_r < n$, be a strictly increasing sequence of integers. The *partial flag variety* $\mathrm{Gr}_{\underline{d},n}$ is the set of all partial flags of subspaces

$$\mathrm{Gr}_{\underline{d},n} := \{0 \subset U_1 \subset U_2 \subset \ldots \subset U_r \subset V \mid \dim U_j = d_j \text{ for } j = 1, \ldots, r\}.$$

If we consider only the case $r = n-1$, i.e., $\underline{d} = (1, 2, \ldots, n-1)$, then we refer to $\mathrm{Gr}_{\underline{d},n}$ as the *flag variety*.

The partial flag variety can be viewed as a subset of a product of Graßmann varieties:

$$\mathrm{Gr}_{\underline{d},n} = \{(U_1, \ldots, U_r) \in \mathrm{Gr}_{d_1,n} \times \ldots \times \mathrm{Gr}_{d_r,n} \mid U_1 \subset U_2 \subset \ldots \subset U_r\}.$$

Note that the condition $U_1 \subset U_2 \subset \ldots \subset U_r$ is a “closed” condition, so $\mathrm{Gr}_{\underline{d},n}$ is naturally equipped with the structure of a projective variety.

We have also a description of $\mathrm{Gr}_{\underline{d},n}$ as a homogeneous space. Let (as before) F_j be the j-dimensional subspace spanned by $e_1, e_2, \ldots, e_j$, then $\mathcal{F} : 0 \subset F_{d_1} \subset \ldots \subset F_{d_r} \subset k^n$ is an element in $\mathrm{Gr}_{\underline{d},n}$. As in the case of the Graßmann variety it is easy to see that given $\mathcal{F}' \in \mathrm{Gr}_{\underline{d},n}$, there exists an element $g \in SL_n(k)$ such that $g\mathcal{F} = \mathcal{F}'$. So we get:

$$\mathrm{Gr}_{\underline{d},n} = SL_n(k)/Q_{\underline{d}},$$

where $Q_{\underline{d}}$ is the isotropy group of $\mathcal{F}$. i.e., $Q_{\underline{d}}$ consists of all $g \in SL_n(k)$ such that $gF_{d_j} \subseteq F_{d_j}$ for all $j = 1, \ldots, r$. In other words:

$$Q_{\underline{d}} = P_{d_1} \cap \ldots \cap P_{d_r}, \tag{2.1}$$

where the maximal parabolic subgroups P_d are defined as in section 1.1.3.

By the Bruhat decomposition (Theorem **??**, Corollary **??**), the partial flag variety $\mathrm{Gr}_{\underline{d},n}$ is the disjoint union of a finite number of B-orbits. Let

$$\mathfrak{S}_{\underline{d}} = \mathfrak{S}_{d_1} \times \mathfrak{S}_{d_2-d_1} \times \ldots \times \mathfrak{S}_{d_r-d_{r-1}} \times \mathfrak{S}_{n-d_r},$$

and denote by $e_{\mathrm{id}} \in \mathrm{Gr}_{\underline{d},n}$ the class of $Q_{\underline{d}}$. For $\tau \in \mathfrak{S}_n/\mathfrak{S}_{\underline{d}}$ we denote by e_τ the point $\tau.e_{\mathrm{id}} \in \mathrm{Gr}_{\underline{d},n}$. Then

$$\mathrm{Gr}_{\underline{d},n} = \bigcup_{\tau \in \mathfrak{S}_n/\mathfrak{S}_{\underline{d}}} B.e_\tau$$

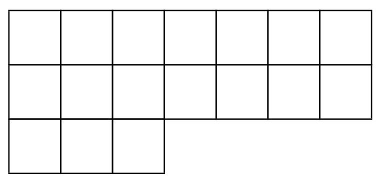
Given $\tau \in \mathfrak{S}_n/\mathfrak{S}_{\underline{d}}$, the B-orbit $B.e_\tau$ is the Schubert cell $C_\tau \subset \mathrm{Gr}_{\underline{d},n}$ and the Zariski closure $\overline{C_\tau}$ of the cell is the Schubert variety $X(\tau)$ (see for example Chapter **??**).

The Schubert cell is an affine cell isomorphic to some k^m, where m is the length of a minimal representative of τ in $\mathfrak{S}_n$ (see Lemma **??**, Lemma **??**, Corollary **??**).

2.1.2. Partitions and Young diagrams. Let $\underline{d} = (d_1, \dots, d_r)$ be as above. A $\underline{d}$-partition $\underline{\lambda}$ is a weakly decreasing sequence of d_r integers, such that

$$\underline{\lambda} = \big(\underbrace{\lambda_1, \dots, \lambda_1}_{d_1}, \underbrace{\lambda_2, \dots, \lambda_2}_{d_2 - d_1}, \dots, \underbrace{\lambda_r, \dots, \lambda_r}_{d_r - d_{r-1}}\big), \quad \lambda_1 \geq \dots \geq \lambda_r \geq 0.$$

A *Young diagram* of shape $\underline{\lambda}$ is a left justified sequence of rows of boxes such that there are λ_1 boxes in the first d_1 rows, there are λ_2 boxes in the next $d_2 - d_1$ rows etc. For example, let $n = 4$ and $\underline{d} = (2,3)$, then $\underline{\lambda} = (7,7,3)$ is a $\underline{d} = (2,3)$-partition and the associated diagram is:



Definition 2.1.3. By a *Young tableau* $\mathcal{T}$ of shape $\underline{\lambda}$ we mean a filling of the boxes of the diagram with numbers $\{1, \dots, n\}$. $\mathcal{T}$ is called *column standard* if the entries in the columns are strictly increasing (top to bottom). In the following we will always only consider column standard tableaux, so by abuse of notation we write in the following just Young tableau for a column standard Young tableau.

A tableau $\mathcal{T}$ is called *semi-standard* if the entries are strictly increasing in the columns and weakly increasing in the rows.

Example 2.1.4. The following tableau of shape $\underline{\lambda} = (7,7,3)$ is a semi-standard Young tableau

$$\mathcal{T} = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 2 & 3 & 3 \\ \hline 2 & 3 & 3 & 3 & 4 & 4 & 4 \\ \hline 3 & 4 & 4 \\ \cline{1-3} \end{array} .$$

For a fixed natural number n and a $\underline{d}$-partition $\underline{\lambda}$ let $\mathrm{Y}_n(\underline{\lambda})$ be the set of all Young tableaux of shape λ whose entries belong to $\{1, \dots, n\}$, and let $\mathrm{Y}_n^{\mathrm{std}}(\underline{\lambda})$ be the set of all *semistandard Young tableaux of shape* $\underline{\lambda}$ (*i.e.* the columns are strictly increasing and the rows are weakly increasing) whose entries belong to $\{1, \dots, n\}$.

2.1.5. The line bundle $\mathcal{L}_{\underline{\lambda}}$. Let $\underline{d} = (d_1, \dots, d_r)$, $1 \leq d_1 < \dots < d_r < n$, be a strictly increasing sequence of integers. The associated group $Q_{\underline{d}}$ (see

(2.1)) is a subgroup of P_{d_j}, so we have surjective maps for all $j = 1, \dots, r$:

$$\pi_{d_j} : \mathrm{Gr}_{\underline{d},n} \to \mathrm{Gr}_{d_j,n} .$$

As before, let $\mathcal{L}_{\omega_{d_j}}$ be the restriction of the canonical line bundle on $\mathrm{Gr}_{d_j,n}$ obtained as the restriction of $\mathcal{O}_{\mathbb{P}(\Lambda^{d_j} V)}$ to $\mathrm{Gr}_{d_j,n} \hookrightarrow \mathbb{P}(\Lambda^{d_j} V)$.

By abuse of notation, we denote by $\mathcal{L}_{\omega_{d_j}}$ also the line bundle on $\mathrm{Gr}_{\underline{d},n}$ obtained as the pull back $\pi^*_{d_j}(\mathcal{L}_{\omega_{d_j}})$. The restriction of the bundle to the fibres of π_{d_j} becomes trivial, so we have natural isomorphisms

$$H^k(\mathrm{Gr}_{\underline{d},n}, \mathcal{L}_{\omega_{d_j}}) \simeq H^k(\mathrm{Gr}_{d_j,n}, \mathcal{L}_{\omega_{d_j}})$$

for all $k \geq 0$ and all $j = 1, \dots r$. We view the sections $p_{\underline{i}} \in H^0(\mathrm{Gr}_{d_j,n}, \mathcal{L}_{\omega_{d_j}})$ considered in Chapter 1 also as sections on $\mathrm{Gr}_{\underline{d},n}$, by abuse of notation we just write

$$p_{\underline{i}} \in H^0(\mathrm{Gr}_{\underline{d},n}, \mathcal{L}_{\omega_{d_j}}).$$

Let $\underline{\lambda}$ be a $\underline{d}$-partition. The line bundle $\mathcal{L}_{\underline{\lambda}}$ on $\mathrm{Gr}_{\underline{d},n}$ is the tensor product of line bundles

$$\mathcal{L}_{\underline{\lambda}} = \mathcal{L}_{\omega_{d_1}}^{\otimes(\lambda_1-\lambda_2)} \otimes \mathcal{L}_{\omega_{d_2}}^{\otimes(\lambda_2-\lambda_3)} \otimes \cdots \otimes \mathcal{L}_{\omega_{d_{r-1}}}^{\otimes(\lambda_{r-1}-\lambda_r)} \otimes \mathcal{L}_{\omega_{d_r}}^{\otimes\lambda_r}$$

For a (column standard) Young tableau $\mathcal{T}$ of shape $\underline{\lambda}$ let $\underline{i}^1, \dots, \underline{i}^{\lambda_1}$ be the columns of the tableaux (indexed from the right to the left). Let $\underline{i}^j$ be such a column, say of length d_j. The entries of the column define an element in $I_{d_j,n}$ and hence an associated section $p_{\underline{i}^j} \in H^0(\mathrm{Gr}_{\underline{d},n}, \mathcal{L}_{\omega_{d_j}})$. We define the monomial $p_{\mathcal{T}}$ as the product of the sections corresponding to the columns of $\mathcal{T}$:

$$p_{\mathcal{T}} = \prod_{j=1}^{\lambda_1} p_{\underline{i}^j}$$

By definition, $p_{\mathcal{T}} \in H^0(\mathrm{Gr}_{\underline{d},n}, \mathcal{L}_{\underline{\lambda}})$.

Definition 2.1.6. We say that $p_{\mathcal{T}}$ is a *standard monomial* on $\mathrm{Gr}_{\underline{d},n}$ if $\mathcal{T}$ is semistandard.

Remark 2.1.7. The odd looking couple *standard monomial* and *semistandard tableau* comes from the fact that the notion of a standard tableau is already reserved for a special class of tableau related to the representation theory of the symmetric group.

The SMT-basis for the Graßmann variety has the nice property that the restriction of basis elements to a Schubert variety either vanish or remain linearly independent. We will see later (Chapter **??**) that it is possible to obtain such a basis in a much more general context for a cone over a generalized partial flag variety. But the aim in this section is to define a SMT for the multi-cone over the variety. The following example shows that we have to pay a price for the fact that we are looking at the multi-cone.

Example 2.1.8. Let $G = SL_3(k)$, $Q_{\underline{d}} = B$, $w = s_2 s_1$ and

$$\mathcal{T}_1 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}\ , \qquad \mathcal{T}_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}\ .$$

Recall that the Schubert cell $B.e_w = U.e_w$ corresponds to the set of flags obtained by the first two columns of the matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} y & 1 & x \\ z & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The image of such a flag $\mathcal{F}$ in $\mathrm{Gr}_{1,3}$ is the subspace spanned by the first column, and the image in $\mathrm{Gr}_{2,3}$ is the subspace spanned by the first two columns. It is now easy to see that $p_{\mathcal{T}_1}(\mathcal{F}) = p_{\mathcal{T}_2}(\mathcal{F}) = -z$.

The set $\{p_{\mathcal{T}_1}, p_{\mathcal{T}_2}\}$ is linearly independent on G/B, but the restrictions to $X(w)$ become obviously linearly dependent.

To define a standard monomial, standard on a Schubert variety, recall the bijection between the set $I(d,n)$ and $\mathfrak{S}_n/\mathfrak{S}_d \times \mathfrak{S}_{n-d}$:

Given $w \in \mathfrak{S}_n$ let $\underline{i}_w \in I(d,n)$ be the element obtained by writing the tuple $(w(1), \ldots, w(d))$ in increasing order. The map

$$\mathfrak{S}_n \longrightarrow I(d,n), \quad w \mapsto \underline{i}_w,$$

induces a bijection $\mathfrak{S}_n/\mathfrak{S}_d \times \mathfrak{S}_{n-d} \to I(d,n)$.

Definition 2.1.9. Given am element $\underline{i} \in I(d,n)$, an element $w \in \mathfrak{S}_n$ is called a *lift* if $\underline{i}_w = \underline{i}$.

Let $\mathcal{T}$ be a Young tableau of shape $\underline{\lambda}$ with columns $\underline{\tau} = (\underline{i}^1, \ldots, \underline{i}^{\lambda_1})$ (indexed from right to left), each column $\underline{i}^j$ defines an element (denoted by the same symbol) $\underline{i}^j \in I_{d_j,n}$ for some $1 \leq d_j < n$.

Definition 2.1.10. A *lift* for $\mathcal{T}$ is a sequence $\underline{w} = (w^1, \ldots, w^{\lambda_1})$ of elements in $\mathfrak{S}_n$ such that $\underline{i}_{w^j} = \underline{i}^j$ for all $j = 1, \ldots, \lambda_1$.

Definition 2.1.11. A sequence $(w_1, \ldots, w_r)$ of elements in $\mathfrak{S}_n$ is called *standard* if the sequence is linearly ordered, i.e., $w_1 \geq w_2 \geq \ldots \geq w_r$ (with respect to the Bruhat-Chevalley ordering, see Chapter **??**).

Example 2.1.12. The sequence of tuples associated to the tableau in Example 2.1.4 is

$$\underline{\tau} = \big((3,4), (3,4), (2,4), (2,3), (2,3,4), (1,3,4), (1,2,3)\big),$$

a lift is for example the following standard sequence:

$$\underline{w} = (s_2 s_1 s_3 s_2 s_3, s_2 s_1 s_3 s_2 s_3, s_1 s_3 s_2 s_3, s_1 s_2 s_3, s_1 s_2 s_3, s_2 s_3, 1).$$

Definition 2.1.13. A lift $\underline{w} = (w^1, \ldots, w^{\lambda_1})$ for $\mathcal{T}$ is called a *defining chain* for $\mathcal{T}$ if the lift is a standard sequence.

Remark 2.1.14. Recall that we assume a Young tableau $\mathcal{T}$ to be always column standard. We leave it as an **excercise** to show:

$$\mathcal{T} \text{ is semistandard} \Longleftrightarrow \mathcal{T} \text{ admits a defining chain.}$$

Definition 2.1.15. The monomial $p_{\mathcal{T}}$ associated to a Young tableau $\mathcal{T}$ is called *semistandard on a Schubert variety* $X(w) \subset \mathrm{Gr}_{\underline{d},n}$ if $\mathcal{T}$ is semistandard and there exists a defining chain $\underline{w} = (w_1, \ldots, w_r)$ for $\mathcal{T}$ such that $w \geq w_1 \mod \mathfrak{S}_{\underline{d}}$.

Example 2.1.16. Consider the Schubert variety $X(s_2s_1) \subset SL_3(k)/B$ (see Example 2.1.8). The tableau

$$\mathcal{T}_2 = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array}\ ,$$

has a unique defining chain: $\underline{w} = (s_1s_2, s_2)$, in particular, $p_{\mathcal{T}_2}$ is standard on $SL_3(k)/B$. Further, $X(s_1s_2s_1)$ and $X(s_1s_2)$ are the only two Schubert varieties on which $p_{\mathcal{T}_2}$ is standard.

The tableau $\mathcal{T}_1$ has four possible defining chains (Exercise), and $p_{\mathcal{T}_1}$ is standard on $X(s_2s_1)$. By Example 2.1.8, $p_{\mathcal{T}_1}\big|_{X(s_2s_1)} = p_{\mathcal{T}_2}\big|_{X(s_2s_1)}$, so the definition of standardness can be understood as making a choice among the monomials $p_{\mathcal{T}_1}$ and $p_{\mathcal{T}_2}$ whose restrictions give the same function.

Remark 2.1.17. Lifts and defining chains (if they exist) for a Young tableau $\mathcal{T}$ are in general not unique. We will see later (Chapter **??**, Deodhar's lemma) that one can define a partial order on the set of defining chains such that one has a unique minimal defining chain $\underline{w}^-$and a unique maximal defining chain $\underline{w}^+$. Let $v_1^- \in \mathfrak{S}_n/\mathfrak{S}_{\underline{d}}$ denote the class of w_1^-, then $X(v_1^-)$ is the smallest Schubert variety in $\mathrm{Gr}_{\underline{d},n}$ on which $p_{\mathcal{T}}$ is standard.

Definition 2.1.18. The monomial $p_{\mathcal{T}}$ associated to a Young tableau $\mathcal{T}$ is called *standard on a union of Schubert varieties* $Z = \bigcup_{i=1}^t X(\phi_i)$ in $\mathrm{Gr}_{\underline{d},n}$ if $p_{\mathcal{T}}$ is standard on $X(\phi_i)$ for some i, $1 \leq i \leq t$.

2.1.19. Linear independence of standard monomials.

Definition 2.1.20. Let $\phi_1, \ldots, \phi_s \in \mathfrak{S}/\mathfrak{S}_{\underline{d}}$ and let $Y = X(\phi_1) \cup \cdots \cup X(\phi_s)$ be a scheme-theoretic union of Schubert varieties (note that Y is reduced). For a $\underline{d}$-partition $\underline{\lambda}$ set $a_i = \lambda_i - \lambda_{i+1}$, $i = 1, \ldots, r-1$, and set $a_r = \lambda_r$. The tuple $\underline{a} = (a_1, \ldots, a_r)$ is called the *multidegree of the sections* in $H^0(\mathrm{Gr}_{\underline{d},n}, \mathcal{L}_{\underline{\lambda}})$. Let $S(Y, \underline{a})$ denote the set of standard monomials on Y of multidegree $\underline{a}$, denote by $s(Y, \underline{a})$ the cardinality of $S(Y, \underline{a})$ and set $h^0(Y, \underline{a}) = \dim H^0(Y, \mathcal{L}_{\underline{\lambda}})$.

Proposition 2.1.21. *The set of standard monomials on Y is linearly independent; in particular, for all $\underline{a} = (a_1, \ldots, a_r) \in \mathbb{Z}_+^r$, we have*

$$h^0(Y, \underline{a}) \geq s(Y, \underline{a}).$$

Proof. We have $Y = X(\phi_1) \cup \cdots \cup X(\phi_s)$, $\phi_i \in W$. The proof is by induction on a_1 and on r. If $r = 1$, then $\mathrm{Gr}_{\underline{d},n}$ is a Graßmann variety and the proposition holds by Theorem 1.6.4.

Suppose now the proposition holds for $r-1$. If $a_1 = 0$, then set $\underline{d}' = (d_2, \ldots, d_r)$, and let $X(\phi_i')$ be the image of $X(\phi_i)$ under $\eta : \mathrm{Gr}_{\underline{d},n} \to \mathrm{Gr}_{\underline{d}',n}$. Let $Y' = X(\phi_1') \cup \cdots \cup X(\phi_s') = \eta(Y)$. We find that any standard monomial of multidegree $(0, a_2, \ldots, a_r)$ on $X(\phi_i)$ can be canonically identified with a standard monomial of multidegree $(a_2, \ldots, a_r)$ on $X(\phi_i')$. This is true for all $i = 1, \ldots, s$. Thus, in order to prove the result for Y, it suffices to prove it for Y'. But Y' is a union of Schubert varieties in $\mathrm{Gr}_{\underline{d}',n}$, and so by induction on r, the result is true for Y'.

Now we assume $a_1 \geq 1$. We have, by induction hypothesis, that

$$h^0(Y, \underline{a}') \geq s(Y, \underline{a}'), \text{where } \underline{a}' = (a_1 - 1, a_2, \ldots, a_r).$$

Let $p_{\mathcal{T}^k}$, $1 \leq k \leq t$, be a minimal set of linearly dependent standard monomials on Y. We fix for each tableau a defining chain $\underline{w}^k = (w_1^k, \ldots)$ such that $w_1^k \leq \phi_m$ for some m. We distinguish two cases.

Case 1:

All tableaux $\mathcal{T}^1, \ldots, \mathcal{T}^t$ have the same first column $\underline{i}$. By definition, each $p_{\mathcal{T}^k}$, $1 \leq k \leq t$, is standard on a Schubert variety Y_k which is an irreducible component of Y. Let $Z = Y_1 \cup \cdots \cup Y_t$. Observe that $p(\mathcal{T}^k)$, $1 \leq k \leq t$, is standard on Z.

Let $\mathcal{T'}^k$ be the tableau obtained by omitting the first column of $\mathcal{T}^k$. We can write $p_{\mathcal{T}^k}$ as a product $p_{\mathcal{T}^k} = p_{\underline{i}} p_{\mathcal{T}'^k}$ for all $k = 1, \ldots, t$.

Observe that the $p_{\mathcal{T}'^k}$ are standard monomials of multidegree $\underline{a}' = (a_1 - 1, a_2, \ldots, a_l)$ on Z, and hence are linearly independent (by induction). Since Z is reduced and $p_{\underline{i}} \neq 0$ on Y_k $(1 \leq k \leq t)$, we find that a non-trivial dependency relation for the $p_{\mathcal{T}^k}$'s gives one for the $p_{\mathcal{T}'^k}$'s, which is a contradiction.

Case 2:

Let $\underline{i}^k$ be the first column of the tableau $\mathcal{T}^k$, fix a minimal element $\underline{i}$ in the set $\{\underline{i}^1, \ldots, \underline{i}^t\}$. Without loss of generality we may assume that the enumeration is such that $\underline{i} = \underline{i}^1 = \ldots = \underline{i}^{t_0}$ and $\underline{i} \not\geq \underline{i}^k$ for $k \geq t_0 + 1$.

Let $\theta^k \in \mathfrak{S}_n / \mathfrak{S}_{\underline{d}}$ be the class of the first element $w_1^k \in \mathfrak{S}_n$ in the defining chain for $\mathcal{T}^k$. We set $Y_0 = X(\theta^1) \cup \cdots \cup X(\theta^{t_0})$. By definition, $Y_0 \subseteq Y$ and $\eta(Y_0) = X_{\underline{i}}$ in G/P_{d_1}, where $\eta : SL_n(k)/Q_{\underline{d}} \to SL_n(k)/P_{d_1}$.

By the choice of $\underline{i}$ we have $p_{\underline{i}^k} = 0$ on $X_{\underline{i}}$ for $t_0 + 1 \leq k \leq t$, so we know $p_{\underline{i}^k} = 0$ on Y_0 for $t_0 + 1 \leq k \leq t$ and hence we get

$$p_{\mathcal{T}^k} = 0, \text{ on } Y_0, \text{ for } t_0 + 1 \leq k \leq t.$$

The monomials $p_{\mathcal{T}^k}$, $1 \leq k \leq t_0$, are standard on Y_0 and are linearly independent on Y_0, by case 1. Now start with a dependency relation on Y, say

$$c_1 p_{\mathcal{T}^1} + \cdots + c_t p_{\mathcal{T}^t} = 0, \;\; c_i \in k^*.$$

Restricting this relation to Y_0, we get the nontrivial relation

$$c_1 p_{\mathcal{T}^1} + \cdots + c_{t_0} p_{\mathcal{T}^{t_0}} = 0 \text{ on } Y_0,$$

which implies $c_i = 0$, $1 \leq i \leq t_0$. But this contradicts the minimality assumption on $\{p_{\mathcal{T}^k}\}_{k=1,\dots,t}$. This completes the proof. □

2.1.22. A generalization: arbitrary enumerations of fundamental weights.

So far we have used the standard enumeration of the fundamental weights. This was actually nowhere necessary except that we could use the well known semistandard Young tableaux as an indexing system for the standard monomials. Actually, everything we have proved so far also holds for the standard monomials which are standard in the following sense:

Let now $\omega_1, \dots, \omega_{n-1}$ be an arbitrary enumeration of the fundamental weights. The stabilizer of ω_i in $\mathfrak{S}_n$ is denoted $\mathfrak{S}_{\omega_i}$.

Let $\lambda = \sum_{j=1}^{n-1} a_j \omega_j$ be a dominant weight. The following is a straightforward generalization of the notion of a Young tableau given above.

Definition 2.1.23. A *tableau* of shape λ is a sequence of $|\lambda| = a_1 + a_2 + \ldots + a_{n-1}$ classes of elements $\mathcal{T} = (\tau^i_j)_{\substack{1 \leq i \leq n-1 \\ 1 \leq j \leq a_i}}$ such that $\tau^i_j \in \mathfrak{S}_n/\mathfrak{S}_{\omega_i}$:

$$\mathcal{T} = (\underbrace{\tau^1_1, \dots, \tau^1_{a_1}}_{\in \mathfrak{S}_n/\mathfrak{S}_{\omega_1}}, \underbrace{\tau^2_1, \dots, \tau^2_{a_2}}_{\in \mathfrak{S}_n/\mathfrak{S}_{\omega_2}}, \dots, \underbrace{\tau^{n-1}_1, \dots, \tau^{n-1}_{a_{n-1}}}_{\in \mathfrak{S}_n/\mathfrak{S}_{\omega_{n-1}}}).$$

As before, denote by P_i the maximal parabolic subgroup associated to ω_i. For a strictly increasing sequence $\underline{d}$, $1 \leq d_1 < \ldots < d_r \leq n-1$, set $Q_{\underline{d}} = P_{d_1} \cap \ldots \cap P_{d_r}$ and let $\mathfrak{S}_{\underline{d}} = \mathfrak{S}_{\omega_{d_1}} \cap \ldots \cap \mathfrak{S}_{\omega_{d_r}}$.

Let $SL_n(k)/Q_{\underline{d}}$ be the associated (possibly) partial flag variety. Fix a dominant weight $\lambda = \sum_{j \in \{d_1,\dots,d_r\}}^{n-1} a_j \omega_j$, then λ gives rise to a line bundle $\mathcal{L}_\lambda$ on $SL_n(k)/Q_{\underline{d}}$.

For a class $\tau \in \mathfrak{S}_n/\mathfrak{S}_{\omega_j}$ we have the section $p_\tau \in H^0(SL_n(k)/P_i, \mathcal{L}_{\omega_i})$. We associate to the tableau $\mathcal{T}$ the monomial *of shape* λ

$$p_{\mathcal{T}} = \prod_{\substack{1 \leq i \leq n-1 \\ 1 \leq j \leq a_i}} p_{\tau^i_j} \in H^0(SL_n(k)/Q_{\underline{d}}, \mathcal{L}_\lambda).$$

Definition 2.1.24. A tableau $\mathcal{T}$ is called a *semistandard tableau* if the tableau admits a defining chain, i.e. there exists a sequence $(w^i_j)_{\substack{1 \leq i \leq n-1 \\ 1 \leq j \leq a_i}}$ of elements in $\mathfrak{S}_n$ such that the sequence is linearly ordered:

$$w^1_1 \geq w^1_2 \geq \ldots \geq w^1_{a_1} \geq w^2_1 \geq \ldots w^2_{a_2} \geq w^3_1 \geq \ldots \geq w^{n-1}_{a_{n-1}}$$

and $w_1^1 \equiv \tau_1^1 \bmod \mathfrak{S}_{\omega_1}$, $w_2^1 \equiv \tau_2^1 \bmod \mathfrak{S}_{\omega_1}$, …, $w_1^2 \equiv \tau_1^2 \bmod \mathfrak{S}_{\omega_2}$, …, $w_{a_{n-1}}^{n-1} \equiv \tau_{a_{n-1}}^{n-1} \bmod \mathfrak{S}_{\omega_{n-1}}$.

The monomial $p_{\mathcal{T}}$ is called *standard* if the tableau is semistandard.

Definition 2.1.25. The tableau $\mathcal{T}$ of shape λ is called *semistandard on a Schubert variety* $X(\kappa) \subset SL_n(k)/Q_{\underline{d}}$ if the tableau is standard and admits a defining chain such that $\kappa \geq w_1^1 \bmod \mathfrak{S}_{\underline{d}}$. The tableau is called *semistandard on a union of Schubert varieties* if the tableau is standard on at least one of the irreducible components.

Similarly, the monomial $p_{\mathcal{T}}$ is called *standard on a Schubert variety respectively a union of Schubert varieties* if the tableau is semistandard on the Schubert variety respectively the union of Schubert varieties.

2.2 A basis for $H^0(X,\mathcal{L})$ and vanishing of higher cohomology groups

2.2.1. SMT-basis for unions of Schubert varieties

Let $Q \subset SL_n(k)$ be a parabolic subgroup, say $Q = Q_{\underline{d}}$. Fix a dominant weight $\lambda = \sum_{j\in\{d_1,\ldots,d_r\}}^{n-1} a_j\omega_j$, so λ gives rise to a line bundle $\mathcal{L}_\lambda$ on $SL_n(k)/Q$. We want to prove:

Theorem 2.2.2. *The standard monomials of shape λ, standard on a union of Schubert varieties $Y \subset SL_n(k)/Q$, form a basis for the space of sections $H^0(Y,\mathcal{L}_\lambda)$. In other words:*

$$\dim H^0(Y,\mathcal{L}_\lambda) = \sharp\{\textit{standard monomials } p_{\mathcal{T}} \textit{ of shape } \lambda\}. \tag{2.2}$$

The proof needs some preparation. Note that to fix an enumeration of the fundamental weights is equivalent to fix an enumeration of the maximal parabolic subgroups $P_1,\ldots,P_{n-1}$. Set $Q_i = P_i \cap \ldots \cap P_{n-1}$, then we get a sequence

$$SL_n(k) \supset Q_{n-1} \supset Q_{n-2} \supset \ldots \supset Q_2 \supset Q_1 = B$$

of parabolic subgroups and maps

$$SL_n(k)/Q_{n-1} \leftarrow SL_n(k)/Q_{n-2} \leftarrow \ldots \leftarrow SL_n(k)/Q_2 \leftarrow SL_n(k)/B.$$

Given a parabolic subgroup Q of $SL_n(k)$, then one can always find an enumeration of the fundamental weights such that Q is conjugate to some Q_i in the sequence above. The theorem has been already proved for $Q = Q_{n-1}$ a maximal parabolic subgroup, the proof will be by an inductive procedure using the maps above.

2.2.3. Chow ring of $SL_n(k)/Q$.

We recall first the Chevalley formula for the Chow ring (or intersection, see [85]). In the following let $G = SL_n(k)$. Let $\mathrm{Chow}(G/B)$ denote the

Chow ring of G/B. For $\tau \in W$, let $[X(\tau)]$ denote the element of $\mathrm{Chow}(G/B)$ determined by the Schubert variety $X(\tau)$. We have from [51] that the elements $[X(\tau)]$, $\tau \in W$, form a basis of $\mathrm{Chow}(G/B)$ considered as a $\mathbb{Z}$-module. Let $P \supset B$ be a maximal parabolic subgroup with ω (resp. α) as the associated fundamental weight (resp. simple root). Let H_P be the unique codimension one Schubert subvariety of G/P. Then, $X(s_\alpha w_0)$ (w_0 being the unique element of maximal length in $W = \mathfrak{S}_n$) is the inverse image of H_P under the canonical projection $\pi : G/B \to G/P$. Consider now the multiplication $[X(\tau)] \cdot [X(s_\alpha w_0)]$ in the ring $\mathrm{Chow}(G/B)$. Let $[X(\tau)] \cdot [X(s_\alpha w_0)] = \sum d_i [X(\tau_i)]$, where the sum runs over all the Schubert divisors $X(\tau_i)$ in $X(\tau)$. Let $\tau_i = \tau s_{\beta_i}$, for some positive root β_i. Chevalley showed in [51] that $d_i = (\omega, \beta_i^\vee)$. Since all the fundamental weights of $SL_n(k)$ are minuscule, it follows that $0 \leq d_i \leq 1$. Thus we obtain

$$[X(\tau)] \cdot [X(s_\alpha w_0)] = \sum d_i [X(\tau_i)],\ d_i \neq 0 \tag{2.3}$$

where the summation runs over a certain set of Schubert divisors in $X(\tau)$.

Let $\pi(X(\tau)) = X_P(\theta)$, where $\theta \in \mathfrak{S}_n / \mathfrak{S}_\omega$. We use the index P to avoid a possible confusion between Schubert varieties in G/P and G/Q. Suppose that $\dim X_P(\theta) > 0$. Then one knows from 1.6.2 that $X_P(\theta) \cap \{p_\theta = 0\}$ is precisely the union of all Schubert divisors in $X_P(\theta)$. Hence we obtain $X(\tau) \cap \{p_\theta = 0\} = X(\tau) \cap \pi^{-1}(X_P(\theta) \cap \{p_\theta = 0\})$. Thus we get Pieri's formula,

$$X(\tau) \cap \{p_\theta = 0\} = \bigcup_i X(\tau_i),\ \tau_i \text{ as in (2.3).} \tag{2.4}$$

Further, from (2.3) it follows that the intersection multiplicity of the left hand side of (2.4) along any $X(\tau_i)$ is 1 (cf. [107] for the definition of intersection multiplicity). This together with the normality of $X(\tau)$ implies that the scheme-theoretic intersection $X(\tau) \cap \{p_\theta = 0\}$ is reduced (cf. [245]). Thus we obtain that the equality in (2.4) is scheme-theoretic.

These arguments generalize to the setting G/Q, where $Q = Q_{\underline{d}}$ for some $\underline{d}$. A codimension 1 Schubert variety in G/Q is of the form $X(s_\alpha u_0)$, where we write u_0 for the image of w_0 in $\mathfrak{S}/\mathfrak{S}_{\underline{d}}$. Let $X(\tau)$ be a Schubert variety in G/Q not contained in $X(s_\alpha u_0)$, then, as in the case $Q = B$,

$$[X(\tau)] \cdot [X(s_\alpha w_0)] = \sum d_i [X(\tau_i)],\ d_i \neq 0, \tag{2.5}$$

where the $X(\tau_i)$ are codimension 1 subvarieties of $X(\tau)$ and d_i is the intersection multiplicity. As above, let $\tau_i = \tau s_{\beta_i}$, for some positive root β_i. Since $X(s_\alpha w_0) \subset G/B$ is the preimage of $X(s_\alpha u_0) \subset G/Q$ with respect to the projection $\phi : G/B \to G/Q$, the projection formula implies $d_i = (\omega, \beta_i^\vee)$, and, since all fundamental weights of $SL_n(k)$ are minuscule, it follows that $0 \leq d_i \leq 1$. Now as above, the same arguments show that $X(\tau) \cap \{p_\theta = 0\}$ is reduced.

Remark 2.2.4. We assume in this section the normality of the Schubert varieties, see for example [208]. The normality can also be proved using SMT, but at this point, to simplify the proof, we omit this part. In the discussion of the general case (section **??**) we will present the proof of the normality of Schubert varieties as a consequence of SMT.

2.2.5. A reduction procedure.

Let $Q \subset SL_n(k)$ be a parabolic subgroup. We assume the enumeration of the fundamental weights is such that say $Q = Q_i$ for some i. Set $\mathfrak{S}_Q = \mathfrak{S}_{\omega_i} \cap \mathfrak{S}_{\omega_{i+1}} \cap \ldots \cap \mathfrak{S}_{\omega_{n-1}}$, then the Schubert varieties in $SL_n(k)$ are indexed by the classes in $\mathfrak{S}_n/\mathfrak{S}_Q$.

Let $\lambda = \sum_{j=i}^{n-1} a_j\omega_j$ be a dominant weight, we often write just $\underline{a} = (a_i, \ldots, a_{n-1})$ for λ. Then λ gives rise to a line bundle $\mathcal{L}_\lambda$ on $SL_n(k)/Q$. We write $s(Y,\underline{a})$ for the number of standard monomials standard on a union of Schubert varieties $Y \subset G/Q$, and we write $h^0(Y,\underline{a})$ for the dimension of the space of global sections of $\mathcal{L}_\lambda$ on Y.

Lemma 2.2.6. *Let $\phi_i, \psi_j \in \mathfrak{S}_n/\mathfrak{S}_Q$, $1 \le i \le s$, $1 \le j \le t$, $Y_1 = \bigcup_{i=1}^s X(\phi_i)$, $Y_2 = \bigcup_{j=1}^t X(\psi_j)$ and $\underline{a} = (a_i, \ldots, a_{n-1}) \in \mathbb{Z}_+^{n-i}$ as above. Then we have*

$$s(Y_1 \cup Y_2, \underline{a}) = s(Y_1, \underline{a}) + s(Y_2, \underline{a}) - s((Y_1 \cap Y_2)_{\mathrm{red}}, \underline{a}).$$

Proof. The intersection $(Y_1 \cap Y_2)_{\mathrm{red}}$ is closed and B-stable and hence a union of Schubert varieties, say

$$(Y_1 \cap Y_2)_{\mathrm{red}} = \bigcup_{i,j,\ell} X(\sigma_{ij}^\ell),$$

for suitable $\sigma_{ij}^\ell \in \mathfrak{S}_n/\mathfrak{S}_Q$ such that $\phi_i, \psi_j \ge \sigma_{ij}^\ell$. Let $\mathcal{T}$ be a semistandard tableau of shape λ. We see that we only have to check that if $p_\mathcal{T}$ is standard on $X(\phi_k)$ and also on $X(\psi_l)$ for some k, l with $1 \le k \le s$ and $1 \le l \le t$, then $p_\mathcal{T}$ is standard on $X(\sigma_{kl}^q)$ for some σ_{kl}^q. Let us look at the minimal defining sequence $\underline{w}^- = (w_1^-, \ldots)$ for the standard Young tableau $\mathcal{T}$, which depends only on $\mathcal{T}$, but not on ϕ_k or ψ_l (cf. Remark 2.1.17 and Lemma **??**). We have

$$\phi_k \ge w_1^- \bmod \mathfrak{S}_Q \text{ and } \psi_l \ge w_1^- \bmod \mathfrak{S}_Q \quad \text{in } \mathfrak{S}_n/\mathfrak{S}_Q.$$

But then we know that $\phi_k, \psi_l \ge \sigma_{kl}^q \ge w_1^- \bmod \mathfrak{S}_Q$ for some σ_{kl}^q, hence $p_\mathcal{T}$ is standard on $X(\sigma_{kl}^q)$ as required. □

Lemma 2.2.7. *Let $\underline{a} = (a_i, \ldots, a_{n-1}) \in \mathbb{Z}_+^{n-i}$ be such that $a_i \ge 1$. Let $\underline{a}' = (a_i - 1, a_2, \ldots, a_{n-1})$. and τ, τ_i be as in (2.5). Then we have*

$$s(X(\tau), \underline{a}) - s(X(\tau), \underline{a}') = s(\bigcup_{i=1}^r X(\tau_i), \underline{a}).$$

Proof. Let $\pi_G/Q \to G/P_i$ the projection and set $\pi(X(\tau)) = X_{P_i}(\theta)$. We write $S(X(\tau), \underline{a})$ for the set of standard monomials of multidegree $\underline{a}$. It is clear that by the mapping $q \mapsto p_\theta q$, we can identify $S(X(\tau), \underline{a}')$ with the subset of elements of $S(X(\tau), \underline{a})$ which begin with p_θ. Set $\pi(X(\tau_j) = X_{P_i}(\theta_j)$, note that $\theta > \theta_j$ for all j.

Let now $p_{\mathcal{T}}$ be a standard monomial of multidegree $\underline{a}$, standard on $X(\tau)$, but such that $p_{\mathcal{T}}$ does not start with p_θ. So $p_{\mathcal{T}}$ starts with p_σ such that $\sigma < \theta$ (otherwise p_σ would vanish on X_{P_i} and hence of $X(\tau)$), and hence $\sigma \leq \theta_j$ for some j. Let $\underline{w}^- = (w_1^-, \ldots)$ be the minimal defining chain for $\mathcal{T}$, then $w_1^- \leq \tau \bmod \mathfrak{S}_Q$. In fact, since $w_1^- = \theta_j < \theta \bmod \mathfrak{S}_{\omega_i}$, we have $w_1^- < \tau \bmod \mathfrak{S}_Q$. Now since $\tau_j \in \mathfrak{S}_n/\mathfrak{S}_Q$ is maximal with the properties $\tau_j = \theta_j \bmod \mathfrak{S}_{\omega_i}$ and $\tau_j < \tau$ (see Deodhar's Lemma, Lemma **??**), it follows that $w_1^- \leq \tau_j \bmod \mathfrak{S}_Q$, so $p_{\mathcal{T}}$ is standard on the union of the $X(\tau_j)$. □

If Z is a closed subscheme of G/Q, let us denote by $\mathcal{O}_Z(\underline{a})$ and $\mathcal{O}_Z(m)$ the invertible sheaves on Z, associated respectively to the line bundle $\mathcal{L}_\lambda|_Z$ and $\mathcal{L}_{m\rho_Q}|_Z$ where $\lambda = \sum_{j=i}^{n-1} a_j\omega_i$, $\rho_Q = \sum_{j=i}^{n-1} \omega_j$ and $m \in \mathbb{Z}$. Note that $\mathcal{O}_Z(1)$ is a very ample sheaf on Z. If Y is a union of Schubert varieties in G/Q endowed with its canonical reduced structure, we shall write $S(Y, m\rho_Q)$, $s(Y, m\rho_Q)$ for $S(Y, (m, \ldots, m))$ and $s(Y, (m, \ldots, m))$ respectively for the set (the number) of standard monomials of the corresponding multidegree.

Lemma 2.2.8. *Let Y_1, Y_2 be unions of Schubert varieties of G/Q. Suppose that for $m \gg 0$ we have*

$$h^0(Y_i, m\rho_Q) = s(Y_i, m\rho_Q), \quad i = 1, 2.$$

Then we have

(a) the scheme-theoretic intersection $Y_1 \cap Y_2$ is reduced.

(b) $h^0(Y_1 \cup Y_2, m\rho_Q) = s(Y_1 \cup Y_2, m\rho_Q)$, and $h^0(Y_1 \cap Y_2, m\rho_Q) = s(Y_1 \cap Y_2, m\rho_Q)$, for $m \gg 0$.

Proof. We have the following exact sequence of sheaves of $\mathcal{O}_{G/Q}$-modules

$$0 \to \mathcal{O}_{Y_1 \cup Y_2} \to \mathcal{O}_{Y_1} \oplus \mathcal{O}_{Y_2} \to \mathcal{O}_{Y_1 \cap Y_2} \to 0$$

where $Y_1 \cap Y_2$ (resp. $Y_1 \cup Y_2$) denotes the scheme-theoretic intersection (resp. union). This gives the exact sequence

$$0 \to \mathcal{O}_{Y_1 \cup Y_2}(m) \to \mathcal{O}_{Y_1}(m) \oplus \mathcal{O}_{Y_2}(m) \to \mathcal{O}_{Y_1 \cap Y_2}(m) \to 0.$$

Consider the cohomology exact sequence, and using Serre's vanishing theorem (cf. [244]), we obtain

$$h^0(Y_1 \cap Y_2, m\rho_Q) = h^0(Y_1, m\rho_Q) + h^0(Y_2, m\rho_Q) - h^0(Y_1 \cup Y_2, m\rho_Q), \text{for } m \gg 0. \tag{2.6}$$

We have, by Lemma 2.1.21

$$h^0(Y_1\cup Y_2, m\rho_Q) \geq s(Y_1\cup Y_2, m\rho_Q).$$

Thus, in view of our hypothesis, we have for $m \gg 0$ that,

$$\begin{aligned} h^0(Y_1, m\rho_Q) + h^0(Y_2, m\rho_Q) - h^0(Y_1\cup Y_2, m\rho_Q) \\ \leq s(Y_1, m\rho_Q) + s(Y_2, m\rho_Q) - s(Y_1\cup Y_2, m\rho_Q). \end{aligned}$$

Now by Lemma 2.2.6, we have,

$$s(Y_1, m\rho_Q) + s(Y_2, m\rho_Q) - s(Y_1\cup Y_2, m\rho_Q) = s((Y_1\cap Y_2)_{\mathrm{red}}, m\rho_Q).$$

Thus we conclude that

$$h^0(Y_1\cap Y_2, m\rho_Q) \leq s((Y_1\cap Y_2)_{\mathrm{red}}, m\rho_Q). \tag{2.7}$$

On the other hand, we see that if $Y_1\cap Y_2$ is not reduced, then

$$h^0(Y_1\cap Y_2, m\rho_Q) > h^0((Y_1\cap Y_2)_{\mathrm{red}}, m\rho_Q), \text{ for } m \gg 0.$$

Also, by Lemma 2.1.21, we have

$$s((Y_1\cap Y_2)_{\mathrm{red}}, m\rho_Q) \leq h^0((Y_1\cap Y_2)_{\mathrm{red}}, m\rho_Q),$$

and hence we obtain

$$h^0(Y_1\cap Y_2, m\rho_Q) > s((Y_1\cap Y_2)_{\mathrm{red}}, m\rho_Q), \text{ for } m \gg 0.$$

This contradicts (2.7), and we conclude that $Y_1\cap Y_2$ is reduced.

The fact that $Y_1\cap Y_2$ is reduced together with (2.7) and linear independence of standard monomials on $Y_1\cap Y_2$ (cf. 2.1.21) implies that $h^0(Y_1\cap Y_2, m\rho_Q) = s(Y_1\cap Y_2, m\rho_Q)$, for $m \gg 0$. Also (2.6) is satisfied with h^0 replaced by s (cf. Lemma 2.2.6). From this we obtain, by our hypothesis, that $h^0(Y_1\cup Y_2, m\rho_Q) = s(Y_1\cup Y_2, m\rho_Q)$, $m \gg 0$. □

Lemma 2.2.9. *Let S_d denote the set of all reduced subschemes Y of G/Q such that every irreducible component of Y is a Schubert variety of dimension $\leq d$. Suppose that for every Schubert variety $X \in S_d$ we have*

$$s(X, \underline{a}) = h^0(X, \underline{a}),\ \ \underline{a} = (a_i, \ldots, a_{n-1}), a_j \geq 0. \tag{2.8}$$

Then

(*i*) *for any $Y_1, Y_2 \in S_d$, $Y_1\cap Y_2$ is reduced,*

(*ii*) *the assertion* (2.8) *holds for any $Y \in S_d$.*

Proof. Suppose that $Y \in S_d$. By induction on the number of components of Y, it follows, (as an immediate extension of the argument of Lemma 2.2.8), that $h^0(Y, m\rho_Q) = s(Y, m\rho_Q)$ for $m \gg 0$, and by Lemma 2.2.8, assertion (i) follows. Thus, it only remains to prove assertion (ii).

We observe that (ii) obviously holds for S_0. We now prove (ii) by induction on t. Let $t \leq d$ and suppose that (ii) holds for every $X \in S_m$, $m < t$; then we shall show that (ii) holds also for every $X \in S_t$. We shall make another inductive argument, namely, if $C(X)$ denotes the number of irreducible components of $X \in S_t$, we suppose that (ii) is true for $X \in S_t$ such that $C(X) \leq r-1$. We now take $X \in S_t$ such that $C(X) = r$ and will show that (ii) holds for X. Let

$$X = X_1 \cup \cdots \cup X_r$$

where X_i are the distinct irreducible components of X. We set

$$Y_1 = X_1 \cup \cdots \cup X_{r-1} \text{ and } Y_2 = X_r.$$

By (i), $Y_1 \cap Y_2$ is reduced, and we get the following exact sequence of $\mathcal{O}_{G/Q}$-modules

$$0 \to \mathcal{O}_X(\underline{a}) \to \mathcal{O}_{Y_1}(\underline{a}) \oplus \mathcal{O}_{Y_2}(\underline{a}) \to \mathcal{O}_{Y_1 \cap Y_2}(\underline{a}) \to 0. \tag{2.9}$$

We see that $Y_1 \cap Y_2 \in S_{t-1}$ since the X_i are the distinct irreducible components of X. By our inductive hypothesis, (ii) holds for Y_1, Y_2 and $Y_1 \cap Y_2$. Hence $H^0(Y_1 \cap Y_2, \mathcal{L}_{\underline{a}})$ has a basis of standard monomials. This implies that the canonical mapping

$$H^0(\mathcal{O}_{Y_1}(\underline{a}) \oplus \mathcal{O}_{Y_2}(\underline{a})) \to H^0(\mathcal{O}_{Y_1 \cap Y_2}(\underline{a}))$$

is surjective. Writing the cohomology exact sequence of (2.9), we obtain

$$h^0(Y_1, \underline{a}) + h^0(Y_2, \underline{a}) = h^0(X, \underline{a}) + h^0(Y_1 \cap Y_2, \underline{a}). \tag{2.10}$$

On the other hand, we have

$$h^0(Y_i, \underline{a}) = s(Y_i, \underline{a}), i = 1, 2, \;\; h^0(Y_1 \cap Y_2, \underline{a}) = s(Y_1 \cap Y_2, \underline{a}).$$

Hence, from (2.10) we get

$$h^0(X, \underline{a}) = s(Y_1, \underline{a}) + s(Y_2, \underline{a}) - s(Y_1 \cap Y_2, \underline{a}). \tag{2.11}$$

By Lemma 2.2.6, we have

$$s(Y_1, \underline{a}) + s(Y_2, \underline{a}) - s(Y_1 \cap Y_2, \underline{a}) = s(Y_1 \cup Y_2, \underline{a}).$$

Since $X = Y_1 \cup Y_2$, we get that

$$h^0(X, \underline{a}) = s(X, \underline{a}).$$

This completes the proof of the Lemma. □

Remark 2.2.10. In Lemma 2.2.9, if in addition to (2.8) we suppose also that for every Schubert variety $X \in S_d$,

$$H^i(X, \mathcal{L}_{\underline{a}}) = 0,\ i > 0, \underline{a} \geq 0,$$

then the same proof gives also, in addition,

(*iii*) For $Y \in S_d$, we have $H^i(Y, \mathcal{L}_{\underline{a}}) = 0$ for $i > 0$, $\underline{a} = (a_i, \ldots, a_{n-1})$, $a_j \geq 0$.

2.2.11. Proof of Theorem 2.2.2.

We shall now prove the main basis theorem:

Proof. We prove the Theorem using various induction procedures. Recall that we assume the enumeration of the fundamental weights is such that $Q = Q_i$ for some $1 \leq i \leq n-1$.

One induction procedure is on $\dim Y$. If $\dim Y = 0$, then clearly the Theorem is true. By Lemma 2.2.9, it suffices to prove (2.2) for the case when Y is a Schubert variety. So let us suppose that $Y = X(\tau)$.

Let now $\underline{a} = (a_i, \ldots, a_{n-1}) \in \mathbb{Z}_+^{n-i}$. Suppose there is a $j > i$ such that $a_k = 0$ for $k < j$. Let Q_j be the parabolic subgroup $Q_j = P_j \cap P_{j+1} \cap \cdots \cap P_{n-1}$ and $\pi : G/Q_i \to G/Q_j$ the canonical morphism of G/Q_i onto G/Q_j. We see that $\mathcal{L}_{\underline{a}}$ descends to a line bundle on G/Q_j, which again we denote by $\mathcal{L}_{\underline{a}}$. Let Y be the Schubert variety in G/Q_j which is the image of X (under π). We see that standard monomials on X of type $\underline{a}$ are precisely standard monomials on Y of type $\underline{a}$. One knows that Y is normal (cf. see Remark 2.2.4). From this it follows that if $f : X \to Y$ is the canonical morphism, then $f_*\mathcal{O}_X = \mathcal{O}_Y$ (there is an open subset of X, namely the big cell which is a "product" over the big cell in Y, which shows that the function field of Y is algebraically closed in that of X, etc.; see also [247]). This fact implies that $H^0(X, \mathcal{L}_{\underline{a}})$ is isomorphic to $H^0(Y, \mathcal{L}_{\underline{a}})$. Thus it suffices to prove the theorem for the Schubert variety Y in G/Q_j. Suppose that $j = n-1$ so that $Q_{n-1} = P_{n-1}$. Then we are reduced to the case of one maximal parabolic subgroup, where (2.2) is known (cf. Theorem 1.6.4). We do one more inductive argument, namely assume (2.2) for $\underline{a}$ of the form

$$a_k = 0,\ k < j+1$$

or what can be termed as assuming (2.2) for Schubert varieties in G/Q_k, $k \geq (i+1)$. Thus we may suppose that (2.2) holds when $a_i = 0$.

Let $X_{P_i}(\theta)$ be the image of $X(\tau)$ in G/P_i and let $X(\tau_\ell)$ be the Schubert divisors in $X(\tau) \cap \{p_\theta = 0\}$. Let $H(\tau)$ be the union of the Schubert varieties $X(\tau_\ell)$

$$H(\tau) = \bigcup_\ell X(\tau_\ell).$$

Then by the scheme-theoretic Pieri's formula, we get the exact sequence

$$0 \to \mathcal{O}_{X(\tau)}(-1,0,0,\dots,0) \to \mathcal{O}_{X(\tau)} \to \mathcal{O}_{H(\tau)} \to 0. \tag{2.12}$$

Tensoring with $\mathcal{L}_{\underline{a}}$, we obtain the exact sequence

$$0 \to \mathcal{O}_{X(\tau)}(\underline{a}') \to \mathcal{O}_{X(\tau)}(\underline{a}) \to \mathcal{O}_{H(\tau)}(\underline{a}) \to 0 \tag{2.13}$$

where $\underline{a}' = (a_1 - 1, a_2, \dots, a_{n-1})$. By our induction hypothesis, we have $s(H(\tau), \underline{a}) = h^0(H(\tau), \underline{a})$, which implies, in particular, that the canonical mapping

$$H^0(X(\tau), \mathcal{L}_{\underline{a}}) \to H^0(H(\tau), \mathcal{L}_{\underline{a}})$$

is surjective. Writing the cohomology exact sequence of (2.13), we get the exact sequence

$$0 \to H^0(X(\tau), \mathcal{L}_{\underline{a}'}) \to H^0(X(\tau), \mathcal{L}_{\underline{a}}) \to H^0(H(\tau), \mathcal{L}_{\underline{a}}) \to 0.$$

This gives

$$h^0(X(\tau), \underline{a}) = h^0(X(\tau), \underline{a}') + h^0(H(\tau), \underline{a}). \tag{2.14}$$

Now we prove (2.2) by induction on a_1. If $a_1 = 0$, as we observed above, (2.2) holds. Let $a_1 > 0$. Then by induction, we may suppose that

$$h^0(X(\tau), \underline{a}') = s(X(\tau), \underline{a}'),$$

On the other hand, by our induction hypothesis, we also have (as observed above)

$$h^0(H(\tau), \underline{a}) = s(H(\tau), \underline{a}).$$

Hence (2.14) gives

$$h^0(X(\tau), \underline{a}) = s(X(\tau), \underline{a}') + s(H(\tau), \underline{a}). \tag{2.15}$$

On the other hand, by Lemma 2.2.7 we have

$$s(X(\tau), \underline{a}) = s(X(\tau), \underline{a}') + s(H(\tau), \underline{a}),$$

which implies that

$$s(X(\tau), \underline{a}) = h^0(X(\tau), \underline{a}).$$

This proves (2.2) for $X(\tau)$ and concludes the proof of the theorem. □

Corollary 2.2.12. *Let Y_1, Y_2 be unions of Schubert varieties in G/Q. Then the scheme-theoretic intersection $Y_1 \cap Y_2$ is reduced.*

This is an immediate consequence of Lemma 2.2.9 and Theorem 2.2.2.

2.2.13. Vanishing Theorems

Theorem 2.2.14. *Let X be a union of Schubert varieties in G/Q, endowed with its canonical reduced structure. Then we have for all $i > 0$,*

$$H^i(X, \mathcal{L}_{\underline{a}}) = 0, \ \underline{a} \in \mathbb{Z}_+^{n-i}. \tag{2.16}$$

Proof. In view of Remark 2.2.10, we see that it suffices to prove (2.16) when X is a Schubert variety $X(\tau)$, $\tau \in W$. The proof is carried out in the same spirit as that of Theorem 1.6.4. Consider the exact sequence (2.13) in the proof of Theorem 2.2.2, we assume again that the enumeration of the fundamental weights is such that $Q = Q_i$ for some i:

$$0 \to \mathcal{O}_{X(\tau)}(\underline{a}') \to \mathcal{O}_{X(\tau)}(\underline{a}) \to \mathcal{O}_{H(\tau)}(\underline{a}) \to 0, \tag{2.17}$$

where $\underline{a}' = (a_1 - 1, a_2, \ldots, a_{n-1})$. By a suitable induction hypothesis, we can suppose that

(a) $H^i(X(\tau), \mathcal{L}_{\underline{a}'}) = 0$, $i > 0$,

(b) $H^i(H(\tau), \mathcal{L}_{\underline{a}}) = 0$, $i > 0$.

Then writing the cohomology sequence for (2.17), (2.16) would follow: but there is one crucial point, namely that we may not have $\underline{a}' \in \mathbb{Z}_+^{n-i}$. But this is only possible in the case $a_i = 0$, and this case could be treated by taking the image of $X(\tau)$ in G/Q_j for some $j > i$, and proceed on the same lines as in the proof of Theorem 2.2.2. □

2.3 Ideal theory of Schubert varieties

In view of Corollary 2.2.12, we obtain that the intersection of a family of Schubert varieties is reduced. Note also that a union of Schubert varieties is obviously reduced. In particular, this enables us to compute the ideal sheaves of Schubert varieties as given by Theorem 2.3.10 below.

Let P_i, $1 \leq i \leq n-1$ be the maximal parabolic subgroups of $G = SL_n(k)$ with ω_i as the associated fundamental weight (standard enumeration). We have the Plücker embedding of the Grassmannian, namely

$$G/P_i \hookrightarrow \mathbb{P}(H^0(G/P_i, \mathcal{L}_{\omega_i})^*),$$

where $H^0(G/P_i, \mathcal{L}_{\omega_i})^*$ is the dual of $H^0(G/P_i, \mathcal{L}_{\omega_i})$. The homogeneous coordinate ring R_i of G/P_i for the above embedding is given by $R_i = \bigoplus_{r \geq 0} H^0(G/P_i, \mathcal{L}_{r\omega_i})$ (cf. Corollary 1.6.9). By the "diagonal embedding" of the flag variety

$$G/B \to \prod_{1 \leq i \leq n-1} G/P_i,$$

we consider the closed immersion

$$G/B \hookrightarrow \prod_{1 \leq i \leq n-1} \mathbb{P}^{m_i}, \ \mathbb{P}^{m_i} = \mathbb{P}(H^0(G/P_i, L_{\omega_i})^*).$$

We denote by $\mathcal{L}_i$ the ample generator of $\mathrm{Pic}(\mathbb{P}^{m_i})$. The restriction of $\mathcal{L}_i$ to G/P_i is $\mathcal{L}_{\omega_i}$. We denote by S the multi-homogeneous coordinate ring of $Z := \prod_i \mathbb{P}^{m_i}$, *i.e.*

$$S = \bigoplus_{\underline{a}} S_{\underline{a}},\ \ S_{\underline{a}} = H^0(Z, \mathcal{L}_1^{a_1} \otimes \cdots \otimes \mathcal{L}_{n-1}^{a_{n-1}}),\ \ \underline{a} = (a_1, \ldots, a_{n-1}) \in \mathbb{Z}_+^{n-1}.$$

Let $\mathbb{A} = \prod_i \mathbb{A}^{m_i+1}$. We have

$$\begin{aligned} S &= k[\ldots, x_0^i, \ldots, x_{m_i}^i, \ldots] \\ \mathbb{A} &= \operatorname{Spec} S \\ \mathbb{A}^{m_i+1} &= \operatorname{Spec} k[x_0^i, \ldots, x_{m_i}^i]. \end{aligned}$$

We denote by T the torus group $T = (t_1, \ldots, t_{n-1})$, $t_i \in \mathbb{G}_m$. We have a canonical action of T on $\mathbb{A}$, namely multiplication by t_i on the component $\mathbb{A}^{m_i+1}$. We denote by $\mathbb{A}^\circ$ the open subscheme formed of points $x = (x_i)$, $x_i \in \mathbb{A}^{m_i+1}$ such that $x_i \neq 0$ for all i. Then T operates freely on $\mathbb{A}^\circ$ and Z identifies with the orbit space $\mathbb{A}^\circ/T$.

Let X be a closed subscheme of the multi-projective space Z. We denote by $I(X)$ the ideal of S generated by all $f \in S_{\underline{a}}$ (for varying $\underline{a}$) such that f vanishes on X (f considered, canonically as above, as a section of a line bundle on Z). We call $I(X)$ the *ideal of* X in S. Obviously, $I(X)$ is a multigraded ideal in S. If $\widehat{X} = \operatorname{Spec} S/I(X)$, we call $\widehat{X}$ the *multicone over* X. On the other hand, let J be a multigraded ideal in S. We can then canonically associate to J a closed subscheme $V(J)$ of Z, called the *variety* associated to J, in the following way. The ideal J determines a T-stable sheaf $\tilde{J}$ of ideals in $\mathbb{A}$. The restriction of $\tilde{J}$ to $\mathbb{A}^\circ$ goes down to a sheaf of ideals in Z and hence defines a closed subscheme $V(J)$ of Z. More concretely, Z can be covered by open subsets of the form $U_1 \times \cdots \times U_l$, where U_i is the open subset of $\mathbb{P}^{m_i}$ defined by $x_{k_i}^i \neq 0$, *i.e.* $U_i = \operatorname{Spec} k[x_0^i/x_{k_i}^i, x_1^i/x_{k_i}^i, \ldots]$ and the restriction of $V(J)$ to $U_1 \times \cdots \times U_l$ is given by the ideal in the coordinate ring of $U_1 \times \cdots \times U_l$, generated by elements of the form

$$\frac{F}{\prod_{1 \leq i \leq l} (x_{k_i}^i)^{a_i}}, \quad F \in S_{\underline{a}}.$$

The mapping $J \mapsto V(J)$ has the property that a union of ideals is taken to the corresponding scheme-theoretic intersection (in Z) and an intersection of ideals to the corresponding scheme-theoretic union. We observe that $J \subseteq I(V(J))$. Note however that if J_1, J_2 are two multigraded ideals of S such that $V(J_1) = V(J_2)$, it does *not* necessarily follow that $(J_1)_{\underline{a}} = (J_2)_{\underline{a}}$ for all but a finite number of $\underline{a}$'s.

Let $R = \bigoplus_{\underline{a} \in \mathbb{Z}_+^{n-1}} R_{\underline{a}}$ be the multigraded ring where

$$R_{\underline{a}} = H^0(G/B, \mathcal{L}_{\underline{a}}).$$

One knows that the canonical mapping

$$H^0(Z, \mathcal{L}_1^{a_1} \otimes \cdots \otimes \mathcal{L}_l^{a_l}) \to H^0(G/B, \mathcal{L}_{\underline{a}})$$

is surjective (for example, by Theorem 2.2.2). Hence we see that the ideal $I(G/B)$ of G/B is precisely the kernel of the canonical homomorphism $S \to R$. Now let I be a multigraded ideal of R. Then $V(I)$ is a closed subscheme of G/B; conversely, if X is a closed subscheme of G/B, the ideal in R generated by all multihomogeneous $f \in R$ vanishing on X is called the *ideal of* X *in* R and it is the image of $I(X)$ under the canonical map $S \to R$. We denote this ideal in R by just $I(X)$.

Now for maximal standard parabolic subgroups P_i, let J_i be a graded ideal of the homogeneous coordinate ring R_i of G/P_i and $\tilde{J}_i$ the multigraded ideal of R generated by J_i. We denote by $V(J_i)$ the closed subscheme G/P_i defined by J_i. Then we observe that if $\eta : G/B \to G/P_i$ is the canonical projection, we have

$$V(\tilde{J}_i) = \eta^{-1}(V(J_i))$$

in the scheme-theoretic sense. This is simple to check: for example, use the concrete description, given above, for the restriction of $V(\tilde{J}_i)$ to affine open subsets of a suitable covering of Z.

A subset T^i of $\mathfrak{S}_n/\mathfrak{S}_{\omega_i}$ is called a *right half space* if for $\alpha, \beta \in \mathfrak{S}_n/\mathfrak{S}_{\omega_i}$, $\alpha \in T^i$ and $\beta \geq \alpha$, then $\beta \in T^i$.

Theorem 2.3.1. *Let* J *be an ideal of* R *generated by Plücker coordinates* p_ϕ, $\phi \in T$ *where* $T = \bigcup_{1 \leq i \leq l} T^i$ *such that* T^i, $1 \leq i \leq l$ *is a right half space in* $\mathfrak{S}_n/\mathfrak{S}_{\omega_i}$ *(some* T^i *could be empty). Then the closed subscheme* $V(J)$ *of* G/B *is reduced and is in fact a union of Schubert varieties.*

Proof. Let J_i (resp. $\tilde{J}_i$) be the ideal in the homogeneous coordinate ring R_i (resp. R), generated by p_ϕ, $\phi \in T^i$. One knows by the results of Section 1.6 that the closed subscheme $V(J_i)$ of G/P_i is reduced and is a union of Schubert varieties (in G/P_i). Then obviously $\eta^{-1}(V(J_i))$ is reduced. As we observed above, we have $V(\tilde{J}_i) = \eta^{-1}(V(J_i))$, so $V(\tilde{J}_i)$ is reduced and is a union of Schubert varieties in G/B. Now one has

$$V(J) = V(\tilde{J}_1) \cap \cdots \cap V(\tilde{J}_l).$$

By Corollary 2.2.12, it follows that $V(J)$ is reduced and is a union of Schubert varieties in G/B. □

Remark 2.3.2. If Q is any parabolic subgroup of G, then, as we did above for the case of B, we can define the multigraded ring of G/Q. We see also that Theorem 2.3.1 has an obvious extension to this case.

Remark 2.3.3. Note that we do *not* claim that $R/I(V(J))$ or $R/I(V(\tilde{J}_i))$ is reduced. This is probably false, unlike the case of G/P, P being a maximal parabolic subgroup. Examples of other difficulties which crop up in the case when Q is not maximal are as follows:

(i) Let X_1, X_2 be two Schubert varieties in G/B. Then do we have

$$I(X_1 \cap X_2) = I(X_1) + I(X_2) \ ?$$

(ii) Let $X = X(\tau)$ be a Schubert variety in G/B and let $X_P(\theta)$ be the image of X in G/P, P being a maximal parabolic subgroup. Let $R(\tau) = R/I(X(\tau))$. Let $H(\tau) = X(\tau) \cap \{p_\theta = 0\}$. Let $I_\tau(H(\tau))$ be the ideal of $H(\tau)$ in $R(\tau)$, *i.e.* the image of $I(H(\tau))$ in $R(\tau)$. Then do we have

$$I_\tau(H(\tau)) = p_\theta R(\tau) \ ?$$

If $f \in R(\tau)$ and $\underline{a} = (a_1, \ldots, a_l)$ such that $a_1 > 0$, then by applying Theorem 2.2.2, it follows that $f = p_\theta g$, $g \in R(\tau)$. However, it seems very likely that $I_\tau(H(\tau)) \cap R(\tau)_{\underline{a}} \neq 0$ with $a_1 = 0$, in which case the above equality does not hold.

Thus, while the ideal theory of Schubert varieties in G/B is as good as in the case of G/P where P is a maximal parabolic subgroup, it doesn't seem to extend in the same manner to the multicone over G/B.

Remark 2.3.4. A closed subscheme X of G/B is said to be of *product type* if it is of the form $V(J)$ as in Theorem 2.3.1. We shall now show that a Schubert variety in G/B is always of product type. This is a consequence of Theorem 2.3.10 below.

Remark 2.3.5. Note that an intersection of subschemes of G/B of product type is of product type and that the subscheme $H(\tau)$ of $X(\tau)$ (see (ii) of Remark 2.3.3 above) is of product type. An arbitrary union of Schubert varieties in G/B is perhaps not of product type.

2.3.6. Equations defining Schubert varieties

Lemma 2.3.7. *Let $w \in \mathfrak{S}_n$. For $1 \le i \le n-1$, let $X(\varphi_i)$ be the pull-back under $\pi_i : G/B \to G/P_i$ of $\pi_i(X(w))$. Let $\theta \in \mathfrak{S}_n$ be such that $\theta \le \varphi_i$, $1 \le i \le n-1$. Then $\theta \le w$.*

Proof. By induction on $\ell(w)$. If $\ell(w) = 0$, then $w = \mathrm{id}$, and φ_i is simply the element of maximal length in $\mathfrak{S}_{\omega_i}$. Now if $\theta \in \mathfrak{S}_n$ is such that $\theta \le \varphi_i$, $1 \le i \le l$, then θ is in fact the identity element since $\bigcap_{i=1}^{n-1} P_i = B$. Thus $\theta = w$.

Now let $\ell(w) \ge 1$. For $1 \le i \le n-1$, let $\sigma_i \in \mathfrak{S}_{\omega_i}$ be such that $\varphi_i = w\sigma_i$ with $\ell(w\sigma_i) = \ell(w) + \ell(\sigma_i)$.

Let $\theta \le \varphi_i$, $1 \le i \le n-1$.

We now use the following:

Lemma 2.3.8. ([163], Lemma 3.11)

Let σ, $\sigma_1, \sigma_2 \in \mathfrak{S}_n$ be such that $\ell(\sigma\sigma_i) = \ell(\sigma) + \ell(\sigma_i)$, $i = 1, 2$. Then for all $\theta \in \mathfrak{S}_n$, we have $\theta \le \sigma\sigma_i$, $i = 1, 2$ if and only if $\theta \le \sigma\sigma_0$ for some σ_0 such that $\sigma_0 \le \sigma_i$, $i = 1, 2$, and $\ell(\sigma\sigma_0) = \ell(\sigma) + \ell(\sigma_0)$.

By Lemma 2.3.8, there exists σ such that $\theta \leq w\sigma$, and $\sigma \leq \sigma_i$, $1 \leq i \leq n-1$. Hence $\sigma \in \mathfrak{S}_{\omega_i}$, $1 \leq i \leq n-1$, and so $\sigma = \mathrm{id}$. This implies that $\theta \leq w$. □

As an immediate consequence of the above Lemma, we have

Corollary 2.3.9. *Let the notations be as in Lemma 2.3.7, we have set-theoretically* $X(w) = \bigcap_{i=1}^{n-1} X(\varphi_i)$.

Theorem 2.3.10. *For a Schubert variety $X(w)$ in G/B, the ideal sheaf of $X(w)$ in G/B is generated by $\{p_\tau,\ \tau \in \mathfrak{S}_n/\mathfrak{S}_{\omega_i},\ 1 \leq i \leq n-1$ and $\tau \not\leq w_i\}$, where for $1 \leq i \leq n-1$, w_i is the image of w in $\mathfrak{S}_n/\mathfrak{S}_{\omega_i}$.*

Proof. By Corollary 2.3.9, we have

$$X(w) = \bigcap_{i=1}^{l} \pi_i^{-1}(X(w_i)),\ \text{set-theoretically.} \tag{2.18}$$

If J_i denotes the ideal in R_i generated by $\{p_\tau,\ \tau \in \mathfrak{S}_n/\mathfrak{S}_{\omega_i},\ 1 \leq i \leq l \mid \tau \not\geq w_i\}$, then we have by Theorem 1.5.6 that

$$V(J_i) = X(w_i),\ V(\tilde{J}_i) = \pi_i^{-1}(X(w_i))$$

where $\tilde{J}_i$ denotes the multigraded ideal of R generated by J_i. This implies (by Theorem 2.3.1) that the relation (2.18) above is in fact scheme-theoretic, from which the required result follows. □

2.4 Lexicographic shellability and Cohen-Macaulayness

This following discussion on lexicographic shellability is à la Björner-Wachs (cf. [25]).

Definition 2.4.1. A *graded poset* is a finite partially ordered set P with a unique maximal and a unique minimal element, denoted by $\widehat{1}$ and $\widehat{0}$ respectively, in which all maximal chains $\widehat{1} = x_0 > x_1 > \cdots > x_r = \widehat{0}$ have the same length r. This common length r is called the *rank of* P.

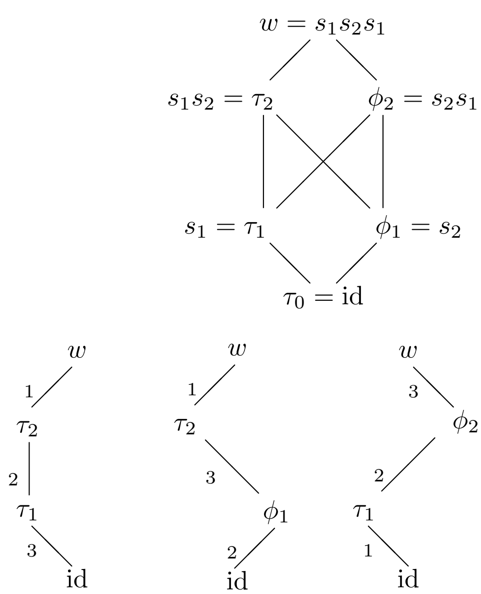
Remark 2.4.2. Given a finite partially ordered set A, there is a canonically defined partial order on A^n for $n \in \mathbb{N}$, namely the lexicographic order: $(\alpha_1, \ldots, \alpha_n)$ is lexicographically greater than $(\beta_1, \ldots, \beta_n)$, if there exists a $t \leq n$ such that $\alpha_i = \beta_i$, $i < t$, and $\alpha_t > \beta_t$.

Definition 2.4.3. A graded poset P is said to be *L-shellable or lexicographically shellable* if every maximal chain $\mathbf{m}$: $\widehat{1} = x_0 > x_1 > \cdots > x_r = \widehat{0}$ can be labeled, say, $\lambda(\mathbf{m}) = (\lambda_1(\mathbf{m}), \lambda_2(\mathbf{m}), \ldots, \lambda_r(\mathbf{m}))$, where $\lambda_i(\mathbf{m})$ are elements of another partially ordered set in such a way that the following two conditions hold:

(L1) If two maximal chains $\mathbf{m}$, $\mathbf{m}'$ have the same first d-edges, then the corresponding labels of the first d-edges are the same. Here, we think of $\lambda_i(\mathbf{m})$ as being associated with the edge $x_{i-1} \to x_i$.

(L2) Given an interval $[x, y]$ together with a path $\underline{\xi}$ from $\widehat{1}$ to y (we refer to this as a *rooted interval*), among all the maximal chains in $[x, y]$ there exists a unique chain whose label is increasing; further, this unique increasing label is lexicographically smaller than the label of any other chain. Here, a label for a maximal chain in $[x, y]$ is that induced by the maximal chain obtained by following $\underline{\xi}$ by the maximal chain from y to x under consideration, followed by an arbitrary chain from x to $\widehat{0}$.

Example 2.4.4. The Bruhat-Chevalley order in $\mathcal{S}_3$. Let $s_1 = (1, 2)$, $s_2 = (2, 3)$. Then $(321) = s_1 s_2 s_1$. We label the maximal chains as shown in the diagram below:



Remark 2.4.5. The same edge occurring in two different maximal chains may have different indexing. For instance, in the above example the edge $\tau_1 \to \tau_0$ has the index 3 when considered as an edge in $\mathbf{m}$: $w > \tau_2 > \tau_1 > \tau_0$ and has index 1 when considered as an edge in $\mathbf{m}'$: $w > \phi_2 > \tau_1 > \tau_0$.

Let (W, S) be a Weyl (Coxeter) group. For $J \subset S$, let W^J be the set of minimal representatives of W/W_J. Recall from Chapter **??** that $W^J = \{w \in W \mid w(\alpha) > 0, \text{ for } \alpha \in J\}$. For $w, w' \in W^J$, $w' \geq w$, we shall describe a labeling of the maximal chains in the interval $[w, w'] = \{u \in W^J \mid w \leq u \leq w'\}$. Let us fix a reduced expression $w' = s_1 s_2 \ldots s_q$. Let

$\mathbf{m}$: $w' = w_0 > w_1 > \cdots > w_r = w$ be a maximal chain in $[w, w']$, where $r = \ell(w') - \ell(w)$. Now since $w_1 < w'$ and $\ell(w_1) = \ell(w') - 1$, a reduced expression for w_1 is obtained by omitting a reflection s_i in the above reduced expression for w', and the deleted reflection is uniquely determined. We set $\lambda_1(\mathbf{m}) = i$. Proceeding thus, at each step we label an edge by the position of the reflection that is deleted in the chosen reduced expression for w'.

Theorem 2.4.6. [25] *With notations as above, the interval* $[w, w']$ *in* W^J *is lexicographic shellable for the labeling of the maximal chains in* $[w, w']$ *as described above.*

Definition 2.4.7. Given a finite poset P, the simplicial complex obtained by taking a q-simplex to be a chain of length q in P is called *the order complex of* P, and is denoted by $\Delta(P)$.

Let Δ be a simplicial complex. A maximal face of Δ is called a *facet*. Δ is said to be of *pure dimension* d if all facets are of dimension d. For a facet F, $\overline{F}$ denotes $\{G \mid G \subseteq F\}$, and ∂F denotes $\{G \mid G \subsetneq F\}$

Definition 2.4.8. Let Δ be a pure d-dimensional simplicial complex. An ordering $F_1, F_2, \ldots, F_t$ of the facets of Δ is said to be a *shelling* if $\overline{F_j} \cap (\bigcup_{i=1}^{j-1} \overline{F_i})$, $2 \leq j \leq t$ is a $(d-1)$-complex having a shelling which extends to a shelling of ∂F_j, *i.e.* ∂F_j has a shelling in which the facets of $\overline{F_j} \cap (\bigcup_{i=1}^{j-1} \overline{F_i})$ come first. We say that Δ is *shellable* if it admits a shelling.

Theorem 2.4.9. *If* P *is lexicographic shellable, then the order complex* $\Delta(P)$ *is shellable.*

Proof. Let P be L-shellable. We consider the total ordering $\prec$ of the maximal chains in $\Delta(P)$ given by $\mathbf{m}_1 \prec \mathbf{m}_2 \prec \cdots \prec \mathbf{m}_s$, where $\lambda(\mathbf{m}_1) < \lambda(\mathbf{m}_2) < \cdots < \lambda(\mathbf{m}_r)$. It can be shown that this total ordering gives a shelling for $\Delta(P)$ (see [24]). □

Let G be a connected semisimple algebraic group, T a maximal torus in G, B a Borel subgroup of G, $B \supset T$, and Q a parabolic subgroup containing B. Let W be the Weyl group of G relative to T. For $\tau \in W^Q$, let $X(\tau)$ be the Schubert variety in G/Q associated to τ.

Definition 2.4.10. Let Z be a union of Schubert varieties in G/Q. Further, let Z be pure of dimension d, *i.e.* all irreducible components of Z have dimension d. We say that Z *admits a nice indexing* if there exists an indexing, say $X_1, \ldots, X_r$, of the components of Z in such a way that

1. $X_j \cap (\bigcup_{i=1}^{j-1} X_i)$ is of pure dimension $d-1$ for $2 \leq j \leq r$.
2. $X_j \cap (\bigcup_{i=1}^{j-1} X_i)$ admits a nice indexing which extends to a nice indexing of the union Y of Schubert divisors in X_j for $2 \leq j \leq r$ (*i.e.* Y admits a nice indexing in which the components of $X_j \cap (\bigcup_{i=1}^{j-1} X_i)$ come first).

Theorem 2.4.11. *Let $\tau \in W^Q$ and let Z be the union of the Schubert divisors in $X(\tau)$. Then Z admits a nice indexing.*

The result follows immediately from Theorems 2.4.6 and 2.4.9.

Theorem 2.4.12. *Let $\tau \in W^Q$ and let P be a parabolic subgroup of G containing Q. Let $\pi : G/Q \to G/P$ be the canonical projection. Let Y be the union of all Schubert divisors $X(w)$, $w \in W^Q$, in $X(\tau)$ such that $\pi(X(w)) \subsetneq \pi(X(\tau))$. Then Y admits a nice indexing.*

Proof. Let

$$\mathcal{B}_\tau = \{w \in W^Q \mid X(w) \text{ is a divisor in } X(\tau) \text{ and } \pi(X(w)) \subsetneq \pi(X(\tau))\}.$$

so that $Y = \bigcup_{w\in\mathcal{B}_\tau} X(w)$. Let us write $\tau = \tau_0\theta_0$, where $\tau_0 \in W^P$, $\theta_0 \in W_P$ and $\ell(\tau) = \ell(\tau_0)+\ell(\theta_0)$. Let $\tau_0 = s_1 \dots s_t$ and $\theta_0 = s_{t+1} \dots s_{t+k}$ be reduced expressions for τ_0 and θ_0 so that $\tau = s_1 \dots s_t s_{t+1} \dots s_{t+k}$ is a reduced expression for τ. Working with this reduced expression for τ, we index the maximal chains in $[\mathrm{id}, \tau]$. Now if $X(w)$ is a divisor in $X(\tau)$, then $w \in \mathcal{B}_\tau$ if and only if $w = s_1 \dots \widehat{s_m} \dots s_{t+k}$ for some $1 \le m \le t$. Hence the maximal chains in $[\mathrm{id}, \tau]$ which start with an edge $\tau \to w$, $w \in \mathcal{B}_\tau$, come earlier (in the lexicographic order) than those which start with an edge $\tau \to w$, $w \notin \mathcal{B}_\tau$. Now the result follows from Theorems 2.4.9 and 2.4.11. □

2.4.13. Cohen-Macaulayness of multi-cones over Schubert varieties in $SL_n(k)/B$

Theorem 2.4.14. *Let $\tau \in W$. Then the ring $R(\tau) = \bigoplus_{\lambda\geq 0} H^0(X, \mathcal{L}_\lambda)$ is Cohen-Macaulay. Thus the multicone for the multi-projective embedding*

$$X(\tau) \hookrightarrow \prod_{1\leq i\leq n-1} \mathbb{P}(H^0(G/P_i, \mathcal{L}_{\omega_i})^*)$$

is Cohen-Macaulay.

Proof. We prove the result by induction on $\dim X(\tau)$. Let α be a simple root such that $\tau > \tau s_\alpha$. Let $P = P_{\widehat{\alpha}}$ be the maximal parabolic subgroup corresponding to α. Let $X(\overline{\tau})$ be the image of $X(\tau)$ under $G/B \to G/P$, and let $H(\tau) = X(\tau) \cap \{p_{\overline{\tau}} = 0\}$. Let us take an indexing of the simple roots so that $\alpha_1 = \alpha$. Let us first show that

$$I(H(\tau)) = p_{\overline{\tau}} R(\tau). \tag{2.19}$$

We clearly have $I(H(\tau)) \supset p_{\overline{\tau}} R(\tau)$. Now let $f \in R(\tau)_{\underline{a}}$ where $\underline{a} = (a_1, \dots, a_{n-1})$, be such that $f = 0$ on $H(\tau)$. The facts that $f \neq 0$ on $X(\tau)$ and $f = 0$ on $H(\tau)$ imply that $a_1 \neq 0$; note that $X(\tau s_\alpha) \subseteq H(\tau)$, and that for all $\underline{b} = (0, b_2, \dots, b_{n-1})$, we have $H^0(X(\tau), L_{\underline{b}}) \cong H^0(X(\tau s_\alpha), L_{\underline{b}})$. Let us write $f = F_1 + F_2$, where each of the F_1 and F_2 is a sum of standard monomials on $X(\tau)$ of type $\underline{a}$ and each monomial in F_1 involves $p_{\overline{\tau}}$, while each monomial in F_2 does not involve $p_{\overline{\tau}}$. Hence we obtain $F_1 \in I(H(\tau))$, and therefore $F_2 \in I(H(\tau))$, *i.e.* $F_2 = 0$ on $H(\tau)$. On the other hand, each monomial in F_2 remains standard on $H(\tau)$ since none of them involves $p_{\overline{\tau}}$. Hence the linear independence of standard monomials on $H(\tau)$ implies that $F_2 = 0$. This implies $f = F_1$, and hence $f \in p_{\overline{\tau}}R(\tau)$ since $F_1 \in p_{\overline{\tau}}R(\tau)$. This establishes the equality (2.19).

Next we shall show that the multigraded ring $R(\tau)/p_{\overline{\tau}}R(\tau)$ is Cohen-Macaulay. In fact we prove the following more general statement: Let Z be a union of Schubert varieties such that Z is pure of dimension less than $\dim X(\tau)$, admitting a nice indexing. Then $R(\tau)/I(Z)$ is Cohen-Macaulay.

We prove this by induction on the number of components of Z and on $\dim Z$. Let $X_1, \ldots, X_r$ be a "nice" indexing of the components of Z. Let $Z = X \cup Y$, where $X = \bigcup_{i=1}^{r-1} X_i$ and $Y = X_r$. Denoting the ideals of X, Y, Z in $X(\tau)$ by $I(X)$, $I(Y)$, $I(Z)$ respectively, the assertion that $R(\tau)/I(Z)$ is Cohen-Macaulay follows from Lemma 2.4.15, by taking $A = R(\tau)$, $I = I(X)$, $J = I(Y)$.

Now, taking $Z = H(\tau)$, we have $R(\tau)/I(Z) = R(\tau)/p_{\overline{\tau}}R(\tau)$, and $p_{\overline{\tau}}$ being a nonzero divisor in $R(\tau)$, we conclude that $R(\tau)$ is Cohen-Macaulay. □

Lemma 2.4.15. *Let $A = \bigoplus_{n\geq 0} A_n$ be a Noetherian graded ring with $A_0 = k$. Let I, J be two homogeneous ideals of A such that* $\dim A/I = \dim A/J = d = \dim(A/(I+J)) + 1$. *If A/I, A/J and $A/(I+J)$ are Cohen-Macaulay, then $A/I \cap J$ is Cohen-Macaulay of dimension d.*

Proof. Let $\mathfrak{m} = A^+$ be the irrelevant ideal. It suffices to show (cf. [205]) that $(A/I \cap J)_\mathfrak{m}$ is Cohen-Macaulay. Consider the exact sequence

$$0 \to (A/I \cap J)_\mathfrak{m} \to (A/I)_\mathfrak{m} \oplus (A/J)_\mathfrak{m} \to (A/I + J)_\mathfrak{m} \to 0.$$

Taking local cohomology, we get,

$$\begin{aligned} \cdots \to H^{i-1}_\mathfrak{m}((A/I+J)_\mathfrak{m}) &\to H^i_\mathfrak{m}((A/I \cap J)_\mathfrak{m}) \\ &\to H^i_\mathfrak{m}((A/I)_\mathfrak{m}) \oplus H^i_\mathfrak{m}((A/J)_\mathfrak{m}) \to \cdots \end{aligned}$$

The depth of a finitely generated $A_\mathfrak{m}$-module N is characterized by

$$\operatorname{depth} N = \min\{i \mid H^i_\mathfrak{m}(N) \neq 0\}.$$

By hypothesis, $H^i_\mathfrak{m}((A/I)_\mathfrak{m}) = H^i_\mathfrak{m}((A/J)_\mathfrak{m}) = 0$, $i < d$, and $H^i_\mathfrak{m}((A/I + J)_\mathfrak{m}) = 0$, $i < d-1$. Hence we obtain $H^i_\mathfrak{m}((A/I \cap J)_\mathfrak{m}) = 0$, $i < d$. This implies that

$$d \leq \operatorname{depth}(A/I \cap J)_\mathfrak{m} \leq \dim(A/I \cap J)_\mathfrak{m} \leq d.$$

It follows that $(A/I \cap J)_\mathfrak{m}$ is Cohen-Macaulay. □

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