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A Generalization of the Littlewood–Richardson Rule

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INTRODUCTION

For the general linear group Gl_m the Littlewood–Richardson rule (see [18, 19]) gives a method to calculate the decomposition of a tensor product of two irreducible Gl_m-representations. The aim of this article is to give a generalization of this rule for all simple, simply connected algebraic groups of type $\mathbf{A}_m$, $\mathbf{B}_m$, $\mathbf{C}_m$, $\mathbf{D}_m$, $\mathbf{G}_2$, and $\mathbf{E}_6$. We obtain also partial results for G of type $\mathbf{F}_4$, $\mathbf{E}_7$, and $\mathbf{E}_8$. The restrictions in the last three cases come from the fact that for the formulation of the decomposition rules we need the notion of a standard Young tableau. Such a notion has been developed by Seshadri, Lakshmibai, Musili, and Rajeswari in a series of articles (see [11, 13, 14, 16]), but not yet for all representations of the last three exceptional groups.

The advantage of the notion of a Young tableau developed by Seshadri *et al.* is that it is independent of the type of the group. For the convenience of the reader not used to this notion we give first a seperate proof for $G = Sl_n$ using the classical notion of a standard Young tableau. Of course, for applications the classical notion is much more appropriate. In the Appendix we give a "translation" of the notion of a standard Young tableau in the sense of Seshadri *et al.* into the classical notion of a Young tableau for $G = Sp_{2m}$ and $Spin_m$. In 3.8 we give also such a translation for $G = \mathbf{G}_2$. For the other exceptional groups such a translation is not known to the author.

Let μ be a dominant weight for G and denote by V_μ the corresponding simple G-module. We assume that the character of V_μ is given by $\sum_{\mathcal{T}} e^{\nu(\mathcal{T})}$, where the sum has to be taken over all standard Young tableaux $\mathcal{T}$ of shape $p(\mu)$ and $\nu(\mathcal{T})$ is the weight of the tableau $\mathcal{T}$. Now by the

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definition of a standard Young tableau we can associate to each tableau $\mathscr{T}$ in a natural way a sequence of weights

$$\nu_1(\mathscr{T}), \nu_2(\mathscr{T}), \ldots, \nu(\mathscr{T}).$$

If λ is a dominant weight, then we say that a standard Young tableau $\mathscr{T}$ of shape $p(\mu)$ is *λ-dominant*, if all the weights

$$\lambda + \nu_1(\mathscr{T}), \lambda + \nu_2(\mathscr{T}), \ldots, \lambda + \nu(\mathscr{T})$$

are contained in the dominant Weyl chamber of G. In particular we know then that $\lambda + \nu(\mathscr{T})$ is a dominant weight.

The generalized Littlewood–Richardson rule can be stated as follows: The decomposition of the tensor product $V_\lambda \otimes V_\mu$ into simple G-modules is given by

$$V_\lambda \otimes V_\mu = \bigoplus_{\mathscr{T}} V_{\lambda + \nu(\mathscr{T})},$$

where $\mathscr{T}$ runs over all standard Young tableaux of shape $p(\mu)$ that are λ-dominant.

It is easy to see that the notion of a λ-dominant Young tableau corresponds for $G = Sl_m$ to the notion of a lattice permutation in the usual formulation of the Littlewood–Richardson rule (see [19]). Hence our approach, which does not use the representation theory of the permutation group S_m, gives also a new proof of the Littlewood–Richardson rule.

Independently of our approach, the generalization of the Littlewood–Richardson rule in the way stated above has been conjectured by Weyman in [22] for G of type $\mathbf{A}_m$, $\mathbf{B}_m$, $\mathbf{C}_m$, and $\mathbf{D}_m$. In fact, using Klimyk's formula (see [9] or 1.5 below) he proves the decomposition formula above for the case where μ is a fundamental weight and G is a group of the type above.

Using the description of a G-standard Young tableau for $G = Sp_{2m}$ and $Spin_m$ in the Appendix, for applications the advantages of our formula in comparison with the decomposition formula in [10] are first of all that we obtain a decomposition formula which also holds for *Spin*-representations, and second, the formula in [10] involves alternating signs, which is not the case in our formula. Also the more general decomposition formulas in [9, 8] involve alternating signs (compare Remark 1.7). Corollary A.6 has already been pointed out in [10].

We obtain a similar decomposition formula for the following problem: Let V_μ be a simple G-module and let L be a Levi subgroup of G. Since L is reductive we know that the L-module $\operatorname{res}_L V_\mu$ obtained by restricting the representation of G on V_μ to L, decomposes into the direct sum of simple

L-modules. We give a method to calculate the decomposition if the character of V_μ can be computed by standard Young tableaux as above.

Part of the results have been announced in [17].

In Section 1 we present the general methods. The idea to use the Euler characteristic to obtain decomposition formulas has been taken from [7]. In fact, the starting point for this note has been the idea to combine the results of Kempf with the standard monomial theory to make the decomposition formulas in [7] more efficient for calculations if G is of classical type (that means G is of type $\mathbf{A}_m$, $\mathbf{B}_m$, $\mathbf{C}_m$, or $\mathbf{D}_m$).

In Section 2 we discuss the case where $G = Sl_m$; in Section 3 we treat the general case. We recall the notion of a standard Young tableau in the sense of Seshadri *et al.* and formulate the decomposition formulas in this notion. For $\mathbf{G}_2$ we show how to reformulate the results in the classical notion of a Young tableau. In Section 4 we give the proof of the decomposition formulas.

In the Appendix we give (without a proof) the reformulation of the decomposition formulas for $G = Sp_{2m}$ and $Spin_m$ in the classical notion of a Young tableau.

We assume throughout the following that the base field k is algebraically closed and of characteristic zero.

1. SOME REMARKS ON THE EULER CHARACTERISTIC AND THE DEMAZURE OPERATOR

1.1 Let G be a connected, simply connected reductive algebraic group. Fix a Borel subgroup B and denote by $X(B)$ the group of characters of B. For a maximal torus T in B denote by $W := \mathrm{Nor}_G(T)/T$ the Weyl group of G. Note that W operates on $X(B)$ via the natural isomorphism $X(B) \xrightarrow{\sim} X(T)$ induced by the restriction map. Fix a W-invariant scalar product $(\,,\,)$ on $X(B) \otimes_{\mathbf{Z}} \mathbf{R}$. For $\lambda, \mu \in X(B)$ let $\langle \lambda, \mu \rangle := 2(\lambda, \mu)/(\mu, \mu)$.

1.2. Let $\mathbf{Z}[X(B)]$ be the group ring on $X(B)$. For every simple root α denote by M_α the linear operator on $\mathbf{Z}[X(B)]$ defined by

$$M_\alpha \colon \mathbf{Z}[X(B)] \to \mathbf{Z}[X(B)], \qquad e^\mu \mapsto e^{-\rho}(e^{\mu+\rho} - e^{s_\alpha(\mu+\rho)})(1 - e^{-\alpha})^{-1},$$

where ρ denotes half the sum of positive roots. It is easy to see that

$$M_\alpha(e^\mu) = \begin{cases} e^\mu + \cdots + e^{\mu - \langle \mu, \alpha \rangle \alpha} & \text{if } \langle \mu, \alpha \rangle \geqslant 0; \\ 0 & \text{if } \langle \mu, \alpha \rangle = -1; \\ -e^{\mu+\alpha} - \cdots - e^{\mu + (-\langle \mu, \alpha \rangle - 1)\alpha} & \text{if } \langle \mu, \alpha \rangle \leqslant -2. \end{cases} \tag{1.1}$$

We will need the following simple lemma. Let λ and μ be elements of $X(B)$.

LEMMA. $M_\alpha(e^\lambda M_\alpha(e^\mu)) = M_\alpha(e^\lambda)\, M_\alpha(e^\mu)$.

Proof. Using (1.1) it is easy to see that the string of weights given by $M_\alpha(e^\lambda)$ is invariant under the reflection s_α. That means if $M_\alpha(e^\lambda) = \sum_i a_i e^{\nu_i}$ then $M_\alpha(e^\lambda) = \sum_i a_i e^{s_\alpha(\nu_i)}$. Now it follows by the definition of M_α that

$$\begin{aligned} M_\alpha(e^\lambda M_\alpha(e^\mu)) &= e^{-\rho}(e^{\lambda+\rho} M_\alpha(e^\mu) - e^{s_\alpha(\lambda+\rho)} M_\alpha(e^\mu))(1-e^{-\alpha})^{-1} \\ &= M_\alpha(e^\mu)\, e^{-\rho}(e^{\lambda+\rho} - e^{s_\alpha(\lambda+\rho)})(1-e^{-\alpha})^{-1} \\ &= M_\alpha(e^\lambda)\, M_\alpha(e^\mu), \end{aligned}$$

which proves the lemma.

COROLLARY. $M_\alpha(M_\alpha(e^\lambda)) = M_\alpha(e^\lambda)$.

1.3. For an element w of the Weyl group W let $w = s_{\alpha_1} \cdots s_{\alpha_r}$ be a reduced decomposition. We denote by M_w the operator $M_{\alpha_1} \circ \cdots \circ M_{\alpha_r}$. Note that M_w is independent of the reduced decomposition (see [4]). For $\lambda \in X(B)$ denote by $\mathscr{L}_\lambda$ the line bundle on G/B corresponding to the character λ. For $w \in W$ denote by $X(w)$ the Schubert variety in G/B corresponding to w. We denote by $\mathscr{L}_\lambda$ also the line bundle restricted to $X(w)$.

THEOREM (Demazure's Character Formula [4]). *If λ is a dominant weight, then the character* $\operatorname{Char} H^0(X(w), \mathscr{L}_\lambda)^*$ *of the B-module $H^0(X(w), \mathscr{L}_\lambda)^*$ is given by $M_w(e^\lambda)$.*

Remark. (a) Let D be the linear operator on $\mathbf{Z}[X(B)]$ defined by $D(e^\mu) = e^{-\mu}$. Then $DM_\alpha D$ is the Demazure operator defined in [4, Théorème 2]. The definition of the operator M_α can be found, for example, in [7] as well as in [15].

(b) Since every nonzero B-module has a B-fixed line we know that the class of a B-module in the Grothendieck group $\mathscr{M}_B$ of B-modules is uniquely determined by its character. Hence we can identify $\mathscr{M}_B$ with $\mathbf{Z}[X(B)]$. In this context we may look at the operator M_α in the following way: Denote by k_λ the one-dimensional B-module corresponding to the character λ. Denote by $P(\alpha)$ the minimal parabolic subgroup of G associated to a simple root α. If $\langle \lambda, \alpha \rangle \geqslant 0$, then let $k_\lambda|^{P(\alpha)}$ be the induced $P(\alpha)$-module $\operatorname{Map}_B(P(\alpha), k_\lambda)$. Then $M_\alpha(e^\lambda)$ is the character $\operatorname{Char}((k_\lambda|^{P(\alpha)})^*)$. (This follows from [3] and Sl_2-theory.) Note that $P(\alpha)/B$ is complete and hence if U is a $P(\alpha)$-module and $U|_B$ denotes the module restricted to B, then $U \simeq U|_B|^{P(\alpha)}$ (see [2]). This is one way to look at Corollary 1.2 in this context. And Lemma 1.2 corresponds for $\langle \lambda, \alpha \rangle$, $\langle \mu, \alpha \rangle \geqslant 0$ to $k_\lambda|^{P(\alpha)} \otimes k_\mu|^{P(\alpha)} \simeq (k_\lambda \otimes k_\mu|^{P(\alpha)}|_B)|^{P(\alpha)}$ (see [2]).

(c) A more geometrical interpretation is given in [7]. Identify the Grothendieck group of G-linearized coherent sheaves on G/B with $\mathbf{Z}[X(B)]$. This can be done because a G-linearised coherent sheaf $\mathscr{S}$ on G/B has a composition series with line bundles $\mathscr{L}_{\lambda_i}$ as factors for $1 \leqslant i \leqslant rk\mathscr{S} = n$. Hence the class of such a sheaf is uniquely determined by the character $\sum_{i=1}^{n} e^{\lambda_i}$. Let π be the projection $\pi\colon G/B \to G/P(\alpha)$. The fibres of π are isomorphic to $\mathbf{P}^1$. The map π is flat and $R^i\pi_* = 0$ for $i \geqslant 2$. Hence the map $\pi^*\pi_*\colon \mathbf{Z}[X(B)] \to \mathbf{Z}[X(B)]$ which sends the class $[\mathscr{S}]$ of a coherent sheaf $\mathscr{S}$ to the class $[\pi^*\pi_*(\mathscr{S})] - [\pi^*R^1\pi_*(\mathscr{S})]$ is well defined and additive. Kempf shows in [7] that $M_\alpha(e^\lambda)$ is the class $\pi^*\pi_*[\mathscr{L}_\lambda] := [\pi^*\pi_*(\mathscr{L}_\lambda)] - [\pi^*R^1\pi_*(\mathscr{L}_\lambda)]$.

1.4. For $\mu \in X(B)$ let $\mathscr{L}_\mu$ be the corresponding line bundle on $Y := G/B$. Since $H^i(G/B, \mathscr{L}_\mu)$ is a G-module for all $i \geqslant 0$ we may regard the Euler characteristic

$$\chi_Y(e^\mu) := \sum_{i \geqslant 0} (-1)^i H^i(Y, \mathscr{L}_\mu)$$

as an element of the Grothendieck group $\mathscr{M}_G$ of G-modules. Extending χ_Y linearly, we get a map

$$\chi_Y\colon \mathbf{Z}[X(B)] \to \mathscr{M}_G, \qquad e^\mu \mapsto \chi_Y(e^\mu).$$

If μ is a dominant weight for G, then denote by V_μ the corresponding simple G-module of highest weight μ. By the Borel–Weil Theorem, we know that in this case the Euler characteristic $\chi_Y(e^\mu)$ is the class $[V_\mu^*]$ in $\mathscr{M}_G$, where V_μ^* denotes the dual space of V_μ. If μ is arbitrary, then $\chi_Y(e^\mu)$ is the class $\pm[V_\nu]$ of a simple G-module V_ν in $\mathscr{M}_G$. The highest weight ν and the sign can be determined using the following formula. Let l be the length function on W. For any τ in the Weyl group W of G one has [5, 7]

$$\chi_Y(e^\mu) = (-1)^{l(\tau)} \chi_Y(e^{\tau(\mu+\rho)-\rho}). \tag{1.2}$$

Let μ be a singular weight, that means $\langle \mu+\rho, \beta \rangle = 0$ for some root β. There exists an element $\tau \in W$ such that $\tau(\beta) = \alpha$ is a simple root. Then $\langle \tau(\mu+\rho), \alpha \rangle = 0$ by the Weyl group invariance of $(\,,)$. It follows by (1.2) that $\pm\chi_Y(e^\mu) = \chi_Y(e^{\tau(\mu+\rho)-\rho}) = -\chi_Y(e^{s_\alpha\tau(\mu+\rho)-\rho}) = -\chi_Y(e^{\tau(\mu+\rho)-\rho})$, and hence $\chi_Y(e^\mu) = 0$. If μ is not singular, then there exists a unique element $\tau \in W$ such that $\tau(\lambda+\rho) - \rho = \mu$, where λ is dominant and hence by (1.2), $\chi_Y(e^\mu) = (-1)^{l(\tau)} [V_\lambda^*]$.

If α is a simple root then we know by (1.2) that $\chi_Y(e^\mu) = -\chi_Y(e^{s_\alpha(\mu+\rho)-\rho}) = -\chi_Y(e^{\mu-(\langle\mu,\alpha\rangle+1)\alpha})$. Now using the symmetry with respect to the reflection s_α in the formula (1.1) for the operator M_α (for example, if $\langle \lambda, \alpha \rangle \geqslant 1$ then $\chi_Y(e^{\lambda-\alpha}) = -\chi_Y(e^{\lambda-\langle\lambda,\alpha\rangle}))$ it is easy to see that

$$\chi_Y(e^\mu) = \chi_Y(M_\alpha(e^\mu)). \tag{1.3}$$

More generally we get

Lemma. *If α is a simple root then $\chi_Y(M_\alpha(e^\lambda)e^\mu) = \chi_Y(e^\lambda M_\alpha(e^\mu))$.*

Proof. We know by (1.3) and Lemma 1.2 that

$$\chi_Y(e^\lambda M_\alpha(e^\mu)) = \chi_Y(M_\alpha(e^\lambda M_\alpha(e^\mu))) = \chi_Y(M_\alpha(e^\lambda)\, M_\alpha(e^\mu)).$$

The lemma follows now by the symmetry of the right hand side of the equation.

1.5 Let λ and μ be two dominant weights of G and denote by V_λ, V_μ two simple G-modules of highest weight λ and μ. We will also write V_λ and $V_\lambda \otimes V_\mu$ for the corresponding classes in $\mathscr{M}_G$.

Proposition. *The decomposition of the tensor product $V_\lambda^* \otimes V_\mu^*$ is given by*

$$V_\lambda^* \otimes V_\mu^* = \chi_Y(e^\lambda \operatorname{Char} V_\mu).$$

Proof. Denote by w_G the longest word in the Weyl group W of G and let $w_G = s_{\alpha_1} \cdots s_{\alpha_r}$ be a reduced decomposition of w_G. If

$$V_\lambda \otimes V_\mu = V_{\nu_1} \oplus \cdots \oplus V_{\nu_s}$$

is a decomposition of the tensor product into simple G-modules of highest weight $\nu_1, ..., \nu_s$, then we know by Demazure's character formula and (1.3)

$$\begin{aligned} V_\lambda^* \otimes V_\mu^* &= \sum_{i=1}^{s} \chi_Y(e^{\nu_i}) = \sum_{i=1}^{s} \chi_Y(M_{w_G} e^{\nu_i}) \\ &= \chi_Y(\operatorname{Char} V_\lambda \otimes V_\mu) = \chi_Y(M_{w_G}(e^\lambda)\, M_{w_G}(e^\mu)). \end{aligned}$$

Now using the reduced decomposition of w_G we get by Lemma 1.4 and Corollary 1.2

$$\begin{aligned} &\chi_Y(M_{w_G}(e^\lambda)\, M_{w_G}(e^\mu)) \\ &\quad = \chi_Y(M_{\alpha_1}(M_{\alpha_2} \circ \cdots \circ M_{\alpha_r}(e^\lambda))\, M_{\alpha_1}(M_{\alpha_2} \circ \cdots \circ M_{\alpha_r}(e^\mu))) \\ &\quad = \chi_Y(M_{\alpha_2} \circ \cdots \circ M_{\alpha_r}(e^\lambda)\, M_{\alpha_1}(M_{\alpha_1}(M_{\alpha_2} \circ \cdots \circ M_{\alpha_r}(e^\mu)))) \\ &\quad = \chi_Y(M_{\alpha_2} \circ \cdots \circ M_{\alpha_r}(e^\lambda)\, M_{w_G}(e^\mu)). \end{aligned}$$

Note that for every simple root α there exists a reduced decomposition of w_G which begins with s_α. Hence we can complete the proof of the proposition by repeating the argument above.

Remark. Using (1.2) one might actually use Proposition 1.5 to compute the decomposition of a tensor product. The formula one obtains then is due to Brauer (see [6, Sect. 24, Exercise 9] or [9]). An approach closer to the proof in [9] (which uses the character formula of Weyl) is the following.

We know $\operatorname{Char} V_\lambda^* \otimes V_\mu^* = M_{w_G}(e^\lambda)\, M_{w_G}(e^\mu) = \sum_{i=1}^s M_{w_G}(e^{\nu_i}) = \sum_{i=1}^s \operatorname{Char} V_{\nu_i}$. Using Lemma 1.2 and similar arguments as in the proof above it is easy to see that $M_{w_G}(e^\lambda)\, M_{w_G}(e^\mu) = M_{w_G}(e^\lambda M_{w_G}(e^\mu))$. Taking the Euler characteristic we get by the equality above

$$\chi_Y(M_{w_G}(e^\lambda M_{w_G}(e^\mu))) = \sum_{i=1}^s \chi_Y(M_{w_G}(e^{\nu_i})).$$

Using (1.3) we see

$$\begin{aligned}\chi_Y(e^\lambda M_{w_G}(e^\mu)) &= \sum_{i=1}^s \chi_Y(e^{\nu_i})\\ &= V_{\nu_1}^* \oplus \cdots \oplus V_{\nu_s}^* = V_\lambda^* \otimes V_\mu^*.\end{aligned}$$

A more "geometrical" approach is the following: Let λ and μ be dominant weights. Denote by $\mathscr{L}_\lambda \boxtimes \mathscr{L}_\mu$ the line bundle $\pi_1^* \mathscr{L}_\lambda \otimes \pi_2^* \mathscr{L}_\mu$ on $G/B \times G/B$, where π_i denotes the projection map $\pi_i\colon G/B \times G/B \to G/B$ on the first resp. second factor. Let G act on $G/B \times G/B$ via the diagonal action. By the Künneth formula we know that

$$H^0(G/B \times G/B, \mathscr{L}_\lambda \boxtimes \mathscr{L}_\mu) \simeq H^0(G/B, \mathscr{L}_\lambda) \otimes H^0(G/B, \mathscr{L}_\mu) \simeq V_\lambda^* \otimes V_\mu^*$$

and $H^i(G/B \times G/B, \mathscr{L}_\lambda \boxtimes \mathscr{L}_\mu) = 0$ for all $i > 0$. The variety $Z := G \times_B G/B$ with the canonical G-action on the left is canonically G-isomorphic to $G/B \times G/B$. If $\mathscr{L}$ is a line bundle on $G/B \times G/B$, then we denote the corresponding line bundle on Z also by $\mathscr{L}$. Let $\pi\colon Z \to G/B$ be the bundle map. The fibres of π are isomorphic to G/B. Since μ is a dominant weight, using the Borel–Weil Theorem it is easy to see that $R^i\pi_*(\mathscr{L}_\lambda \boxtimes \mathscr{L}_\mu) = 0$ for $i > 0$ and

$$\pi_*(\mathscr{L}_\lambda \boxtimes \mathscr{L}_\mu) = G \times_B (k_{-\lambda} \otimes H^0(G/B, \mathscr{L}_\mu)).$$

Hence in $\mathscr{M}_G$ we get the equality

$$\begin{aligned}[V_\lambda^* \otimes V_\mu^*] &= [H^0(G/B \times G/B, \mathscr{L}_\lambda \boxtimes \mathscr{L}_\mu)]\\ &= [H^0(G/B, \pi_*(\mathscr{L}_\lambda \boxtimes \mathscr{L}_\mu))]\\ &= \chi_Y(e^\lambda \operatorname{Char} H^0(G/B, \mathscr{L}_\mu)^*)\\ &= \chi_Y(e^\lambda M_{w_G}(e^\mu)).\end{aligned}$$

1.6. Denote by L a Levi subgroup of G containing the maximal torus T. Let $\mathcal{M}_L$ be the Grothendieck group of L-modules. Denote by B' the Borel subgroup $B \cap L$ of L (note that $X(B) \simeq X(B')$.) For $\lambda \in X(B')$ denote by $\mathcal{L}_\lambda$ the corresponding line bundle on $Y' := L/B'$ and let $\chi_{Y'}(e^\lambda)$ be the Euler characteristic $\chi_{Y'}(e^\lambda) = \sum_{i \geqslant 0} (-1)^i H^i(Y', \mathcal{L}_\lambda)$. (We consider $\chi_{Y'}$ as a map from $\mathbf{Z}[X(B')]$ to $\mathcal{M}_L$.) If $\nu \in X(B')$ is a dominant weight, then let U_ν be a simple L-module of heighest weight ν. We will denote the corresponding class in $\mathcal{M}_L$ also by U_ν. We consider the Weyl group $W_L = \mathrm{Nor}_L(T)/T$ of L as a subgroup of W via the natural inclusion $\mathrm{Nor}_L(T) \subset \mathrm{Nor}_G(T)$. If w_L is the longest element in W_L, then let $w' \in W$ be such that $l(w_L) + l(w') = l(w_G)$ and $w_L w' = w_G$ (the longest word in W).

If V_λ is a simple G-module then denote by $\mathrm{res}_L\, V_\lambda$ the L-module obtained by restricting the representation of G on V_λ to L.

Proposition. *The decomposition of* $\mathrm{res}_L\, V_\lambda^*$ *into simple L-modules is given by*

$$\mathrm{res}_L\, V_\lambda^* = \chi_{Y'}(M_{w'}(e^\lambda)).$$

Proof. Let $\mathrm{res}_L\, V_\lambda = U_{\nu_1} \oplus \cdots \oplus U_{\nu_s}$ be a decomposition into simple L-modules. It follows by Demazure's character formula and Lemma 1.4 (applied now to $\chi_{Y'}$) that

$$\mathrm{res}_L\, V_\lambda^* = \sum_{i=1}^{s} \chi_{Y'}(e^{\nu_i}) = \sum_{i=1}^{s} \chi_{Y'}(M_{w_L} e^{\nu_i}) = \chi_{Y'}(\mathrm{Char}\, V_\lambda). \tag{1.4}$$

Now $\chi_{Y'}(\mathrm{Char}\, V_\lambda) = \chi_{Y'}(M_{w_G}(e^\lambda)) = \chi_{Y'}(M_{w_L}(M_{w'}(e^\lambda))) = \chi_{Y'}(M_{w'}(e^\lambda))$, again by Lemma 1.4.

Remark. The decomposition formula of Proposition 1.6 is due to Kempf (see [7]).

1.7. The disadvantage of the decomposition formulas in the propositions above is that in general one has to compute the Euler characteristic for non-dominant weights (using (1.2)). As a consequence during the computation of the right side there will be considerable cancellations. The aim of the following paragraphs is to show that if the character of V_μ (resp. of V_λ in Proposition 1.6) is given by $\sum_{\mathcal{T}} e^{\nu(\mathcal{T})}$, where $\mathcal{T}$ runs over all standard Young tableaux of shape $p(\mu)$ (resp. $p(\lambda)$), then it is possible to make an a priori choice among the tableaux such that the Euler characteristic has only to be taken over a sum of dominant weights.

2. THE CASE $G = Sl_{m+1}$

2.1. We want to recall briefly the notion of a Young tableau. Let $p = (p_1, p_2, ...)$ with $p_1 \geqslant p_2 \geqslant \cdots$ be a partition of a natural number n. We will identify p with its Young diagram which consists of left justified rows of boxes with p_1 boxes in the first column, p_2 boxes in the second column, etc. By a Young tableau $\mathscr{T}$ of shape p we mean a filling of the boxes of the corresponding diagram with positive integers. We identify a row or a column of a Young tableau with the sequence of integers filled in the boxes of the corresponding row or column. The Young tableau is called *row-standard,* if the integers are strictly increasing in the rows and are smaller or equal to $m+1$. We say that the tableau $\mathscr{T}$ is *standard,* if the tableau is row-standard and the integers are non-decreasing in the columns (from the top to the bottom). We will enumerate the rows from the *bottom* to the *top.* For $1 \leqslant l \leqslant p_1$ we denote by $\mathscr{T}(l)$ the Young tableau obtained from $\mathscr{T}$ by deleting the $(l+1)$st row up to the top row. If i is a positive integer and $\mathscr{T}$ is a given Young tableau then we denote by $c_{\mathscr{T}}(i)$ the number of boxes of $\mathscr{T}$ containing the integer i.

2.2. For $i = 1, ..., m$ denote by $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$ the ith fundamental weight of Sl_{m+1}. If $\mu = \sum_{i=1}^m a_i \omega_i$ is a dominant weight, then we associate to μ the partition $p(\mu) = (p_1, ..., p_m)$ with $p_i := \sum_{j=i}^m a_j$. (Note that $\mu = \sum_{i=1}^m p_i \varepsilon_i$.)

If $\mathscr{T}$ is a standard Young tableau of shape $p(\mu)$ then we define the weight of the tableau $\mathscr{T}$ as

$$v(\mathscr{T}) := c_{\mathscr{T}}(1)\varepsilon_1 + \cdots + c_{\mathscr{T}}(m+1)\,\varepsilon_{m+1}.$$

For $1 \leqslant l \leqslant p_1$ denote by $v_l(\mathscr{T})$ the weight $v(\mathscr{T}(l))$ of the tableau $\mathscr{T}(l)$.

DEFINITION. If L is a Levi subgroup of Sl_{m+1} then a standard Young tableau $\mathscr{T}$ of shape $p(\mu)$ is called *L-dominant* if all the weights

$$v_1(\mathscr{T}), v_2(\mathscr{T}), ..., v(\mathscr{T})$$

are contained in the dominant Weyl chamber of L.

If λ is a dominant weight for Sl_{m+1} then a standard Young tableau $\mathscr{T}$ of shape $p(\mu)$ is called *λ-dominant* if all the weights

$$\lambda + v_1(\mathscr{T}), \lambda + v_2(\mathscr{T}), ..., \lambda + v(\mathscr{T})$$

are contained in the dominant Weyl chamber of Sl_{m+1}.

THEOREM (Littlewood–Richardson Rule [19]). (a) *The decomposition of a tensor product* $V_\lambda \otimes V_\mu$ *of two simple* Sl_{m+1}*-modules of highest weights* λ *and* μ *is given by*

$$V_\lambda \otimes V_\mu = \bigoplus_{\mathcal{T}} V_{\lambda + \nu(\mathcal{T})},$$

where $\mathcal{T}$ *runs over all standard young tableaux of shape* $p(\mu)$ *that are* λ*-dominant.*

(b) *The decomposition of a simple* Sl_{m+1}*-module* V_μ *of highest weight* μ *into simple L-modules* U_ν *of highest weight* ν *is given by*

$$\operatorname{res}_L V_\mu = \bigoplus_{\mathcal{T}} U_{\nu(\mathcal{T})},$$

where $\mathcal{T}$ *runs over all standard Young tableaux of shape* $p(\mu)$ *that are L-dominant.*

2.3. The rest of this paragraph is devoted to the proof of the theorem.

It is well known that the character of the simple Sl_{m+1}-module V_μ is given by (see [21, 15])

$$\operatorname{Char} V_\mu = \sum_{\mathcal{T}} e^{\nu(\mathcal{T})},$$

where $\mathcal{T}$ runs over all standard Young tableaux of shape $p(\mu)$. Hence to prove part (a) of the theorem, by Proposition 1.5 it suffices to show that

$$\chi_Y\left(e^\lambda \sum_{\mathcal{T}'} e^{\nu(\mathcal{T}')}\right) = 0, \tag{2.1}$$

where $\mathcal{T}'$ runs over all not λ-dominant standard Young tableaux of shape $p(\mu)$. And by (1.4), to prove part (b) of the theorem it suffices to show that

$$\chi_{Y'}\left(\sum_{\mathcal{T}'} e^{\nu(\mathcal{T}')}\right) = 0, \tag{2.2}$$

where $\mathcal{T}'$ runs over all not L-dominant standard Young tableaux of shape $p(\mu)$.

2.4. If τ is an element of the Weyl group W of Sl_{m+1} then denote by $X(\tau)$ the corresponding Schubert variety in Sl_{m+1}/B. We know by [15] (or [20]) that the restriction map

$$H^0(Sl_{m+1}/B, \mathcal{L}_\mu) \to H^0(X(\tau), \mathcal{L}_\mu)$$

is surjective. Hence we have a natural inclusion

$$H^0(X(\tau), \mathscr{L}_\mu)^* \subsetneq H^0(Sl_{m+1}/B, \mathscr{L}_\mu)^* = V_\mu.$$

To get a character formula for the B-module $H^0(X(\tau), \mathscr{L}_\mu)^*$ we need to introduce the notion of a standard Young tableau on a Schubert variety.

2.5. Let $\mathscr{T}$ be a row-standard Young tableau of shape $p(\mu)$. For $1 \leqslant l \leqslant p_1$ let $(i_1, ..., i_s)$ be the lth row of $\mathscr{T}$. We associate to this row the weight $\varepsilon_{i_1} + \cdots + \varepsilon_{i_s}$. Since $\mathscr{T}$ is row-standard this is a weight of the fundamental representation V_{ω_s}. Recall that all fundamental representations of Sl_{m+1} are minuscule (see [1]) and hence we can find a $w_l \in W$ such that $w_l(\omega_s) = \varepsilon_{i_1} + \cdots + \varepsilon_{i_s}$. We say that w_l is a lift for the lth row of $\mathscr{T}$.

Remark. Let P_s be the maximal parabolic subgroup of Sl_{m+1} corresponding to the fundamental weight ω_s and let W_s be its Weyl group. Since W_s is the stabilizer in W of ω_s we can naturally identify the weights in V_{ω_s} with the cosets in W/W_s. Consider the projection π: $W \to W/W_s$. Then w_l is a lift with respect to π for the coset σ in W/W_s corresponding to the weight $\varepsilon_{i_1} + \cdots + \varepsilon_{i_s}$.

We say that the sequence $(w_1, ..., w_{p_1})$ is a lift for $\mathscr{T}$ if w_l is a lift for the lth row of $\mathscr{T}$ for $l = 1, ..., p_1$. The lift $(w_1, ..., w_{p_1})$ is called a *defining chain* for $\mathscr{T}$, if $w_1 \geqslant \cdots \geqslant w_{p_1}$. It is shown in [15] that such a defining chain exists if and only if $\mathscr{T}$ is standard, and if $\mathscr{T}$ is standard then there exists a unique minimal defining chain. Here minimal means that if $(w_1, ..., w_{p_1})$ is the minimal defining chain then $w'_i \geqslant w_i$ for all $i = 1, ..., p_1$ if $(w'_1, ..., w'_{p_1})$ is a defining chain for $\mathscr{T}$.

2.6. Since we will use extensively the existence of a minimal defining chain for a standard Young tableau we will recall the construction of a minimal defining chain. Let $\mathscr{T}$ be a standard Young tableau of shape $p(\mu)$ with defining chain $(w'_1, ..., w'_{p_1})$. If $(i_1, ..., i_s)$ is the p_1st row of $\mathscr{T}$ then denote by θ_{p_1} the corresponding coset in W/W_s. Let $X(\theta_{p_1})$ be the corresponding Schubert variety in Sl_{m+1}/P_s. For the projection π: $Sl_{m+1}/B \to Sl_{m+1}/P_s$ there exists a unique Schubert variety $X(\tau)$ in Sl_{m+1}/B such that π: $X(\tau) \to X(\theta_{p_1})$ is birational. Then τ is the minimal lift in W for θ_{p_1} and hence for the p_1st row. We set $w_{p_1} := \tau$. The minimal defining chain $(w_1, ..., w_{p_1})$ for $\mathscr{T}$ can be obtained by recurrence using a lemma due to Deodhar (see [16]). For our purpose the lemma can be stated as follows.

Let $\gamma, \sigma \in W/W_s$ be such that $\gamma \geqslant \sigma$. If $\Sigma \in W$ is a lift for σ with respect to the projection π: $W \to W/W_s$ then there exists an element $\Gamma \in W$ such that Γ is a lift for γ, $\Gamma \geqslant \Sigma$, and Γ is minimal for this property and unique.

Now assume that $w_l, ..., w_{p_1}$ of the minimal defining chain for $\mathscr{T}$ have already been constructed. Then $w'_{l-1} \geqslant w'_l \geqslant w_l$ since $(w'_1, ..., w'_{p_1})$ is a defining chain and $w'_l \geqslant w_l$ by assumption. If $(i_1, ..., i_t)$ is the $(l-1)$st row of $\mathscr{T}$ then denote by θ_{l-1} the corresponding coset in W/W_t. For $\tau \in W$ denote by $\bar{\tau}$ the image $\pi(\tau)$ of τ in W/W_t. It follows that $\theta_{l-1} = \bar{w}'_{l-1} \geqslant \bar{w}_l$. Since w_l is a lift for $\bar{w}_l$ we know by Deodhar's Lemma that there exists a unique element $\Gamma \in W$ which is a lift for θ_{l-1} and minimal with the property $\Gamma \geqslant w_l$. Since Γ is minimal we know $w'_{l-1} \geqslant \Gamma$. For the minimal defining chain set $w_{l-1} := \Gamma$. It follows now from Deodhar's Lemma and the construction that the minimal defining chain is unique and "minimal" in the sense stated in Section 2.5.

2.7. Let $\mathscr{T}$ be a standard Young tableau of shape $p(\mu)$ with minimal defining chain $(w_1, ..., w_{p_1})$. We say that $\mathscr{T}$ is standard on a Schubert variety $X(\tau)$, $\tau \in W$, if $\tau \geqslant w_1$. The main theorem for standard monomial theory for Sl_{m+1} states the following:

THEOREM (Basis Theorem, see [16]). *If $\mu = a_1\omega_1 + \cdots + a_m\omega_m$ is a dominant weight and $X(\tau) \subseteq G/B$ is a Schubert variety, then $H^0(X(\tau), \mathscr{L}_\mu)$ has a basis of T-eigenvectors $p_{\mathscr{T}}$ called standard monomials. They are indexed by the Young tableaux $\mathscr{T}$ of shape $p(\mu)$ standard on $X(\tau)$. The weight of $p_{\mathscr{T}}$ is $-\nu(\mathscr{T})$.*

2.8. For a dominant weight $\mu = \sum_{i=1}^m a_i\omega_i$ and $\tau \in W$ let $SD(\tau, \mu)$ be the set of all standard Young tableaux $\mathscr{T}$ of shape $p(\mu)$ such that $\tau = w_1$ for the minimal defining chain $(w_1, ..., w_{p_1})$ of $\mathscr{T}$. Let $\mathscr{S}\mathscr{D}(\tau, \mu)$ be defined by

$$\mathscr{S}\mathscr{D}(\tau, \mu) := \sum_{\mathscr{T} \in SD(\tau, \mu)} e^{\nu(\mathscr{T})}.$$

We will need the following "character formula":

LEMMA. *If α is a simple root and $\tau \in W$ is such that $s_\alpha\tau < \tau$, then*

$$M_\alpha(\mathscr{S}\mathscr{D}(s_\alpha\tau, \mu)) = \mathscr{S}\mathscr{D}(s_\alpha\tau, \mu) + \mathscr{S}\mathscr{D}(\tau, \mu).$$

Proof. We know by Demazure's character formula that

$$M_\alpha(\text{Char } H^0(X(s_\alpha\tau), \mathscr{L}_\mu)^*) = \text{Char } H^0(X(\tau), \mathscr{L}_\mu)^*.$$

The proof of the lemma is by induction on $l(\tau)$.

If $l(\tau) = 1$, then $\tau = s_\alpha$,

$$\mathscr{S}\mathscr{D}(id, \mu) = \text{Char } H^0(X(id), \mathscr{L}_\mu)^*$$

and

$$\text{Char}\, H^0(X(s_\alpha), \mathcal{L}_\mu)^* = \mathcal{SD}(s_\alpha, \mu) + \mathcal{SD}(id, \mu)$$

by the Basis Theorem, which proves the lemma for this case.

Assume that $l(\tau) > 1$. By the Basis Theorem and by induction we know

$$\begin{aligned}\text{Char}\, H^0(X(\tau), \mathcal{L}_\mu)^* &= \sum_{w \leqslant \tau} \mathcal{SD}(w, \mu) \\ &= \mathcal{SD}(s_\alpha\tau, \mu) + \mathcal{SD}(\tau, \mu) + \sum_{\substack{w < s_\alpha\tau \\ s_\alpha w > w}} M_\alpha(\mathcal{SD}(w, \mu)).\end{aligned}$$

(Recall that if $\eta < \tau$, but $\eta \nleqslant s_\alpha\tau$, then $s_\alpha\eta < s_\alpha\tau$; see [16, Lemma 1.5].) Now note that if $w < s_\alpha w < s_\alpha\tau$, then $M_\alpha(\mathcal{SD}(w, \mu) + \mathcal{SD}(s_\alpha w, \mu)) = M_\alpha(M_\alpha(\mathcal{SD}(w, \mu))) = M_\alpha(\mathcal{SD}(w, \mu))$, since $M_\alpha^2 = M_\alpha$ (Corollary 1.2). Hence we get by the Basis Theorem

$$\begin{aligned}M_\alpha(\text{Char}\, H^0(X(s_\alpha\tau), \mathcal{L}_\mu)^*) &= M_\alpha\left(\sum_{w \leqslant s_\alpha\tau} \mathcal{SD}(w, \mu)\right) \\ &= M_\alpha(\mathcal{SD}(s_\alpha\tau, \mu)) + \sum_{\substack{w < s_\alpha\tau \\ s_\alpha w > w}} M_\alpha(\mathcal{SD}(w, \mu)),\end{aligned}$$

which proves the lemma.

Corollary. *Let L be a Levi subgroup of Sl_{m+1} and let v be a weight. If α is a simple root of the root system of L such that $s_\alpha\tau < \tau$ and $\langle v, \alpha \rangle = 0$, then*

$$\chi_{Y'}(e^v \mathcal{SD}(\tau, \mu)) = 0.$$

Proof. We know by Lemma 1.4, Section 2.8, and (1.1) that

$$\begin{aligned}\chi_{Y'}(e^v\mathcal{SD}(s_\alpha\tau, \mu)) + \chi_{Y'}(e^v\mathcal{SD}(\tau, \mu)) &= \chi_{Y'}(e^v(M_\alpha(\mathcal{SD}(s_\alpha\tau, \mu)))) \\ &= \chi_{Y'}((M_\alpha(e^v))\, \mathcal{SD}(s_\alpha\tau, \mu)) \\ &= \chi_{Y'}(e^v\mathcal{SD}(s_\alpha\tau, \mu)),\end{aligned}$$

and hence $\chi_{Y'}(e^v\mathcal{SD}(\tau, \mu)) = 0$.

2.9. Now we come to the proof of part (a) of the theorem. If $\mathcal{T}$ is a standard Young tableau of shape $p(\mu)$ then recall that for $1 \leqslant l \leqslant p_1$, $\mathcal{T}(l)$ denotes the Young tableau obtained from $\mathcal{T}$ by deleting the last $(p_1 - l)$ rows. (Recall that we enumerate the rows from the bottom to the top.)

Denote by $\check{\mathcal{T}}(l)$ the Young tableau obtained from $\mathcal{T}$ by deleting the first l rows of $\mathcal{T}$. Let i_0, j_0 be such that

$$l = a_1 + \cdots + a_{i_0 - 1} + j_0,$$

where $1 \leqslant j_0 \leqslant a_{i_0}$. Then $\mathcal{T}(l)$ is a standard Young tableau of shape $p(\mu_l)$, where $\mu_l := \sum_{i=1}^{i_0-1} a_i \omega_i + j_0 \omega_{i_0}$, and $\check{\mathcal{T}}(l)$ is a standard Young tableau shape $p(\check{\mu}_l)$, where $\check{\mu}_l := \mu - \mu_l$. If $(w_1, ..., w_{p_1})$ is the minimal defining chain for $\mathcal{T}$ then $(w_1, ..., w_l)$ is a defining chain for $\mathcal{T}(l)$. And, by the construction of the minimal defining chain in 2.6 it follows that $(w_{l+1}, ..., w_{p_1})$ is the minimal defining chain for $\check{\mathcal{T}}(l)$.

For $0 \leqslant l < p_1$ denote by $\Omega_{\lambda, l}$ the set of all standard Young tableaux of shape $p(\mu)$ such that $\mathcal{T}(l)$ is λ-dominant but $\mathcal{T}(l+1)$ is not λ-dominant. Since this induces a partition of all not λ-dominant tableaux, by (2.1) it suffices to show that

$$\chi_Y\Bigl(\sum_{\mathcal{T} \in \Omega_{\lambda, l}} e^{\lambda + \nu(\mathcal{T})}\Bigr) = 0.$$

For $\mathcal{T} \in \Omega_{\lambda, l}$ with minimal defining chain $(w_1, ..., w_{p_1})$ denote by $\Omega_{\lambda, l}(\mathcal{T})$ the set of all standard Young tableaux $\mathcal{T}'$ of shape $p(\mu)$ such that $(w_1, ..., w_{l+1}) = (w'_1, ..., w'_{l+1})$ for the minimal defining chain $(w'_1, ..., w'_{p_1})$ of $\mathcal{T}'$. It is obvious that $\Omega_{\lambda, l}(\mathcal{T}) \subseteq \Omega_{\lambda, l}$ and one can find $\mathcal{T}_1, ..., \mathcal{T}_r \in \Omega_{\lambda, l}$ such that $\Omega_{\lambda, l} = \Omega_{\lambda, l}(\mathcal{T}_1) \cup \cdots \cup \Omega_{\lambda, l}(\mathcal{T}_r)$ is a disjoint union. Hence to prove (a) it suffices to show that

$$\chi_Y\Bigl(\sum_{\mathcal{T}' \in \Omega_{\lambda, l}(\mathcal{T})} e^{\lambda + \nu(\mathcal{T}')}\Bigr) = 0.$$

2.10. Now let $\mathcal{T} \in \Omega_{\lambda, l}$ be with minimal defining chain $(w_1, ..., w_{p_1})$. If $(w'_1, ..., w'_{p_1})$ is the minimal defining chain for $\mathcal{T}' \in \Omega_{\lambda, l}(\mathcal{T})$, then we know that $\mathcal{T}'(l) = \mathcal{T}(l)$ and $w'_{l+1} = w_{l+1}$ (since $(w'_1, ..., w'_{l+1}) = (w_1, ..., w_{l+1})$). It follows that $\check{\mathcal{T}}'(l) \in SD(w_{l+1}, \check{\mu}_l)$. (Recall that $SD(w_{l+1}, \check{\mu}_l)$ is the set of all standard Young tableaux of shape $p(\check{\mu}_l)$ such that the minimal defining chain starts with w_{l+1}.) It is obvious that the map

$$\Omega_{\lambda, l}(\mathcal{T}) \to SD(w_{l+1}, \check{\mu}_l), \mathcal{T}' \to \check{\mathcal{T}}'(l)$$

is injective. In fact, it is a bijection: indeed, let $\mathcal{T}'' \in SD(w_{l+1}, \check{\mu}_l)$ be with minimal defining chain $(w''_1, ..., w''_{p_1 - l})$. Let the Young tableau $\mathcal{T}'$ be defined by $\mathcal{T}'(l) := \mathcal{T}(l)$ and $\check{\mathcal{T}}'(l) := \mathcal{T}''$. Then $\mathcal{T}'$ is a Young tableau of shape $p(\mu)$. It is easy to see that $(w'_1, ..., w'_{p_1})$ defined by $w'_i := w_i$ for $1 \leqslant i \leqslant l$ and $w'_i := w''_{i-l}$ for $l + 1 \leqslant i \leqslant p_1$ is a minimal defining chain for $\mathcal{T}'$

with $(w_1, \ldots, w_{l+1}) = (w'_1, \ldots, w'_{l+1})$. (Use the construction of the minimal defining chain described in 2.6.)

Hence it follows that

$$\chi_Y\left(\sum_{\mathcal{T}' \in \Omega_{\lambda,l}(\mathcal{T})} e^{\lambda+\nu(\mathcal{T}')}\right) = \chi_Y(e^{\lambda+\nu(\mathcal{T}(l))}\mathcal{S}\mathcal{D}(w_{l+1}, \check{\mu}_l)).$$

Furthermore, since $\mathcal{T}(l)$ is λ-dominant but not $\mathcal{T}(l+1)$ we know that there exists a simple root α such that

$$\langle \lambda + \nu(\mathcal{T}(l)), \alpha \rangle \geqslant 0$$

and

$$\langle \lambda + \nu(\mathcal{T}(l+1)), \alpha \rangle = \langle \lambda + \nu(\mathcal{T}(l)) + w_{l+1}(\omega_{i_0}), \alpha \rangle < 0.$$

Here i_0 is such that the $(l+1)$st row of $\mathcal{T}$ is of length i_0. The fundamental representations of Sl_{m+1} are all minuscule, and hence $|\langle w_{l+1}(\omega_s), \alpha \rangle| \leqslant 1$. It follows that

$$\langle \lambda + \nu(\mathcal{T}(l)), \alpha \rangle = 0 \qquad \text{and} \qquad \langle w_{l+1}(\omega_{i_0}), \alpha \rangle = -1.$$

But this implies that $s_\alpha w_{l+1} < w_{l+1}$. Hence we are in the situation of Corollary 2.8 (with $L = G$) and

$$\chi_Y(e^{\lambda+\nu(\mathcal{T}(l))}\mathcal{S}\mathcal{D}(w_{l+1}, \check{\mu}_l)) = 0,$$

which proves (a).

2.11. The proof of (b) is in the same spirit. For $0 \leqslant l < p_1$ let $\Omega_{L,l}$ denote the set of all standard Young tableaux $\mathcal{T}$ of shape $p(\mu)$ such that $\mathcal{T}(l)$ is L-dominant but $\mathcal{T}(l+1)$ is not L-dominant. For $\mathcal{T} \in \Omega_{L,l}$ with minimal defining chain $(w_1, \ldots, w_{p_1})$ denote by $\Omega_{L,l}(\mathcal{T})$ the set of all standard Young tableaux $\mathcal{T}'$ of shape $p(\mu)$ such that $(w_1, \ldots, w_{l+1}) = (w'_1, \ldots, w'_{l+1})$ for the minimal defining chain $(w'_1, \ldots, w'_{p_1})$ of $\mathcal{T}'$. As above, by (2.2), to prove (b) it suffices to show that

$$\chi_{Y'}\left(\sum_{\mathcal{T}' \in \Omega_{L,l}(\mathcal{T})} e^{\nu(\mathcal{T}')}\right) = 0.$$

The argument is the same as above. The map $\Omega_{L,l}(\mathcal{T}) \to SD(w_{l+1}, \check{\mu}_l)$ given by $\mathcal{T}' \mapsto \mathcal{T}'(l)$ is a bijection and hence

$$\chi_{Y'}\left(\sum_{\mathcal{T}' \in \Omega_{L,T}(\mathcal{T})} e^{\nu(\mathcal{T}')}\right) = \chi_{Y'}(e^{\nu(\mathcal{T}(l))}\mathcal{S}\mathcal{D}(w_{l+1}, \check{\mu}_l)).$$

Since $\mathscr{T}(l)$ is L-dominant and $\mathscr{T}(l+1)$ is not L-dominant there exists a simple root α of the root system of L such that

$$\langle \nu(\mathscr{T}(l)), \alpha \rangle \geqslant 0$$

and

$$\langle \nu(\mathscr{T}(l+1)), \alpha \rangle = \langle \nu(\mathscr{T}(l)) + w_{l+1}(\omega_{i_0}), \alpha \rangle < 0.$$

Since α is also a simple root Sl_{m+1} we know $|\langle w_{l+1}(\omega_{i_0}), \alpha \rangle| \leqslant 1$. It follows that

$$\langle \nu(\mathscr{T}(l)), \alpha \rangle = 0 \qquad \text{and} \qquad \langle \omega_{l+1}(\omega_{i_0}), \alpha \rangle = -1,$$

and hence $s_\alpha w_{l+1} < w_{l+1}$. Using Corollary 2.8 we see

$$\chi_{Y'}(e^{\nu(\mathscr{T}(l))} \mathscr{S}\mathscr{D}(w_{l+1}, \check{\mu}_l)) = 0,$$

which proves (b).

3. The General Case

3.1. To generalize the decomposition formulas of Section 2 we will first recall the notion of a Young tableau $\mathscr{T}$ for the other simple groups developed in [16, 14]. For $G = Sl_{m+1}$ we have already pointed out in Remark 2.5 that if we associate to each row $(i_1, ..., i_s)$ of a standard Young tableau the corresponding coset σ in W/W_s, then we can look at a standard Young tableau $\mathscr{T}$ as a sequence of cosets. We will use from now on this notion of a Young tableau. We introduce the notion of a λ-dominant resp. L-dominant Young tableau, which is a straightforward generalization of the notion in Section 2. The main result in this paragraph will be to show that for the decomposition formulas stated in the propositions of 1.5 and 1.6 it suffices to take the Euler characteristic over all weights which correspond to λ-dominant (resp. L-dominant) Young tableaux. The proof of Theorem 3.7 will be given in Section 4.

In Section 3.8 we show how to translate the results for $G = \mathbf{G}_2$ back into the classical notion of a Young tableau in the sense of 2.1. Using similar ideas, one obtains the definition of a Sp_{2m}- resp. $Spin_m$-standard Young tableau $\mathscr{T}$ (where $\mathscr{T}$ is a Young tableau in the sense of 2.1) given in the Appendix. There Theorem 3.7 will also be reformulated for these groups in the language of Sp_{2m}- and $Spin_m$-standard Young tableaux. As already pointed out in the Introduction the notion of a Young tableau which will be developed in the following has the advantage of being independent of the type of the group.

3.2. Let G be a simple, simply connected algebraic group of rank m. We enumerate the fundamental weights $w_1, ..., w_m$ as in [1]. If $\lambda = \sum_{i=1}^{m} a_i\omega_i$ is a dominant weight then, unless otherwise stated, we assume throughout the following that $a_2 = 0$ if G is of type F_4, $a_4 = 0$ if G is of type E_7, and $a_3 = a_4 = a_5 = a_b = 0$ if G is of type E_8. Note that this means $a_i \geqslant 1$ only if $|\langle\omega_i, \beta\rangle| \leqslant 3$ for all roots β. The geometrical meaning of this condition is the following.

Let P_i be the maximal parabolic subgroup of G corresponding to the fundamental weight ω_i and let W_i be its Weyl group. Denote by H the unique maximal Schubert variety of G/P_i of codimension 1 and let $[H]$ be the class in the Chow ring $Ch(G/P_i)$ of G/P_i. For $\tau \in W/W_i$ let $X(\tau)$ be the corresponding Schubert variety. By the formula of Chevalley [4] we know that

$$[X(\tau)] \cdot [H] = \sum_j d_j [X(\eta_j)].$$

Here $\cdot$ denotes the multiplication in $Ch(G/P_i)$. The sum is taken over all Schubert subvarieties of $X(\tau)$ of codimension 1 and $d_j = |\langle\omega_i, \beta_j\rangle|$, where $\eta_j = \tau s_{\beta_j}$ and β_j is a root. The number d_j is called the multiplicity of $X(\eta_j)$ in $X(\tau)$ and we will denote it by $m(\tau, \eta_j)$. Now the condition $|\langle\omega_i, \beta\rangle| \leqslant 3$ means that $d_j \leqslant 3$ for all j.

3.3. A *k-chain* for a pair (τ, κ), $\tau, \kappa \in W/W_i$, is a sequence $(w_0, ..., w_r, \beta_1, ..., \beta_r)$, where $w_j \in W/W_i$ for $0 \leqslant j \leqslant r$ and β_j is a root for $1 \leqslant j \leqslant r$. And either $r = 0$ and $\tau = w_0 = \kappa$ or

$$\tau = w_0 > \cdots > w_r = \kappa, \qquad l(w_j) = l(w_{j-1}) - 1,$$

$$s_{\beta_j} w_j = w_{j-1} \qquad \text{and} \qquad |\langle w_j(w_i), \beta_j\rangle| = k \quad \text{for } j = 1, ..., r.$$

If $r > 0$ this is the same as to say that there exists a sequence of Schubert varieties $X(\tau) = X(w_0) \supset \cdots \supset X(w_r) = X(\kappa)$ in G/P_i such that $X(w_j)$ is of codimension 1 in $X(w_{j-1})$ and the multiplicity $m(w_{j-1}, w_j)$ is k for $j = 1, ..., r$.

3.4. Let i be such that $|\langle\omega_i, \beta\rangle| \leqslant 3$ for all roots β. A quadruple $\theta := (\gamma, \delta, \sigma, \varphi)$, $\gamma, \delta, \sigma, \varphi \in W/W_i$, is called *admissible*, if $\gamma \geqslant \delta \geqslant \sigma \geqslant \varphi$ and there exist 3-chains for the pairs (γ, δ) and (σ, φ) and a 2-chain for the pair (δ, σ). Denote by $v(\theta)$ the weight

$$v(\theta) := 2\gamma(\omega_i) + \delta(\omega_i) + \sigma(\omega_i) + 2\varphi(\omega_i).$$

DEFINITION. Let $\mu = \sum_{i=k}^{m} a_i\omega_i$, $a_k \geqslant 1$, be a dominant weight. A *Young tableau of shape* $p(\mu)$ is a sequence $\mathscr{T} = (\theta_{i,j})$, $k \leqslant i \leqslant m$, $1 \leqslant j \leqslant a_i$, where $\theta_{i,j} = (\gamma_{i,j}, \delta_{i,j}, \sigma_{i,j}, \varphi_{i,j})$ is an admissible quadruple with $\gamma_{i,j}, \delta_{i,j}, \sigma_{i,j}, \varphi_{i,j} \in W/W_i$. (We assign admissible quadruples only if $a_i \geqslant 1$.) The Young tableau is called *standard* if there exists a sequence $\Theta = (\Theta_{i,j})$, $k \leqslant i \leqslant m$, $1 \leqslant j \leqslant a_i$, such that

$$\Theta_{i,j} = (\Gamma_{i,j}, \Delta_{i,j}, \Sigma_{i,j}, \Phi_{i,j}), \qquad \Gamma_{i,j}, \Delta_{i,j}, \Sigma_{i,j}, \Phi_{i,j} \in W$$

and

$$\Gamma_{i,j} \equiv \gamma_{i,j} \bmod W_i, \qquad \Delta_{i,j} \equiv \delta_{i,j} \bmod W_i,$$
$$\Sigma_{i,j} \equiv \sigma_{i,j} \bmod W_i, \qquad \Phi_{i,j} \equiv \varphi_{i,j} \bmod W_i$$

and $\Gamma_{k,1} \geqslant \Delta_{k,1} \geqslant \Sigma_{k,1} \geqslant \Phi_{k,1} \geqslant \cdots \geqslant \Phi_{r,a_r}$ $(r = \max\{i \mid a_i > 0\})$. If $\mathscr{T}$ is standard then the sequence Θ is called a *defining chain* for $\mathscr{T}$.

Note that if $\mathscr{T}$ is standard, then there exists a unique minimal defining chain Θ for $\mathscr{T}$ (see [14, 16], the construction is the same as in Section 2.6). As in Section 2 we say that $\mathscr{T}$ is *standard on a Schubert variety* $X(\tau) \subset G/B$, $\tau \in W$, if $\tau \geqslant \Gamma_{k,1}$ for the minimal defining chain Θ. The weight $\nu(\mathscr{T})$ of the tableau $\mathscr{T}$ is defined as

$$\begin{aligned}\nu(\mathscr{T}) &:= \tfrac{1}{6} \sum_{i=k}^{m} \sum_{j=1}^{a_i} \nu(\theta_{i,j}) \\ &= \tfrac{1}{6} \sum_{i=k}^{m} \sum_{j=1}^{a_i} (2\gamma_{i,j}(\omega_i) + \delta_{i,j}(\omega_i) + \sigma_{i,j}(\omega_i) + 2\varphi_{i,j}(\omega_i)).\end{aligned}$$

Remark. Recall that if $G = Sl_{m+1}$ then $|\langle \omega_i, \beta \rangle| \leqslant 1$ for all roots β. Hence $\gamma = \delta = \sigma = \varphi$ for all admissible quadruples $\theta = (\gamma, \delta, \sigma, \varphi)$. This is how the definition of a Young tableau above specialises to the one given for Sl_{m+1} in Section 2. If G is of type $\mathbf{B}_m$, $\mathbf{C}_m$, or $\mathbf{D}_m$ then $|\langle \omega_i, \beta \rangle| \leqslant 2$ for all roots and hence $\gamma = \delta$ and $\sigma = \varphi$ for all admissible quadruples $\theta = (\gamma, \delta, \sigma, \varphi)$. If one is only interested in these cases one might rather talk about admissible pairs then quadruples (see [16]).

3.5. The main theorem of standard monomial theory states the following (see [14, 16]):

THEOREM (Basis Theorem). *Let G be a simple, simply connected algebraic group and let $\mu = \sum_{i=1}^{m} a_i \omega_i$ be a dominant weight (such that $a_i \geqslant 1$ only if $|\langle \omega_i, \beta \rangle| \leqslant 3$ for all roots). If $X(\tau) \subset G/B$ is a Schubert variety then $H^0(X(\tau), \mathscr{L}_\lambda)$ has a basis of T-eigenvectors $p_{\mathscr{T}}$ called standard monomials. They are indexed by the Young tableaux $\mathscr{T}$ of shape $p(\mu)$ standard on $X(\tau)$. The weight of $p_{\mathscr{T}}$ is $-\nu(\mathscr{T})$.*

3.6. For $1 \leqslant l \leqslant a_k + \cdots + a_m$ let i_0, j_0 be such that $l = a_k + a_{k+1} + \cdots + a_{i_0-1} + j_0$, where $1 \leqslant j_0 \leqslant a_{i_0}$. Denote by $\nu_{l,1}(\mathscr{T})$, $\nu_{l,2}(\mathscr{T})$, $\nu_{l,3}(\mathscr{T})$, and $\nu_{l,4}(\mathscr{T})$ the weights

$$\nu_{l,1}(\mathscr{T}) := \sum_{i=k}^{i_0-1} \sum_{j=1}^{a_i} \nu(\theta_{i,j}) + \sum_{j=1}^{j_0-1} \nu(\theta_{i_0,j}) + 2\gamma_{i_0,j_0}(\omega_{i_0})$$

$$\nu_{l,2}(\mathscr{T}) := \nu_{l,1}(\mathscr{T}) + \delta_{i_0,j_0}(\omega_{i_0})$$

$$\nu_{l,3}(\mathscr{T}) := \nu_{l,2}(\mathscr{T}) + \sigma_{i_0,j_0}(\omega_{i_0})$$

$$\nu_{l,4}(\mathscr{T}) := \nu_{l,3}(\mathscr{T}) + 2\varphi_{i_0,j_0}(\omega_{i_0}) = \sum_{i=k}^{i_0-1} \sum_{j=1}^{a_i} \nu(\theta_{i,j}) + \sum_{j=1}^{j_0} \nu(\theta_{i_0,j}).$$

The Young tableau $\mathscr{T}$ is uniquely determined by the sequence

$$\nu_{1,1}(\mathscr{T}), \nu_{1,2}(\mathscr{T}), \nu_{1,3}(\mathscr{T}), \nu_{1,4}(\mathscr{T}), ..., \nu_{(a_k + \cdots + a_m),4}(\mathscr{T}) = 6\nu(\mathscr{T}).$$

3.7. Let $\mu = \sum_{i=k}^{m} a_i\omega_i$, $a_k \geqslant 1$, be a dominant weight such that $a_i > 0$ only if $|\langle \omega_i, \beta \rangle| \leqslant 3$ for all roots.

DEFINITION. If L is a Levi subgroup of G then a standard Young tableau $\mathscr{T}$ of shape $p(\mu)$ is called *L-dominant*, if all the weights

$$\nu_{1,1}(\mathscr{T}), \nu_{1,2}(\mathscr{T}), \nu_{1,3}(\mathscr{T}), \nu_{1,4}(\mathscr{T}), ..., \nu_{(a_k + \cdots + a_m),4}(\mathscr{T}) = 6\nu(\mathscr{T})$$

are contained in the dominant Weyl chamber of L.

If λ is an arbitrary dominant weight then a standard Young tableau of shape $p(\mu)$ is called *λ-dominant* if all the weights

$$6\lambda + \nu_{1,1}(\mathscr{T}), 6\lambda + \nu_{1,2}(\mathscr{T}), 6\lambda + \nu_{1,3}(\mathscr{T}), 6\lambda + \nu_{1,4}(\mathscr{T}), ..., 6\lambda + \nu_{(a_k + \cdots + a_m),4}(\mathscr{T}) = 6\lambda + 6\nu(\mathscr{T})$$

are contained in the dominant Weyl chamber of G.

Let λ and μ be as above.

THEOREM. (a) *The decomposition of a tensor product $V_\lambda \otimes V_\mu$ of two simple G-modules of highest weights λ and μ is given by*

$$V_\lambda \otimes V_\mu = \bigoplus_{\mathscr{T}} V_{\lambda + \nu(\mathscr{T})},$$

where $\mathscr{T}$ runs over all standard Young tableaux of shape $p(\mu)$ that are λ-dominant.

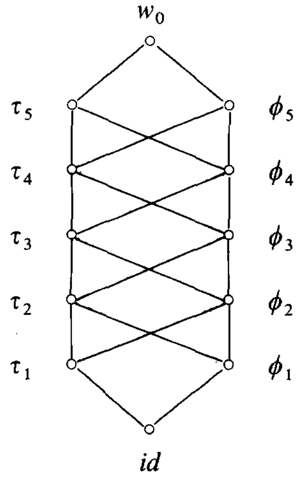
(b) *The decomposition of a simple G-module V_μ of highest weight μ into simple L-modules U_ν of highest weight ν is given by*

$$\operatorname{res}_L V_\lambda = \bigoplus_{\mathcal{T}} U_{\nu(\mathcal{T})},$$

where $\mathcal{T}$ runs over all standard Young tableaux of shape $p(\mu)$ that are L-dominant.

3.8. In the following example we want to show how to "translate" the notion of a standard Young tableau in the sense of 3.4 into the usual notion of a Young tableau.

Let G be the simple algebraic group of type $\mathbf{G}_2$ with maximal torus T. Let α_1, α_2 be a basis of the root system such that α_1 is a short root. Denote the reflections s_{α_1} by s_1 and s_{α_2} by s_2. The Weyl group of G has the elements



$w_0 = s_1 s_2 s_1 s_2 s_1 s_2$

$\tau_5 = s_2 s_1 s_2 s_1 s_2$ $\qquad \phi_5 = s_1 s_2 s_1 s_2 s_1$

$\tau_4 = s_1 s_2 s_1 s_2$ $\qquad \phi_4 = s_2 s_1 s_2 s_1$

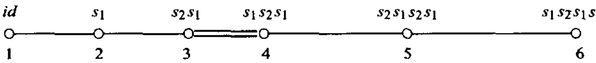
$\tau_3 = s_2 s_1 s_2$ $\qquad \phi_3 = s_1 s_2 s_1$

$\tau_2 = s_1 s_2$ $\qquad \phi_2 = s_2 s_1$

$\tau_1 = s_2$ $\qquad \phi_1 = s_1,$

and $\kappa \geqslant \delta$ for $\kappa, \delta \in W$ if and only if κ is above δ in the diagram.

The simple G-module V_{ω_1} is 7-dimensional. The Weyl group operates transitively on the nonzero weights. Note that the weight space corresponding to the trivial weight has dimension one. We choose a basis $\{e_1, ..., e_7\}$ of V_{ω_1} of weight vectors such that e_1 is a highest weight vector and e_7 is the weight vector corresponding to the trivial weight. Since W/W_1 is linearly ordered, we can identify W/W_1 with the set $\{1, ..., 6\}$ such that if i_1, i_2 correspond to $\kappa_1, \kappa_2 \in W/W_1$, then e_{i_j} is of weight $\kappa_j(\omega_1)$ and $\kappa_1 \geqslant \kappa_2$ if and only if $i_1 \geqslant i_2$:



.

Note that $m(4, 3) = 2$. Hence the admissible quadruples are

$$(1, 1, 1, 1),\ (2, 2, 2, 2),\ (3, 3, 3, 3),\ (4, 4, 3, 3),\ (4, 4, 4, 4),$$
$$(5, 5, 5, 5),\ (6, 6, 6, 6).$$

To every admissible quadruple we associate a Young tableau $\mathscr{T}$ (in the sense of 2.1) of shape $p(6)$. We list the associated tableaux in the same order as the admissible quadruples above:

$$\begin{array}{ccccccc}
1 & 2 & 3 & 3 & 4 & 5 & 6 \\
1 & 2 & 3 & 3 & 4 & 5 & 6 \\
1 & 2 & 3 & 3 & 4 & 5 & 6 \\
1 & 2 & 3 & 4 & 4 & 5 & 6 \\
1 & 2 & 3 & 4 & 4 & 5 & 6 \\
1 & 2 & 3 & 4 & 4 & 5 & 6.
\end{array}$$

Note that if $\mathscr{T}$ is the associated tableau of an admissible quadruple θ, then the weight of θ is

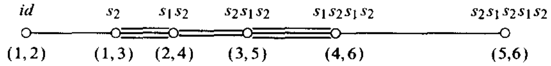
$$\begin{aligned} v(\theta) = \tfrac{1}{6}(c_{\mathscr{T}}(1) - c_{\mathscr{T}}(6))\omega_1 + (c_{\mathscr{T}}(2) - c_{\mathscr{T}}(5))(-\omega_1 + \omega_2) \\ + (c_{\mathscr{T}}(3) - c_{\mathscr{T}}(4))(2\omega_1 - \omega_2)). \end{aligned}$$

We will refer to these tableaux as *admissible sixtuples of rows.*

The G-module $\Lambda^2 V_{\omega_1}$ decomposes into the direct sum $V_{\omega_1} \oplus V_{\omega_2}$ and $e_1 \wedge e_2$ is a highest weight vector of weight ω_2. A vector $e_i \wedge e_j$, $1 \leqslant i < j \leqslant 6$, is a weight vector in V_{ω_2} of weight $\kappa(\omega_2)$ for some $\kappa \in W/W_2$ if and only if

$$(i, j) = (1, 2), (1, 3), (2, 4), (3, 5), (4, 6), (5, 6).$$

We identify W/W_2 with the pairs above. Note that W/W_2 is linearly ordered and $\kappa_1 \geqslant \kappa_2$ if and only if $i_1 \geqslant i_2$ and $j_1 \geqslant j_2$ for the corresponding pairs (i_1, j_1), (i_2, j_2):



Here a pair (i_1, j_1) is connected with $m((i_1, j_1), (i_2, j_2))$ lines with the pair (i_2, j_2). Hence the admissible quadruples are

$$\begin{array}{ll}
((1, 2), (1, 2), (1, 2), (1, 2)) & ((1, 3), (1, 3), (1, 3), (1, 3)) \\
((2, 4), (1, 3), (1, 3), (1, 3)) & ((2, 4), (2, 4), (2, 4), (1, 3)) \\
((2, 4), (2, 4), (2, 4), (2, 4)) & ((3, 5), (3, 5), (2, 4), (1, 3)) \\
((3, 5), (3, 5), (2, 4), (2, 4)) & ((3, 5), (3, 5), (3, 5), (3, 5)) \\
((4, 6), (3, 5), (2, 4), (1, 3)) & ((4, 6), (3, 5), (2, 4), (2, 4)) \\
((4, 6), (3, 5), (3, 5), (3, 5)) & ((4, 6), (4, 6), (4, 6), (3, 5)) \\
((4, 6), (4, 6), (4, 6), (4, 6)) & ((5, 6), (5, 6), (5, 6), (5, 6)).
\end{array}$$

To every admissible quadruple we associate a Young tableau (in the sense of 2.1) of shape $p(6, 6)$. We will refer to them as admissible sixtuples of rows. We list them in the same order as the admissible quadruples above:

$$\begin{array}{cccccccccccccc}
12 & 13 & 13 & 13 & 24 & 13 & 24 & 35 & 13 & 24 & 35 & 35 & 46 & 56 \\
12 & 13 & 13 & 13 & 24 & 13 & 24 & 35 & 13 & 24 & 35 & 35 & 46 & 56 \\
12 & 13 & 13 & 24 & 24 & 24 & 24 & 35 & 24 & 24 & 35 & 46 & 46 & 56 \\
12 & 13 & 13 & 24 & 24 & 35 & 35 & 35 & 35 & 35 & 35 & 46 & 46 & 56 \\
12 & 13 & 24 & 24 & 24 & 35 & 35 & 35 & 46 & 46 & 46 & 46 & 46 & 56 \\
12 & 13 & 24 & 24 & 24 & 35 & 35 & 35 & 46 & 46 & 46 & 46 & 46 & 56.
\end{array}$$

Note that if $\mathcal{T}$ is the associated tableau of an admissible quadruple θ then

$$\begin{aligned} v(\theta) = {} & \tfrac{1}{6}(c_{\mathcal{T}}(1) - c_{\mathcal{T}}(6))\omega_1 + (c_{\mathcal{T}}(2) - c_{\mathcal{T}}(5))(-\omega_1 + \omega_2) \\ & + (c_{\mathcal{T}}(3) - c_{\mathcal{T}}(4))(2\omega_1 - \omega_2)). \end{aligned}$$

If $\mu = a_1\omega_1 + a_2\omega_2$ is a dominant weight and $\mathcal{T} = (\theta_{i,j})$ is a standard Young tableau of shape $p(\mu)$ in the sense of 3.4 then we associate to $\mathcal{T}$ a Young tableau $\mathcal{T}'$ of shape $p(6a_1 + 6a_2, 6a_1)$ in the sense of 2.1. We define $\mathcal{T}'$ to be the tableau having the admissible sixtuple of rows corresponding to $\theta_{1,l}$ as $(6l-5)$th up to the $6l$th row and having the admissible sixtuple corresponding to $\theta_{2,l}$ as $(6(a_1+l)-5)$th up to the $6(a_1+l)$th row. Using the description of the order in the Weyl group above it is easy to see that the map $\mathcal{T} \mapsto \mathcal{T}'$ induces a bijection between the standard Young tableaux $\mathcal{T}$ of shape $p(\mu)$ in the sense of 3.4 and the Young tableaux of shape $p(6a_1 + 6a_2, 6a_2)$ in the sense of 2.1 having the following properties:

(i) The entries in the columns are not decreasing (from the top to the bottom).

(ii) For all $l = 1, ..., a_1 + a_2$ the subtableau consisting of the $(6l-5)$th up to the $6l$th row is an admissible sixtuple of rows.

We refer to these Young tableaux as $\mathbf{G}_2$-standard Young tableaux.

Now we are able to translate the decomposition formulas in the language of $\mathbf{G}_2$-standard Young tableaux.

Let $\mu = a_1\omega_1 + a_2\omega_2$ be a dominant weight and let $\mathcal{T}$ be a $\mathbf{G}_2$-standard Young tableau of shape $p(6a_1 + 6a_2, 6a_2)$. For $1 \leqslant l \leqslant 6a_1 + 6a_2$ denote by $v_l(\mathcal{T})$ the weight

$$\begin{aligned} v_l(\mathcal{T}) = {} & (c_{\mathcal{T}(l)}(1) - c_{\mathcal{T}(l)}(6))\omega_1 + (c_{\mathcal{T}(l)}(2) - c_{\mathcal{T}(l)}(5))(-\omega_1 + \omega_2) \\ & + (c_{\mathcal{T}(l)}(3) - c_{\mathcal{T}(l)}(4))(2\omega_1 - \omega_2)). \end{aligned}$$

The weight of the tableau is defined as $v(\mathcal{T}) := v_{6a_1 + 6a_2}(\mathcal{T})/6$.

If L is a Levi subgroup of G, then we say that $\mathscr{T}$ is L-dominant if all the weights

$$\nu_1(\mathscr{T}), \ldots, \nu_{6a_1+6a_2}(\mathscr{T})$$

are contained in the dominant Weyl chamber of L.

If λ is a dominant weight, then we say that $\mathscr{T}$ is λ-dominant if all the weights

$$6\lambda + \nu_1(\mathscr{T}), \ldots, 6\lambda + \nu_{6a_1+6a_2}(\mathscr{T})$$

are contained in the dominant Weyl chamber of $\mathbf{G}_2$.

By Theorem 3.7 we get

$$V_\lambda \otimes V_\mu = \bigoplus_{\mathscr{T}} V_{\lambda+\nu(\mathscr{T})},$$

where $\mathscr{T}$ runs over all $\mathbf{G}_2$-standard Young tableaux of shape $p(6a_1+6a_2, 6a_2)$ that are λ-dominant and

$$\operatorname{res}_L V_\mu = \bigoplus_{\mathscr{T}} U_{\nu(\mathscr{T})},$$

where $\mathscr{T}$ runs over all $\mathbf{G}_2$-standard Young tableaux of shape $p(6a_1+6a_2, 6a_2)$ that are L-dominant.

For example, if $\lambda = \omega_1$ and $\mu = \omega_1 + \omega_2$ then the λ-dominant $\mathbf{G}_2$-standard Young tableaux are

$$\begin{array}{ll} 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 6 & \\ 6 & \\ 6 & \\ 6 & \\ 6 & \\ 6 & \end{array} \quad \begin{array}{ll} 1 & 3 \\ 1 & 3 \\ 1 & 3 \\ 1 & 3 \\ 2 & 4 \\ 2 & 4 \\ 3 & \\ 3 & \\ 3 & \\ 4 & \\ 4 & \\ 4 & \end{array} \quad \begin{array}{ll} 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 3 & \\ 3 & \\ 3 & \\ 4 & \\ 4 & \\ 4 & \end{array} \quad \begin{array}{ll} 1 & 3 \\ 1 & 3 \\ 1 & 3 \\ 1 & 3 \\ 1 & 3 \\ 1 & 3 \\ 2 & \\ 2 & \\ 2 & \\ 2 & \\ 2 & \\ 2 & \end{array} \quad \begin{array}{ll} 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 2 & \\ 2 & \\ 2 & \\ 2 & \\ 2 & \\ 2 & \end{array} \quad \begin{array}{ll} 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & \\ 1 & \\ 1 & \\ 1 & \\ 1 & \\ 1 & \end{array}$$

We get the decomposition

$$V_{\omega_1} \otimes V_{\omega_1+\omega_2} = V_{2\omega_1+\omega_2} \oplus V_{\omega_1+\omega_2} \oplus V_{3\omega_1} \oplus V_{2\omega_2} \oplus V_{2\omega_1} \oplus V_{\omega_2}.$$

4. The Proof

4.1. The idea of the proof will be the same as in Section 2: namely to give a partition of those tableaux which are not λ-dominant (resp. not L-dominant), such that the corresponding character is of the form $e^{\lambda'}\mathcal{S}\mathcal{D}(\tau, \mu')$ as in 2.10. And then we use the "character formula" of Lemma 2.8 and Lemma 1.4 to show that the Euler characteristic over the weights corresponding to not λ-dominant (resp. not L-dominant) Young tableaux is zero.

4.2. In fact, we are going to prove a more general decomposition formula. Let λ be an arbitrary dominant weight for L and let $\mu = \sum_{i=1}^{m} a_i \omega_i$ be a dominant weight for G such that $a_i > 0$ only if $|\langle \omega_i, \beta \rangle| \leq 3$ for all roots β.

We say that a standard Young tableau of shape $p(\mu)$ is (L, λ)*-dominant* if all the weights

$$6\lambda + \nu_{1,1}(\mathcal{T}), 6\lambda + \nu_{1,2}(\mathcal{T}), 6\lambda + \nu_{1,3}(\mathcal{T}), 6\lambda$$
$$+ \nu_{1,4}(\mathcal{T}), \ldots, 6\lambda + \nu_{(a_k + \cdots + a_m),4}(\mathcal{T}) = 6\lambda + 6\nu(\mathcal{T})$$

are contained in the dominant Weyl chamber of L.

Claim. Denote by U_λ a simple L-module of highest weight λ. The decomposition of the L-module $U_\lambda \otimes \operatorname{res}_L V_\mu$ into simple L-modules is given by

$$U_\lambda \otimes \operatorname{res}_L V_\mu = \bigoplus_{\mathcal{T}} U_{\lambda + \nu(\mathcal{T})},$$

where $\mathcal{T}$ runs over all standard Young tableaux of shape $p(\mu)$ that are (L, λ)-dominant.

Note that part (a) of the theorem follows from the caim for $L = G$ and part (b) follows from the claim for $\lambda = 0$.

4.3. Proof of the claim. First we proceed as in Section 1. Let

$$U_\lambda \otimes \operatorname{res}_L V_\mu = U_{\nu_1} \oplus \cdots \oplus U_{\nu_s}$$

be a decomposition into simple L-modules. Denote by w_L the longest word in the Weyl group W_L of L. Let w_G be the longest word in W and let $w' \in W$ be such that $l(w_L) + l(w') = l(w_G)$ and $w_L w' = w_G$. Again, as in

Section 1, using the Euler characteristic, Demazure's character formula, and Lemma 1.4 we obtain

$$\begin{aligned} U_\lambda^* \otimes \operatorname{res}_L V_\lambda^* &= \sum_{i=1}^{s} \chi_{Y'}(e^{\nu_i}) \\ &= \sum_{i=1}^{s} \chi_{Y'}(M_{w_L}(e^{\nu_i})) \\ &= \chi_{Y'}(\operatorname{Char} U_\lambda \otimes \operatorname{res}_L V_\mu) \\ &= \chi_{Y'}(M_{w_L}(e^\lambda)\, M_{w_L}(M_{w'}(e^\mu))). \end{aligned}$$

Now using the same arguments as in the proof of Proposition 1.5 we get

$$\begin{aligned} \chi_{Y'}(M_{w_L}(e^\lambda)\, M_{w_L}(M_{w'}(e^\mu))) &= \chi_{Y'}(e^\lambda M_{w_L}(M_{w'}(e^\mu))) \\ &= \chi_{Y'}(e^\lambda M_{w_G}(e^\mu)) \\ &= \chi_{Y'}(e^\lambda \operatorname{Char} V_\mu). \end{aligned}$$

Now it follows by the Basis Theorem that

$$U_\lambda^* \otimes \operatorname{res}_L V_\lambda^* = \chi_{Y'}\left(\sum_{\mathscr{T}} e^{\lambda + \nu(\mathscr{T})}\right),$$

where in the last sum $\mathscr{T}$ runs over all standard Young tableaux of shape $p(\mu)$. To prove the claim it remains to show that

$$\chi_{Y'}\left(\sum_{\mathscr{T}'} e^{\lambda + \nu(\mathscr{T}')}\right) = 0,$$

where $\mathscr{T}'$ runs over all not (L, λ)-dominant standard Young tableaux of shape $p(\mu)$. Before we can do this we need some facts about admissible quadruples (some of them can be found in [14]).

4.4. Let $\theta = (\gamma, \delta, \sigma, \varphi)$ be an admissible quadruple where $\gamma, \delta, \sigma, \varphi \in W/W_i$ (again i is such that $|\langle \omega_i, \beta \rangle| \leqslant 3$ for all roots). For a simple root α denote by (a, b, c, d) the quadruple

$$(a, b, c, d) := (\langle \gamma(\omega_i), \alpha \rangle, \ldots, \langle \varphi(\omega_i), \alpha \rangle).$$

Note that if $\tau = s_\beta w$, where $w, \tau \in W/W_i$ and β is a root, then

$$\langle \tau(\omega_i), \alpha \rangle = \langle w(\omega_i), \alpha \rangle - \langle w(\omega_i), \beta \rangle \langle \beta, \alpha \rangle.$$

Hence the admissible condition on $(\gamma, \delta, \sigma, \varphi)$ implies that

$$a \equiv b \bmod 3, \qquad c \equiv d \bmod 3, \qquad \text{and} \qquad b \equiv c \bmod 2. \tag{4.1}$$

LEMMA. *If there exists a k-chain for the pair* (τ, η), $\tau, \eta \in W/W_i$, *and* α *is a simple root such that*

$$\begin{Bmatrix} \langle \tau(\omega_i), \alpha\rangle \leqslant 0 \text{ and } \langle \eta(\omega_i), \alpha\rangle > 0 \\ \langle \tau(\omega_i), \alpha\rangle < 0 \text{ and } \langle \eta(\omega_i), \alpha\rangle \geqslant 0 \end{Bmatrix}$$

then there exists also a k-chain for

$$\begin{Bmatrix} (\tau, s_\alpha \eta) \\ (s_\alpha \tau, \eta) \end{Bmatrix}.$$

Proof. Assume first that $\langle \eta(\omega_i), \alpha\rangle > 0$ and $\langle \tau(\omega_i), \alpha\rangle \leqslant 0$. If $(w_0, ..., w_r, \beta_1, ..., \beta_r)$ is a k-chain for (τ, η), then let p be such that $\langle w_l(\omega_i), \alpha\rangle > 0$ for $l \geqslant p$ and $\langle w_{p-1}(\omega_i), \alpha\rangle \leqslant 0$. Note that if $l \geqslant p+1$, then we have for $\beta'_l := s_\alpha(\beta_l)$

$$l(s_\alpha w_{l-1}) = l(s_\alpha w_l) + 1, \qquad s_{\beta'_l}(s_\alpha w_l) = s_\alpha w_{l-1}$$

and

$$|\langle s_\alpha w_l(\omega_i), \beta'_l\rangle| = |\langle s_\alpha w_l(\omega_i), s_\alpha(\beta_l)\rangle| = |\langle w_l(\omega_i), \beta_l\rangle| = k.$$

It follows that $(s_\alpha w_p, ..., s_\alpha w_r, \beta'_{p+1}, ..., \beta'_r)$ is a k-chain for $(s_\alpha w_p, s_\alpha \eta)$.

Now $w_{p-1} > w_p$ and $\langle w_{p-1}(\omega_i), \alpha\rangle \leqslant 0$ implies that $s_\alpha w_p \leqslant w_{p-1}$. But $l(s_\alpha w_p) = l(w_p) + 1 = l(w_{p-1})$ and hence $s_\alpha w_p = w_{p-1}$. The k-chain for $(\tau, s_\alpha \eta)$ is obtained by taking the k-chains for (τ, w_{p-1}) and $(w_{p-1}, s_\alpha \eta)$.

The proof for the case $\langle \eta(\omega_i), \alpha\rangle \geqslant 0$ and $\langle \tau(\omega_i), \alpha\rangle < 0$ is similar.

4.5. LEMMA. *If* β *is a root such that* $\langle \eta(\omega_i), \beta\rangle = 3$ *and* $l(s_\beta \eta) = l(\eta) + 1$ *for some* $\eta \in W/W_i$, *then there exists a simple root* α *such that* $s_\alpha \eta = s_\beta \eta$ *and* $\langle \eta(\omega_i), \alpha\rangle = 3$.

Proof. Note first that $s_\alpha \eta = s_\beta \eta$ implies that $\langle \eta(\omega_i), \alpha\rangle = \langle \eta(\omega_i), \beta\rangle$ (recall that $|\langle \eta(\omega_i), \beta\rangle|$ is the multiplicity of $X(\eta)$ in $X(s_\beta \eta)$ (see 3.2).

The proof is by induction on $l(\eta)$ (compare [16, Lemma 2.6]. Let ζ be a minimal element in W/W_i such that there exists a root β with $\tau = s_\beta \zeta$, $l(\tau) = l(\zeta) + 1$, and $\langle \zeta(\omega_i), \beta\rangle = 3$. If there exists no simple root α' with $\tau = s_{\alpha'} \zeta$, then choose $\tau' \in W/W_i$ and a simple root α such that $\tau' < \tau$ and $s_\alpha \tau' = \tau$. There exists a $\zeta' \in W/W_i$ such that $\zeta' < \tau'$ and $s_\alpha \zeta' = \zeta$ (see [16, Lemma 1.5]. We get for $\beta' := s_\alpha(\beta)$

$$s_{\beta'} \zeta' = \tau', \qquad l(\tau') = l(\zeta') + 1, \qquad \text{and} \qquad \langle \zeta'(\omega_i), \beta'\rangle = \langle \zeta(\omega_i), \beta\rangle = 3,$$

contradicting the minimality assumption on ζ.

Now assume that η is an element of W/W_i and β is a root such that $\tau = s_\beta \eta$, $l(\tau) = l(\eta)+1$, and $\langle \eta(\omega_i), \beta \rangle = 3$. There exists a simple root α with $\varphi := s_\alpha \tau < \tau$. If $\varphi \neq \eta$, then there exists a $\theta < \varphi$ with $s_\alpha \theta = \eta$. Denote $s_\alpha(\beta)$ by γ. Then $s_\gamma \theta = \varphi$, $l(\theta)+1 = l(\varphi)$, and $\langle \theta(\omega_i), \gamma \rangle = \langle \eta(\omega_i), \beta \rangle = 3$. By induction hypothesis we know that we can choose γ to be a simple root. Now $\tau = s_\alpha s_\gamma \theta = s_\beta s_\alpha \theta$ and hence

$$\tau(\omega_i) = s_\alpha(s_\gamma(\theta(\omega_i))) = \theta(\omega_i) + 3\gamma + p\alpha,$$

where $p = \langle \varphi(\omega_i), \alpha \rangle$,

$$\tau(\omega_i) = s_\beta(s_\alpha(\theta(\omega_i))) = \theta(\omega_i) + q\alpha + 3\beta,$$

where $q = \langle \theta(\omega_i), \alpha \rangle$. It follows that $3\beta = 3\gamma + (p-q)\alpha$. Since γ and α are simple roots, we know that $p \equiv q \bmod 3$. But $1 \leqslant p, q \leqslant 3$ implies that $p = q$ and hence $\gamma = \beta$, proving the lemma.

4.6. COROLLARY. *Assume that there exists a 3-chain for* (τ, η), $\tau, \eta \in WW_i$. *If* α *is a simple root such that* $-1 \geqslant \langle \tau(\omega_i), \alpha \rangle \geqslant -2$ *or* $1 \leqslant \langle \eta(\omega_i), \alpha \rangle \leqslant 2$, *then* $\langle \tau(\omega_i), \alpha \rangle = \langle \eta(\omega_i), \alpha \rangle$ *and there exists a 3-chain for* $(s_\alpha \tau, s_\alpha \eta)$.

Proof. If $(w_0, ..., w_r, \beta_1, ..., \beta_r)$ is a 3-chain for (τ, η), where $\beta_1, ..., \beta_r$ are simple roots (because of Lemma 4.5), then we know

$$\langle w_{j-1}(\omega_i), \alpha \rangle = \langle w_j(\omega_i), \alpha \rangle - 3\langle \beta_j, \alpha \rangle,$$

and hence

$$\langle w_j(\omega_i), \alpha \rangle \equiv \langle \tau(\omega_i), \alpha \rangle \equiv \langle \eta(\omega_i), \alpha \rangle \qquad \text{mod } 3.$$

Since $|\langle \kappa(\omega_i), \alpha \rangle| \leqslant 3$ for all $\kappa \in W/W_i$ it follows that

$$1 \leqslant |\langle w_j(\omega_i), \alpha \rangle| \leqslant 2 \qquad \text{and} \qquad \alpha \neq \beta_j \text{ for all } j.$$

Since α and β_j are simple roots we know $\langle \beta_j, \alpha \rangle \leqslant 0$. Since $\langle w_r(\omega_i), \alpha \rangle > 0$ (resp. $\langle w_0(\omega_i), \alpha \rangle < 0$), this implies that $\langle \beta_j, \alpha \rangle = 0$ for $j = 1, ..., r$. Hence $\langle w_j(\omega_i), \alpha \rangle = \langle \eta(\omega_i), \alpha \rangle = \langle \tau(\omega_i), \alpha \rangle$ for $j = 0, ..., r$ and $(s_\alpha w_0, ..., s_\alpha w_r, \beta_1, ..., \beta_r)$ is a 3-chain for $(s_\alpha \tau, s_\alpha \eta)$.

4.7 For an admissible quadruple $\theta = (\gamma, \delta, \sigma, \varphi)$, $\gamma, \delta, \sigma, \varphi \in W/W_i$, and a simple root α denote by (a, b, c, d) the quadruple $(\langle \gamma(\omega_i), \alpha \rangle, ..., \langle \varphi(\omega_i), \alpha \rangle)$.

COROLLARY. (i) *If* $c \leqslant 0$ *and* $d \geqslant 0$, *then* $(c, d) = (-3, 0), (-3, 3), (0, 0)$, *or* $(0, 3)$.

(ii) *If* $b \leqslant 0$, *then* $c \neq 3$.

(iii) *If* $b \leqslant 0$ *and* $c, d > 0$, *then* $(c, d) = (2, 2)$.

Proof. (i) is a consequence of Corollary 4.6 and (4.1). Now assume that $b \leqslant 0$ and $c = 3$. It follows by Lemma 4.4 that $\theta' = (\gamma, \delta, s_\alpha \sigma, \varphi)$ is an admissible quadruple. Since $\nu(\theta)$ and $\nu(\theta')$ are weights in V_{ω_i} we know that $\nu(\theta) - \nu(\theta') = \frac{1}{2}\alpha$ is an element of the root lattice, which is a contradiction. If $b \leqslant 0$ and $c, d > 0$, then $c \neq 3$ by (ii) and $c = d$ (for $c \equiv d \bmod 3$). Hence $\theta' := (\gamma, \delta, s_\alpha \sigma, s_\alpha \varphi)$ is an admissible quadruple by Corollary 4.6 and Lemma 4.4. Since $\nu(\theta) - \nu(\theta') = \frac{1}{6}(c + 2d)\alpha$ is an element of the root lattice, we know that $c + 2d = 3c \equiv 0 \bmod 6$.

4.8. Now we come back to the proof of the claim. As in 2.8 denote for a dominant weight $\mu = \sum_{i=k}^{m} a_i \omega_i$, $a_k \geqslant 1$, and $\tau \in W$ by $SD(\tau, \mu)$ the set of all standard Young tableaux $\mathscr{T}$ of shape $p(\mu)$ such that $\tau = \Gamma_{k,1}$ for the minimal chain Θ of $\mathscr{T}$. And let $\mathscr{S}\mathscr{D}(\tau, \mu)$ be defined by

$$\mathscr{S}\mathscr{D}(\tau, \mu) = \sum_{\mathscr{T} \in SD(\tau, \mu)} e^{\nu(\mathscr{T})}.$$

If α is a simple root such that $s_\alpha \tau < \tau$ then the "character formula" of Lemma 2.8

$$M_\alpha(\mathscr{S}\mathscr{D}(s_\alpha \tau, \mu)) = \mathscr{S}\mathscr{D}(s_\alpha \tau, \mu) + \mathscr{S}\mathscr{D}(\tau, \mu) \tag{4.2}$$

still holds, since for the proof we needed a Basis Theorem and Demazure's character formula (which holds in general). As a consequence, Corollary 2.8 also holds.

If ν is a weight and α is a simple root of the root system of the Levi subgroup L such that $\langle \nu, \alpha \rangle = 0$ and $s_\alpha \tau < \tau$, then

$$\chi_{Y'}(e^{\nu} \mathscr{S}\mathscr{D}(\tau, \mu)) = 0. \tag{4.3}$$

4.9. As in Section 2 for $0 \leqslant l \leqslant a_k + \cdots + a_m$ we break a standard Young tableau $\mathscr{T}$ and its minimal defining chain Θ into two parts:

$$\mathscr{T}(l), \Theta(l) \qquad \text{and} \qquad \check{\mathscr{T}}(l), \check{\Theta}(l).$$

To be precise, if $l = 0$ then set $\check{\mathscr{T}}(0) = \mathscr{T}$ and $\check{\Theta}(0) = \Theta$. Denote by $\mathscr{T}(0)$ and $\Theta(0)$ the empty sequences. For $1 \leqslant l \leqslant a_k + \cdots + a_m$ let i_0, j_0 be such that $l = a_k + a_{k+1} + \cdots + a_{i_0 - 1} + j_0$, where $1 \leqslant j_0 \leqslant a_{i_0}$. Denote by μ_l the weight $\sum_{i=k}^{i_0 - 1} a_i \omega_i + j_0 \omega_{i_0}$ and by $\check{\mu}_l$ the weight $\mu - \mu_l$. Then $\mathscr{T}(l) = (\theta(l)_{i,j})$

is a Young tableau of shape $p(\mu_l)$ with defining chain $\Theta(l)=(\Theta(l)_{i,j})$ where $\theta(l)_{i,j}$ and $\Theta(l)_{i,j}$ are given by

$$\begin{array}{l} \theta(l)_{i,j}:=\theta_{i,j} \\ \Theta(l)_{i,j}:=\Theta_{i,j} \end{array} \qquad \text{for} \quad k\leqslant i\leqslant i_0-1,\ 1\leqslant j\leqslant a_i, \text{ and } 1\leqslant j\leqslant j_0 \text{ for } i=i_0.$$

The Young tableau $\check{\mathcal{T}}(l)=(\check{\theta}(l)_{i,j})$ is of shape $p(\check{\mu}(l))$ with minimal defining chain $\check{\Theta}(l)=(\check{\Theta}(l)_{i,j})$ where $\check{\theta}(l)_{i,j}$ and $\check{\Theta}_{i,j}$ are defined by

$$\begin{array}{l} \check{\theta}(l)_{i,j}:=\theta_{i,j} \\ \check{\Theta}(l)_{i,j}:=\check{\Theta}_{i,j} \end{array} \qquad \text{for} \quad i_0+1\leqslant i\leqslant m,\ 1\leqslant j\leqslant a_i$$

and

$$\begin{array}{l} \check{\theta}(l)_{i_0,j}:=\theta_{i_0,j_0+j} \\ \check{\Theta}(l)_{i_0,j}:=\check{\Theta}_{i_0,j_0+j} \end{array} \qquad \text{for} \quad 1\leqslant j\leqslant a_{i_0}-j_0.$$

Note that $\nu_{l,4}(\mathcal{T})=6\nu(\mathcal{T}(l))$.

For $0\leqslant l<a_k+\cdots+a_m$ denote by Ω_l the set of all standard Young tableaux of shape $p(\mu)$ such that $\mathcal{T}(l)$ is (L,λ)-dominant but $\mathcal{T}(l+1)$ is not (L,λ)-dominant. Since this induces a partition of all standard Young tableaux of shape $p(\mu)$ that are not (L,λ)-dominant, to prove the claim it suffices to show that

$$\chi_{Y'}\left(e^{\lambda}\sum_{\mathcal{T}\in\Omega_l}e^{\nu(\mathcal{T})}\right)=0.$$

4.10. To prove this we will give a partition $\Omega_{l-1}=\Omega_{l-1}(\mathcal{T}^1)\cup\cdots\cup\Omega_{l-1}(\mathcal{T}^s)$ of Ω_{l-1}. In the general case it will not be true anymore that $\chi_{Y'}(e^{\lambda}\sum_{\mathcal{T}'\in\Omega_{l-1}(\mathcal{T}^i)}e^{\nu(\mathcal{T}')})=0$ as in the case of $G=Sl_{m+1}$. But we will show that there exists a partition $p_1\cup\cdots\cup p_t$ of $\{1,\ldots,s\}$ such that $\#p_i\leqslant 3$ and

$$\chi_{Y'}\left(e^{\lambda}\sum_{j\in p_i}\ \sum_{\mathcal{T}'\in\Omega_{l-1}(\mathcal{T}^j)}e^{\nu(\mathcal{T}')}\right)=0$$

for all $1\leqslant j\leqslant t$.

Consider $\mathcal{T}\in\Omega_{l-1}$ with minimal defining chain Θ. Let i_0, j_0 be such that $l=a_k+a_{k+1}+\cdots+a_{i_0-1}+j_0$, where $1\leqslant j_0\leqslant a_{i_0}$. Set $\theta_{i_0,j_0}=(\gamma,\delta,\sigma,\varphi)$ and $\Theta_{i_0,j_0}=(\Gamma,\Delta,\Sigma,\Phi)$. Since $\frac{1}{6}|\langle\nu(\theta_{i_0,j_0}),\beta\rangle|\leqslant 3$ and $\mathcal{T}(l)$ is not (L,λ)-dominant we know that there exists a simple root α of the root system of L such that

$$0 \leqslant \langle \lambda + \nu(\mathscr{T}(l-1)), \alpha \rangle \leqslant 2,$$

$$\begin{aligned} \text{and} \qquad & \text{either } \langle \lambda + \nu(\mathscr{T}(l-1)) + \tfrac{1}{6}(2\gamma(\omega_{i_0})), \alpha \rangle < 0 \\ & \text{or } \langle \lambda + \nu(\mathscr{T}(l-1)) + \tfrac{1}{6}(2\gamma(\omega_{i_0}) + \delta(\omega_{i_0})), \alpha \rangle < 0 \\ & \text{or } \langle \lambda + \nu(\mathscr{T}(l-1)) + \tfrac{1}{6}(2\gamma(\omega_{i_0}) + \delta(\omega_{i_0}) + \sigma(\omega_{i_0})), \alpha \rangle < 0 \\ & \text{or } \langle \lambda + \nu(\mathscr{T}(l)), \alpha \rangle < 0. \end{aligned} \tag{4.4}$$

4.11. We consider first the case where

$$\langle \lambda + \nu(\mathscr{T}(l-1)), \alpha \rangle = 0 \qquad \text{and} \qquad \langle \lambda + \nu(\mathscr{T}(l-1)) + \tfrac{1}{3}\gamma(\omega_{i_0}), \alpha \rangle < 0,$$

which implies that $s_\alpha \gamma < \gamma$. Denote by $\Omega_{l-1}(\mathscr{T})$ the set of all standard Young tableaux $\mathscr{T}'$ of shape $p(\mu)$ such that $\Theta'(l-1) = \Theta(l-1)$ and $\Gamma'_{i_0, j_0} = \Gamma$ for the minimal defining chain Θ' of $\mathscr{T}'$. Then $\Omega_{l-1}(\mathscr{T}) \subset \Omega_{l-1}$. As in Section 2 it is easy to see that the map $\Omega_{l-1}(\mathscr{T}) \to SD(\Gamma, \check{\mu}_{l-1})$ defined by $\mathscr{T}' \mapsto \check{\mathscr{T}}'(l-1)$ is a bijection. Hence it follows by (4.3) that

$$\chi_{Y'}\Bigg(\sum_{\mathscr{T}' \in \Omega_{l-1}(\mathscr{T})} e^{\lambda + \nu(\mathscr{T}')}\Bigg) = \chi_{Y'}(e^{\lambda + \nu(\mathscr{T}(l-1))} \mathscr{S}\mathscr{D}(\Gamma, \check{\mu}_{l-1})) = 0,$$

since by assumption we know that $s_\alpha \gamma < \gamma$ and hence $s_\alpha \Gamma < \Gamma$ (because Γ is a lift for γ).

4.12. Throughout the following we may assume that

$$s_\alpha \gamma \geqslant \gamma \qquad \text{if} \quad \langle \lambda + \nu(\mathscr{T}(l-1)), \alpha \rangle = 0.$$

Since $s_\alpha \gamma \geqslant \gamma$ implies that $\langle \gamma(\omega_{i_0}), \alpha \rangle \geqslant 0$, using Corollary 4.7 and (4.1) it easy to see that

$$0 \geqslant \langle \lambda + \nu(\mathscr{T}(l)), \alpha \rangle \geqslant -2$$

and

$$\langle \lambda + \nu(\mathscr{T}(l-1)) + \tfrac{1}{3}\gamma(\omega_{i_0}), \alpha \rangle \geqslant 0 \qquad \text{for} \quad \mathscr{T} \in \Omega_{l-1}. \tag{4.5}$$

We will first discuss the case where $\langle \lambda + \nu(\mathscr{T}(l)), \alpha \rangle = 0$ or -2.

Let i'_0, j'_0 be such that $l + 1 = a_k + a_{k+1} + \cdots + a_{i'_0} + j'_0$, where $1 \leqslant j'_0 \leqslant a_{i'_0}$. Denote $\Gamma_{i'_0, j'_0}$ by Ψ. If

$$\langle \lambda + \nu(\mathscr{T}(l)), \alpha \rangle = 0 \qquad \text{and} \qquad s_\alpha \Psi < \Psi,$$

then denote by $\Omega_{l-1}(\mathscr{T})$ the set of all standard Young tableaux $\mathscr{T}'$ such that $\Theta'(l)=\Theta(l)$ and $\Gamma'_{i'_0,j'_0}=\Psi$ for the minimal defining chain Θ' of $\mathscr{T}'$. It follows that $\Omega_{l-1}(\mathscr{T})\subset\Omega_{l-1}$. Using the same arguments as above we get

$$\chi_{Y'}\left(\sum_{\mathscr{T}'\in\Omega_{l-1}(\mathscr{T})} e^{\lambda+\nu(\mathscr{T}')}\right)=\chi_{Y'}(e^{\lambda+\nu(\mathscr{T}(l))}\mathscr{S}\mathscr{D}(\Psi,\check{\mu}_l))=0.$$

4.13. We will see that the cases

$$\langle\lambda+\nu(\mathscr{T}(l)),\alpha\rangle=0,\qquad s_\alpha\Psi>\Psi$$

and

$$\langle\lambda+\nu(\mathscr{T}(l)),\alpha\rangle=-2$$

belong closely together. Let i_0, j_0, i'_0, j'_0 be as above. Let $\mathscr{T}^1=(\theta^1_{i,j})\in\Omega_{l-1}$ be a standard Young tableau of shape $p(\mu)$ such that $\langle\lambda+\nu(\mathscr{T}(l)),\alpha\rangle=0$ and $s_\alpha\Psi>\Psi$ (where $\Psi=\Gamma^1_{i'_0,j'_0}$ for the minimal defining chain $\Theta^1=(\Theta^1_{i,j})$ of $\mathscr{T}^1$). Set $\theta^1_{i_0,j_0}=(\gamma^1,\delta^1,\sigma^1,\varphi^1)$. Using Corollary 4.7 (and (4.5)) one can show that $\langle\sigma^1(\omega_{i_0}),\alpha\rangle=-3$ and $\langle\varphi^1(\omega_{i_0}),\alpha\rangle=3$. Denote by $\mathscr{T}^2$ the Young tableau defined by

$$\mathscr{T}^2(l-1):=\mathscr{T}^1(l-1),\qquad \theta^2_{i_0,j_0}:=(\gamma^1,\delta^1,\sigma^1,s_\alpha\varphi^1),$$

and

$$\check{\mathscr{T}}^2(l):=\check{\mathscr{T}}^1(l).$$

($(\gamma^1,\delta^1,\sigma^1,s_\alpha\varphi^1)$ is an admissible quadruple by Lemma 4.4.) Then $\mathscr{T}^2$ is a standard Young tableau of shape $p(\mu)$ with minimal defining chain Θ^2 defined by

$$\Theta^2(l-1):=\Theta^1(l-1),\qquad \Theta^2_{i_0,j_0}:=(\Gamma^1,\Delta^1,\Sigma^1,s_\alpha\Phi^1),$$

and

$$\check{\Theta}^2(l):=\check{\Theta}^1(l).$$

(Note that $s_\alpha\Phi^1>\Phi^1$, but since $\Sigma^1\geqslant\Phi^1$ and $s_\alpha\Sigma^1<\Sigma^1$ we know that $s_\alpha\Phi^1\leqslant\Sigma^1$ and hence Θ^2 is a defining chain. Using the following remark one can easily see that Θ^2 is the minimal defining chain.) The standard Young tableau $\mathscr{T}^2$ has the properties

$$\mathscr{T}^2\in\Omega_{l-1},\qquad \lambda+\nu(\mathscr{T}^2(l))=\lambda+\nu(\mathscr{T}^1(l))-\alpha,$$

$$\langle\lambda+\nu(\mathscr{T}^2(l)),\alpha\rangle=-2,$$

and

$$s_\alpha\Gamma^2_{i'_0,j'_0}(=s_\alpha\Psi)>\Gamma^2_{i'_0,j'_0}.$$

Remark. We should point out certain properties of the minimal lifts in Deodhar's Lemma which will be used several times. They follow easily from the uniqueness of the lifts: For $\gamma, \sigma \in W/W_s$, $\gamma \geqslant \sigma$, let $\Sigma \in W$ be the lift for σ and let $\Gamma \in W$ be the unique minimal lift for γ such that $\Gamma \geqslant \Sigma$. If α is a simple root such that $s_\alpha \gamma > \gamma$ or $\gamma > s_\alpha \gamma \geqslant \sigma$ and $s_\alpha \Gamma > \Sigma$ then $s_\alpha \Gamma$ is a minimal lift for $s_\alpha \gamma$ such that $s_\alpha \Gamma \geqslant \Sigma$. If $\gamma \geqslant s_\alpha \sigma > \sigma$ and $\Gamma > s_\alpha \Sigma$, then Γ is the minimal lift for γ such that $\Gamma \geqslant s_\alpha \Sigma$. And if α is a simple root such that $s_\alpha \sigma < \sigma$ and $s_\alpha \gamma < \gamma$ then Γ is the minimal lift for γ such that $\Gamma \geqslant s_\alpha \Sigma$.

4.14. On the other hand let $\mathcal{T}^2 = (\theta^2_{i,j}) \in \Omega_{l-1}$ be a standard Young tableau with minimal defining chain $\Theta^2 = (\Theta^2_{i,j})$ such that $\langle \lambda + \nu(\mathcal{T}^2(l)), \alpha \rangle = -2$ and $s_\alpha \Psi > \Psi$ (where $\Psi = \Gamma^2_{i'_0, j'_0}$). Set $\theta^2_{i_0, j_0} = (\gamma^2, \delta^2, \sigma^2, \varphi^2)$. Then using Corollary 4.7 (and (4.5)) one can chow that $\langle \sigma^2(\omega_{i_0}), \alpha \rangle = \langle \varphi^2(\omega_{i_0}), \alpha \rangle = -3$. Denote by $\mathcal{T}^1$ the Young tableau of shape $p(\mu)$ defined by

$$\mathcal{T}^1(l-1) := \mathcal{T}^2(l-1), \qquad \theta^1_{i_0, j_0} := (\gamma^2, \delta^2, \sigma^2, s_\alpha \varphi^2),$$

and

$$\check{\mathcal{T}}^1(l) := \check{\mathcal{T}}^2(l).$$

Note that $\langle \phi(\omega_{i_0}), \alpha \rangle = -3$ and hence $(\gamma^2, \delta^2, \sigma^2, s_\alpha \varphi^2)$ is an admissible quadruple. $\mathcal{T}^1$ is standard with minimal defining chain Θ^1 defined by

$$\Theta^1(l-1) := \Theta^2(l-1), \qquad \Theta^1_{i_0, j_0} := (\Gamma^2, \Delta^2, \Sigma^2, s_\alpha \Phi^2),$$

and

$$\check{\Theta}^1(l) := \check{\Theta}^2(l).$$

(Note that $s_\alpha \Phi < \Phi$, but since $\Phi \geqslant \Psi$ and $s_\alpha \Psi \geqslant \Psi$ we know that $s_\alpha \Phi \geqslant \Psi$ and hence Θ^1 is a defining chain. To show that Θ^1 is minimal use Remark 4.13.) The standard Young tableau $\mathcal{T}^1$ has the properties

$$\mathcal{T}^1 \in \Omega_{l-1}, \qquad \lambda + \nu(\mathcal{T}^1(l)) = \lambda + \nu(\mathcal{T}^2(l)) + \alpha, \qquad \langle \lambda + \nu(\mathcal{T}^1(l)), \alpha \rangle = 0,$$

and

$$s_\alpha \Gamma^1_{i'_0, j'_0} (= s_\alpha \Psi) > \Gamma^1_{i'_0, j'_0}.$$

4.15. It follows that the map $\mathcal{T}^1 \mapsto \mathcal{T}^2$ (defined by the construction above) from the set of all standard Young tableaux $\mathcal{T}^1$ in Ω_{l-1} such that

$$\langle \lambda + \nu(\mathcal{T}^1(l)), \alpha \rangle = 0 \qquad \text{and} \qquad s_\alpha \Gamma^1_{i'_0, j'_0} (= s_\alpha \Psi) > \Gamma^1_{i'_0, j'_0}$$

to the set of all standard Young tableaux $\mathcal{T}^2$ in Ω_{l-1} such that

$$\langle \lambda + \nu(\mathcal{T}^2(l)), \alpha \rangle = -2 \qquad \text{and} \qquad s_\alpha \Gamma^2_{i'_0, j'_0} (= s_\alpha \Psi) > \Gamma^2_{i'_0, j'_0}$$

is bijective.

Denote by $\Omega_{l-1}(\mathcal{T}^1)$ (resp. $\Omega_{l-1}(\mathcal{T}^2)$) the set of all standard Young tableaux $\mathcal{T}'$ such that $\Theta'(l) = \Theta^1(l)$ (resp. $\Theta'(l) = \Theta^2(l)$) and $\Gamma'_{i'_0, j'_0} = \Psi$ for the minimal defining chain Θ' of $\mathcal{T}'$. If there exists a standard Young tableau $\mathcal{T}^3$ such that

$$\Theta^3(l) = \Theta^2(l) \qquad \text{and} \qquad \Gamma^3_{i'_0, j'_0} = s_\alpha \Psi, \tag{4.6}$$

then denote by $\Omega_{l-1}(\mathcal{T}^3)$ the set of all standard Young tableaux $\mathcal{T}'$ such that $\Theta'(l) = \Theta^3(l)$ and $\Gamma'_{i'_0, j'_0} = s_\alpha \Psi$. If such a Young tableau $\mathcal{T}^3$ does not exist then denote by $\Omega_{l-1}(\mathcal{T}^3)$ the empty set. Note that in this case $SD(s_\alpha \Psi, \check{\mu}_l)$ is empty for otherwise let $\xi \in SD(s_\alpha \Psi, \check{\mu}_l)$ be with minimal defining chain Ξ. Then the Young tableau $\mathcal{T}^3$ of shape $p(\mu)$ defined by $\mathcal{T}^3(l) := \mathcal{T}^2(l)$ and $\check{\mathcal{T}}^3(l) := \xi$ would be standard with minimal defining chain Θ^3 defined by $\Theta^3(l) := \Theta^2(l)$ and $\check{\Theta}^3(l) := \Xi$ and of the choosen kind. Note that

$$\Omega_{l-1}(\mathcal{T}^{1,2,3}) := \bigcup_{i=1}^{3} \Omega_{l-1}(\mathcal{T}^i) \subset \Omega_{l-1}.$$

Now using again the map $\mathcal{T}' \mapsto \check{\mathcal{T}}'(l)$, (4.2), and Lemma 1.4 we get

$$\begin{aligned}
\chi_{Y'}\left(\sum_{\mathcal{T}' \in \Omega_{l-1}(\mathcal{T}^{1,2,3})} e^{\lambda + \nu(\mathcal{T}')} \right)
&= \chi_{Y'}(e^{\lambda + \nu(\mathcal{T}^2(l))}(\mathcal{S}\mathcal{D}(\Psi, \check{\mu}_l) + \mathcal{S}\mathcal{D}(s_\alpha \Psi, \check{\mu}_l))) \\
&\quad + \chi_{Y'}((e^{\lambda + \nu(\mathcal{T}^1(l))}\mathcal{S}\mathcal{D}(\Psi, \check{\mu}_l)) \\
&= \chi_{Y'}(e^{\lambda + \nu(\mathcal{T}^2(l))}M_\alpha(\mathcal{S}\mathcal{D}(\Psi, \check{\mu}_l))) + \chi_{Y'}(e^{\lambda + \nu(\mathcal{T}^1(l))}\mathcal{S}\mathcal{D}(\Psi, \check{\mu}_l)) \\
&= \chi_{Y'}(M_\alpha(e^{\lambda + \nu(\mathcal{T}^2(l))})\, \mathcal{S}\mathcal{D}(\Psi, \check{\mu}_l)) + \chi_{Y'}(e^{\lambda + \nu(\mathcal{T}^1(l))}\mathcal{S}\mathcal{D}(\Psi, \check{\mu}_l)).
\end{aligned}$$

Since $\langle \lambda + \nu(\mathcal{T}^2(l)), \alpha \rangle = -2$ we know by (1.1) that

$$\chi_{Y'}(M_\alpha(e^{\lambda + \nu(\mathcal{T}^2(l))})\, \mathcal{S}\mathcal{D}(\Psi, \check{\mu}_l)) = -\chi_{Y'}(e^{\lambda + \nu(\mathcal{T}^2(l)) + \alpha}\mathcal{S}\mathcal{D}(\Psi, \check{\mu}_l)).$$

But $\lambda + \nu(\mathcal{T}^2(l)) + \alpha = \lambda + \nu(\mathcal{T}^1(l))$ and hence

$$\chi_{Y'}\left(\sum_{\mathcal{T}' \in \Omega_{l-1}(\mathcal{T}^{1,2,3})} e^{\lambda + \nu(\mathcal{T}')} \right) = 0.$$

4.16. Finally we remark that if $\mathcal{T}^3 \in \Omega_{l-1}$ with minimal defining chain Θ^3 is such that

$$\langle \lambda + \nu(\mathcal{T}^3(l)), \alpha \rangle = -2 \qquad \text{and} \qquad s_\alpha \Gamma^3_{i'_0, j'_0} < \Gamma^3_{i'_0, j'_0},$$

then set $\Psi := s_\alpha \Gamma^3_{i'_0, j'_0}$. Since $SD(s_\alpha \Psi, \check{\mu}_l)$ is not empty (note that $\check{\mathcal{T}}^3(l)$ is an element of $SD(s_\alpha \Psi, \check{\mu}_l)$) we know by (4.2) that $SD(\Psi, \check{\mu}_l)$ is not empty. For $\Upsilon \in SD(\Psi, \check{\mu}_l)$ define the Young tableau $\mathcal{T}^2$ of shape $p(\mu)$ by $\mathcal{T}^2(l) := \mathcal{T}^3(l)$ and $\check{\mathcal{T}}^2(l) := \Upsilon$. Again, one can check that $\mathcal{T}^2$ is standard, $\langle \lambda + \nu(\mathcal{T}^2(l)), \alpha \rangle = -2$, and $s_\alpha \Gamma^2_{i'_0, j'_0} > \Gamma^2_{i'_0, j'_0}$ for the minimal defining chain Θ^2 of $\mathcal{T}^2$. But this means that $\mathcal{T}^2$ and $\mathcal{T}^3$ are a pair of the kind considered in (4.6). Using the construction in Section 4.14 we can find a standard Young tableau $\mathcal{T}^1$ such that the triple $\mathcal{T}^1, \mathcal{T}^2, \mathcal{T}^3$ is of the kind above, which finishes this case.

Hence, by (4.5) and the assumptions made at the beginning of Section 4.12 it remains to consider the case where $\mathcal{T} \in \Omega_{l-1}$ is such that $\langle \lambda + \nu(\mathcal{T}(l)), \alpha \rangle = -1$.

4.17. Now assume that $\langle \lambda + \nu(\mathcal{T}(l)), \alpha \rangle = -1$. Let i_0, j_0, i'_0, j'_0 be as above and denote φ_{i_0, j_0} by $(\gamma, \delta, \sigma, \varphi)$, Θ_{i_0, j_0} by $(\Gamma, \Delta, \Sigma, \Phi)$, and $\Gamma_{i'_0, j'_0}$ by Ψ, where Θ is the minimal defining chain for $\mathcal{T}$. Let $\Omega_{l-1}(\mathcal{T})$ be the set of all standard Young tableaux $\mathcal{T}'$ of shape $p(\mu)$ such that $\Theta'(l) = \Theta(l)$ and $\Gamma'_{i'_0, j'_0} = \Psi$ for the minimal defining chain Θ' of $\mathcal{T}'$. Note that $\Omega_{l-1}(\mathcal{T}) \subset \Omega_{l-1}$. Again, using the map $\mathcal{T}' \mapsto \check{\mathcal{T}}'(l)$ we see that

$$\sum_{\mathcal{T}' \in \Omega_{l-1}(\mathcal{T})} e^{\lambda + \nu(\mathcal{T}')} = e^{\lambda + \nu(\mathcal{T}(l))} \mathcal{S}\mathcal{D}(\Psi, \check{\mu}_l).$$

If $SD(s_\alpha \Psi, \check{\mu}_l)$ is empty, then we know by (4.2) that $s_\alpha \Psi > \Psi$ and $M_\alpha(\mathcal{S}\mathcal{D}(\Psi, \check{\mu}_l)) = \mathcal{S}\mathcal{D}(\Psi, \check{\mu}_l)$. Hence we get by (1.1)

$$\begin{aligned} \chi_{Y'}\Bigg(\sum_{\mathcal{T}' \in \Omega_{l-1}(\mathcal{T})} e^{\lambda + \nu(\mathcal{T}')}\Bigg) &= \chi_{Y'}(e^{\lambda + \nu(\mathcal{T}(l))} \mathcal{S}\mathcal{D}(\Psi, \check{\mu}_l)) \\ &= \chi_{Y'}(e^{\lambda + \nu(\mathcal{T}(l))} M_\alpha(\mathcal{S}\mathcal{D}(\Psi, \check{\mu}_l))) \\ &= \chi_{Y'}(M_\alpha(e^{\lambda + \nu(\mathcal{T}(l))})\, \mathcal{S}\mathcal{D}(\Psi, \check{\mu}_l)) = 0, \end{aligned}$$

since $\langle \lambda + \nu(\mathcal{T}(l)), \alpha \rangle = -1$.

If $SD(s_\alpha \Psi, \check{\mu}_l)$ is not empty, then let ξ be an element of $SD(s_\alpha \Psi, \check{\mu}_l)$ with minimal defining chain Ξ. Denote by $\mathcal{T}^2$ the Young tableau of shape $p(\mu)$ defined by $\mathcal{T}^2(l) := \mathcal{T}(l)$ and $\check{\mathcal{T}}(l) := \xi$. Then $\mathcal{T}^2$ is standard with minimal defining chain Θ^2 defined by

$$\Theta^2(l-1) := \Theta(l-1) \qquad \text{and} \qquad \Theta^2(l) := \Xi$$

and

$$\Theta^2_{i_0,j_0} := \begin{cases} (\Gamma, \Delta, \Sigma, s_\alpha\Phi), & \text{if } \langle \varphi(\omega_i), \alpha\rangle = 0 \\ & \text{and either } s_\alpha\Phi > \Phi \text{ and } s_\alpha\Psi \not\leqslant \Phi \\ & \text{or } s_\alpha\Phi < \Phi \text{ and } s_\alpha\Psi \leqslant s_\alpha\Phi; \\ \Theta_{i_0,j_0} & \text{else.} \end{cases}$$

To check that $\mathscr{T}^2$ is standard note that using Corollary 4.7 one can show that

$$\begin{aligned} &\text{either } \langle \varphi(\omega_{i_0}), \alpha\rangle < 0 \text{ and hence } s_\alpha\Phi < \Phi; \\ &\text{or } \langle \sigma(\omega_{i_0}), \alpha\rangle < 0,\ \langle \varphi(\omega_{i_0}), \alpha\rangle = 0 \\ &\text{and hence } s_\alpha\Sigma < \Sigma \text{ and } s_\alpha\Phi \text{ is a lift for } \varphi. \end{aligned}$$

Now in the first case $s_\alpha\Phi < \Phi$ and $\Psi < \Phi$ implies that $s_\alpha\Psi \leqslant \Phi$. In the second case $\Sigma \geqslant \Phi$ and $s_\alpha\Sigma < \Sigma$ implies that $\Sigma \geqslant s_\alpha\Phi$ and hence either $\Sigma \geqslant s_\alpha\Phi \geqslant s_\alpha\Psi$ or $\Sigma \geqslant \Phi \geqslant s_\alpha\Psi$. This shows that Θ^2 is a defining chain for $\mathscr{T}^2$. The minimality follows by Remark 4.13.

Denote by $\Omega_{l-1}(\mathscr{T}^2)$ the set of all standard Young tableaux $\mathscr{T}'$ of shape $p(\mu)$ such that $\mathscr{T}'(l) = \mathscr{T}^2(l)$ and $\Gamma'_{i'_0,j'_0} = s_\alpha\Psi$ for the minimal defining chain Θ' of $\mathscr{T}'$. Then

$$\sum_{\mathscr{T}' \in \Omega_{l-1}(\mathscr{T}^2)} e^{\lambda + \nu(\mathscr{T}')} = e^{\lambda + \nu(\mathscr{T}(l))} \mathscr{S}\mathscr{D}(s_\alpha\Psi, \check{\mu}_l).$$

By (4.2), Lemma 1.4, and (1.1) we get (we assume without loss of generality $s_\alpha\Psi > \Psi$)

$$\begin{aligned} &\chi_{Y'}\Bigg(\sum_{\mathscr{T}' \in \Omega_{l-1}(\mathscr{T}) \cup \Omega_{l-1}(\mathscr{T}^2)} e^{\lambda + \nu(\mathscr{T}')}\Bigg) \\ &\quad = \chi_{Y'}(e^{\lambda + \nu(\mathscr{T}(l))}(\mathscr{S}\mathscr{D}(\Psi, \check{\mu}_l) + \mathscr{S}\mathscr{D}(s_\alpha\Psi, \check{\mu}_l))) \\ &\quad = \chi_{Y'}(e^{\lambda + \nu(\mathscr{T}(l))} M_\alpha(\mathscr{S}\mathscr{D}(\Psi, \check{\mu}_l))) \\ &\quad = \chi_{Y'}(M_\alpha(e^{\lambda + \nu(\mathscr{T}(l))})\, \mathscr{S}\mathscr{D}(\Psi, \check{\mu}_l)) = 0 \end{aligned}$$

by (1.1) since $\langle \lambda + \nu(\mathscr{T}(l)), \alpha\rangle = -1$.

4.18. Now 4.11–4.17 show that it is possible to find $\mathscr{T}^1, ..., \mathscr{T}^s \in \Omega_{l-1}$ such that $\Omega_{l-1} = \Omega_{l-1}(\mathscr{T}^1) \cup \cdots \Omega_{l-1}(\mathscr{T}^s)$ is a partition (where $\Omega_{l-1}(\mathscr{T}^i)$ is defined according to the cases 4.11–4.17). And, it is possible to choose $\mathscr{T}^1, ..., \mathscr{T}^s$ in such a way that there exists a partition $p_1, ..., p_t$ of $\{1, ..., s\}$ such that for

$$\Omega_{l-1}(p_i) := \bigcup_{j \in p_i} \Omega_{l-1}(\mathscr{T}^j)$$

we have

$$\chi_{Y'}\left(\sum_{\mathscr{T}'\in\Omega_{l-1}(p_i)} e^{\lambda+\nu(\mathscr{T}')}\right)=0$$

for all $i=1, ..., t$. But this shows that

$$\chi_{Y'}\left(\sum_{\mathscr{T}\in\Omega_{l-1}} e^{\lambda+\nu(\mathscr{T})}\right)=0$$

which proves the claim.

APPENDIX

A.1. In the following we will give a translation of the notion of a standard Young tableau in the sense of 3.4 in the usual notion of a Young tableau (as in 2.1) for $G=Sp_{2m}$ and $Spin_m$. We will not prove the equivalence of the notion of a G-standard Young tableau defined in the following and the notion defined in 3.4. This can be derived using similar ideas as in 3.8 and Lemma 12.4 and 12.6 in [15]. Such a translation has partially been done in [12].

In the following we mean by a Young tableau always a Young tableau in the sense of 2.1. We say that a Young tableau is standard if the entries are strictly increasing in the rows and not decreasing in the columns. (We drop the condition that the entries are smaller or equal to $m+1$ in 2.1.)

A.2. Sp_{2m}- and $Spin_{2m+1}$-standard Young tableau. For the fundamental weights we use the same notation as in [1]. To a dominant weight $\lambda=\sum_{i=1}^{m} a_i\omega_i$ we associate the partition $p(\lambda)=(p_1, ..., p_m)$ with $p_i=\sum_{j=i}^{m} 2a_j$ for $G=Sp_{2m}$ and $p_i=\sum_{j=i}^{m-1} 2a_j+a_m$ for $G=Spin_{2m+1}$.

Let $r=(i_1, ..., i_t)$ be a row of length $t\leqslant m$ such that if i_j is an entry of the row then $2m+1-i_j$ is not an entry of this row and the entries are smaller or equal to $2m$. For $i=1, ..., m$ denote by $s_i(r)$ the row defined as follows:

> If $i<m$ and $i+1$ and $2m+1-i$ are entries of the row r then $s_i(r)$ is the row obtained from r by replacing the entry $i+1$ by i and the entry $2m+1-i$ by $2m-i$. Else we set $s_i(r):=r$. If $i=m$ and $G=Spin_{2m+1}$ and $m+1$ is an entry of the row r, then denote by $s_i(r)$ the row obtained from r by replacing the entry $m+1$ by m. Else we set $s_m(r):=r$.

We say that a pair of rows (r, r') is *admissible*, if $r = r'$ or if there exists a sequence of different rows $(r_0, ..., r_k)$ such that

$$r = r_0, \qquad r' = r_k, \qquad \text{and} \qquad s_{i_l}(r_{l-1}) = r_l \qquad \text{for} \quad l = 1, ..., k$$

and some integers $i_1, ..., i_k \in \{1, ..., m\}$.

For $G = Sp_{2m}$ or $Spin_{2m+1}$ a Young tableau $\mathscr{T}$ of shape $p(\lambda)$ is called *G-standard* if the following holds:

(1) The tableau $\mathscr{T}$ is standard, contains only integers smaller or equal to $2m$, and the integers i and $(2m + 1 - i)$ do not occur in the same row.

(2) If $G = Sp_{2m}$ then set $\overline{p_1} = p_1/2$ and if $G = Spin_{2m+1}$ then set $\overline{p_1} = (p_1 - a_m)/2$. Denote by r_i the ith row of the tableau. For all $i = 1, ..., \overline{p_1}$ the pair of rows (r_{2i-1}, r_{2i}) is admissible.

A.3. *Spin*$_{2m}$-standard Young tableaux. For the fundamental weights we use the same notation as in [1]. To a dominant weight $\lambda = \sum_{i=1}^{m} a_i \omega_i$ we associate the partition $p(\lambda) = (p_1, ..., p_m)$ with $p_i = \sum_{j=i}^{m-2} 2a_j + a_{m-1} + a_m$ for $1 \leqslant i \leqslant m - 1$ and $p_m = p_{m-1}$.

Let $r = (i_1, ..., i_t)$ be a row of length $t < m$ such that if i_j is an entry of the row then $2m + 1 - i_j$ is not an entry of this row and the entries are smaller or equal to $2m$. For $i = 1, ..., m$ denote by $s_i(r)$ the row defined as follows:

> If $i < m$ and $i + 1$ and $2m + 1 - i$ are entries of the row r then $s_i(r)$ is the row obtained from r by replacing the entry $i + 1$ by i and the entry $2m + 1 = i$ by $2m - i$. Else we set $s_i(r) := r$. If $i = m$ and $m + 1$ and $m + 2$ are entries of the row r, then denote by $s_i(r)$ the row obtained from r by replacing the entry $m + 1$ by $m - 1$ and the entry $m + 2$ by m. Else we set $s_m(r) := r$.

We say that a pair of rows (r, r') is *admissible*, if $r = r'$ or if there exists a sequence of different rows $(r_0, ..., r_k)$ such that

$$r = r_0, \qquad r' = r_k, \qquad \text{and} \qquad s_{i_l}(r_{l-1}) = r_l \qquad \text{for} \quad l = 1, ..., k$$

and some integers $i_1, ..., i_k \in \{1, ..., m\}$.

A Young tableau $\mathscr{T}$ of shape $p(\lambda)$ is called *Spin*$_{2m}$*-standard* if the following holds:

(1) The integers contained in $\mathscr{T}$ are smaller or equal to $2m$ and the integers i and $(2m + 1 - i)$ do not occur in the same row. Denote by $\mathscr{T}_1$ the tableau $\mathscr{T}(\overline{p_1})$ where $\overline{p_1} := p_1 - a_m - a_{m-1}$, let $\mathscr{T}_2$ be the tableau consisting of the $(\overline{p_1} + 1)$st row up to the $(\overline{p_1} + a_{m-1})$th row of $\mathscr{T}$ and let $\mathscr{T}_3$ be the tableau consisting of the top a_m rows of $\mathscr{T}$. The tableaux $\mathscr{T}_1, \mathscr{T}_2, \mathscr{T}_3$ are

standard and in $\mathscr{T}_2$ (resp. $\mathscr{T}_3$) the number of integers greater than m in a row is odd (resp. even).

(2) For $1 \leqslant i \leqslant (\overline{p_1}/2) - 1$ let the $2i$th row of $\mathscr{T}_1$ be equal to $(k_1, ..., k_s)$ and let the $(2i+1)$st row be equal to $(l_1, ..., l_t)$, $s \leqslant t$. For all sequences $1 \leqslant j_1 < \cdots < j_q \leqslant s$ such that

$$m+1-q \leqslant k_{j_1} < \cdots < k_{j_q} \leqslant m+q$$

and

$$m+1-q \leqslant l_{j_1} < \cdots < l_{j_q} \leqslant m+q$$

one has $k_{j_1} + \cdots + k_{j_q} \equiv l_{j_1} + \cdots + l_{j_q} \bmod 2$. (Note that this condition is empty if neither m nor $m+1$ is an entry in one of the rows.) Furthermore, let r_i be the ith row of $\mathscr{T}_1$. Then the pairs (r_{2i-1}, r_{2i}) are admissible for $i = 1, ..., \overline{p_1}$.

(3) This last condition is only needed if either $a_{m-1} > 0$ and $a_m > 0$ or $\sum_{i=1}^{m-2} a_i > 0$ and $a_{m-1} + a_m > 0$. Let $(k_1, ..., k_s)$, $s < m$, be the top row of $\mathscr{T}_1$. Denote by R the set $\{k_1, ..., k_s, l_1, ..., l_{m-s-1}, x\}$ with the following properties:

> $2m \geqslant l_1 > \cdots > l_{m-s-1} > m$, $l_{m-s-1} > x$, $l_{m-s-1} > 2m+1-x$, $l_i \neq k_j$ and $x \neq k_j$ for all $1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant m-s-1$, and if $r \in R$ then $2m+1-r \notin R$. Furthermore, if the number of integers strictly greater than m in $R \backslash \{x\}$ is odd then $x > m$ else $x \leqslant m$. Note that the set R is uniquely defined by these properties.

Denote by $\mathscr{T}'_2$ the tableau obtained from $\mathscr{T}_2$ by adding one row of length m at the bottom of $\mathscr{T}_2$ and filling the boxes of that row with the elements of $\{2m+1-x\} \cup R \backslash \{x\}$ in increasing order. Then the tableau $\mathscr{T}'_2$ is standard.

Denote by $\mathscr{T}'_3$ the tableau obtained from $\mathscr{T}_3$ by adding $(a_{m-1}+1)$ rows of length m at the bottom, the filling of these rows being defined inductively as follows: The boxes of the bottom row of $\mathscr{T}'_3$ are filled with the elements of R in increasing order. Assume now that $2 \leqslant i \leqslant a_{m-1}+1$ and the filling of the $(i-1)$st row has already been defined. Let $(j_1, ..., j_m)$ be the $(i-1)$st row of $\mathscr{T}_2$. For $1 \leqslant l \leqslant m$ let R_l denote the m-tuple $(i_1, ..., i_m)$ such that $i_1 < \cdots < i_m$ and $\{i_1, ..., i_m\} = \{j_1, ..., 2m+1-j_l, ..., j_m\}$. Note that $R_1, ..., R_m$ are linearly ordered with respect to the lexicographic order. The ith row of $\mathscr{T}'_3$ is equal to $R_{l'}$ where $R_{l'}$ is maximal among those R_j's for which $\mathscr{T}'_3(i)$ with $R_{l'}$ as top row is standard. Then the tableau $\mathscr{T}'_3$ defined above is standard.

A.4. The decomposition formulas. Let μ be a dominant weight and let $\mathscr{T}$ be a Sp_{2m}, $Spin_{2m+1}$, or $Spin_{2m}$-standard Young tableau of shape $p(\mu)=(p_1, ..., p_m)$. Define the weight of the tableau $\mathscr{T}$ as

$$v(\mathscr{T}) := \tfrac{1}{2}((c_{\mathscr{T}}(1)-c_{\mathscr{T}}(2m))\varepsilon_1 + \cdots + (c_{\mathscr{T}}(m)-c_{\mathscr{T}}(m+1))\varepsilon_m).$$

For $1 \leqslant l \leqslant p_1$ denote by $v_l(\mathscr{T})$ the weight $2v(\mathscr{T}(l))$.

Definition. If L is a Levi subgroup of G then a G-standard Young tableau of shape $p(\mu)$ is called *L-dominant* if all the weights

$$v_1(\mathscr{T}), v_2(\mathscr{T}), ..., v_{p_1}(\mathscr{T})$$

are contained in the dominant Weyl chamber of L.

If λ is a dominant weight for G, then a G-standard Young tableau of shape $p(\mu)$ is called *λ-dominant* if all the weights

$$2\lambda + v_1(\mathscr{T}), 2\lambda + v_2(\mathscr{T}), ..., 2\lambda + v_{p_1}(\mathscr{T})$$

are contained in the dominant Weyl chamber of G.

Theorem. (a) *The decomposition of the tensor product $V_\lambda \otimes V_\mu$ into irreducible G-modules is given by*

$$V_\lambda \otimes V_\mu = \bigoplus_{\mathscr{T}} V_{\lambda + v(\mathscr{T})},$$

where $\mathscr{T}$ runs over all G-standard Young tableaux of shape $p(\mu)$ that are λ-dominant.

(b) *The decomposition of the L-module* $\operatorname{res}_L V_\lambda$ *into irreducible L-modules U_v of highest weight v is given by*

$$\operatorname{res}_L V_\lambda = \bigoplus_{\mathscr{T}} U_{v(\mathscr{T})},$$

where $\mathscr{T}$ runs over all G-standard Young tableau of shape $p(\lambda)$ that are L-dominant.

A.5. Example. For $G = Sp_4$ and $\lambda = \mu = \omega_1 + \omega_2$ the λ-dominant Sp_4-standard Young tableaux are

$$\begin{array}{ll}3&4\\3&4\\4&\\4&\end{array}\qquad\begin{array}{ll}1&3\\1&3\\4&\\4&\end{array}\qquad\begin{array}{ll}1&2\\1&2\\4&\\4&\end{array}\qquad\begin{array}{ll}2&4\\2&4\\3&\\3&\end{array}\qquad\begin{array}{ll}1&3\\2&4\\3&\\3&\end{array}$$

$$\begin{array}{ll}1&2\\1&2\\3&\\3&\end{array}\qquad\begin{array}{ll}1&3\\1&3\\2&\\2&\end{array}\qquad\begin{array}{ll}1&2\\1&2\\2&\\2&\end{array}\qquad\begin{array}{ll}1&3\\1&3\\1&\\1&\end{array}\qquad\begin{array}{ll}1&2\\1&2\\1&\\1&\end{array}$$

Hence we get the decomposition

$$\begin{aligned} V_{\omega_1+\omega_2} \otimes V_{\omega_1+\omega_2} = {} & V_{4\omega_1} \oplus V_{2\omega_1+2\omega_2} \oplus 2V_{2\omega_1+\omega_2} \\ & \oplus 2V_{2\omega_1} \oplus V_{3\omega_2} \oplus V_{2\omega_2} \oplus V_{\omega_2} \oplus k. \end{aligned}$$

A.6. Let $\lambda = \sum_{i=1}^{m} a_i\omega_i$ be a dominant weight for $G_m = Sp_{2m}$, $Spin_{2m+1}$, or $Spin_{2m}$. Denote $\max_i\{i \mid a_i > 0\}$ by $\deg\lambda$. Let $\mu = \sum_{i=1}^{m} b_i\omega_i$ be a nontrivial dominant weight and assume that

$$\deg\mu \leqslant \deg\lambda \qquad \text{and} \qquad \deg\lambda + \deg\mu < m.$$

In particular we have $b_{m-1} = b_m = 0$ and hence the partition $p(\mu)$ is independent of the type of the Dynkin diagram of G_m. Note the following:

(i) If $\mathscr{T}$ is a λ-dominant G_m-standard Young tableau of shape $p(\mu)$, then neither m nor $m+1$ is an entry of $\mathscr{T}$. Hence it follows by the description of the G_m-standard Young tableaux above that the tableau is standard (and λ-dominant) for $G_m = Sp_{2m}$, $Spin_{2m+1}$, and $Spin_{2m}$.

(ii) If $m' > \deg\lambda + \deg\mu$, then the shift which replaces all entries i strictly greater to m in a λ-dominant G_m-standard Young tableau by $i + 2(m'-m)$ induces a bijection between the λ-dominant G_m-standard Young tableaux and the λ-dominant $G_{m'}$-standard Young tableaux.

As a consequence we get:

COROLLARY. *Let λ, μ be dominant weights. The decomposition*

$$V_\lambda \otimes V_\mu = V_{\nu_1} \oplus \cdots \oplus V_{\nu_s}$$

is independent of the type and the rank of G if $\operatorname{rk} G > \deg\lambda + \deg\mu$.

REFERENCES

1. N. BOURBAKI, "Groupes et algèbres de Lie," Chaps. 4–6, Hermann, Paris, 1968.
2. E. CLINE, B. PARSHALL, AND L. SCOTT, Induced modules and affine quotients, *Math. Ann.* **230** (1977), 1–13.
3. E. CLINE, B. PARSHALL, AND L. SCOTT, Induced modules and extensions of representations, *Invent. Math.* **47** (1978), 41–51.
4. M. DEMAZURE, Désingularisation des variétés de Schubert généralisées, *Ann. École Nat. Sup.* **7** (1974), 53–88.
5. M. DEMAZURE, A very simple proof of Bott's Theorem, *Invent. Math.* **33** (1976), 271–272.
6. J. E. HUMPHREYS, Introduction to Lie algebras and representation theory, *in* "Graduate Texts in Mathematics," Vol. 40, Springer-Verlag, New York/Berlin, 1972.
7. G. KEMPF, A decomposition formula for representations, *Nagoya Math. J.* **107** (1987), 63–68.

8. G. KEMPF, Tensorproducts of representations, *Amer. J. Math.* **109** (1987), 401–416.
9. A. U. KLIMYK, Decomposition of a tensorproduct of irreducible representations of a semisimple Lie algebra into a direct sum of irreducible representations, *Amer. Math. Soc. Transl.* **76** (1968), 63–73.
10. K. KOIKE AND I. TERADA, Young-diagrammatic methods for the representations theory of the classical groups of type $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$, *J. Algebra* **107** (1987), 466–511.
11. V. LAKSHMIBAI, Standard monomial theory for $\mathbf{G}_2$, *J. Algebra* **98** (1986), 281–318.
12. V. LAKSHMIBAI, Bases pour les représentations fondamentales des groupes classiques I, *C.R. Acad. Sci. Paris* **302** (1986), 387–390.
13. V. LAKSHMIBAI AND K. N. RAJESWARI, Bases pour les représentations fondamentales des groupes exceptionels $\mathbf{E}_6$ et $\mathbf{F}_4$, *C.R. Acad. Sci. Paris* **302** (1986), 575–577.
14. V. LAKSHMIBAI AND K. N. RAJESWARI, Standard monomial theory for exceptional groups, preprint, 1987.
15. V. LAKSHMIBAI, C. MUSILI, AND C. S. SESHADRI, Geometry of G/P, IV, *Proc. Indian Acad. Sci. Part A* **87** (1979), 279–362.
16. V. LAKSHMIBAI AND C. S. SESHADRI, Geometry of G/P, V, *J. Algebra* **100** (1986), 462–557.
17. P. LITTELMANN, A Littlewood–Richardson rule for classical groups, *C.R. Acad. Sci. Paris* **306** (1988), 299–303.
18. D. E. LITTLEWOOD AND A. R. RICHARDSON, Group characters and algebra, *Philos. Trans. Roy. Soc. London Ser. A* **233** (1934), 99–141.
19. I. MACDONALD, "Symmetric Functions and Hall Polynomials," Oxford Univ. Press (Clarendon), London/New York, 1979.
20. S. RAMANAN AND A. RAMANATHAN, Projective normality of flag varieties and Schubert varieties, *Invent. Math.* **79** (1985), 217–224.
21. H. WEYL, "The Classical Groups," Princeton Univ. Press, Princeton, NJ, 1946.
22. J. WEYMAN, Pieri's formulas for classical groups, preprint, Northeastern University, 1987.