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A Littlewood-Richardson rule for symmetrizable Kac-Moody algebras

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Introduction.

Let $\mathfrak{G}$ be a complex symmetrizable Kac-Moody algebra. In this article we prove a Littlewood-Richardson type rule to calculate the decomposition of the tensor product of two simple, integrable, highest weight modules of $\mathfrak{G}$ into irreducible components.

In the representation theory of the group $GL_n(\mathbb{C})$, an important tool are the Young tableaux. The irreducible representations are in one-to-one correspondence with the shapes of these tableaux. Let T be the subgroup of diagonal matrices in $GL_n(\mathbb{C})$. Then there is a canonical way to assign a weight of T to any Young tableau such that the sum over the weights of all tableaux of a fixed shape is the character $\mathrm{Char}\, V$ of the corresponding $GL_n(\mathbb{C})$-module V. Note that this gives not only a way to compute the character, it gives also a possibility to describe the multiplicity of a weight in the representation: It is the number of *different* tableaux of the same weight. Eventually, the Littlewood-Richardson rule describes the decomposition of tensor products of $GL_n(\mathbb{C})$-modules purely in terms of the combinatoric of these Young tableaux.

Our main concern will be to generalize these tableaux for symmetrizable Kac-Moody algebras. Let $\mathfrak{H}$ be the Cartan subalgebra of $\mathfrak{G}$ and denote by $\mathcal{X}$ the weight lattice. In [7], section 4, Lakshmibai and Seshadri conjectured that a basis of $\mathfrak{H}$-eigenvectors of V_μ can be indexed by a certain set of sequences of elements in the Weyl group. Our new approach to this conjecture is to interpret these sequences as piecewise linear paths $\pi : [0,1] \to \mathcal{X}_\mathbb{R}$, where $\mathcal{X}_\mathbb{R} := \mathcal{X} \otimes_\mathbb{Z} \mathbb{R}$. In the following, we call these paths the *Lakshmibai-Seshadri* paths of shape μ (see 2.2).

More generally, we consider the set Π of all piecewise linear paths $\pi : [0,1] \to \mathcal{X}_\mathbb{R}$ such that $\pi(0) = 0$, where we identify π and π' if $\pi = \pi'$ up to a reparametrization. By the product $\pi := \pi_1 * \pi_2$ of two such paths we mean the concatenation of π_1 and the shifted path $\pi_1(1) + \pi_2$. For each simple root α we introduce operators e_α and f_α on $\Pi \cup \{0\}$: We cut π into three well-defined parts, i.e., $\pi = \pi_1 * \pi_2 * \pi_3$. The new path $e_\alpha(\pi)$ (or $f_\alpha(\pi)$) is then either equal to 0, or it is equal to $\pi_1 * s_\alpha(\pi_2) * \pi_3$. Here s_α denotes the simple reflection with respect to the root α. It follows by the construction

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that if $f_\alpha(\pi)$ (respectively $e_\alpha(\pi)$) is not equal to 0, then the endpoint of the new path is equal to $\pi(1)-\alpha$ (respectively $\pi(1)+\alpha$).

It turns out that these operators have properties very similar to the operators considered by Kashiwara (see [3] and [4]) on the crystal basis: For example, if $e_\alpha(\pi)\neq 0$, then $f_\alpha\big(e_\alpha(\pi)\big)=\pi$ and vice versa. In fact, the starting point for this article had been the effort to understand better the connection between the crystal basis and the generalized Young tableaux found in [10]. We prove that the set of Lakshmibai-Seshadri paths of shape μ can be viewed as a set of paths generated by these operators:

Character formula. *For a dominant weight μ let $\pi_\mu:[0,1]\to\mathcal{X}_\mathbb{R}$ be the straight line connecting 0 with μ. The set $\mathcal{P}_\mu$ of Lakshmibai-Seshadri paths of shape μ is equal to the set of all paths π of the form*

$$\pi=f_{\alpha_1}\circ\ldots\circ f_{\alpha_s}(\pi_\mu)$$

where $\alpha_1,\ldots,\alpha_s$ are simple roots. Moreover, the union $\mathcal{P}_\mu\cup\{0\}$ is stable under the operators e_α for all simple roots, and $\mathcal{P}_\mu$ provides a character formula for the module V_μ:

$$\operatorname{Char}V_\mu=\sum_{\pi\in\mathcal{P}_\mu}e^{\pi(1)}.$$

Note that (as in the case of the Young tableaux) the set of Lakshmibai-Seshadri paths carries more information than just the character of the representation: The multiplicity of a weight in the representation is equal to the number of *different* paths ending in the weight. This possibility to "split" the multiplicities enables us to prove the following decomposition rule:

Let π be a Lakshmibai-Seshadri path of shape μ. If λ is a dominant weight, then we call π a λ-*dominant* path if the shifted image $\{x\in\mathcal{X}_\mathbb{R}\mid x=\lambda+\pi(t),\ t\in[0,1]\}$ of the path is contained in the dominant Weyl chamber.

Decomposition rule. *The decomposition of the tensor product of two integrable, simple, highest weight modules of $\mathfrak{G}$ is given by:*

$$V_\lambda\otimes V_\mu\simeq\bigoplus_\pi V_{\lambda+\pi(1)},$$

where π runs over all λ-dominant Lakshmibai-Seshadri paths of shape μ.

The condition of λ-dominance can also be expressed in terms of the operators e_α: π is λ-dominant if and only if for all simple roots $e_\alpha^{\langle\lambda,\alpha^\vee\rangle+1}(\pi)=0$. In this terminology the decomposition rule of Kashiwara (in terms of the crystal basis, see [3] and [4]) is identical with our rule above.

To show how the rule can be used to prove existence results, we give a new proof of the P-R-V conjecture. We obtain also a branching rule for the restriction of a simple $\mathfrak{G}$-module to a Levi subalgebra $\mathfrak{L}$ of $\mathfrak{G}$: Denote by U_ν the simple $\mathfrak{L}$-module corresponding to a (for $\mathfrak{L}$) dominant weight, and call a path π $\mathfrak{L}$-*dominant* if its image is contained in the dominant Weyl chamber of the root system of $\mathfrak{L}$.

Branching rule. *The decomposition of V_μ into simple $\mathfrak{L}$-modules is given by*

$$\mathrm{res}_{\mathfrak{L}} V_\mu = \bigoplus_\pi U_{\pi(1)},$$

where π runs over all $\mathfrak{L}$-dominant Lakshmibai-Seshadri paths of shape μ.

Further, these operators enable us to associate in a natural way a colored oriented graph $\mathcal{G}$ to $\mathfrak{G}$: The set of vertices is Π, and we put an arrow $\pi \xrightarrow{\alpha} \pi'$ between π and π' if $f_\alpha(\pi) = \pi'$. For $\pi \in \Pi$ let $\mathcal{G}(\pi)$ be the connected component of $\mathcal{G}$ containing π. We conjecture that if $\pi(1)$ is a dominant weight and the image of π is contained in the dominant Weyl chamber, then $\mathcal{G}(\pi)$ is the crystal graph of the module $V_{\pi(1)}$ constructed by Kashiwara. Note, that this would imply that the graph $\mathcal{G}(\pi)$ depends only on the endpoint $\pi(1)$ and is otherwise independent of the choice of π. Moreover, the three theorems above (Character formula, Decomposition rule and the Branching rule) could be reformulated for the set of paths $\mathcal{P}(\pi)$ generated from π by applying successively the operators f_α. We give a short sketch in section 8 of how the three theorems can be generalized for paths of the form $\pi = \pi_{\lambda_1} * \ldots * \pi_{\lambda_r}$, where $\lambda_1, \ldots, \lambda_r$ are dominant weights.

1. Paths and roots.

1.0. Let Π be the set of all piecewise linear paths $\pi : [0,1] \to \mathcal{X}_{\mathbb{R}}$ such that $\pi(0) = 0$, modulo the equivalence relation $\pi \sim \pi'$ if $\pi = \pi'$ up to a reparametrization. For each simple root α we define operators e_α and f_α on $\Pi \cup \{0\}$ such that the "root-string" of paths

$$\ldots,\ e_\alpha^2(\pi),\ e_\alpha(\pi),\ \pi,\ f_\alpha(\pi),\ f_\alpha^2(\pi),\ \ldots$$

generated by an element $\pi \in \Pi$ has properties similar to the root-strings through a weight of a $\mathfrak{G}$-module.

1.1. Let $\pi_1, \pi_2 \in \Pi$ be two paths. By the product $\pi := \pi_1 * \pi_2$ we mean the path defined by

$$\pi(t) := \begin{cases} \pi_1(2t), & \text{if } 0 \le t \le 1/2; \\ \pi_1(1) + \pi_2(2t-1), & \text{if } 1/2 \le t \le 1. \end{cases}$$

For a simple root α let $s_\alpha(\pi)$ be the path given by $s_\alpha(\pi)(t) := s_\alpha(\pi(t))$.

1.2. Fix a simple root α. According to the behavior of the function

$$h_\alpha : [0,1] \to \mathbb{R}, \quad t \mapsto \langle \pi(t), \alpha^\vee \rangle$$

we cut a path $\pi \in \Pi$ into three parts: Choose the minimal integer

$$Q := \min(\mathrm{Im}\, h_\alpha \cap \mathbb{Z}) \le 0$$

attained by h_α, and let $q := \min\{t \in [0,1] \mid h_\alpha(t) = Q\}$ be the smallest real number such that Q is attained at q. If $Q \leq -1$, then let $y < q$ be such that:

$$h_\alpha(y) = Q + 1 \text{ and } Q < h_\alpha(t) < Q + 1 \text{ for } y < t < q.$$

Denote by π_1, π_2 and π_3 the paths in Π defined by

$$\pi_1(t) := \pi(ty);\ \pi_2(t) := \pi\big(y + t(q-y)\big) - \pi(y);\ \pi_3(t) := \pi\big(q + t(1-q)\big) - \pi(q)$$

for $t \in [0,1]$. By the definition of the π_i we have $\pi = \pi_1 * \pi_2 * \pi_3$.

Definition. If $Q = 0$, then let $e_\alpha(\pi)$ be equal to 0. path. If $Q < 0$, then let $e_\alpha(\pi)$ be equal to $\pi_1 * s_\alpha(\pi_2) * \pi_3$.

Example. In the figure below we give an example in the rank two case. The part of the new path $e_\alpha(\pi)$ different from π is drawn as a dashed line.

1.3. The definition of the operator f_α is similar. Let $p \in [0,1]$ be maximal such that $h_\alpha(p) = Q$, and denote by P the integral part of $h_\alpha(1) - Q$. If $P \geq 1$, then let $x > p$ be such that:

$$h_\alpha(x) = Q + 1 \text{ and } Q < h_\alpha(t) < Q + 1 \text{ for } p < t < x.$$

Denote by π_1, π_2 and π_3 the paths in Π defined by

$$\pi_1(t) := \pi(tp);\ \pi_2(t) := \pi\big(p + t(x-p)\big) - \pi(p);\ \pi_3(t) := \pi\big(x + t(1-x)\big) - \pi(x)$$

for $t \in [0,1]$. By the definition of the π_i we have $\pi = \pi_1 * \pi_2 * \pi_3$.

Definition. If $P = 0$, then let $f_\alpha(\pi)$ be equal to 0. If $P > 0$, then let $f_\alpha(\pi)$ be equal to $\pi_1 * s_\alpha(\pi_2) * \pi_3$.

Example. Suppose $\mathfrak{G}$ is of type $\mathtt{A}_2$ and V_μ is the adjoint representation. If we start with the path $\pi_\mu : t \mapsto t\mu$, then the paths obtained from π_μ by applying the operators f_α and e_α to π_μ, are either of the form $\pi_\beta(t) := t\beta$, where β is an arbitrary root, or of the form

$$\pi_i(t) := \begin{cases} -t\alpha, & \text{for } 0 \leq t \leq 1/2; \\ (t-1)\alpha, & \text{for } 1/2 \leq t \leq 1, \end{cases}$$

where α is a simple root. So if λ is a dominant weight, then the decomposition rule states that a representation V_ν occurs in the tensor product $V_\lambda \otimes V_\mu$ if and only if either $\nu = \lambda + \beta$ for some root β, or $\nu = \lambda$ is such that $\nu - (\alpha/2)$ is a point in the dominant Weyl chamber for some simple root α.

1.4. The following properties are obvious by the definition of the operators:

Lemma. *a) If $\nu(\pi)$ is a weight, then $P + Q = \langle \nu(\pi), \alpha^\vee \rangle$.*

b) If $e_\alpha(\pi) \neq 0$, then $\nu(e_\alpha(\pi)) = \nu(\pi) + \alpha$, and if $f_\alpha(\pi) \neq 0$, then $\nu(f_\alpha(\pi)) = \nu(\pi) - \alpha$.

c) Let $\rho \in \mathcal{X}$ be such that $\langle \rho, \gamma \rangle = 1$ for all simple roots γ. Then $e_\gamma(\pi) = 0$ for all simple roots if and only if the shifted path $\rho + \pi$ is completely contained in the interior of the dominant Weyl chamber. ♢

1.5. The formulas in *a*) and *b*) above show already a certain resemblance with well-known formulas in the representation theory of the group $SL_2(\mathbb{C})$. In fact, in the next proposition we show that the action of the operators e_α and f_α on the set $\Pi \cup \{0\}$ is similar to the action of the operators Kashiwara considers on the crystal basis of $SL_2(\mathbb{C})$-modules.

Proposition. *Let $\pi \in \Pi$ be a path and let α be a simple root.*

a) $e_\alpha^n(\pi) = 0$ if and only if $n > -Q$, and $f_\alpha^n(\pi) = 0$ if and only if $n > P$.

b) If $\pi' \neq 0$ is a second path, then $e_a(\pi) = \pi'$ if and only if $f_\alpha(\pi') = \pi$.

Proof. We give the proof only for the action of f_α, the proof for e_α is analogue. Suppose that $\pi' = f_\alpha(\pi)$. For π let Q, P, p and x be as in 1.3. The minimal integer attained by the function $h'_\alpha : t \mapsto \langle \pi'(t), \alpha^\vee \rangle$ is then $Q - 1$, and the smallest real number such that this value is attained is x. Further, by the definition of $\pi' = f_\alpha(\pi)$, the next smaller real number such that h' attains the value Q is p. But this implies $e_\alpha(f_\alpha(\pi)) = \pi$. Finally, since

$$\langle \pi'(1), \alpha^\vee \rangle - (Q - 1) = \langle \pi(1) - \alpha, \alpha^\vee \rangle - (Q - 1) = P - 1,$$

this proves *a*) by induction on P. ♢

1.6. Fix a weight $\eta \in \mathcal{X}$ and denote by $\Pi(\eta)$ the subset of paths π in Π of weight $\nu(\pi) = \eta$. For $j \in \mathbb{Z}$ consider the map $\Phi_j : \Pi(\eta) \to \Pi(\eta - j\alpha)$ defined by $\pi \mapsto f_\alpha^j(\pi)$ for $j \geq 0$ and $\pi \mapsto e_\alpha^{-j}(\pi)$ for $j \leq 0$. As an immediate consequence of Lemma 1.4 and Proposition 1.5 one obtains:

Corollary. *Set $m := \langle \eta, \alpha^\vee \rangle$. If $m \geq 0$, then Φ_j is injective for $0 \leq j \leq m$, and if $m \leq 0$, then Φ_j is injective for $m \leq j \leq 0$.* ♢

2. Lakshmibai-Seshadri paths.

2.0. In the following let V_λ be the simple highest weight module corresponding to a dominant weight λ, and let W be the Weyl group of $\mathfrak{G}$. In this section we introduce the notion of a W-path of shape λ. The Lakshmibai-Seshadri paths can then be described as W-paths satisfying certain integrality conditions. The definition given here is just a "translation" of the definition in [7] into the language of paths.

2.1. In $\mathcal{X}_{\mathbb{R}}$ let $\mathcal{C}(\lambda)$ be the convex hull of the orbit $W\cdot\lambda$. We consider pairs of sequences representing a path in $\mathcal{X}_{\mathbb{R}}$:

Let W_λ be the stabilizer of λ, and let "$\leq$" be the Bruhat order on W/W_λ. Suppose

- $\underline{\tau}: \tau_1 > \tau_2 > \ldots > \tau_r$ is a sequence of linearly ordered cosets in W/W_λ and
- $\underline{a}: a_0 := 0 < a_1 < \ldots < a_r := 1$ is a sequence of rational numbers.

We call the pair $\pi = (\underline{\tau}, \underline{a})$ a *rational W-path of shape* λ. We identify π with the path $\pi : [0,1] \to \mathcal{X}_{\mathbb{R}}$ given by

$$\pi(t) := \sum_{i=1}^{j-1}(a_i - a_{i-1})\tau_i(\lambda) + (t - a_{j-1})\tau_j(\lambda) \quad \text{for} \quad a_{j-1} \leq t \leq a_j.$$

The endpoint $\pi(1)$ of the path is called the *weight* $\nu(\pi)$ of π.

2.2. Recall that a weight μ in $\mathcal{X}$ is a weight of V_λ if and only if $\mu \in \mathcal{C}(\lambda)$ and $\lambda - \mu$ is a sum of positive roots (see [2], 11.3). Since the τ_i are linearly ordered, the differences $\tau_{i+1}(\lambda) - \tau_i(\lambda)$ are sums of positive roots. Note that

$$\lambda - \nu(\pi) = \lambda - \sum_{i=1}^{r}\big(a_i - a_{i-1}\big)\tau_i(\lambda) = \big(\lambda - \tau_r(\lambda)\big) + \sum_{i=1}^{r-1} a_i\big(\tau_{i+1}(\lambda) - \tau_i(\lambda)\big),$$

so if the a_i are chosen such that the $a_i(\tau_{i+1}(\lambda) - \tau_i(\lambda))$ are still in the root lattice, then $\nu(\pi)$ is a weight of V_λ. To ensure that $\nu(\pi)$ is a weight of V_λ, we introduce now the notion of an a-chain. Note that the condition below is stronger than just demanding that the $a_i(\tau_{i+1}(\lambda) - \tau_i(\lambda))$ are in the root lattice. We use the usual notation $l(\cdot)$ for the length function on W/W_λ and $\beta^\vee$ for the coroot of a positive real root β:

Let $\tau > \sigma$ be two elements of W/W_λ and let $0 < a < 1$ be a rational number. By an *a-chain* for the pair (τ, σ) we mean a sequence of cosets in W/W_λ:

$$\kappa_0 := \tau > \kappa_1 := s_{\beta_1}\tau > \kappa_2 := s_{\beta_2}s_{\beta_1}\tau > \ldots > \kappa_s := s_{\beta_s}\cdot\ldots\cdot s_{\beta_1}\tau = \sigma,$$

where $\beta_1, \ldots, \beta_s$ are positive real roots such that for all $i = 1, \ldots, s$:

$$l(\kappa_i) = l(\kappa_{i-1}) - 1 \quad \text{and} \quad a\langle\kappa_i(\lambda), \beta_i^\vee\rangle \in \mathbb{Z}.$$

The last condition can be expressed as follows: Each summand in

$$a\big(\tau(\lambda) - \sigma(\lambda)\big) = \sum_{i=0}^{s-1} a\big(\kappa_i(\lambda) - \kappa_{i-1}(\lambda)\big) = \sum_{i=1}^{s} a\langle\kappa_i(\lambda), \beta_i^\vee\rangle\beta_i$$

is an element in the root lattice. This is obviously stronger than just to demand that $a\big(\tau(\lambda) - \sigma(\lambda)\big)$ is an element of the root lattice.

Definition. A rational W-path π of shape λ is called a *Lakshmibai-Seshadri* path, if for all $i = 1, \dots, r-1$ there exists an a_i-chain for the pair (τ_i, τ_{i+1}).

Remark. If $\pi = (\underline{\tau}, \underline{a})$ is a rational W-path of shape λ, then there exists an $n \geq 1$ such that π is a Lakshmibai-Seshadri path of shape $n\lambda$.

3. Some integrality properties.

3.0. To prove that the set of Lakshmibai-Seshadri paths is stable under the action of e_α and f_α, we need to derive criterions under which conditions we can replace certain entries τ_i by $s_\alpha \tau_i$ in $\pi = (\underline{\tau}, \underline{a})$ such that the new path π' is a Lakshmibai-Seshadri path. We begin with two simple observations:

Lemma 3.1. *Let $\pi = (\underline{\tau}, \underline{a})$ be a Lakshmibai-Seshadri path of shape λ.*

a) The W-path $\pi' := (\tau_i, \dots, \tau_l; 0, a_i, \dots, a_{l-1}, 1)$ is a Lakshmibai-Seshadri path for any pair (i, l), $1 \leq i \leq l \leq r$.

b) Suppose $\mathfrak{G}$ is finite dimensional and w_0 is the longest word in W. The W-path $\pi' := (\mathrm{w}_0\tau_r, \dots, \mathrm{w}_0\tau_1; 1 - a_r, \dots, 1 - a_0)$ is a Lakshmibai-Seshadri path, and the weight $\nu(\pi')$ is equal to $\mathrm{w}_0(\nu(\pi))$. ♢

Lemma 3.2. *Let $\tau = \kappa_0 > \ldots > \kappa_s = \sigma$ be an a-chain such that $s > 1$.*

a) If $s_\alpha \tau < \tau$ but $s_\alpha \kappa_l \geq \kappa_l$ for some l, then $s_\alpha \tau > \sigma$, and there exists an a-chain for the pair $(s_\alpha \tau, \sigma)$.

b) If $s_\alpha \sigma > \sigma$ but $s_\alpha \kappa_l \leq \kappa_l$ for some l, then $\tau > s_\alpha \sigma$, and there exists an a-chain for the pair $(\tau, s_\alpha \sigma)$.

Proof. The proofs of a) and b) are similar, so we give only the proof of a). In the finite dimensional case it is in fact easy to see that b) follows from a) by Lemma 3.1.

Assume that $\kappa, \xi \in W/W_\lambda$ are such that $\kappa > \xi$ and $l(\kappa) = l(\xi) + 1$. Let β be a positive real root such that $s_\beta \kappa = \xi$. If $s_\alpha \kappa < \kappa$, then either $s_\alpha \kappa = \xi$, or $s_\alpha \xi < \xi$. Further, if we set $\gamma = s_\alpha(\beta)$ in the last case, then

$$s_\gamma(s_\alpha \kappa) = (s_\alpha \xi) \quad \text{and} \quad \langle \kappa(\lambda), \beta^\vee \rangle = \langle s_\alpha \kappa(\lambda), \gamma^\vee \rangle.$$

These considerations show that there exists an $k \leq l$ such that $s_\alpha \kappa_{k-1} = \kappa_k$, and the chain $s_\alpha \tau = s_\alpha \kappa_0 > \ldots > s_\alpha \kappa_{k-1} > \kappa_{k+1} > \ldots > \kappa_s = \sigma$ is an a-chain for $(s_\alpha \tau, \sigma)$. ♢

Lemma 3.3. *Let $\tau = \kappa_0 > \ldots > \kappa_s = \sigma$ be an a-chain.*

a) If $s_\alpha \tau < \tau$ but $s_\alpha \kappa_l \geq \kappa_l$ for some l, then $a\langle \tau(\lambda), \alpha^\vee \rangle$ and $a\langle \sigma(\lambda), \alpha^\vee \rangle$ are integers.

b) If $s_\alpha \sigma > \sigma$ but $s_\alpha \kappa_l \leq \kappa_l$ for some l, then $a\langle \sigma(\lambda), \alpha^\vee \rangle$ and $a\langle \sigma(\lambda), \alpha^\vee \rangle$ are integers.

Proof. We give only the proof for a). If $s_\alpha\tau = \sigma$, then a) is true by the definition of an a-chain. Else consider the Lakshmibai-Seshadri paths (Lemma 3.2):

$$\pi_1 := (\tau, \sigma; 0, a, 1) \text{ and } \pi_2 := (s_\alpha\tau, \sigma; 0, a, 1)$$

of shape λ. Since the difference $\nu(\pi_1) - \nu(\pi_2) = a\langle\tau(\lambda), \alpha^\vee\rangle\alpha$ is an element of the root lattice (see 2.2), this proves $a\langle\tau(\lambda), \alpha^\vee\rangle \in \mathbb{Z}$. But $\nu(\pi_1) = a\tau(\lambda) + (1-a)\sigma(\lambda)$ is a weight, which implies also $a\langle\sigma(\lambda), \alpha^\vee\rangle \in \mathbb{Z}$. $\diamondsuit$

3.4. For a fixed simple root α let $h_\alpha : [0,1] \to \mathbb{R}$ be the function $t \mapsto \langle\pi(t), \alpha^\vee\rangle$.

Lemma. *Let $\tau_i = \kappa_0 > \ldots > \kappa_s = \tau_{i+1}$ be an a_i-chain for the pair (τ_i, τ_{i+1}).*
a) If $s_\alpha\tau_i < \tau_i$ but $s_\alpha\kappa_l \geq \kappa_l$ for some l, then $h_\alpha(a_i) \in \mathbb{Z}$.
b) If $s_\alpha\tau_{i+1} > \tau_{i+1}$ but $s_\alpha\kappa_l \leq \kappa_l$ for some l, then $h_\alpha(a_i) \in \mathbb{Z}$.

Proof. By Lemma 3.1, $\pi_1 = (\tau_1, \ldots, \tau_i; a_0, \ldots, a_{i-1}, 1)$ is a Lakshmibai-Seshadri path. Since $h_\alpha(a_i) = \langle\pi_1(1), \alpha^\vee\rangle - (1-a_i)\langle\tau_i(\lambda), \alpha^\vee\rangle$ is an integer by Lemma 3.3, this proves a). The proof of b) is analogue. $\diamondsuit$

3.5. As in 1.2 and 1.3, let Q be the minimal integer attained by the function h_α, and let P be equal to $h_\alpha(1) - Q$.

Lemma. *a) Q is the absolute minimum attained by the function h_α.*
b) P is the absolute maximum attained by the function $h_\alpha(1) - h_\alpha$.

Corollary. *Let π_λ be the Lakshmibai-Seshadri-path $(\overline{1}; 0, 1)$, where $\overline{1}$ is the coset of the identity in W/W_λ. Then π_λ is the unique Lakshmibai-Seshadri path of shape λ such that $e_\alpha(\pi) = 0$ for all simple roots.*

Proof of the Corollary. If $\pi = (\tau_1, \ldots, \tau_r; a_0, \ldots, a_r)$ is a Lakshmibai-Seshadri path, then $\pi = \pi_\lambda$ if $\tau_1 = \overline{1}$. If $\tau_1 \neq \overline{1}$, then let α be a simple root such that $s_\alpha\tau_1 < \tau_1$. But this implies $\langle\tau_1(\lambda), \alpha^\vee\rangle < 0$, so $h_\alpha(a_1) < 0$ and hence $e_\alpha(\pi) \neq 0$ by Lemma 3.5. Since $e_\alpha(\pi_\lambda) = 0$ for all simple roots, this proves the corollary. $\diamondsuit$

Proof of the Lemma. The statement b) is a consequence of a). To prove a), note that π is a piecewise linear path and hence h_α attains its minimum in a point $t = a_i$ for some $i \leq r$. Let $0 \leq q \leq r$ be minimal such that

$$h_\alpha(a_q) = \min\{h_\alpha(t) \mid t \in [0,1]\}.$$

In particular, we have $\langle\tau_q(\lambda), \alpha^\vee\rangle < 0$ and $\langle\tau_{q+1}(\lambda), \alpha^\vee\rangle \geq 0$. But this implies $s_\alpha\tau_q < \tau_q$ and $s_\alpha\tau_{q+1} \geq \tau_{q+1}$, so by Lemma 3.4 we have $h_\alpha(a_q) \in \mathbb{Z}$ and hence $h_\alpha(a_q) = Q$. $\diamondsuit$

3.6. Let $0 \leq p \leq r$ be maximal such that $h_\alpha(a_p) = Q$ and let $0 \leq q \leq r$ be minimal such that $h_\alpha(a_q) = Q$.

Proposition. *a) If $P > 0$, then there exists an integer $x \geq p$ such that h_α is a strictly increasing function on $[a_p, a_x]$, and for any a_j-chain*

$$\tau_j = \kappa_0 > \ldots > \kappa_s = \tau_{j+1}, \quad p < j < x,$$

the chain $s_\alpha\tau_j = s_\alpha\kappa_0 > \ldots > s_\alpha\kappa_s = s_\alpha\tau_{j+1}$ *is an* a_j*-chain for* $(s_\alpha\tau_j, s_\alpha\tau_{j+1})$*. Further,* $h_\alpha(t) \geq Q+1$ *for* $t \geq a_x$*.*

b) If $Q < 0$*, then there exists an integer* $y \leq q$ *such that* h_α *is a strictly decreasing function on* $[a_y, a_q]$*, and for any* a_j*-chain*

$$\tau_j = \kappa_0 > \ldots > \kappa_s = \tau_{j+1}, \quad y < j < q,$$

the chain $s_\alpha\tau_j = s_\alpha\kappa_0 > \ldots > s_\alpha\kappa_s = s_\alpha\tau_{j+1}$ *is an* a_j*-chain for* $(s_\alpha\tau_j, s_\alpha\tau_{j+1})$*. Further,* $h_\alpha(t) \geq Q+1$ *for* $t \leq y$*.*

Proof. We give again only the proof of a). Let $j > p$ be such that $h_\alpha(a_j) < Q+1$, and suppose for the a_j-chain

$$\tau_j = \kappa_0 > \ldots > \kappa_s = \tau_{j+1}$$

there exists an l such that $\langle \kappa_l(\lambda), \alpha^\vee \rangle \leq 0$ and hence $s_\alpha\kappa_l \leq \kappa_l$, $0 \leq l \leq s$. We may assume that j is maximal with these properties, so $\langle \tau_{j+1}(\lambda), \alpha^\vee \rangle > 0$ (since $P \geq 1$), and hence $s_\alpha\tau_{j+1} > \tau_{j+1}$. But this implies $h_\alpha(a_j) \in \mathbb{Z}$ by Lemma 3.4, which contradicts the assumption that j is such that $Q < h_\alpha(a_j) < Q+1$. So there exists an element $x \geq p$ such that h_α is a strictly increasing function on $[a_p, a_x]$ and $h_\alpha(t) \geq Q+1$ for $t \geq x$. The claim on the a_j-chains follows as in the proof of Lemma 3.2. ♢

4. The action on the Lakshmibai-Seshadri paths.

4.0. The aim of this section is to prove that the (union of the set $\{0\}$ and the) set of Lakshmibai-Seshadri paths is stable under the operators e_α and f_α. We give an explicit description (in terms of rational W-paths) of the image of a Lakshmibai-Seshadri path π under these operators.

4.1. Let $\pi = (\tau_1, \ldots, \tau_r; a_0, \ldots, a_r)$ be a Lakshmibai-Seshadri path of shape λ and fix a simple root α. As in 3.5 and 3.6 let:

· Q be the minimal integer attained by the function h_α, q is minimal such that $h_\alpha(a_q) = Q$, p is maximal such that $h_\alpha(a_p) = Q$, and P is equal to $h_\alpha(1) - Q$.

Further, choose y and x as in Proposition 3.6, i.e.:

· If $Q \leq -1$, then $y \leq q$ is maximal such that $h_\alpha(t) \geq Q+1$ for $t \leq a_y$.
  If $P \geq 1$, then $x \geq p$ is minimal such that $h_\alpha(t) \geq Q+1$ for $t \geq a_x$.

Proposition 4.2. *a) If $P > 0$, then $f_\alpha(\pi)$ is equal to the Lakshmibai-Seshadri path*

$$\begin{cases} (\tau_1,\ldots,\tau_{p-1},s_\alpha\tau_{p+1},\ldots,s_\alpha\tau_x,\tau_{x+1},\ldots,\tau_r;a_0,\ldots,a_{p-1},a_{p+1},\ldots,a_r), \\ \hfill \text{if } h_\alpha(a_x)=Q+1 \text{ and } s_\alpha\tau_{p+1}=\tau_p; \\ (\tau_1,\ldots,\tau_p,s_\alpha\tau_{p+1},\ldots,s_\alpha\tau_x,\tau_{x+1},\ldots,\tau_r;a_0,\ldots,a_r), \\ \hfill \text{if } h_\alpha(a_x)=Q+1 \text{ and } s_\alpha\tau_{p+1}<\tau_p; \\ (\tau_1,\ldots,\tau_{p-1},s_\alpha\tau_{p+1},\ldots,s_\alpha\tau_x,\tau_x,\ldots,\tau_r;a_0,\ldots,a_{p-1},a_{p+1},\ldots,a_{x-1},a,a_x,\ldots,a_r), \\ \hfill \text{if } h_\alpha(a_x)>Q+1 \text{ and } s_\alpha\tau_{p+1}=\tau_p; \\ (\tau_1,\ldots,\tau_p,s_\alpha\tau_{p+1},\ldots,s_\alpha\tau_x,\tau_x,\ldots,\tau_r;a_0,\ldots,a_{x-1},a,a_x,\ldots,a_r), \\ \hfill \text{if } h_\alpha(a_x)>Q+1 \text{ and } s_\alpha\tau_{p+1}<\tau_p; \end{cases}$$

where $a_{x-1} < a < a_x$ is such that $h_\alpha(a) = Q+1$.
b) If $Q < 0$, then $e_\alpha(T)$ is equal to the Lakshmibai-Seshadri path

$$\begin{cases} (\tau_1,\ldots,\tau_y,s_\alpha\tau_{y+1},\ldots,s_\alpha\tau_q,\tau_{q+2},\ldots,\tau_r;a_0,\ldots,a_q,a_{q+2},\ldots,a_r), \\ \hfill \text{if } h_\alpha(a_y)=Q+1 \text{ and } s_\alpha\tau_q=\tau_{q+1}; \\ (\tau_1,\ldots,\tau_y,s_\alpha\tau_{y+1},\ldots,s_\alpha\tau_q,\tau_{q+1},\ldots,\tau_r;a_0,\ldots,a_r), \\ \hfill \text{if } h_\alpha(a_y)=Q+1 \text{ and } s_\alpha\tau_q>\tau_{q+1}; \\ (\tau_1,\ldots,\tau_{y+1},s_\alpha\tau_{y+1},\ldots,s_\alpha\tau_q,\tau_{q+2},\ldots,\tau_r;a_0,\ldots,a_y,a,a_{y+1},\ldots,a_q,a_{q+2},\ldots,a_r), \\ \hfill \text{if } h_\alpha(a_y)>Q+1 \text{ and } s_\alpha\tau_q=\tau_{q+1}; \\ (\tau_1,\ldots,\tau_{y+1},s_\alpha\tau_{y+1},\ldots,s_\alpha\tau_q,\tau_{q+1},\ldots,\tau_r;a_0,\ldots,a_y,a,a_{y+1},\ldots,a_r), \\ \hfill \text{if } h_\alpha(a_y)>Q+1 \text{ and } s_\alpha\tau_q>\tau_{q+1}; \end{cases}$$

where $a_y < a < a_{y+1}$ is such that $h_\alpha(a) = Q+1$.

4.3. Proof. The rest of this section is devoted to the proof of the proposition.

To see that only the cases considered above occur, note that by the choice of p $\langle\tau_p(\lambda),\alpha^\vee\rangle \leq 0$, so $s_\alpha\tau_p \leq \tau_p$ and hence $s_\alpha\tau_{p+1} \leq \tau_p$. Similarly, $\langle\tau_{q+1}(\lambda),\alpha^\vee\rangle \geq 0$ and hence $s_\alpha\tau_{q+1} \geq \tau_{q+1}$, which implies $s_\alpha\tau_q \geq \tau_{q+1}$.

By the choice of P, Q, p, q, x, y and a, Proposition 3.6 implies that $e_\alpha(\pi)$ respectively $f_\alpha(\pi)$ is the rational W-path described in the proposition. It remains to show that these paths are Lakshmibai-Seshadri paths. Since the proofs for e_α and f_α are similar, we give only the proof for f_α.

Consider now the first two cases in a). Since $\langle\tau_p(\lambda),\alpha\rangle \leq 0$ and hence $s_\alpha\tau_p \leq \tau_p$, in the second case there exists by Lemma 3.2 an a_p-chain for the pair $(\tau_p, s_\alpha\tau_{p+1})$. Now by Proposition 3.6, to prove that π' is a Lakshmibai-Seshadri path, it remains to prove (in both cases) that there exists an a_x-chain for the pair $(s_\alpha\tau_x, \tau_{x+1})$. Consider the chain

$$s_\alpha\tau_x > \tau_x = \kappa_0 > \ldots > \kappa_s = \tau_{x+1},$$

where $\tau_x = \kappa_0 > \ldots > \kappa_s = \tau_{x+1}$ is an a_x-chain for the pair (τ_x, τ_{x+1}). All we have to prove is that $a_x\langle\tau_x(\lambda), \alpha^\vee\rangle \in \mathbb{Z}$. But $h_\alpha(a_x) = Q + 1$ implies

$$a_x\langle\tau_x(\lambda), \alpha^\vee\rangle = Q + 1 - h_\alpha(a_{x-1}) + a_{x-1}\langle\tau_x(\lambda), \alpha^\vee\rangle = Q + 1 + \langle\tau_x(\lambda) - \nu(\pi'), \alpha^\vee\rangle,$$

where π' is the Lakshmibai-Seshadri path $\pi' := (\tau_1, \ldots, \tau_x; a_0, \ldots, a_{x-1}, 1)$ (Lemma 3.1). Since the right side is an integer, this proves that $f_\alpha(\pi)$ is a Lakshmibai-Seshadri path.

To prove in the remaining cases that $f_\alpha(\pi)$ is a Lakshmibai-Seshadri path, we proceed as before. By Lemma 3.2 and Proposition 3.6, it is easy to see that all that remains to prove is the existence of an a-chain for the pair $(s_\alpha\tau_x, \tau_x)$. So one has to show that $a\langle\tau_x(\lambda), \alpha^\vee\rangle \in \mathbb{Z}$. Since $h_\alpha(a) = Q + 1$, we know that

$$a\langle\tau_x(\lambda), \alpha^\vee\rangle = Q + 1 - h_\alpha(a_{x-1}) + a_{x-1}\langle\tau_x(\lambda), \alpha^\vee\rangle = Q + 1 + \langle\tau_x(\lambda) - \nu(\pi'), \alpha^\vee\rangle,$$

where π' is as above. Since the right side is an integer, this proves that $f_\alpha(\pi)$ is a Lakshmibai-Seshadri path. ♢

5. Proof of the Character formula.

5.0. Denote by $\mathcal{P}_\lambda$ the set of Lakshmibai-Seshadri paths of shape λ. The aim of this section is to prove the character formula presented in the introduction. The idea of the proof is the following: Let Π_{int} be the subset of paths in Π such that $\nu(\pi) \in \mathcal{X}$. Denote by $\mathbb{Z}[\Pi_{\text{int}}]$ the free $\mathbb{Z}$-module with the set Π_{int} as basis. We define an operator on $\mathbb{Z}[\Pi_{\text{int}}]$ analogue to the usual Demazure operator on the group ring $\mathbb{Z}[\mathcal{X}]$, and show that we get a Demazure type character formula for $\mathcal{P}_\lambda$.

5.1. For a simple root α denote by Λ_α the linear operator on $\mathbb{Z}[\Pi_{\text{int}}]$ defined by

$$\Lambda_\alpha(\pi) := \begin{cases} \pi + f_\alpha(\pi) + \ldots + f_\alpha^n(\pi), & \text{if } n := \langle\nu(\pi), \alpha^\vee\rangle \geq 0; \\ 0, & \text{if } \langle\nu(\pi), \alpha^\vee\rangle = -1; \\ -e_\alpha(\pi) - \ldots - e_\alpha^{-n-1}(\pi), & \text{if } n := \langle\nu(\pi), \alpha^\vee\rangle \leq -2; \end{cases}$$

By the character $\nu(m)$ of an element $m = a_1\pi_1 + \ldots + a_s\pi_s$ in $\mathbb{Z}[\Pi_{\text{int}}]$ we mean the sum $a_1e^{\nu(\pi_1)} + \ldots + a_se^{\nu(\pi_s)}$ in the group ring $\mathbb{Z}[\mathcal{X}]$.

Let $\rho \in \mathcal{X}$ be such that $\langle\rho, \alpha^\vee\rangle = 1$ for all simple roots α. In the following we denote by Λ_α also the usual Demazure operator on $\mathbb{Z}[\mathcal{X}]$:

$$\Lambda_\alpha(e^\mu) := \frac{e^{\mu+\rho} - e^{s_\alpha(\mu+\rho)}}{1 - e^{-\alpha}} e^{-\rho}$$

One checks easily that $\nu\big(\Lambda_\alpha(\pi)\big) = \Lambda_\alpha(e^{\nu(\pi)})$.

5.2. Denote by $\phi : \mathcal{P}_\lambda \to W/W_\lambda$ the map defined by $\phi(\pi) = \phi\big((\underline{\tau}, \underline{a})\big) := \tau_1$, and for $\tau \in W/W_\lambda$ set

$$\mathcal{P}_{\lambda,\tau} := \{\pi \in \mathcal{P}_\lambda \mid \phi(\pi) \leq \tau\}.$$

Let $\pi_\lambda := (\overline{1}; 0, 1)$ be the unique W-path in $\mathcal{P}_\lambda$ such that $e_\alpha(\pi_\lambda) = 0$ for all simple roots (see Corollary 3.5).

Theorem. *For any reduced decomposition $\tau = s_{\alpha_1} \cdot \ldots \cdot s_{\alpha_r}$ we have the following equalities in $\mathbb{Z}[\Pi_{\rm int}]$ respectively $\mathbb{Z}[\mathcal{X}]$:*

$$\Lambda_{\alpha_1} \circ \cdots \circ \Lambda_{\alpha_r}(\pi_\lambda) = \sum_{\pi \in \mathcal{P}_{\lambda,\tau}} \pi \qquad \text{and} \qquad \Lambda_{\alpha_1} \circ \cdots \circ \Lambda_{\alpha_r}(e^\lambda) = \sum_{\pi \in \mathcal{P}_{\lambda,\tau}} e^{\nu(\pi)}.$$

5.3. The proof of the theorem will be given in 5.5. For a fixed root α and a Lakshmibai-Seshadri path π let Q, q and p be as in 3.5 and 3.6, so Q is the minimal integer attained by the function h_α, q is minimal such that $\pi(a_q) = Q$ and p is maximal such that $\pi(a_p) = Q$.

Lemma. *a) If $s_\alpha \phi(\pi) < \phi(\pi)$, then $e_\alpha(\pi) \neq 0$.*

b) If $f_\alpha(\pi) \neq 0$, then either $\phi(f_\alpha(\pi)) = \phi(\pi)$, or $\phi(f_\alpha(\pi)) = s_\alpha\phi(\pi) > \phi(\pi)$ and $e_\alpha(\pi) = 0$.

c) If $e_\alpha(\pi) \neq 0$, then either $\phi(e_\alpha(\pi)) = \phi(\pi)$, or $\phi(e_\alpha(\pi)) = s_\alpha\phi(\pi) < \phi(\pi)$ and $e_\alpha^2(\pi) = 0$.

Proof. By Proposition 4.2 we have $\phi(f_\alpha(\pi)) \neq \phi(\pi)$ if and only if $p = 0$. And in this case we have $\phi(f_\alpha(\pi)) = s_\alpha\phi(\pi) > \phi(\pi)$. Further, $p = 0$ implies $q = 0$ and hence $Q = 0$, so $e_\alpha(\pi) = 0$, which proves *b*). It is now easy to see that *c*) is an immediate consequence of *b*) by Proposition 1.5.

To prove *a*) note that $s_\alpha\phi(\pi) < \phi(\pi)$ implies for $\phi(\pi) = \tau_1$ that $\langle \tau_1(\lambda)\alpha \rangle < 0$. It follows that $h_\alpha(a_1) < 0$ and hence $Q \leq -1$, so $e_\alpha(\pi) \neq 0$. ♢

5.4. Let π be a Lakshmibai-Seshadri path of shape λ such that $e_\alpha(\pi) = 0$. Denote by $\mathcal{S}_\alpha(\pi)$ the string

$$\mathcal{S}_\alpha(\pi) := \{\pi, f_\alpha(\pi), \ldots, f^{\langle \nu(\pi), \alpha^\vee \rangle}(\pi)\}$$

in $\mathcal{P}_\lambda$ generated by π under the operators f_α and e_α. The following result is an easy consequence of Lemma 5.3 and the definition of Λ_α:

Lemma. *For $\tau \in W/W_\lambda$ the intersection $\mathcal{S}_\alpha(\pi) \cap \mathcal{P}_{\lambda,\tau}$ is either empty, or $\mathcal{S}_\alpha(\pi) \subset \mathcal{P}_{\lambda,\tau}$, or $\mathcal{S}_\alpha(\pi) \cap \mathcal{P}_{\lambda,\tau} = \pi$. Further,*

$$\sum_{\pi' \in \mathcal{S}_\alpha(\pi)} \pi' = \Lambda_a(\pi) = \Lambda_\alpha \circ \Lambda_\alpha(\pi).$$

♢

5.5. Proof of the theorem. We proceed by induction over $l(\tau)$. If $l(\tau) = 0$, then $\mathcal{P}_{\lambda,\bar{1}} = \{(\bar{1}; 0, 1)\}$, which proves the theorem in this case.

Suppose now $l(\tau) > 0$ and choose a simple root α such that $s_\alpha\tau < \tau$. By Lemma 5.3 and Lemma 5.4, the set $\mathcal{P}_{\lambda,s_\alpha\tau}$ has a decomposition $\mathcal{P}_0 \cup \mathcal{P}_+$ such that $\mathcal{P}_0$ is the union of strings $\mathcal{S}_\alpha(\pi)$ for some $\pi \in \mathcal{P}_{\lambda,s_\alpha\tau}$, and $\mathcal{S}_\alpha(\pi) \cap \mathcal{P}_{\lambda,s_\alpha\tau} = \pi$ for $\pi \in \mathcal{P}_+$.

Now if $\pi' \in \mathcal{P}_{\lambda,\tau}$ is such that $\pi' \notin \mathcal{P}_{\lambda,s_\alpha\tau}$, then by Lemma 5.3 there exists an element $\pi \in \mathcal{P}_{\lambda,s_\alpha\tau}$ such that $e_\alpha(\pi) = 0$ and $\pi' \in \mathcal{S}_\alpha(\pi)$. It follows now by Lemma 5.3 and Lemma 5.4 that

$$\Lambda_\alpha\big(\sum_{\pi\in\mathcal{P}_{\lambda,s_\alpha\tau}} \pi\big) = \sum_{\pi\in\mathcal{P}_0} \pi \cup \sum_{\pi\in\mathcal{P}_+} \Lambda_\alpha(\pi) = \sum_{\pi\in\mathcal{P}_{\lambda,\tau}} \pi,$$

which proves the theorem. ♢

5.6. Proof of the Character formula. Proposition 4.2 shows that the set of Lakshmibai-Seshadri paths is stable under the operators e_α and f_α for all simple roots. Since π_λ is the only Lakshmibai-Seshadri path such that $\phi(\pi) = \overline{1}$, Lemma 5.3 *a*) and Proposition 1.5 imply that the Lakshmibai-Seshadri paths are of the required form.

It follows by Demazure's character formula (see [12]) and Theorem 5.2 that the sum $\sum_\pi e^{\nu(\pi)}$ over all $\pi \in \mathcal{P}_\lambda$ is the character $\mathrm{Char}\, V_\lambda$ of the simple $\mathfrak{G}$-module V_λ. ♢

6. Proof of the decomposition formulas.

6.0. The proof of the formulas is based on the Brauer-Klimyk decomposition formula (see [1], §24, Exercise 9, or [5] for the finite dimensional case), which we recall in the following. Using the Character formula and the operators e_α and f_α, we show that the contributions in the formula, which do not correspond to λ-dominant paths, cancel each other.

6.1. For $\eta \in \mathcal{X}$ let $m(\eta)$ be the dimension of the weight space $(V_\mu)_\eta$ in V_μ, and denote by $n(\nu)$ the multiplicity of V_ν in the tensor product $V_\lambda \otimes V_\mu$. The equality

$$\mathrm{Char}\, V_\lambda \cdot \mathrm{Char}\, V_\mu = \sum_{\nu\in\mathcal{X}^+} n(\nu)\, \mathrm{Char}\, V_\nu$$

implies by Weyl's character formula (see [2], §10)):

$$\left(\frac{\sum_{\sigma\in W} \mathrm{sgn}\,\sigma e^{\sigma(\lambda+\rho)}}{\sum_{\sigma\in W} \mathrm{sgn}\,\sigma e^{\sigma(\rho)}}\right)\big(\sum_{\eta\in\mathcal{X}} m(\eta)e^\eta\big) = \sum_{\nu\in\mathcal{X}^+} n(\nu)\left(\frac{\sum_{\sigma\in W} \mathrm{sgn}\,\sigma e^{\sigma(\nu+\rho)}}{\sum_{\sigma\in W} \mathrm{sgn}\,\sigma e^{\sigma(\rho)}}\right).$$

If we multiply the equality above by the denominator of Weyl's character formula, then we get by the W-invariance of $\mathrm{Char}\, V_\mu$:

$$\sum_{\sigma\in W} \mathrm{sgn}\,\sigma \sum_{\eta\in\mathcal{X}} m(\eta)e^{\sigma(\lambda+\rho+\eta)} = \sum_{\nu\in\mathcal{X}^+} n(\nu) \sum_{\sigma\in W} \mathrm{sgn}\,\sigma e^{\sigma(\nu+\rho)}.$$

Now note that $\nu+\rho$ is a strictly dominant weight. So to determine the $n(\nu)$ we have only to compare the coefficients on both sides of the strictly dominant weights. We get the following decomposition formula (Brauer-Klimyk):

For $\eta \in \mathcal{X}$ let $\{\lambda+\eta\}$ and $p(\lambda,\eta)$ be defined as follows: If there exists a real root β such that $\langle \lambda+\eta+\rho, \beta^\vee\rangle = 0$, then set $p(\lambda,\eta) := 0$ and $\{\lambda+\eta\} := 0$. Else let $\sigma \in W$ be the unique element such that $\{\lambda+\eta\} := \sigma(\lambda+\eta+\rho)-\rho$ is a dominant weight, and we set $p(\lambda,\eta) := \operatorname{sgn}\sigma$. Then

$$\sum_{\eta\in\mathcal{X}} p(\lambda,\eta)m(\eta)e^{\{\lambda+\eta\}+\rho} = \sum_{\nu\in\mathcal{X}^+} n(\nu)e^{\nu+\rho}.$$

Using the Character formula we can reformulate this equality as:

$$\sum_{\pi\in\mathcal{P}_\mu} p(\lambda,\nu(\pi))\operatorname{Char} V_{\{\lambda+\nu(\pi)\}} = \sum_{\nu\in\mathcal{X}^+} n(\nu)\operatorname{Char} V_\nu. \qquad (*)$$

6.2. Proof of the decomposition rule. If π is a λ-dominant W-path, then $\nu(\pi)$ is a dominant weight and hence $\{\lambda+\nu(\pi)\} = \lambda+\nu(\pi)$ and $p(\lambda,\nu(\pi)) = 1$. So to prove the decomposition formula it remains to show that the contributions on the left side of $(*)$ coming from not λ-dominant W-paths cancel each other. To do this, we show that this set is the disjoint union of very special subsets.

For $z\in[0,1]$ and $\pi\in\Pi$ denote by $\pi_z : [0,1]\to\mathcal{X}_\mathbb{R}$ the path $t\mapsto\pi(tz)$. We fix a Lakshmibai-Seshadri path $\pi = (\underline{\tau},\underline{a})$ such that π is not λ-dominant. Choose $s\in[0,1]$ minimal such that $\langle\lambda,\alpha^\vee\rangle + h_\alpha(s) = -1$. (Since π is not λ-dominant, by Lemma 3.5 such a simple root α always exists). Consider the set

$$M_\pi := \{\pi' = (\underline{\tau}',\underline{a}')\in\mathcal{P}_\mu \mid \exists\, z\in[0,1] \text{ such that } \pi'_z = \pi_s\}.$$

(In other words, M_π is the set of paths π' in $\mathcal{P}_\mu$ which up to a reparametrization coincide with π between 0 and s.) All elements in M_π are not λ-dominant and $\pi\in M_\pi$. Further, either $M_\pi = M_{\pi'}$ or $M_\pi\cap M_{\pi'} = \emptyset$ for two not λ-dominant Lakshmibai-Seshadri paths. So to prove the decomposition rule, it suffices to show that the contributions on the left side of $(*)$ coming from elements in M_π cancel each other.

Note that $f_\alpha(M_\pi)\subset M_\pi\cup\{0\}$. For $\pi'\in M_\pi$ let Q' and P' be as in Lemma 3.5. Because of the minimal choice of s we have $e_\alpha(\pi')\in M_\pi$ as long as $\langle\lambda,\alpha^\vee\rangle + Q' \le -2$. By replacing π' by $e_\alpha(\pi')$ if necessary, we may assume that $\langle\lambda,\alpha^\vee\rangle + Q' = -1$. Since $\langle\nu(\pi'),\alpha^\vee\rangle = P'+Q'$, this implies $\langle\lambda+\rho+\nu(\pi'),\alpha^\vee\rangle = P'$, so the set of weights

$$\{\lambda+\rho+\nu(\pi'), \lambda+\rho+\nu\big(f_\alpha(\pi')\big), \ldots, \lambda+\rho+\nu\big(f_a^{P'}(\pi')\big)\}$$

is stable under s_α. Because of the alternating sign on the left side in $(*)$, the contributions of the paths $\pi', f_a(\pi'),\ldots, f_a^{P'}(\pi')$ cancel each other. $\diamondsuit$

6.3. Proof of the branching rule. The proof is in the same spirit as the proof above, so we may skip a few details. Let $W(\mathfrak{L})$ be the Weyl group of $\mathfrak{L}$ and denote by $n(\nu)$ the multiplicity of U_ν in V_λ. We write $\mathcal{X}^+(\mathfrak{L})$ for the dominant weights of $\mathfrak{L}$, and we denote by $m(\eta)$ the dimension of the weight space $(V_\lambda)_\eta$ in V_λ. We multiply the equality

$$\mathrm{Char}\, V_\lambda = \sum_{\eta\in\mathcal{X}} m(\eta)e^\eta = \sum_{\nu\in\mathcal{X}^+(\mathfrak{L})} n(\nu)U_\nu = \sum_{\nu\in\mathcal{X}^+(\mathfrak{L})} n(\nu)\left(\frac{\sum_{\sigma\in W(\mathfrak{L})} e^{\sigma(\nu+\rho)}}{\sum_{\sigma\in W(\mathfrak{L})} e^{\sigma(\rho)}}\right)$$

by the denominator of the character formula. Since $\mathrm{Char}\, V_\lambda$ is $W(\mathfrak{L})$-stable we get in the same way as above the following formula:
Let $\{\eta\}$ and $p(\mathfrak{L},\eta)$ be defined as follows: If there exists a real root β in the root system of $\mathfrak{L}$ such that $\langle\lambda+\eta+\rho,\beta^\vee\rangle = 0$, then set $p(\mathfrak{L},\eta) := 0$ and $\{\eta\} := 0$. Else let $\sigma\in W(\mathfrak{L})$ be the unique element such that $\{\eta\} := \sigma(\eta+\rho)-\rho$ is a dominant weight, and we set $p(\mathfrak{L},\eta) := \mathrm{sgn}\,\sigma$. Then

$$\sum_{\pi\in\mathcal{P}_\lambda} p(\mathfrak{L},\nu(\pi))\,\mathrm{Char}\, U_{\{\nu(\pi)\}} = \sum_{\nu\in\mathcal{X}^+(\mathfrak{L})} n(\nu)\,\mathrm{Char}\, U_\nu. \qquad (**)$$

Since $p(\mathfrak{L},\nu(\pi)) = 1$ and $\{\nu(\pi)\} = \nu(\pi)$ for an $\mathfrak{L}$-dominant path, it suffices to show that the contributions in $(**)$ coming from not $\mathfrak{L}$-dominant paths cancel each other.

Fix a Lakshmibai-Seshadri path $\pi = (\underline{\tau},\underline{a})$ such that π is not $\mathfrak{L}$-dominant. Let $s\in[0,1]$ be minimal such that there exists a simple root α in the root system of $\mathfrak{L}$ for which $h_\alpha(s) = -1$. Consider the set

$$M_\pi := \{\pi'\in\mathcal{P}_\lambda \mid \exists\, z\in[0,1] \text{ such that } \pi'_z = \pi_s\}.$$

As above, all elements in M_π are not $\mathfrak{L}$-dominant and $\pi\in M_\pi$. Further, either $M_\pi = M_{\pi'}$ or $M_\pi\cap M_{\pi'} = \emptyset$ for two not $\mathfrak{L}$-dominant Lakshmibai-Seshadri paths. So to prove the branching rule it suffices to show that the contributions on the left side of $(**)$ coming from elements in M_π cancel each other.

The same arguments as above show that $f_\alpha(M_\pi)\subset M_\pi\cup\{0\}$, and if $\pi'\in M_\pi$, then $e_\alpha(\pi')\in M_\pi$ as long as $\langle\lambda,\alpha^\vee\rangle + Q' \le -2$. So we may assume that $\langle\lambda,\alpha^\vee\rangle + Q' = -1$. But this implies $\langle\rho+\nu(\pi'),\alpha^\vee\rangle = P'$. Hence the set of weights

$$\{\rho+\nu(\pi), \rho+\nu\big(f_\alpha(\pi')\big), \ldots, \rho+\nu\big(f_a^{P'}(\pi')\big)\}$$

is stable under s_α, and the contributions in $(**)$ of the paths π',$f_a(\pi')$,..., $f_a^{P'}(\pi')$ cancel each other. ◇

7. A new proof of the P-R-V conjecture.

7.0. Consider the tensor product $V_\lambda\otimes V_\mu$ of two $\mathfrak{G}$-modules of highest weight λ and μ. The Parthasaraty–Ranga-Rao–Varadarajan conjecture states that if $\sigma,\tau\in W$ are such that $\nu := \tau(\lambda)+\sigma(\mu)$ is a dominant weight, then the module V_ν occurs in $V_\lambda\otimes V_\mu$.

Proofs of the conjecture have been given independently in [6] and [11]. To show how the methods developed in this article can be used to prove existence results, we give as an example a new proof of this conjecture.

7.1. For $\eta \in \mathcal{X}$ let $[\eta] \in \mathcal{X}^+$ be the unique dominant element in the Weyl group orbit $W \cdot \eta$ of η.

Suppose $\pi = (\tau_1, \ldots, \tau_r; a_0, \ldots, a_r)$ is a Lakshmibai-Seshadri path of shape μ such that $\lambda + \pi(a_i)$ is a dominant weight for all $i = 1, \ldots, r-1$. So π is not λ-dominant if and only if $\lambda + \pi(a_r)$ is not a dominant weight.

Note that any Lakshmibai-Seshadri path of the form $(\tau; 0, 1)$ satisfies this condition.

Proposition. *There exists a λ-dominant Lakshmibai-Seshadri path π' of shape μ such that $\lambda + \nu(\pi') = [\lambda + \nu(\pi)]$.*

Corollary. (P-R-V conjecture) *If $\nu := \tau(\lambda) + \sigma(\mu)$ is a dominant weight, then the module V_ν occurs in $V_\lambda \otimes V_\mu$.*

7.2. Proof of the Corollary. Note that $\tau(\lambda) + \sigma(\mu) = [\lambda + \tau^{-1}\sigma(\mu)]$. The W-path $\pi := (\tau^{-1}\sigma; 0, 1)$ of shape μ is a Lakshmibai-Seshadri path, and it satisfies the condition of Proposition 7.1 (here $r = 1$). The corollary follows hence by the decomposition rule. $\diamondsuit$

7.3. Proof of the Proposition. For an element $\beta = \sum_\alpha b_\alpha \alpha$ in the root lattice we call the sum of the coefficients $\sum_\alpha b_\alpha$ the height $\mathrm{ht}(\beta)$ of β.

The proof of the proposition is by induction on $\mathrm{ht}\big(\mu - \nu(\pi)\big)$. Note that if $\mu = \nu(\pi)$, then $\pi = (\bar{1}; 0, 1)$ and hence λ-dominant. If π is not λ-dominant, then let $t \in [a_{r-1}, 1]$ be minimal such that

$$\langle \lambda, \alpha^\vee \rangle + h_\alpha(t) = 0$$

for some simple root α, and set $m := \langle \lambda + \nu(\pi), \alpha^\vee \rangle$. (Note that $m < 0$ by the assumptions on π.) By Corollary 1.5 we know that $\pi' := e_\alpha^{-m}\pi \neq 0$, and by the choice of m is $\lambda + \nu(\pi') = s_\alpha(\lambda + \nu(\pi))$.

Moreover, by the definition of e_α, we have $\pi' = (\tau_1, \ldots, \tau_{r-1}, s_\alpha\tau_r; a_0, \ldots, a_r)$ if $t = a_{r-1}$ and $\pi' = (\tau_1, \ldots, \tau_{r-1}, \tau_r, s_\alpha\tau_r; a_0, \ldots, a_{r-1}, t, a_r)$ if $t > a_{r-1}$.

In both cases is $\mathrm{ht}\big(\lambda - \nu(\pi')\big) < \mathrm{ht}\big(\lambda - \nu(\pi)\big)$, and π' satisfies the assumptions of the proposition (by the choice of t). The proof follows hence by induction. $\diamondsuit$

8. Some concluding remarks and conjectures.

8.0. In this section we would like sketch how to extend the results for Lakshmibai-Seshadri paths also to paths of the type $\pi = \pi_{\lambda_1} * \ldots * \pi_{\lambda_m}$, where $\lambda_1, \ldots, \lambda_m$ are dominant weights. We make a also few remarks comparing the results in this article with the results in [9] and the results of Kashiwara in [3] and [4]. We will not give any technical details.

8.1. Let $\lambda_1, \ldots, \lambda_m$ be dominant weights and denote by $\underline{\lambda}$ the sequence $(\lambda_1, \ldots, \lambda_m)$. The endpoint of the path $\pi_{\underline{\lambda}} := \pi_{\lambda_1} * \ldots * \pi_{\lambda_m}$ is the dominant weight $\lambda := \lambda_1 + \ldots + \lambda_m$, and the image $\pi_{\underline{\lambda}}([0,1])$ is contained in the dominant Weyl chamber.

Denote by $\mathcal{P}_{\underline{\lambda}}$ the set of paths $\pi \in \Pi_{\mathrm{int}}$ such that

$$\pi = f_{\alpha_1} \circ \ldots \circ f_{\alpha_s}(\pi_{\underline{\lambda}})$$

for some simple roots $\alpha_1, \ldots, \alpha_s$. It is easy to see that such a path is of the form $\pi = \pi_1 * \ldots * \pi_m$, where π_i is a Lakshmibai-Seshadri path of shape λ_i for $i = 1, \ldots, m$.

In fact, using Deodhar's Lemma and the notion of Young tableau as it has been developed by Lakshmibai and Seshadri (see for example [7] or [8]), one can give a precise combinatorial criterion to decide which product of this type is an element in $\mathcal{P}_{\underline{\lambda}}$. Further, using similar arguments as in section 5 and 6, one can generalize the results for Lakshmibai-Seshadri paths also to this type of paths:

Theorem. *a)* $\mathcal{P}_{\underline{\lambda}}$ *is stable under the operators* e_α *and* f_α *for all simple roots.*

b) $\mathrm{Char}\, V_\lambda = \sum_{\pi \in \mathcal{P}_{\underline{\lambda}}} e^{\pi(1)}$.

c) If μ *is a dominant weight, then* $V_\mu \otimes V_\lambda \simeq \bigoplus_\pi V_{\mu+\pi(1)}$*, where the sum runs over all paths* $\pi \in \mathcal{P}_{\underline{\lambda}}$ *such that the image of the shifted path* $\mu + \pi$ *is contained in the dominant Weyl chamber.*

d) If $\mathfrak{L}$ *is a Levi subalgebra of* $\mathfrak{G}$*, then* $V_\lambda \simeq \bigoplus_\pi U_{\pi(1)}$*, where the sum runs over all paths* $\pi \in \mathcal{P}_{\underline{\lambda}}$ *such that the image of* π *is contained in the dominant Weyl chamber of the root system of* $\mathfrak{L}$.

8.2. Suppose we are in the finite dimensional case. Fix an enumeration of the fundamental weights $\omega_1, \ldots, \omega_n$. If in the situation above $\underline{\lambda}$ is of the form

$$\underline{\lambda} = (\omega_1, \ldots, \omega_1, \omega_2, \ldots, \omega_m),$$

then a product of the form $\pi = \pi_1 * \ldots * \pi_m$ is contained in $\mathcal{P}_{\underline{\lambda}}$ if and only if $(\pi_1, \ldots, \pi_m)$ is a standard Young tableau in the sense of Lakshmibai and Seshadri (see [7] and [8]). Further, the decomposition rules above correspond in these cases precisely to the generalized Littlewood-Richardson rule proved in [9]. In particular, using the description in [9] of the correspondence between "classical" Young tableaux and the generalization by Lakshmibai and Seshadri, we get the Littlewood-Richardson rule for the group $GL_n(\mathbb{C})$ back.

8.3. To compare the results with Kashiwara's approach, note that we can naturally associate a graph $\mathcal{G}$ to $\mathfrak{G}$: The set of vertices is Π_{int}, and we put an arrow $\pi \xrightarrow{\alpha} \pi'$ between π and π' if $f_\alpha(\pi) = \pi'$ (or equivalently $e_\alpha(\pi') = \pi$).

For $\pi \in \Pi_{\mathrm{int}}$ let $\mathcal{G}(\pi)$ be the connected component of $\mathcal{G}$ containing π. If π is a Lakshmibai-Seshadri path of shape λ, then the set of vertices of $\mathcal{G}(\pi)$ is just the set of all Lakshmibai-Seshadri paths of shape λ.

Suppose now λ is a dominant weight and π is a Lakshmibai-Seshadri path of shape μ. For a simple root α let Q_α be as in Lemma 3.5, i.e. Q_α is the absolute minimum of the function

$$h_\alpha : [0,1] \to \mathbb{R},\ t \mapsto \langle \pi(t), \alpha^\vee \rangle.$$

Since π is λ-dominant if and only if $\langle \lambda, \alpha^\vee \rangle + \langle \pi(t), \alpha^\vee \rangle \geq 0$ for all $t \in [0,1]$ and all simple roots α, it follows that π is λ-dominant if and only if $\langle \lambda, \alpha^\vee \rangle \geq -Q_\alpha$ for all simple roots. So we get by Proposition 1.5:

$$\pi \text{ is } \lambda\text{-dominant} \Leftrightarrow e_\alpha^{\langle \lambda, \alpha^\vee \rangle + 1}(\pi) = 0 \text{ for all simple roots } \alpha.$$

In this terminology the decomposition rule in this article and in [4] are identical. This suggests that $\mathcal{G}(\pi_\lambda)$ is isomorphic to the crystal graph of V_λ constructed by Kashiwara.

8.4. More generally, let $\lambda \in \mathcal{X}$ be a dominant weight and let $\pi \in \Pi_{\text{int}}$ be such that $\pi(1) = \lambda$ and $\pi([0,1])$ is contained in the dominant Weyl chamber. We conjecture that $\mathcal{G}(\pi)$ is isomorphic to the crystal graph of the module V_λ. In particular, this would imply that the graph $\mathcal{G}(\pi)$ depends only on the endpoint $\pi(1)$ and is otherwise independent of the choice of π. Moreover, the character formula, the decomposition rule and the branching rule could be reformulated for the set of paths obtained by applying successively the operators f_α to π.

It has been proved in [10] that $\mathcal{G}(\pi_{\underline{\lambda}})$ is the crystal graph if $\underline{\lambda}$ is as in 8.2 and $\mathfrak{G}$ is of type $\mathtt{A}_n$, $\mathtt{B}_n$, $\mathtt{C}_n$, $\mathtt{D}_n$, $\mathtt{E}_n$, or $\mathtt{G}_2$.

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