A plactic algebra for semisimple Lie algebras

P. Littelmann*

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1 Introduction

A plactic algebra can be thought of as a (non-commutative) model for the representation ring of a semisimple Lie algebra $\mathfrak{g}$. This algebra was introduced by Lascoux and Schützenberger in [13], [18] in order to study the representation theory of $GL_n(\mathbb{C})$ and S_n. This new tool enabled them for example to give the first rigouros proof of the Littlewood-Richardson rule to determine the decomposition of tensor products into direct sums of irreducible representations. Using a case by case analysis, such a plactic algebra has been constructed also for some other simple groups, see [1], [8], [19], [20], [21].

Recently, two constructions of isomorphic plactic algebras have been given for symmetrisable Kac-Moody algebras. From the point of view of quantum groups, this algebra is the algebra of crystal bases ([5], [6], [7], [16], [17], [19]). The second construction realizes this algebra as the algebra $\mathbb{Z}\mathcal{P}$ of equivalence classes of paths in the space $X_{\mathbb{Q}}$ of rational weights ([5], [14], [15]).

For simplicity, assume that G is a simple, simply connected algebraic group. To give a description of $\mathbb{Z}\mathcal{P}$ which is more in the spirit of the original work of Lascoux and Schützenberger, let $V = V_{\lambda_1} \oplus \ldots \oplus V_{\lambda_r}$ be a faithful representation and let $\mathbb{D}$ be the associated set of L-S paths, i.e. $\mathbb{D}$ is a basis of the corresponding model of V in $\mathbb{Z}\mathcal{P}$. Let $\mathbb{Z}\{\mathbb{D}\}$ be the free associative algebra generated by $\mathbb{D}$. If $\lambda = \sum a_\omega \omega$ is a dominant weight, then let $|\lambda|$ denote the sum $\sum a_\omega$. The canonical projection which maps a monomial to the concatenation:

$$\psi : \mathbb{Z}\{\mathbb{D}\} \to \mathbb{Z}\mathcal{P}, \quad d_1 \cdots d_s \mapsto [d_1 * \ldots * d_s]$$

is surjective. For $N \in \mathbb{N}$ denote by $\mathbf{R}_N \subset \operatorname{Ker} \psi$ the set

$$\mathbf{R}_N := \{d_1 \cdots d_s - c_1 \cdots c_r \mid \psi(d_1 \cdots d_s) = \psi(c_1 \cdots c_r),\ r, s \le N\}.$$

Main Theorem A *Fix $m_V \in \mathbb{N}$ such that for every fundamental weight ω of G there exists an injection $V_\omega \hookrightarrow V^{\otimes m_\omega}$ for some $m_\omega \le m_V$. Let $I \subset \mathbb{Z}\{\mathbb{D}\}$ be the two-sided ideal generated by*

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$\mathbf{R}_N$ *for* $N = m_V \max\{7, |\lambda_1|, \dots, |\lambda_t|\}$. *The canonical map* $\mathbb{Z}\{\mathbb{D}\} \to \mathbb{Z}\mathcal{P}$ *induces an isomorphism* $\mathbb{Z}\{\mathbb{D}\}/I \simeq \mathbb{Z}\mathcal{P}$.

The theorem is a consequence of the case where $V = \oplus V_\omega$ is the sum of all fundamental representations. To describe $\operatorname{Ker}\psi$ in this case, one introduces the notion of a *standard Young tableau* (sections 7, 8). For every pair $(d, d') \in \mathbb{D} \times \mathbb{D}$ such that $d \cdot d'$ is not a standard Young tableau, let $d_1, \dots, d_r \in \mathbb{D}$ be such that $d_1 \cdots d_r$ is the unique standard tableau with $\psi(d_1 \cdots d_r) = \psi(d \cdot d')$, and denote by $\mathbf{R}$ the corresponding set of "plactic Plücker relations":

$$\mathbf{R} := \{d \cdot d' - d_1 \cdots d_r \mid d \cdot d' \text{ is not a standard Young tableau}\} \subset \operatorname{Ker}\psi.$$

Main Theorem B $\operatorname{Ker}\psi$ *is the two-sided ideal* J *generated by* $\mathbf{R}$.

We also use this opportunity to extend the Demazure type character formula [14] to standard monomials (Corollary 4). The generating system R_N, $N = m_V \max\{7, |\lambda|, \dots, |\mu|\}$, for $\operatorname{Ker}\psi$ given by Theorem A is in general not a minimal system. Using the algebra of root operators $\mathcal{A}$, we prove for the following cases (the enumeration of the fundamental weights is as in [2]):

Main Theorem C $\operatorname{Ker}\psi$ *is generated by*
a) $\mathbf{R}_3$ *for* $(Spin_{2n+1}, V_{\omega_n})$, $(Spin_{2n}, V_{\omega_{n-1}} \oplus V_{\omega_n})$, *and* $(\mathtt{G}_2, V_{\omega_1})$.
b) $\mathbf{R}_3$ *and the relation:* $12\dots n =$ *trivial path, for* (SL_n, V_{ω_1}). *Further,* $\mathbb{Z}\mathcal{P}$ *is the plactic algebra defined by Lascoux and Schützenberger.*
c) $\mathbf{R}_3$ *and the relations:* $\pi - \phi_i(\pi)$, $\pi \in \mathcal{A}[12\dots i(-i)]$, *for* (Sp_{2n}, V_{ω_1}). *Here* ϕ_i *is the isomorphism* $\mathcal{A}[12\dots i(-i)] \to \mathcal{A}[12\dots(i-1)]$ *for* $i = 3, \dots, n$.
The following bounds for the other exceptional groups can possibly be reduced by a more careful case by case analysis: $\operatorname{Ker}\psi$ *is generated by* $\mathbf{R}_6$ *for* $(\mathtt{F}_4, V_{\omega_4})$ *and* $(\mathtt{E}_6, V_{\omega_1} \oplus V_{\omega_6})$, *by* $\mathbf{R}_9$ *for* $(\mathtt{E}_6, V_{\omega_1})$, $\mathbf{R}_{10}$ *for* $(\mathtt{E}_7, V_{\omega_7})$, *and* $\mathbf{R}_{11}$ *for* $(\mathtt{E}_8, V_{\omega_8})$.

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2 The paths

Let X be the weight lattice of a symmetrizable Kac-Moody algebra $\mathbf{g}$. Write $X_{\mathbb{Q}}$ for $X \otimes_{\mathbb{Z}} \mathbb{Q}$, and let $[0,1]_{\mathbb{Q}}$ be the set of rational numbers t such that $0 \le t \le 1$. Denote by Π the set of all piecewise linear paths $\pi : [0,1]_{\mathbb{Q}} \to X_{\mathbb{Q}}$ such that $\pi(0) = 0$ and $\pi(1) \in X$. We consider two paths π_1, π_2 as identical if there exists a piecewise linear, nondecreasing, surjective, continuous map $\phi : [0,1]_{\mathbb{Q}} \to [0,1]_{\mathbb{Q}}$ such that $\pi_1 = \pi_2 \circ \phi$. Let $\mathbb{Z}\Pi$ be the free $\mathbb{Z}$-module with basis Π. By $\pi := \pi_1 * \pi_2$ we mean the concatenation of the paths, i.e. π is the path defined by

$$\pi(t) := \begin{cases} \pi_1(2t), & \text{if } 0 \le t \le 1/2; \\ \pi_1(1) + \pi_2(2t-1), & \text{if } 1/2 \le t \le 1. \end{cases}$$

The concatenation gives $\mathbb{Z}\Pi$ the structure of an associative algebra where the neutral element is the trivial path $\theta(t) := 0$ for all $t \in [0,1]_{\mathbb{Q}}$.

3 The root operators

The aim of this section is to recall the definition of the root operators (see [15]). Fix a simple root α, and for $\pi \in \Pi$ let $s_\alpha(\pi)$ be defined by $s_\alpha(\pi)(t) := s_\alpha(\pi(t))$. Denote by h_α the function $h_\alpha : [0,1]_{\mathbb{Q}} \to \mathbb{Q},\ t \mapsto \langle \pi(t), \alpha^\vee \rangle$, and let m be the minimal value attained by h_α. If $m \le -1$, then fix t_1 minimal such that $h_\alpha(t_1) = m$ and let t_0 be minimal such that $h_\alpha(t) = m+1$. Choose $t_0 = s_0 < s_1 < \ldots < s_r = t_1$ such that either

a) $h_\alpha(s_{i-1}) = h_\alpha(s_i)$ and $h_\alpha(t) \ge h_\alpha(s_{i-1})$ for $t \in [s_{i-1}, s_i]_{\mathbb{Q}}$;

or b) h_α is strictly decreasing on $[s_{i-1}, s_i]_{\mathbb{Q}}$.

Set $s_{-1} := 0$ and $s_{r+1} := 1$, then $\pi = \pi_0 * \pi_1 * \ldots * \pi_{r+1}$ where π_i is defined by

$$\pi_i(t) := \pi((s_{i-1} + t(s_i - s_{i-1})) - \pi(s_{i-1}), \quad i = 0, \ldots, r+1.$$

Definition 1 *If $m > -1$, then $e_\alpha \pi := 0$, else $e_\alpha \pi := \pi_0 * \eta_1 * \ldots * \eta_r * \pi_{r+1}$, where $\eta_i := \pi_i$ if h_α satisfies condition a) on $[s_{i-1}, s_i]_{\mathbb{Q}}$, and $\eta_i := s_\alpha(\pi_i)$ if not.*

The definition of f_α is similar. Fix t_0 maximal such that $h_\alpha(t_0) = m$. If $h_\alpha(1) - m \ge 1$, then let t_1 be maximal such that $h_\alpha(t) = m+1$ and choose $t_0 = s_0 < s_1 < \ldots < s_r = t_1$ such that either

a) $h_\alpha(s_i) = h_\alpha(s_{i-1})$ and $h_\alpha(t) \ge h_\alpha(s_{i-1})$ for $t \in [s_{i-1}, s_i]_{\mathbb{Q}}$;

or b) or h_α is strictly increasing on $[s_{i-1}, s_i]_{\mathbb{Q}}$.

Definition 2 *Let the π_i be as above. If $h_\alpha(1) - m < 1$, then $f_\alpha \pi := 0$. Otherwise $f_\alpha \pi := \pi_0 * \eta_1 * \ldots * \eta_r * \pi_{r+1}$, where $\eta_i := \pi_i$ if h_α is on $[s_{i-1}, s_i]_{\mathbb{Q}}$ as in a), and $\eta_i := s_\alpha(\pi_i)$ if not.*

Remark 1 It is easy to see that if $e_\alpha \pi \ne 0$, then $(e_\alpha \pi)(1) = \pi(1) + \alpha$ and $f_\alpha e_\alpha \pi = \pi$, and if $f_\alpha \pi \ne 0$, then $(f_\alpha \pi)(1) = \pi(1) - \alpha$ and $e_\alpha f_\alpha \pi = \pi$.

4 The path model of a representation

We recall the main results in [14], [15]. Denote by $\mathcal{A} \subset \mathrm{End}_{\mathbb{Z}} \mathbb{Z}\Pi$ the subalgebra generated by the root operators e_α and f_α. Let Π^+ be the set of paths π such that the image is contained in the dominant Weyl chamber and denote by M_π the $\mathcal{A}$-module $\mathcal{A}\pi$. Clearly, $B_\pi := M_\pi \cap \Pi$ is a $\mathbb{Z}$-basis of M_π.

Theorem 1 *i) If $\pi(1) = \pi'(1)$ for $\pi, \pi' \in \Pi^+$, then the $\mathcal{A}$-modules M_π and $M_{\pi'}$ are isomorphic.*

ii) If $\pi \in \Pi^+$, then $\mathrm{Char}\, M_\pi := \sum_{\eta \in B_\pi} e^{\eta(1)}$ is equal to the character $\mathrm{Char}\, V_\lambda$ of the irreducible $\mathbf{g}$-module V_λ of heighest weight $\lambda := \pi(1)$.

iii) For $\pi \in \Pi^+$ let $\eta \in M_\pi$ be an arbitrary path. The minimum $m_\alpha(\eta)$ of the function $h_\alpha : t \mapsto \langle \eta(t), \alpha^\vee \rangle$ is an integer for all simple roots, and $e_\alpha \eta = 0$ for all simple roots if and only if $\eta = \pi$.

Since $m_\alpha(\eta) \in \mathbb{Z}$ one has (see [15]) for $\eta \in M_\pi$ and $\eta' \in M_{\pi'}$:

$$f_\alpha(\eta * \eta') = \begin{cases} (f_\alpha \eta) * \eta', & \text{if } f_\alpha^n \eta \neq 0 \text{ but } e_\alpha^n \eta' = 0 \text{ for some } n \geq 1; \\ \eta * (f_\alpha \eta'), & \text{otherwise.} \end{cases}$$

$$e_\alpha(\eta * \eta') = \begin{cases} \eta * (e_\alpha \eta') & \text{if } e_\alpha^n \eta' \neq 0 \text{ but } f_\alpha^n \eta = 0 \text{ for some } n \geq 1; \\ (e_\alpha \eta) * \eta', & \text{otherwise.} \end{cases}$$

For $\pi_1, \pi_2 \in \Pi^+$ denote by $M_{\pi_1} * M_{\pi_2}$ the $\mathbb{Z}$-module spanned by the concatenations $\eta_1 * \eta_2$, where $\eta_1 \in B_{\pi_1}$ $\eta_2 \in B_{\pi_2}$. This is an $\mathcal{A}$-module (see [15]):

Theorem 2 *Suppose $\pi_1, \pi_2 \in \Pi^+$, then $M_{\pi_1} * M_{\pi_2} = \bigoplus_\eta M_{\pi_1 * \eta}$, where η runs over all paths in B_{π_2} such that $\pi_1 * \eta \in \Pi^+$.*

By the character formula we get immediately (see [15]):

Theorem 3 *For $\pi_1, \pi_2 \in \Pi^+$ set $\lambda = \pi_1(1)$ and $\mu = \pi_2(1)$. Then $V_\lambda \otimes V_\mu$ decomposes into the direct sum $\bigoplus_\eta V_{\lambda + \eta(1)}$ of irreducible $\mathbf{g}$-modules, where η runs over all paths in B_{π_2} such that $\pi_1 * \eta \in \Pi^+$.*

In the following we mean by an $\mathcal{A}$-morphism $\bigoplus_i M_{\pi_i} \to \bigoplus_j M_{\eta_j}$ always a modul homomorphism that maps paths onto paths.

5 The plactic algebra

Denote by $\mathbb{Z}\Pi_0 := \mathcal{A}\Pi^+$ the $\mathcal{A}$-submodule of $\mathbb{Z}\Pi$ generated by the paths in Π^+. Note that, by Theorem 2, $\mathbb{Z}\Pi_0$ is a subalgebra.

Definition 3 *For two paths $\pi, \eta \in \mathbb{Z}\Pi_0$ let $\pi^+, \eta^+ \in \Pi^+$ be the unique paths such that $\pi \in M_{\pi^+}$, $\eta \in M_{\eta^+}$. We call π, η equivalent and write $\pi \sim \eta$, if $\pi^+(1) = \eta^+(1)$ and $\phi(\pi) = \eta$ under the isomorphism $\phi : M_{\pi^+} \to M_{\eta^+}$.*

Set $\mathbb{Z}\mathcal{P} := \mathbb{Z}\Pi_0/\sim$, and for $\pi \in \mathbb{Z}\Pi_0$ let $[\pi] \in \mathbb{Z}\mathcal{P}$ be its equivalence class. $\mathbb{Z}\mathcal{P}$ is an $\mathcal{A}$-module: $f_\alpha[\pi] := [f_\alpha\pi]$, $e_\alpha[\pi] := [e_\alpha\pi]$, and an algebra: $[\pi_1] * [\pi_2] := [\pi_1 * \pi_2]$ (see [15]). We write M_λ for $\mathcal{A}[\pi] \subset \mathbb{Z}\mathcal{P}$, where $\pi \in \Pi^+$ is an arbitrary path such that $\lambda = \pi(1)$.

Definition 4 *The algebra $\mathbb{Z}\mathcal{P}$ is called a plactic algebra for* $\mathbf{g}$.

As before, set $\operatorname{Char} M_\lambda := \sum_{[\pi]\in M_\lambda} e^{\pi(1)}$. The previous results imply:

Theorem 4 *The plactic algebra is a model for the representation ring of* $\mathbf{g}$. *More precisely, $\mathbb{Z}\mathcal{P} = \bigoplus_{\lambda\in X^+} M_\lambda$ is the sum of simple $\mathcal{A}$-modules,* $\operatorname{Char} M_\lambda$ *is the character* $\operatorname{Char} V_\lambda$ *of the corresponding simple* $\mathbf{g}$*-module, and for $\lambda, \mu \in X^+$ one has* $\operatorname{Char}(M_\lambda * M_\mu) = \operatorname{Char}(V_\lambda \otimes V_\mu)$.

6 Lakshmibai-Seshadri paths

In order that we may give a description $\mathbb{Z}\mathcal{P}$ in terms of generators and relations, we recall the description of the basis of the $\mathcal{A}$-module generated by $\pi_\lambda : t \mapsto t\lambda$ for a dominant weight λ. These are called L-S paths (see [14]). Let W_λ be the stabilizer of λ, let "$\leq$" denote the Bruhat order on W/W_λ and let $l(\cdot)$ be the length function on W/W_λ. We identify a pair $\pi = (\underline{\tau}, \underline{a})$ of sequences:

- $\underline{\tau} : \tau_1 > \tau_2 > \ldots > \tau_r$ is a sequence of linearly ordered cosets in W/W_λ,
- $\underline{a} : a_0 := 0 < a_1 < \ldots < a_r := 1$ is a sequence of rational numbers,

with the path:

$$\pi(t) := \sum_{i=1}^{j-1}(a_i - a_{i-1})\tau_i(\lambda) + (t - a_{j-1})\tau_j(\lambda) \quad \text{for} \quad a_{j-1} \leq t \leq a_j.$$

Let $\tau > \sigma$ be two elements of W/W_λ and let $0 < a < 1$ be a rational number. By an *a-chain* for (τ, σ) we mean a sequence of cosets, where $\beta_1, \ldots, \beta_s$ are positive real roots, $l(\kappa_i) = l(\kappa_{i-1}) - 1$, $a\langle\kappa_i(\lambda), \beta_i^\vee\rangle \in \mathbb{Z}$ for all $i = 1, \ldots, s$ and:

$$\kappa_0 = \tau > \kappa_1 := s_{\beta_1}\tau > \kappa_2 := s_{\beta_2}s_{\beta_1}\tau > \ldots > \kappa_s := s_{\beta_s} \cdot \ldots \cdot s_{\beta_1}\tau = \sigma.$$

Definition 5 *A pair $(\underline{\tau}, \underline{a})$ is called a Lakshmibai-Seshadri path (L–S path) of shape λ if, for all $1 \leq i \leq r-1$, there exists an a_i-chain for the pair (τ_i, τ_{i+1}).*

Theorem 5 [*14*] *The set of all L-S paths $(\underline{\tau}, \underline{a})$ of shape λ is a basis for the $\mathcal{A}$-module $\mathcal{A}\pi_\lambda \subset \mathbb{Z}\Pi_0$ generated by the path π_λ.*

Corollary 1 *The set of all equivalence classes $[(\underline{\tau}, \underline{a})] \in \mathbb{Z}\mathcal{P}$ of L-S paths forms a basis for $\mathbb{Z}\mathcal{P}$.*

In general it is quite difficult to find for two L-S paths $(\underline{\tau}_1, \underline{a}_1)$, $(\underline{\tau}_2, \underline{a}_2)$ the unique L-S path $(\underline{\tau}_3, \underline{a}_3)$ such that $[(\underline{\tau}_1, \underline{a}_1)] * [(\underline{\tau}_2, \underline{a}_2)] = [(\underline{\tau}_3, \underline{a}_3)]$ in $\mathbb{Z}\mathcal{P}$.

7 L-S monomials

In this section we consider monomials of L-S paths. A combinatorial description of the "standard monomials" will be given in section refstandardmonomialsanddefiningchains.

Definition 6 *Let $\lambda_1, \ldots, \lambda_k$ be dominant weights and set $\lambda = \lambda_1 + \ldots + \lambda_k$. If for all $i = 1, \ldots, k$, π_i is an L-S path of shape λ_i, then the monomial $m = \pi_1 * \ldots * \pi_k \in \mathbb{Z}\Pi_0$ is called an L-S monomial of shape λ (or $\underline{\lambda} = (\lambda_1, \ldots, \lambda_k)$).*

To give a monomial basis of the plactic algebra, we introduce now the notion of standard monomials. Let m be the L-S monomial $\pi_1 * \ldots * \pi_k$:

Definition 7 *m is called weakly standard of shape $\underline{\lambda} = (\lambda_1, \ldots, \lambda_k)$, if for all $i = 1, \ldots, k-1$, the concatenation $\pi_i * \pi_{i+1}$ is an element of $\mathcal{A}(\pi_{\lambda_i} * \pi_{\lambda_{i+1}})$. The monomial m is called standard of shape $\underline{\lambda}$, if $m \in \mathcal{A}(\pi_{\lambda_1} * \ldots * \pi_{\lambda_k})$.*

For all $1 \leq i, j \leq k$ fix $\mathcal{A}$-isomorphisms $\phi_{i,j} : \mathcal{A}\pi_{\lambda_i} * \mathcal{A}\pi_{\lambda_j} \to \mathcal{A}\pi_{\lambda_j} * \mathcal{A}\pi_{\lambda_i}$. Set $M := \bigoplus_{\sigma \in S_k} \mathcal{A}\pi_{\lambda_{\sigma(1)}} * \ldots * \mathcal{A}\pi_{\lambda_{\sigma(k)}}$, and denote by $\tau_i : M \to M$ the $\mathcal{A}$-isomorphism defined by

$$\tau_i(\pi_1 * \ldots * \pi_i * \pi_{i+1} * \ldots * \pi_k) := \pi_1 * \ldots * \phi_{\sigma(i),\sigma(i+1)}(\pi_i * \pi_{i+1}) * \ldots * \pi_k$$

for $\pi_1 * \ldots * \pi_k \in \mathcal{A}\pi_{\lambda_{\sigma(1)}} * \ldots * \mathcal{A}\pi_{\lambda_{\sigma(k)}} \subset M$. Note that for any choice of $\phi_{l,n}$ one has: $\phi_{l,n}(\pi_{\lambda_l} * \pi_{\lambda_n}) = \pi_{\lambda_n} * \pi_{\lambda_l}$. So if m is a weakly standard monomial, then $\tau_i(m)$ is independent of the choice of the $\phi_{l,n}$ for all i.

Theorem 6 *For every element $\sigma \in S_k$ choose a reduced decomposition $\sigma = s_{i_1} \cdots s_{i_t}$, and let $m = \pi_1 * \ldots * \pi_k \in \mathcal{A}\pi_{\lambda_1} * \ldots * \mathcal{A}\pi_{\lambda_k}$ be an L-S monomial. Then m is a standard monomial if and only if for all $\sigma \in S_k$ the L-S monomial $\tau_{i_1} \circ \ldots \circ \tau_{i_t}(m)$ is a weakly standard L-S monomial.*

Proof. Note first that if an L-S monomial m is a (weakly) standard monomial, then all paths in the $\mathcal{A}$-module $\mathcal{A}m$ generated by m are (weakly) standard monomials. Since the τ_i are $\mathcal{A}$-isomorphisms, it is sufficient to prove the theorem for monomials with the property $e_\alpha m = 0$ for all simple roots. The only standard monomial with this property is $m = \pi_{\lambda_1} * \ldots * \pi_{\lambda_k}$. Since $\tau_{i_1} \circ \ldots \circ \tau_{i_t}(m) = \pi_{\lambda_{\sigma^{-1}(1)}} * \ldots * \pi_{\lambda_{\sigma^{-1}(k)}}$ is a weakly standard monomial for all $\sigma \in S_k$, this proves one direction of the theorem.

Suppose now m is such that $\tau_{i_1} \circ \ldots \circ \tau_{i_t}(m)$ is a weakly standard monomial for all $\sigma \in S_k$. If $k = 2$, then all weakly standard monomials are standard. By induction one can assume that $m = \pi_{\lambda_1} * \ldots * \pi_{\lambda_{k-1}} * \pi_k$. Suppose $\pi_k \neq \pi_{\lambda_k}$. Since $\pi_{\lambda_{k-1}} * \pi_k$ is standard, we know by Remark 1 and Theorem 1:

$$\pi_{\lambda_{k-1}} * \pi_k = f_{\alpha_1} \ldots f_{\alpha_q}(\pi_{\lambda_{k-1}} * \pi_{\lambda_k}) = \pi_{\lambda_{k-1}} * (f_{\alpha_1} \ldots f_{\alpha_q} \pi_{\lambda_k})$$

for some simple roots. By section 4, this is only possible if $\langle\lambda_{k-1},\alpha_j^\vee\rangle = 0$ for all $1 \le j \le q$. By assumption, the monomial $\pi_{\lambda_{\sigma(1)}} * \ldots * \pi_{\lambda_{\sigma(k-1)}} * \pi_k$ is weakly standard for all $\sigma \in S_{k-1}$. This shows that $\langle\lambda_l,\alpha_j^\vee\rangle = 0$ for all $1 \le j \le q$ and $1 \le l \le k-1$. But this implies that $e_{\alpha_1} m = \pi_{\lambda_1} * \ldots * \pi_{\lambda_{k-1}} * (e_{\alpha_1}\pi_k) \neq 0$, contradicting the assumption. So $\pi_k = \pi_{\lambda_k}$, which finishes, the proof. •

8 Young tableaux

Fix an enumeration $\omega_1,\ldots,\omega_n$ of the fundamental weights of $\mathbf{g}$. A Young tableau is an L-S monomial that follows the chosen enumeration:

Definition 8 *A Young tableau of shape $\lambda = \sum_{i=1}^n a_i\omega_i$ is an L-S monomial $T = \pi * \ldots * \eta$ such that the first a_1 paths are of shape ω_1, the next a_2 are of shape ω_2, etc. The tableau is called (weakly) standard if the monomial is (weakly) standard.*

We have by the definition of standard tableaux for $\mathbf{g}$ semisimple:

Proposition 1 *The classes $[T]$ of the standard Young tableaux form a basis for the plactic algebra $\mathbb{Z}\mathcal{P}$.*

9 Main Theorem B

We assume in this section that $\mathbf{g}$ is semisimple. Fix an enumeration $\omega_1,\ldots,\omega_n$ of the fundamental weights. Let $\mathtt{B}_i$ be the set of all L-S paths of shape ω_i, and denote by $\mathtt{B}$ the union $\bigcup_{i=1}^n \mathtt{B}_i$. The free associative algebra $\mathbb{Z}\{\mathtt{B}\}$ generated by $\mathtt{B}$ can be naturally considered as the $\mathcal{A}$-stable subalgebra of $\mathbb{Z}\Pi_0$ generated by $\mathtt{B}$, so it makes sense to talk also about (weakly) standard monomials, tableaux, (weakly) standard tableaux etc. in $\mathbb{Z}\{\mathtt{B}\}$. The canonical map

$$\psi : \mathbb{Z}\{\mathtt{B}\} \to \mathbb{Z}\mathcal{P}, \quad b_1 \cdot \ldots \cdot b_N \mapsto [b_1] * \ldots * [b_N],$$

is surjective (Proposition reftableaucor). Let $\mathbf{R} \subset \operatorname{Ker}\psi$ be the following set of Plücker type relations for all $b_1, b_2 \in \mathtt{B}$ such that $b_1 \cdot b_2$ is not a standard tableau:

$$\mathbf{R} := \{b_1 \cdot b_2 - T \mid T \text{ standard tableau, } [T] = [b_1 * b_2]\}.$$

Main Theorem B. *Let $J \subset \mathbb{Z}\{\mathtt{B}\}$ be the two-sided ideal generated by $\mathbf{R}$. The canonical map $\mathbb{Z}\{\mathtt{B}\} \to \mathbb{Z}\mathcal{P}$ induces an isomorphism $\mathbb{Z}\{\mathtt{B}\}/J \simeq \mathbb{Z}\mathcal{P}$.*

Proof. One has to show that an arbitrary monomial $b_1 \cdots b_k$ in $\mathbb{Z}\{\mathtt{B}\}$ is equivalent modulo J to a standard monomial. Note first that one can "reorder" the factors of a monomial modulo J:

We know that $\mathcal{A}\pi_\omega * \mathcal{A}\pi_{\omega'}$ is isomorphic to $\mathcal{A}\pi_{\omega'} * \mathcal{A}\pi_\omega$ as an $\mathcal{A}$-module. Let $b_1 \in \mathtt{B}_\omega$ and let $b_2 \in \mathtt{B}_{\omega'}$. Then $b_1 \cdot b_2$ is either a standard tableau T, or it is equivalent to a standard tableau T by a relation in $\mathbf{R}$. By the isomorphism, there exist (not necessarily uniquely determined) $d_1 \in \mathtt{B}_\omega$ and $d_2 \in \mathtt{B}_{\omega'}$, such that $d_2 \cdot d_1$ is either equal to T or equivalent to T by a relation in $\mathbf{R}$.

This correspondence can be extended in an $\mathcal{A}$-equivariant way to an isomorphism $\phi_{\omega,\omega'}$: $\mathcal{A}\pi_\omega \cdot \mathcal{A}\pi_{\omega'} \rightarrow \mathcal{A}\pi_{\omega'} \cdot \mathcal{A}\pi_\omega$ such that $b \cdot b' - \phi_{\omega,\omega'}(b \cdot b') \in J$ for all $b \in \mathtt{B}_\omega$, $b' \in \mathtt{B}_{\omega'}$. So $b_1 \cdot b_2 \equiv d_2 \cdot d_1 \bmod J$ for some $d_1 \in \mathtt{B}_\omega, d_2 \in \mathtt{B}_{\omega'}$.

Hence one can assume that $m = b_1 \cdots b_k$ is (modulo J) a tableau of shape λ. Let "$\leq$" be the usual partial order on the weights. If m is not standard, by Theorem 6, there exists a "reordering" $m' = b'_1 \cdots b'_k$ such that $m' \equiv m \pmod J$, m' is an L-S monomial of the same shape λ (but not necessarily a tableau), but $b'_i \cdot b'_{i+1}$ is not a standard monomial for some $1 \leq i \leq k-1$. Replacing $b'_i \cdot b'_{i+1}$ by the corresponding standard tableau T in m', after reordering the factors we get a new tableau m'' of shape λ' such that $m'' \equiv m \pmod J$. But since $b'_i * b'_{i+1} \in \mathcal{A}\pi_\omega * \mathcal{A}\pi_{\omega'}$ is not a standard monomial, the shape of T is strictly smaller then the shape $\omega + \omega'$ of $b'_i \cdot b'_{i+1}$. So $\lambda' < \lambda$, and after repeating the procedure a finite number of times, this algorithm yields a standard Young tableau m'' such that $m'' \equiv m \pmod J$. •

10 Main Theorem A

To give a presentation of the plactic algebra which is more in the original style of the work of Lascoux and Schützenberger, suppose $G = G_1 \times \ldots \times G_r$ is the product of simple, simply connected algebraic groups and with Lie algebra $\mathbf{g}$. Let $V = V_{\lambda_1} \oplus \ldots \oplus V_{\lambda_t}$ be a faithful representation of G and denote by $\mathtt{D} = \mathtt{B}_{\lambda_1} \cup \ldots \cup \mathtt{B}_{\lambda_t}$ the union of all L-S paths of shape $\lambda_1, \ldots, \lambda_t$. Let $\mathbb{Z}\{\mathtt{D}\}$ be the free associative algebra generated by $\mathtt{D}$. The canonical map

$$\psi : \mathbb{Z}\{\mathtt{D}\} \rightarrow \mathbb{Z}\mathcal{P}, \quad d_1 \cdot \ldots \cdot d_s \mapsto [d_1] * \ldots * [d_s],$$

is obviously surjective. Fix $m_V \in \mathtt{N}$ such that for every fundamental weight ω there exists an $m_\omega \leq m_V$ and an injection $V_\omega \hookrightarrow V^{\otimes m_\omega}$.

Example 1 *We use the enumeration of the fundamental weights in [2]. Using [14] or the tables in [3] or the program LiE [4], one sees that:*
a) $m_V = 2$ *for* $(Spin_{2n+1}, V_{\omega_n})$, $(Spin_{2n}, V_{\omega_{n-1}} \oplus V_{\omega_n})$ *and* $(\mathtt{G}_2, V_{\omega_1})$.
b) $m_V = 3$ *for* $(\mathtt{F}_4, V_{\omega_1})$ *and* $(\mathtt{E}_6, V_{\omega_1} \oplus V_{\omega_6})$.
c) $m_V = 4$ *for* $(\mathtt{E}_6, V_{\omega_1})$ *and* $(\mathtt{E}_7, V_{\omega_7})$, $m_V = 5$ *for* $(\mathtt{E}_8, V_{\omega_8})$.
d) $m_V = n - 1$ *for* (SL_n, V_{ω_1}), $m_V = n$ *for* (Sp_{2n}, V_{ω_1}).

Let $\mathbf{R}_N \subset \operatorname{Ker} \psi$ be the set of relations of the form

$$d_1 \cdots d_p - c_1 \cdots c_q, \quad \text{where} \quad 1 \leq p, q \leq N, \; c_1, \ldots, c_q, d_1, \ldots, d_p \in \mathtt{D},$$

and $[d_1 * d_2 * \ldots * d_p] = [c_1 * c_2 * \ldots * c_q]$ in $\mathbb{Z}\mathcal{P}$. For a dominant weight $\lambda = \sum_i a_i\omega_i$ set $|\lambda| := \sum_i a_i$.

Main Theorem A *Let $I \subset \mathbb{Z}\{\mathtt{D}\}$ be the two-sided ideal generated by $\mathbf{R}_N$ for*

$$N = m_V \max\{7, |\lambda_1|, \ldots, |\lambda_t|\}.$$

The canonical map $\mathbb{Z}\{\mathtt{D}\} \to \mathbb{Z}\mathcal{P}$ induces an isomorphism $\mathbb{Z}\{\mathtt{D}\}/I \simeq \mathbb{Z}\mathcal{P}$.

Proof of Main Theorem A. For every fundamental weight ω fix a monomial $\eta_\omega = d_1 \cdots d_r$, $r \leq m_V$, such that the path $d_1 * \ldots * d_r \in \Pi^+$ and ends in ω. Denote by $\mathtt{F}$ the set of monomials in $\oplus_\omega \mathcal{A}\eta_\omega$. The algebra $\mathbb{Z}\{\mathtt{F}\}$ is $\mathcal{A}$-isomorphic to $\mathbb{Z}\{\mathtt{B}\}$ by Theorem 1, let $j : \mathbb{Z}\{\mathtt{F}\} \to \mathbb{Z}\{\mathtt{D}\}$ be the canonical map.

For $N = m_V \max\{7, |\lambda_1|, \ldots, |\lambda_t|\}$ let I be the two-sided ideal in $\mathbb{Z}\{\mathtt{D}\}$ generated by $\mathbf{R}_N$. Since $N \geq m_V \max\{|\lambda_1|, \ldots, |\lambda_t|\}$, π_{λ_i} is by Theorem 1 equivalent to a monomial in $\operatorname{Im} j$ modulo the ideal I. This implies that, modulo I, every monomial in $\mathbb{Z}\{\mathtt{D}\}$ is equivalent to an element in $\operatorname{Im} j$.

In order to prove Theorem A, it is sufficient to show that the ideal $j^{-1}(I)$ satisfies the conditions of Theorem B. Call a monomial in $\mathbb{Z}\{\mathtt{F}\}$ a standard tableau if the corresponding monomial in $\mathbb{Z}\{\mathtt{B}\}$ is a standard tableau. Suppose now that $f, g \in \mathtt{F}$ are such that $f \cdot g$ is not a standard tableau, and let ω, ω' be the fundamental weights such that $f \in \mathcal{A}\eta_\omega$ and $g \in \mathcal{A}\eta_{\omega'}$.

For a monomial $m \in \mathbb{Z}\{\mathtt{F}\}$ let $\deg m$ be the degree of $j(m)$, so $\deg(f \cdot g) \leq 2m_V$. Now $[f * g] \in M_\lambda \subset \mathbb{Z}\mathcal{P}$ for some dominant weight λ such that V_λ occurs in $V_\omega \otimes V_{\omega'}$. Hence the corresponding standard tableau is of degree at most $|\lambda| m_V$. To prove the theorem, one has to show that $|\lambda| \leq 7$. If ω and ω' correspond to different connected components of the Dynkin diagram, then $\lambda = \omega + \omega'$. Hence one may assume that $\mathbf{g}$ is simple.

One knows for the classical groups that $|\lambda| \leq 3$, for $\mathbf{g}$ of type $\mathtt{G}_2$ and $\mathtt{F}_4$ one checks easily that $|\lambda| \leq 4$. Recall that $\lambda = \omega + \mu$ for some weight μ of $V_{\omega'}$. In the remaining cases, all roots are of the same length. Let $\beta^\vee$ be the sum of all simple coroots, so $|\lambda| = \langle \lambda, \beta^\vee \rangle \leq 1 + |\langle \mu, \beta^\vee \rangle|$. Let β_0 be the highest root, then $|\langle \mu, \beta^\vee \rangle| \leq \langle \omega', \beta_0^\vee \rangle$ is bounded by the coefficients of the highest root as a sum of simple roots, which are ≤ 6. So $|\lambda| \leq 7$. •

For $\lambda = \sum_\omega a_\omega \omega$ set $\deg \lambda := \sum_\omega a_\omega \deg \eta_\omega$. The proof shows in fact:

Corollary 2 *Suppose V is a sum of fundamental representations. For two arbitrary fundamental weights ω, ω' let $N(\omega, \omega')$ be the maximum of the degrees $\deg \lambda$ for all λ such that $M_\lambda \subset M_\omega * M_{\omega'}$, and let N be the maximum of the $N(\omega, \omega')$. Then $\operatorname{Ker} \psi$ is the two-sided ideal generated by $\mathbf{R}_N$.*

11 Standard monomials and defining chains

We develop in this section a combinatorial description of standard monomials and standard tableaux using the ideas in [10], [11], and [12]. Another aim is to say for a standard monomial

m a few words about the unique L-S path π such that $[m] = [\pi]$ in $\mathbb{Z}\mathcal{P}$. In this section let $\mathbf{g}$ be again an arbitrary symmetrizable Kac-Moody algebra.

Theorem 7 *An L-S monomial $m = \pi_1 * \ldots * \pi_p$ is standard of shape $\underline{\lambda} = (\lambda_1, \ldots, \lambda_p)$ if and only if there exists a defining chain for m, i.e. for $\pi_1 = (\tau_1, \ldots, \tau_r; a_0, \ldots, a_r), \ldots, \pi_p = (\tau_s, \ldots, \tau_K; b_s, \ldots, b_K)$: there exist elements $w_1, \ldots, w_K \in W$ such that $w_1 \geq w_2 \geq \ldots \geq w_K$, and*

$$w_1 \equiv \tau_1, \ldots, w_r \equiv \tau_r \bmod W_{\lambda_1}; \ldots;\ w_s \equiv \tau_s, \ldots,\ w_K \equiv \tau_K \bmod W_{\lambda_p}.$$

Proof. We first show that the span of the monomials with a defining chain is stable under the operator f_α. The proof for e_α is similar. Let $C(m) := (\tau_1, \ldots, \tau_r, \ldots, \tau_s, \ldots, \tau_K)$ be the list of Weyl group cosets corresponding to m and let $(w_1, \ldots, w_K)$ be a corresponding defining chain. For $\tau_i \in C(m)$ let λ_i be the associated dominant weight. By [14], $C(f_\alpha(m))$ is of the form

$$(\ldots, \tau_i, s_\alpha\tau_{i+1}, \ldots, s_\alpha\tau_j, \tau_{j+1}, \ldots) \text{ or } (\ldots, \tau_i, s_\alpha\tau_{i+1}, \ldots, s_\alpha\tau_j, \tau_j, \ldots).$$

Further, either $s_\alpha\tau_l \equiv \tau_l \bmod W_{\lambda_l}$ for all $l = 1, \ldots, i$ or there exists an $k \leq i$ such that $s_\alpha\tau_k < \tau_k \bmod W_{\lambda_k}$ and $s_\alpha\tau_l = \tau_l \bmod W_{\lambda_l}$ for all $l = k+1, \ldots, i$.

If $i \geq 1$, then we can assume $s_\alpha w_i < w_i$: In the first case, if $s_\alpha w_1 > w_1$, then we may replace w_1 by $s_\alpha w_1$: This is still a lift for τ_1, and $s_\alpha w_1 > w_1 \geq w_2$. So we may assume that $s_\alpha w_l < w_l$ for $l = 1, \ldots, m$ for some $m \leq i$. Suppose now $m < i$ and $s_\alpha w_{m+1} > w_{m+1}$. Since $s_\alpha w_{m+1}$ is a lift for τ_{m+1} and $s_\alpha w_m < w_m$, $w_{m+1} \leq w_m$ implies $s_\alpha w_{m+1} \leq w_m$. So one can replace w_{m+1} by $s_\alpha w_{m+1}$ in the defining chain. In the second case, we have anyway $s_\alpha w_k < w_k$, so, by induction, we may assume $s_\alpha w_l < w_l$ for $l = k, \ldots, m$ for some $m \leq i$. The same arguments as above show that if $m < i$ and $s_\alpha w_{m+1} > w_{m+1}$, then one can replace w_{m+1} by $s_\alpha w_{m+1}$ in the defining chain.

But now the same arguments ($s_\alpha w_i < w_i$ and $w_{i+1} \leq w_i \Rightarrow s_\alpha w_{i+1} \leq w_i$) show that one of the following is a defining chain for $f_\alpha(m)$:

$$(\ldots, w_i, s_\alpha w_{i+1}, \ldots, s_\alpha w_j, w_{j+1}, \ldots) \textit{ or } (\ldots, w_i, s_\alpha w_{i+1}, \ldots, s_\alpha w_j, w_j, \ldots).$$

These arguments show that the module of paths with a defining chain is stable under the root operators. If τ_i is congruent to the coset of the neutral element for all $i = 1, \ldots N$, then the monomial is equal to $\pi_\lambda * \ldots * \pi_\mu$. Suppose now $m \neq \pi_\lambda * \ldots * \pi_\mu$, and fix i minimal such that $\tau_i \not\equiv id$, and let α be a simple root such that $s_\alpha\tau_i < \tau_i$. Recall that this equivalent to saying that, for the dominant weight λ_i one has $\langle \tau_i(\lambda_i), \alpha^\vee \rangle < 0$. The condition also implies that $s_\alpha w_i < w_i$, and hence $s_\alpha w_i \leq w_i \bmod W_\lambda$ for any dominant weight.

In this way one gets for all $j = 1, \ldots i-1$: $w_j \geq w_i \geq s_\alpha w_i \bmod W_{\lambda_j}$. But $w_j \equiv id \bmod W_{\lambda_j}$ for $j < i$, so $w_i \equiv id \bmod W_{\lambda_j}$ and $s_\alpha \equiv id \bmod W_{\lambda_j}$, which can only be if $\langle \lambda_j, \alpha^\vee \rangle = 0$ for all $j < i$. So the function h_α attains strictly negative values for this monomial, and consequently $e_\alpha(m) \neq 0$.

Since the weight of the monomial is smaller or equal to $\lambda_1 + \ldots + \lambda_p$, this shows that for any monomial m with a defining chain one can find simple roots such that $e_{\alpha_1} \ldots e_{\alpha_r}(m) =$

$\pi_\lambda * \ldots * \pi_\mu$. So the module of monomials with a defining chain coincides with the module of standard monomials. •

Let $\mathbf{b}$ be the Borel subalgebra of $\mathbf{g}$ corresponding to the choice of simple roots. Let $\lambda_1, \ldots, \lambda_s$ be dominant weights and suppose that $\mathbf{q} \supset \mathbf{b}$ is a parabolic subalgebra such that the weights can be extended to characters of $\mathbf{g}$. Let $W_\mathbf{q}$ be the Weyl group of $\mathbf{q}$. Recall that the fibres $p^{-1}(w)$ of the projection $p : W \to W/W_\mathbf{q}$ have a unique minimal element $w^{min} \in W$ (respectively unique maximal element $w^{max} \in W$), which is called the minimal (resp. maximal) representative in W of w.

Corollary 3 *A monomial $m = \pi_1 * \ldots * \pi_p$ of shape $\underline{\lambda} = (\lambda_1, \ldots, \lambda_p)$ is standard if and only if there exists a $\mathbf{q}$-defining chain for m, i.e.:*
For $\pi_1 = (\tau_1, \ldots, \tau_r; a_0, \ldots, a_r), \ldots, \pi_p = (\tau_s, \ldots, \tau_K; b_s, \ldots, b_K)$ there exist elements $w_1, \ldots, w_K$ in $W/W_\mathbf{q}$ such that $w_1 \geq \ldots \geq w_K$, and

$$w_1 \equiv \tau_1, \ldots, w_2 \equiv \tau_r \bmod W_{\lambda_1}; \ldots; w_s \equiv \tau_s, \ldots, w_K \equiv \tau_K \bmod W_{\lambda_p}.$$

Proof. If $(w_1, \ldots, w_K)$ is a defining chain, then the projection of the chain into $(W/W_\mathbf{q})^K$ gives the desired $\mathbf{q}$-chain. If $(w_1, \ldots, w_K)$ is a $\mathbf{q}$-chain, then it is easy to see that $(w_1^{min}, \ldots, w_K^{min})$ is a defining chain for m. •

It follows that the notion of a standard Young tableau given here and in [10] and [12] coincide. As there one proves easily (notation as above):

Lemma 1 *For a standard monomial $m = \pi_1 * \ldots * \pi_p$ of shape $\underline{\lambda} = (\lambda_1, \ldots, \lambda_p)$ there exists a unique maximal $\mathbf{q}$-defining chain $(w_1^+, \ldots, w_K^+)$ and a unique minimal $\mathbf{q}$-defining chain $(w_1^-, \ldots, w_K^-)$. I.e. for any $\mathbf{q}$-defining chain $(w_1, \ldots, w_K)$ for m one has $w_1^+ \geq w_1 \geq w_1^-$, ..., $w_K^+ \geq w_K \geq w_K^-$.*

Theorem 8 *Set $\lambda = \lambda_1 + \ldots + \lambda_p$, and suppose that $\mathbf{q}$ is maximal such that λ can be extended to a character of $\mathbf{q}$. For a standard monomial $m = \pi_1 * \ldots * \pi_p$ of shape $\underline{\lambda} = (\lambda_1, \ldots, \lambda_p)$ let $\eta = (\tau_1, \ldots, \tau_r; a_0, \ldots, a_r)$ be the unique L-S path of shape λ such that $[m] = [\eta]$ in $\mathbb{Z}\mathcal{P}$. Let $(w_1^+, \ldots, w_K^+)$ be the maximal $\mathbf{q}$-defining chain for m and let $(w_1^-, \ldots, w_K^-)$ be the minimal $\mathbf{q}$-defining chain for m. Then $\tau_1 = w_1^-$ and $\tau_r = w_K^+$.*

Proof. By the maximality of $\mathbf{q}$ one has $W_\mathbf{q} = W_\lambda$. For $\tau \in W/W_\lambda$ let π_τ be the L-S path $(\tau; 0, 1)$ of shape λ. Now $\eta * \pi_\tau$ is standard by Corollary 3 if and only if $\tau \leq \tau_r$. In the same way one sees that $\pi_\tau * \eta$ is standard if and only if $\tau \geq \tau_1$. By Corollary 3 and Lemma 1, the same arguments imply that $m * \pi_\tau$ is standard if and only if $w_K^+ \geq \tau$, and $\pi_\tau * m$ is standard if and only if $w_1^- \leq \tau$. Since $[m] = [\eta]$ in $\mathbb{Z}\mathcal{P}$, it follows that $\tau_1 = w_1^-$ and $\tau_r = w_K^+$. •

For $\tau \in W/W_\lambda$ let $P_{\underline{\lambda},\tau}$ be the set of standard monomials of shape $\underline{\lambda}$ such that $w_1^- \leq \tau$. Choose $\rho \in X$ such that $\langle \rho, \alpha^\vee \rangle = 1$ for all simple roots, and let $\Lambda_\alpha : e^\mu \mapsto (e^\mu - e^{s_\alpha(\mu+\rho)-\rho})/(1-$

$e^{-\alpha}$) be the Demazure operator. Let $\tau = s_{\alpha_1} \dots s_{\alpha_r}$ be a reduced decomposition. It follows from Theorem 5.2, [14]:

Corollary 4 $\sum_{m \in P_{\underline{\lambda},\tau}} e^{m(1)} = \Lambda_{\alpha_1} \circ \dots \circ \Lambda_{\alpha_r}(e^\lambda)$.

We conclude this section with another version of defining chains: Suppose that $\lambda_1, \dots, \lambda_s$ are dominant weights and let $\mathbf{b} \subseteq \mathbf{q}_1 \subseteq \mathbf{q}_2 \subseteq \dots \subseteq \mathbf{q}_s$ be parabolic subgroups such that λ_i can be extended to a character of $\mathbf{q}_i$. As above one proves:

Proposition 2 *Let $m = \pi_1 * \pi_2 * \dots * \pi_s$ be of shape $\underline{\lambda} = (\lambda_1, \dots, \lambda_s)$. Then m is standard if and only if there exists a defining chain in $\Pi_{i=1}^s W/W_{\mathbf{q}_i}$. I.e. for the paths $\pi_1 = (\tau_1, \dots, \tau_p; a_0, \dots, a_p)$, $\pi_2 = (\delta_1, \dots, \delta_q; b_0, \dots, b_q)$, $\pi_3 = (\kappa_1, \dots, \kappa_r; c_0, \dots, c_r)$ and so, there exist $w_1 \geq \dots \geq w_p$ in $W/W_{\mathbf{q}_1}$, $u_1 \geq \dots \geq u_q$ in $W/W_{\mathbf{q}_2}$, $v_1 \geq \dots \geq v_r$ in $W/W_{\mathbf{q}_3}$, and so, such that $w_p \geq u_1 \bmod W_{\mathbf{q}_2}$, $u_q \geq v_1 \bmod W_{\mathbf{q}_3}$ and so, and*

$$w_1 \equiv \tau_1, \dots, w_p \equiv \tau_p \bmod W_{\lambda_1}; u_1 \equiv \delta_1, \dots, u_p \equiv \delta_q \bmod W_{\lambda_2};\ \textit{and so.}$$

12 A lifting criterium

To make the Young tableaux more compatible with the classical notion of a Young tableau for example for $SL_n(\mathbb{C})$ (compare also [16]), we show that for a "good" enumeration of the fundamental weights in many cases the weakly standard tableaux are standard. Let G be as in section 10.

Fix a Borel subgroup $B \subset G$. Let $\alpha \neq \gamma$ be simple roots, denote by ω_α and ω_γ the fundamental weights and let $P(\alpha), P(\gamma) \supset B$ be the associated minimal parabolic subgroup. Suppose $Q \supset B$ is a parabolic subgroup such that $P(\alpha), P(\gamma) \not\subset Q$. Let Q' be generated by Q and $P(\alpha)$, and let $W_{\mathbf{q}}$, $W_{\mathbf{q}'}$, be the Weyl groups of $\mathbf{q} := \operatorname{Lie} Q$, $\mathbf{q}' := \operatorname{Lie} Q$. Consider the diagram:

$$\begin{array}{ccccc} & \nearrow & W/W_{\mathbf{q}} & \xrightarrow{p} & W/W_{\omega_a} \\ W & & \downarrow j & & \\ & \searrow & W/W_{\mathbf{q}'} & \xrightarrow{p'} & W/W_{\omega_\gamma} \end{array}$$

For $\tau \in W/W_{\omega_a}$ let $\tau^{max} \in W/W_{\mathbf{q}}$ be the unique maximal element in $p^{-1}(\tau)$. Denote by $D - \gamma$ the diagram obtained from the Dynkin diagram D of G after removing (the node of) γ, and let D_α be the irreducible component of $D - \gamma$ containing the node of α.

Lemma 2 *Suppose that $P(\beta) \subset Q'$ for all simple roots β corresponding to a node in D_α. Then, for all elements $\tau \in W/W_{\omega_\alpha}$, there exists an element $\tau' \in W/W_{\omega_\gamma}$ such that $j(\tau^{max}) = \tau'^{max}$.*

Proof. Let $w \in W$ be the maximal lift for $\tau \in W/W_{\omega_\alpha}$, so $l(ws_\beta) < l(w)$ for all simple roots $\beta \neq \alpha$. Let now $w' \in W$ be arbitrary such that $w \equiv w' \bmod W_{\mathbf{q}'}$ and $l(w's_\beta) < l(w')$ for all simple roots $\beta \notin D_\alpha \cup \{\gamma\}$.

If $\beta \in D_\alpha$ is such that $l(w's_\beta) > l(w')$, then set $w'' := w's_\beta$. One has $w'' \equiv w \bmod W_{\mathbf{q}'}$, and for $\delta \notin D_\alpha \cup \{\gamma\}$ one has $l(w''s_\delta) = l(w's_\beta s_\delta) = l(w's_\delta s_\beta) < l(w'')$ since s_β and s_δ commute. So w'' is again of the same type. Since W is finite, one can assume that $w' \in W$ is such that $w' \equiv w \bmod W_{\mathbf{q}'}$ and $l(w's_\beta) < l(w')$ for all simple roots $\beta \neq \gamma$. So $w' \in W$ is the maximal lift of $\tau' \in W/W_{\omega_\gamma}$, where $\tau' := w' \bmod W_{\omega_\gamma}$. Since $w' \equiv w \bmod W_{\mathbf{q}'}$, it follows for $\tau'^{max} \equiv w' \bmod W_{\mathbf{q}'}$ that $j(\tau^{max}) = \tau'^{max}$. •

Corollary 5 *Suppose D_α satisfies the conditions of Lemma 2. Let $\kappa \in W/W_{\omega_\gamma}$ be an arbitrary element. If there exists an element $w \in W$ such that $w \equiv \tau \bmod W_{\omega_\alpha}$ and $w \geq \kappa \bmod W_{\omega_\gamma}$, then $j(\tau^{max}) \geq \kappa^{max}$.*

Let $\omega_1, \ldots, \omega_r$ be fundamental weights and let $\alpha_1, \ldots, \alpha_r$ be the corresponding simple roots. Suppose $Q_0 \supset B$ is a parabolic subgroup such that the ω_i can be extended to characters of Q_0. Let Q_i be the parabolic subgroup generated by Q_0 and the $P(\alpha_j)$, $j \leq i$, and for $1 \leq i \leq r-1$ let D_{α_i} be the irreducible component of $D - \alpha_{i+1}$ containing the node corresponding to α_i.

Definition 9 *The tuple $(Q_0, \omega_1, \ldots, \omega_r)$ is called a good string if the following holds for all $i = 1, \ldots, r-1$: Whenever $\gamma \in D_{\alpha_i}$, then $P(\gamma) \subset Q_{i+1}$.*

One sees immediatly:

Lemma 3 *Suppose $(Q_0, \omega_1, \ldots, \omega_r)$ is a good string. For a subset $I := \{i_1, \ldots, i_s\} \subset \{1, \ldots, r\}$ such that $i_1 < \ldots < i_s$ let Q'_0 be generated by Q_0 and the $P(\alpha_l)$ such that $l \notin I$. Then $(Q'_0, \omega_{i_1}, \ldots, \omega_{i_s})$ is a good string.*

Lemma 4 *If $(Q_0, \omega_1, \ldots, \omega_r)$ is a good string, then all weakly standard monomials of shape $\underline{\lambda}$, $\lambda = a_1\omega_1 + \ldots + a_r\omega_r$, are standard.*

Proof. For $\tau \in W/W_{\omega_i}$ write τ^{max} for the unique maximal representative in $W/W_{\mathbf{q}_{i-1}}$. Suppose $m = \pi \cdots \eta$ is of the shape above and weakly standard. For a factor $(\tau_1, \ldots, \tau_r; a_0, \ldots, a_r)$ of shape ω_i let $\tau_1^{max} \geq \ldots \geq \tau_r^{max}$ be be the corresponding sequence of maximal lifts in $W/W_{\mathbf{q}_{i-1}}$. If the next factor $(\kappa_1, \ldots, \kappa_t, b_1, \ldots, b_t)$ is of the same shape, then $\tau_r^{max} \geq \kappa_1^{max}$. This is because m is weakly standard (Corollary 3). If the type changes, then one can assume that $a_{i+1} \neq 0$ (Lemma 3). Let q_i be the projection $W/W_{\mathbf{q}_{i-1}} \to W/W_{\mathbf{q}_i}$. One finds $q_i(\tau_r^{max}) \geq \kappa_1^{max}$. This is due to Corollary 5 and the fact that m is weakly standard (Theorem 7). So this sequence in $\Pi_{s=0}^{r-1} W/W_{\mathbf{q}_s}$ is a defining chain, and m is standard by Proposition 2. •

Corollary 6 *Suppose G is simple and not of type $\mathtt{D}_n$ or $\mathtt{E}_n$. Let the enumeration $\omega_1, \ldots, \omega_n$ of the fundamental weights be as in [2]. Then every weakly standard Young tableau is a standard Young tableau.*

Proof. Since $(B, \omega_1, \ldots, \omega_n)$ is a good string, by Lemma 4 all weakly standard tableaux are standard. •

Suppose now $\mathbf{g}$ is of type $\mathtt{D}_n$ or $\mathtt{E}_n$. Let the enumeration of the fundamental weights be as in [2]. Using good strings, one proves as above:

Corollary 7 *A weakly standard Young tableau of shape λ such that $a_n = 0$ or $a_{n-1} = 0$ for G of type $\mathtt{D}_n$, respectively $a_2 = 0$ or $a_1 = a_3 = 0$ for G of type $\mathtt{E}_n$, is a standard Young tableau. Further (the different ordering is important), a weakly standard Young tableau of shape $\lambda = a_{n-1}\omega_{n-1} + a_{n-2}\omega_{n-2} + a_n\omega_n$ is standard for G of type $\mathtt{D}_n$, and a weakly standard Young tableau of shape $\lambda = a_1\omega_1 + a_3\omega_3 + a_4\omega_4 + a_2\omega_2$ is standard for G of type $\mathtt{E}_n$.*

To get a criterium for an arbitrary tableau m, let m_1 be the product of the factors of type ω_1, m_2 the product of the factors of type ω_2, and so. Of course, if $a_i = 0$ for some i, then m_i is not supposed to show up in the monomial, so $m = m_1 \cdots m_n$. If we reorder the factors, then we write the factors with a $'$. For example $m_2' m_1' m_3$ is a monomial obtained from the tableau $m_1 m_2 m_3$ by reordering the factors such that all paths of type ω_2 come first.

Suppose now G of type $\mathtt{D}_n$ and $\lambda = \sum_{i=1^n} a_i\omega_i$ is such that $a_{n-1}, a_n > 0$ and $a_i > 0$ for some $i < n-2$. Choose $i \leq n-2$ maximal such that $a_i \neq 0$, and let $(\tau_1, \ldots, \tau_r; a_0, \ldots, a_r)$ be the last factor of m_i. If $i = n-2$, then set $\overline{\tau_r} := \tau_r$. Else let $\tau_r^{max} \in W/W_{\mathbf{q}_{i-1}}$ be its maximal representative, and denote by $\overline{\tau_r}$ its image under the projection $W/W_{\mathbf{q}_{i-1}} \to W/W_{\omega_{n-2}}$. We set $\pi := (\overline{\tau_r}; 0, 1)$.

Corollary 8 *The tableau m is a standard tableau if and only if $m_1 \cdots m_{n-2}$ and the monomial $m_{n-1}' \pi' m_n$ are weakly standard.*

Proof. If m is standard, then also $m_1 \cdots m_{n-2} \pi m_{n-1} m_n$ is standard. This is due to Proposition 2 and the choice of $\overline{\tau_r}$. So also $m_1 \ldots m_{n-2}$ is standard (and hence weakly standard), and $\pi m_{n-1} m_n$ is standard. But then the monomial $m_{n-1}' \pi' m_n$ is standard too.

Now if $m_1 \ldots m_{n-2}$ and $m_{n-1}' \pi' m_n$ are weakly standard, then they are standard by Corollary 7. Hence also the monomial $\pi m_{n-1} m_n$ is standard, and, by the choice of π, the monomial $m_1 \ldots m_{n-2} \pi$ is standard too.

The proof of Lemma 4 shows that in order to get a defining chain in $\Pi_{s=0}^{n-3} W/W_{Q_s}$ for a standard monomial $m_1 \cdots m_{n-2} \pi$ (using the good string $(B, \omega_1, \ldots, \omega_{n-2})$), one has to take for a factor of shape ω_j, $j \leq n-2$, as lifts the maximal representatives in $W/W_{Q_{j-1}}$. Since the monomial $\pi m_{n-1} m_n$ is also standard, there exists a defining chain (Corollary 3) in $\Pi_{s=n-2}^{n} W/W_{Q_s}$. Since π comes first, one can assume without loss of generality that the lifts for π are the maximal representatives in $W/W_{Q_{n-3}}$. Hence the terms for π in the defining chain of $m_1 \cdots m_{n-2}\pi$ coincide with the terms for π of the defining chain of $\pi m_{n-2} m_{n-1} m_n$, so is a defining chain for $m_1 \cdots m_{n-2} \pi m_{n-1} m_n$ in $\Pi_{s=0}^{n-1} W/W_{Q_s}$. It follows that m is a standard tableau. •

Suppose now G of type $\mathtt{E}_n$ and $\lambda = \sum_{i=1^n} a_i\omega_i$ is such that $a_2, a_1 + a_3 > 0$ and $a_i > 0$ for some $i > 4$. We call a monomial a tableau if the factors show up in the *reverse* ordering, i.e. the paths of shape ω_n come first etc., and the terms of shape ω_1 come last. Similarly, let Q_i be the parabolic subgroup generated by B and the $P(\alpha_j)$, $j \geq i$, and let $\mathbf{q}_i$ be its Lie algebra and $W_{\mathbf{q}_i}$ be its Weyl group. As above, if we reorder the factors, then we write the factors with a $'$.

Choose $i \geq 4$ minimal such that $a_i \neq 0$, and let $(\tau_1, \ldots, \tau_r; a_0, \ldots, a_r)$ be the last factor of m_i. If $i = 4$, then set $\overline{\tau_r} := \tau_r$. Else let $\tau_r^{max} \in W/W_{\mathbf{q}_{i-1}}$ be its maximal representative, and let $\overline{\tau_r}$ its image in W/W_{ω_4}. We set $\pi := (\overline{\tau_r}; 0, 1)$. Using the good strings $(B, \omega_n, \ldots, \omega_4)$ and $(Q_5, \omega_2, \omega_4, \omega_3, \omega_1)$, one proves:

Corollary 9 *The tableau $m = m_n \cdots m_1$ is a standard tableau if and only if $m_n \cdots m_4$ and the monomial $m_2'\pi' m_3' m_1$ are weakly standard.*

13 Examples

It remains to prove Theorem C. For a monomial $m \in \mathbb{Z}\{\mathtt{D}\}$ let $\deg m$ be its degree, and for a dominant weight $\lambda = \sum_\omega a_\omega \omega$ set $\deg \lambda := \sum_\omega a_\omega \deg \eta_\omega$.

Using [14], the tables in [3] or the program LiE [4], one checks easily that for the exceptional groups $\neq \mathtt{G}_2$ the number given in Theorem C is the number N given by Corollary 2. We consider now the remaining cases.

Proof. Case $\mathtt{A}_n$ Then $\mathtt{D}$ is the set of paths $\pi_i : t \mapsto t\epsilon_i$. If one identifies π_i with the number i, then $\mathbb{Z}\{\mathtt{D}\}$ is just the word algebra $\mathbb{Z}\{1, \ldots, n\}$ on the alphabet $\{1, \ldots, n\}$. The relations given by $\mathbf{R}_3$ can be written for $a < b < c$ as:

$$aab = aba,\ cab = acb,\ bac = bca,\ bab = abb,$$

which are the well known Knuth relations [9]. So $\mathbb{Z}\{\mathtt{D}\}/I$, where I is the two sided ideal generated by $\mathbf{R}_3$, is the algebra considered by Lascoux and Schützenberger. These relations imply for $j \leq i$: $12\ldots ij = j12\ldots i$. To prove that these relations (together with $\theta = 12\ldots n$) generate $\operatorname{Ker}\psi$, it is sufficient to prove that a monomial $m = n_1 \cdots n_s$ such that $n_1 * \ldots * n_s \in \Pi^+$ is equivalent to a standard tableau: $1\ldots i_1 \ldots 1 \ldots i_s$, where $i_1 \leq \ldots \leq i_s < n$. We prove this by induction, the case where $\deg m = 1$ being obvious. Suppose m is as above. By induction one can assume that $m = 1\ldots i_1 \ldots 1 \ldots i_t j$ for some j and $i_1 \leq \ldots \leq i_s$. Since $n_1 * \ldots * n_s \in \Pi^+$, one has $j \leq i_t + 1$. If $j = i_t + 1$, then m is standard. Else m is equivalent by the Knuth relations to $1 \ldots i_{t-1} j 1 \ldots i_t$, which is by induction equivalent to a standard tableau. •

Proof. Case $\mathtt{C}_n$ Here $\mathtt{D}$ is the set of paths $\pi_{\pm i} : t \mapsto t \pm \epsilon_i$. If one identifies $\pi_{\pm i}$ with the number $\pm i$, then $\mathbb{Z}\{\mathtt{D}\}$ is the word algebra $\mathbb{Z}\{1, \ldots, n, -n, \ldots, -1\}$ on the alphabet $1 < \ldots < n < -n < \ldots < -1$. Let ϕ_i be the isomorphism $\mathcal{A}[12\ldots i(-i)] \to \mathcal{A}[12\ldots(i-1)]$, $2 \leq i \leq n$. The relations given by $\mathbf{R}_3$ are:

$$1(-1) = \theta,\ 1a(-1) = a,\ 12(-2) = 1,\ 2(-2)(-1) = (-1),\quad \textit{for } 1 \leq a \leq -1,$$

$$aab = aba,\ cab = acb,\ bac = bca,\ bab = abb\ \ \textit{for } a < b < c,\ (a,c) \neq (1,-1).$$

To prove that $\mathbf{R}_3$ together with the relations $\pi - \phi_i(\pi)$, $\pi \in \mathcal{A}[12\dots i(-i)]$, generate $\operatorname{Ker}\psi$, it is sufficient to prove that a monomial $n_1 \cdots n_s$ such that $n_1 * \dots * n_s \in \Pi^+$ is equivalent to a standard tableau.

We prove this by induction on the degree of the monomial, the case $\deg m = 1$ being obvious. Suppose $\deg m > 1$, by induction one can assume that $m = 1 \dots i_1 1 \dots i_2 \dots 1 \dots i_s j$ for some $i_1 \leq \dots \leq i_s \leq n$ and some $1 \leq j \leq -1$. Since the corresponding path is in Π^+, one has $|j| \leq i_s$ or $j = i_s + 1$. In the last case, m is a standard tableau. If $1 \leq j \leq i_s$, the same arguments as in the case $\mathtt{A}_n$ show that m is equivalent to a standard tableau. If $-1 \geq j \geq -i_s$, then $i_l = |j|$ for some l and (by induction) m is equivalent to $m' = 1 \dots i_1 \dots 1 \dots i_s 1 \dots |j| j$. Hence m' is equivalent to $1 \dots i_1 \dots 1 \dots i_s 1 \dots (|j|-1)$, which is by induction equivalent to a standard tableau. •

Proof. Case $\mathtt{B}_n, \mathtt{D}_n$ In this case $m_V = 2$. Further, one sees easily by weight considerations that if λ is a dominant weight such that $M_\lambda \subset M_\omega * M_{\omega'}$ for two fundamental weights, then $|\lambda| \geq 3$ only if $\lambda = 2\omega_n + \omega_j$ for some $1 \leq j \leq n-1$ in the case $\mathtt{B}_n$ and $\lambda = 2\omega_n + \omega_j, 2\omega_{n-1} + \omega_j$ or $\omega_{n-1} + \omega_n + \omega_j$ for some $1 \leq j \leq n-2$ in the case $\mathtt{D}_n$. So $\operatorname{Ker}\psi$ is generated by $\mathbf{R}_4$ by Corollary 2.

We consider in the following only the case $\mathtt{B}_n$, the proof for $\mathtt{D}_n$ is similar. To prove that $\operatorname{Ker}\psi$ is already generated by $\mathbf{R}_3$, it is sufficient to show that every monomial $m = d_1 \cdots d_r$ of degree $r \leq 4$ such that $d_1 * \dots * d_r \in \Pi^+$ is equivalent to a standard tableau modulo the ideal I generated by $\mathbf{R}_3$. Since $\deg \lambda \leq 3$ for a dominant weight such that $M_\lambda \subset M_{\omega_n}^{*k}$, $k = 2, 3$, this true for monomials of degree ≤ 3. Suppose now $\deg m = 4$. Using the relations for monomials of degree ≤ 3, one can assume that m is of the form

$$\eta_{\omega_n} \cdot \eta_{\omega_n} \cdot \eta_{\omega_n} \cdot d \text{ or } \eta_{\omega_j} \cdot \eta_{\omega_n} \cdot d, \quad 1 \leq j < n$$

for some $d \in \mathtt{D}$. Now in the first case the corresponding path is in Π^+ if and only if already $\eta_{\omega_n} * d \in \Pi^+$, so this monomial is equivalent to a standard tableau modulo I. In the second case, if already $\eta_{\omega_n} * d \in \Pi^+$ or $\eta_{\omega_j} * d \in \Pi^+$, then the monomial is equivalent to a standard tableau modulo I. Otherwise identify d with its endpoint, then the only possibilities for d are

$$d = \frac{1}{2}(\epsilon_1 + \dots + \epsilon_{k-1} - \epsilon_k - \dots - \epsilon_j + \epsilon_{j+1} + \dots + \epsilon_{l-1} - \epsilon_l - \dots - \epsilon_n)$$

for some $k < j+1 < l$. Now $\eta_{\omega_j} * d = f_{\alpha_n} f_{\alpha_{n-1}} \cdots f_{\alpha_l} \cdots f_{\alpha_n} \pi$ for

$$\pi = \eta_{\omega_j} * \frac{1}{2}(\epsilon_1 + \dots + \epsilon_{k-1} - \epsilon_k - \dots - \epsilon_j + \epsilon_{j+1} + \dots + \epsilon_n),$$

and π is equivalent to $\eta_{\omega_{k-1}} * \eta_{\omega_n}$. So $\eta_{\omega_j} \cdot d$ is equivalent to

$$f_{\alpha_n} f_{\alpha_{n-1}} \cdots f_{\alpha_l} \cdots f_{\alpha_n}(\eta_{\omega_{k-1}} \cdot \eta_{\omega_n}) = \eta_{\omega_{k-1}} \cdot \frac{1}{2}(\epsilon_1 + \dots + \epsilon_{l-1} - \epsilon_l - \dots - \epsilon_n).$$

Hence $\eta_{\omega_j} \cdot \eta_{\omega_n} \cdot d$ is equivalent to $\eta_{\omega_{k-1}} \cdot \eta_{\omega_{l-1}}$, which is a standard tableau. A detailed description of the relations will be given in a forthcoming article. •

Proof. Case $\mathtt{G}_2$ Using Corollary 2, one checks easily that $\operatorname{Ker}\psi$ is generated by $\mathbf{R}_4$. We identify $\mathtt{D}$ with the set $\{1, 2, 3, \mathrm{z}, 4, 5, 6\}$ in the following way:

$$1 := \pi_{\omega_1};\ 2 := f_{\alpha_1}1;\ 3 := f_{\alpha_2}2;\ \mathrm{z} := f_{\alpha_1}3;\ 4 := f_{\alpha_1}\mathrm{z};\ 5 := f_{\alpha_2}4;\ 6 := f_{\alpha_1}5.$$

We define the numerical value of z as $3\frac{1}{2}$. The relations in $\mathbf{R}_2$ are:

$$16 = \theta;\ 1 = 1\mathrm{z},\ 2 = 14,\ 3 = 15,\ \mathrm{z} = 25,\ 4 = 26,\ 5 = 36,\ 6 = \mathrm{z}6.$$

The relations in $\mathbf{R}_3$, which are independant of those in $\mathbf{R}_2$, are coming from the isomorphisms $\mathcal{A}123 \simeq \mathcal{A}11$ and $\mathcal{A}121 \simeq \mathcal{A}112$. In the first case one gets:
$123 = 11,\ 12\mathrm{z} = 21,\ 124 = 22,\ 13\mathrm{z} = 31,\ 134 = 32,\ 23\mathrm{z} = \mathrm{z}1,\ 234 = \mathrm{z}2$
$135 = 33,\ 2\mathrm{zz} = 41,\ 235 = \mathrm{z}3,\ 2\mathrm{z}4 = 42,\ 3\mathrm{zz} = 51,\ 2\mathrm{z}5 = 43,\ 3\mathrm{z}4 = 52,$
$\mathrm{zzz} = 61,\ 245 = 4\mathrm{z},\ 3\mathrm{z}5 = 53,\ \mathrm{zz}4 = 62,\ 345 = 5\mathrm{z},\ 246 = 44,\ \mathrm{zz}5 = 63,$
$346 = 54,\ \mathrm{z}45 = 6\mathrm{z},\ 356 = 55,\ \mathrm{z}46 = 64,\ \mathrm{z}56 = 65,\ 456 = 66.$
The basis of $\mathcal{A}121$ is $\{abc \mid a < b \geq c,\ b - a \leq 2,\ \mathit{or}\ (a, b) = (\mathrm{z}, \mathrm{z}),\ c < \mathrm{z}\}$, and the basis of $\mathcal{A}112$ is $\{abc \mid a \geq b < c,\ c - b \leq 2,\ \mathit{or}\ (b, c) = (\mathrm{z}, \mathrm{z}),\ a > \mathrm{z}\}$. The relations given by the isomorphism are:

$$aab = aba,\ cab = acb,\ bac = bca,\ bab = abb,$$

for the paths ending in an extremal weight. For the other paths one gets:
132=312, 2z2=412, 3z3=513, z44=624, z55=635, z66=645, 231=213, 2z2=22z,
3z3=33z, z44=4z4, z55=5z5, 465=546, 232=z12, 233=z13, 24z=42z, 35z=53z,
454=6z4, 455=6z5, 2z1=223, 3z1=323, z42=z24, z53=z35, 46z=445, 56z=545,
2z3=413, 243=423, 343=523, 344=5z4, 354=534, z54=634, 3z2=512, zz2=612,
zz3=613, z4z=62z, z5z=63z, 45z=6zz, zz1=z23, z41=z2z, z51=z3z, 461=4zz,
561=5zz, 562=5z4, 341=32z, 342=324, 352=334, 452=434, 453=435, 463=4z5,
34z=52z , z43=623 , z52=z34 , 451=43z.
So every monomial of length ≤ 3 can be written as a standard tableau of length ≤ 3. To prove that $\operatorname{Ker}\psi$ is generated by these relations, it is sufficient to show that a monomial $m = d_1 \cdots d_4$ such that $d_1 * \ldots * d_r \in \Pi^+$, is equivalent to a standard tableau. Using the relations above, one sees that such a monomial is either equivalent to one of length ≤ 3, or it has to be of the form

$$\eta_{\omega_1} \cdot \eta_{\omega_1} \cdot \eta_{\omega_1} \cdot d, \quad \eta_{\omega_1} \cdot \eta_{\omega_2} \cdot d, \quad \text{or} \quad \eta_{\omega_2} \cdot \eta_{\omega_2}$$

for some $d \in \mathtt{D}$. The last monomial is already a standard tableau. In the first case the corresponding path is in Π^+ if and only if the path corresponding to $m' = \eta_{\omega_1} \cdot \eta_{\omega_1} \cdot d$ is already in Π^+. This monomial is of degree ≤ 3, so one can assume that it is already standard, but then $\eta_{\omega_1} \cdot m'$ is standard. In the second case one shows similarly that either already $\eta_{\omega_2} * d \in \Pi^+$ or $\eta_{\omega_1} * d \in \Pi^+$. Therefore, the monomial is equivalent to a standard tableau. •

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