Annals of Mathematics, **142** (1995), 499–525

Paths and root operators in representation theory

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Introduction

Let X be the weight lattice of a complex symmetrizable Kac-Moody algebra $\mathfrak{g}$ and denote by Π the set of all piecewise linear paths $\pi : [0,1] \to X_{\mathbb{Q}}$ starting at 0. In [8] we associated to a simple root α linear operators e_α and f_α on the $\mathbb{Z}$-module $\mathbb{Z}\Pi$ spanned by Π. Let $\mathcal{A} \subset \mathrm{End}_{\mathbb{Z}}\, \mathbb{Z}\Pi$ be the subalgebra generated by these operators.

We studied in [8] a special $\mathcal{A}$-submodule of $\mathbb{Z}\Pi$: For a dominant weight λ let π_λ be the path $t \mapsto t\lambda$ and denote by M_λ the $\mathcal{A}$-module $\mathcal{A}\pi_\lambda$ generated by π_λ. Considered as a $\mathbb{Z}$-module, the module M_λ has as a basis the set B_λ consisting of all paths contained in M_λ.

We showed that B_λ has some remarkable properties which are closely related to the representation theory of $\mathfrak{g}$: The sum $\sum e^{\pi(1)}$ over the endpoints of all paths in B_λ is the character of the irreducible representation V_λ of $\mathfrak{g}$ of highest weight λ. Further, the Littlewood-Richardson rule to decompose tensor products of representations of $\mathfrak{g} = \mathfrak{gl}_n$ can be generalized in a straightforward way to all symmetrizable Kac-Moody algebras using the paths in B_λ.

The aim of this article is to show that the results in [8] are independent of the choice of the path connecting the origin with λ. As a consequence one obtains a very interesting interpretation (and a new proof) of the decomposition rules proved in [8]: The concatenation of paths can be viewed as a "model" for the tensor product of representations of $\mathfrak{g}$.

We describe first the operators f_α and e_α: Let $\alpha^\vee$ be the coroot of α. According to the behavior of the function $t \mapsto \langle \pi(t), \alpha^\vee \rangle$ we write a path $\pi = \pi_1 * \cdots * \pi_r$ as a concatenation of "smaller" paths. If $f_\alpha \pi \neq 0$, then

$$f_\alpha \pi = \eta_1 * \cdots * \eta_r,$$

where either $\eta_j = \pi_j$ or $\eta_j = s_\alpha(\pi_j)$, and $f_\alpha\pi(1) = \pi(1) - \alpha$. The definition of e_α is similar, only that $e_\alpha\pi(1) = \pi(1) + \alpha$ (see Section 1).

*Supported by the Schweizerischer Nationalfonds

Let $\mathcal{P}^+$ be the set of paths π such that the image is contained in the dominant Weyl chamber and $\pi(1) \in X$, and for $\pi \in \mathcal{P}^+$ denote by M_π the $\mathcal{A}$-module $\mathcal{A}\pi$. Clearly the set B_π of paths contained in M_π is a basis for M_π. We show that the $\mathcal{A}$-module structure of M_π is invariant under those deformations of π which stay inside the dominant Weyl chamber and fix the starting point and the endpoint of the path:

ISOMORPHISM THEOREM. *For $\pi,\pi' \in \mathcal{P}^+$, the $\mathcal{A}$-modules M_π and $M_{\pi'}$ are isomorphic if and only if $\pi(1) = \pi'(1)$.*

In particular, the isomorphism theorem shows that we always get the same "character" for M_π. The character can be calculated using Weyl's character formula (the proof given here is independent of the proof of the character formula given in [8]): Let $\rho \in X$ be such that $\langle \rho, \alpha^\vee \rangle = 1$ for all simple roots.

CHARACTER FORMULA. *For $\pi \in \mathcal{P}^+$ let $\operatorname{Char} M_\pi$ be the character $\sum_{\eta \in B_\pi} e^{\eta(1)}$ of the $\mathcal{A}$-module M_π. Then:*

$$\sum_{\sigma \in W} \operatorname{sgn}(\sigma) e^{\sigma(\rho)} \operatorname{Char} M_\pi = \sum_{\sigma \in W} \operatorname{sgn}(\sigma) e^{\sigma(\rho+\lambda)}.$$

In particular, $\operatorname{Char} M_\pi$ *is equal to the character of the irreducible, integrable $\mathfrak{g}$-module V_λ of highest weight $\lambda := \pi(1)$.*

To define an analogue of a tensor product for $\mathcal{A}$-modules, note that the concatenation of paths induces a map $* : \Pi \times \Pi \to \Pi$, $(\pi_1, \pi_2) \mapsto \pi_1 * \pi_2$. Let $\mathcal{O}$ be the $\mathcal{A}$-submodule $\mathcal{A}\mathcal{P}^+ \subset \mathbb{Z}\Pi$ generated by $\mathcal{P}^+$, and extend "$*$" to a bilinear map $* : \mathbb{Z}\Pi \times \mathbb{Z}\Pi \to \mathbb{Z}\Pi$.

TENSOR PRODUCT RULE. *The concatenation induces a bilinear map $*: \mathcal{O} \times \mathcal{O} \to \mathcal{O}$ of $\mathcal{A}$-modules such that for $\pi_1,\pi_2 \in \mathcal{P}^+$:*

$$M_{\pi_1} * M_{\pi_2} = \bigoplus_\pi M_\pi,$$

where π runs over all paths in $\mathcal{P}^+$ of the form $\pi = \pi_1 * \eta$ for some $\eta \in B_{\pi_2}$.

By the character formula we get immediately the following Littlewood-Richardson type decomposition rule (proved in [8] for a special choice of π_2):

DECOMPOSITION FORMULA. *If $\pi_1,\pi_2 \in \mathcal{P}^+$ are such that $\lambda = \pi_1(1)$ and $\mu = \pi_2(1)$, then the tensor product $V_\lambda \otimes V_\mu$ of irreducible $\mathfrak{g}$-modules decomposes into the direct sum*

$$V_\lambda \otimes V_\mu \simeq \bigoplus_\pi V_{\pi(1)},$$

where π runs over all paths in $\mathcal{P}^+$ of the form $\pi = \pi_1 * \eta$ for some $\eta \in B_{\pi_2}$.

As described in [8, Section 8], for an appropriate choice of π_2 this rule is for $\mathfrak{g} = \mathfrak{gl}_n$ the Littlewood-Richardson rule. It should be interesting to find a direct correspondence to Lusztig's decomposition formula [9].

For a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ let $\mathcal{A}_{\mathfrak{l}}$ be the subalgebra generated by those e_α, f_α such that α is a simple root of $\mathfrak{l}$. Denote by $\mathcal{P}_{\mathfrak{l}}^+$ the set of paths contained in the dominant Weyl chamber of the root system of $\mathfrak{l}$, and for $\eta \in \mathcal{P}_{\mathfrak{l}}^+$ denote by N_η the $\mathcal{A}_{\mathfrak{l}}$-module generated by η.

RESTRICTION RULE. *The $\mathcal{A}$-module M_π, $\pi \in \mathcal{P}^+$, decomposes as an $\mathcal{A}_{\mathfrak{l}}$-module into the direct sum $M_\pi = \bigoplus_\eta N_\eta$, where η runs over all paths in B_π contained in $\mathcal{P}_{\mathfrak{l}}^+$.*

By the character formula we get for $\lambda = \pi(1)$: V_λ decomposes as an $\mathfrak{l}$-module into the direct sum $\bigoplus_\eta U_{\eta(1)}$ of simple $\mathfrak{l}$-modules, where η runs over all paths in B_π contained in $\mathcal{P}_{\mathfrak{l}}^+$.

Let $\Pi_{\text{int}} \subset \Pi$ be the subset of paths such that $\pi(1) \in X$. Using the operators e_α and f_α, we easily construct for each simple root a Lie subalgebra of $\text{End}_{\mathbb{Z}} \mathbb{Z}\Pi_{\text{int}}$ that is isomorphic to $\mathfrak{sl}_2(\mathbb{Z})$, but these subalgebras (see Section 2) do not satisfy the Serre relations (for different simple roots).

Now we define an action of the Weyl group W of $\mathfrak{g}$ on $\mathbb{Z}\Pi_{\text{int}}$ such that $w(\eta)(1) = w(\eta(1))$ for $w \in W$. We construct also for each simple root an action of the q-analogue $U_q(\mathfrak{sl}_2)$ of the enveloping algebra of $\mathfrak{sl}_2(\mathbb{Z})$ on $\mathbb{Z}[q, q^{-1}]\Pi$.

Another connection between the $\mathcal{A}$-modules M_π and the $\mathfrak{g}$-module $V_{\pi(1)}$ is given as follows: Let $\mathcal{G}(\pi)$ be the oriented, colored graph having as points the elements of the basis B_π, and we put an arrow $\pi_1 \xrightarrow{\alpha} \pi_2$ with color α if and only if $f_\alpha(\pi_1) = \pi_2$. Kashiwara [4] and Lakshmibai [6] have proved (independently):

THE CRYSTAL GRAPH. *For $\pi = \pi_\lambda$ the graph $\mathcal{G}(\pi_\lambda)$ is isomorphic to the crystal graph of the representation V_λ of the q-analogue $U_q(\mathfrak{g})$ of the enveloping algebra of $\mathfrak{g}$.*

The isomorphism has also been proved by Joseph [1] using the isomorphism theorem for $\mathcal{A}$-modules. He gives a list of properties characterizing the crystal graph uniquely up to isomorphism. The most important condition: For all dominant weights λ, μ the graphs $\mathcal{G}(\pi_\lambda * \pi_\mu)$ and $\mathcal{G}(\pi_{\lambda+\mu})$ are isomorphic, is satisfied by the isomorphism theorem.

Acknowledgments. The author would like to thank M. Kashiwara for helpful discussions and the RIMS, Kyoto, for its hospitality. I would also like to thank M. Kashiwara and the referee for pointing out a gap in the proof in a preprint version of this article.

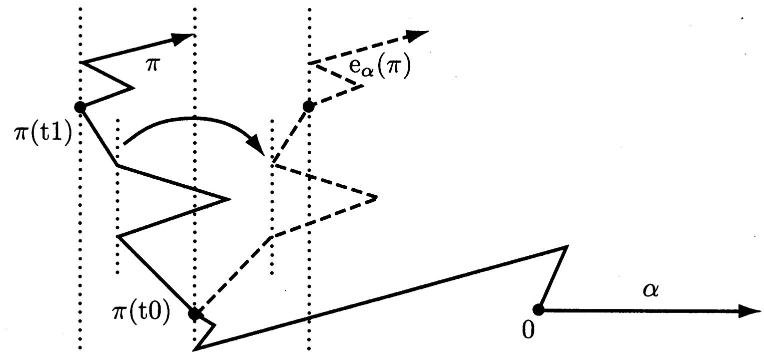

FIGURE 1. The part of the new path $e_\alpha\pi$ different from π is drawn as a dashed line.

1. The root operators

We write $[0,1]$ for the set $\{t \in \mathbb{Q} \mid 0 \leq t \leq 1\}$. Denote by Π the set of all piecewise linear paths $\pi : [0,1] \to X_\mathbb{Q}$ such that $\pi(0) = 0$. We consider two paths π_1, π_2 as identical if there exists a piecewise linear, nondecreasing, surjective, continuous map $\phi : [0,1] \to [0,1]$ such that $\pi_1 = \pi_2 \circ \phi$. Let $\mathbb{Z}\Pi$ be the free $\mathbb{Z}$-module with basis Π. For each simple root α we define linear operators e_α and f_α (the root operators) on $\mathbb{Z}\Pi$.

The definition given here is slightly different from the definition given in [8], but the effect on Lakshmibai-Seshadri paths is the same (see Section 4).

Let $\pi, \pi_1, \pi_2 \in \Pi$ be paths. For a simple root α let $s_\alpha(\pi)$ be the path given by $s_\alpha(\pi)(t) := s_\alpha(\pi(t))$. By $\pi := \pi_1 * \pi_2$ we mean the concatenation of the paths, i.e. π is the path defined by

$$\pi(t) := \begin{cases} \pi_1(2t), & \text{if } 0 \leq t \leq 1/2; \\ \pi_1(1) + \pi_2(2t-1), & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

Fix a simple root α. To define the operator e_α we cut a path $\pi \in \Pi$ into several parts according to the behavior of the function

$$h_\alpha : [0,1] \to \mathbb{Q}, \quad t \mapsto \langle \pi(t), \alpha^\vee \rangle.$$

Let $m_\alpha := \min\{h_\alpha(t) \mid t \in [0,1]\}$ be the minimal value attained by h_α.

If $m_\alpha \leq -1$, then fix $t_1 \in [0,1]$ minimal such that $h_\alpha(t_1) = m_\alpha$ and let $t_0 \in [0, t_1]$ be maximal such that $h_\alpha(t) \geq m_\alpha + 1$ for $t \in [0, t_0]$.

Choose $t_0 = s_0 < s_1 < \cdots < s_r = t_1$ such that either

(1) $h_\alpha(s_{i-1}) = h_\alpha(s_i)$ and $h_\alpha(t) \geq h_\alpha(s_{i-1})$ for $t \in [s_{i-1}, s_i]$;

(2) or h_α is strictly decreasing on $[s_{i-1}, s_i]$ and $h_\alpha(t) \geq h_\alpha(s_{i-1})$ for $t \leq s_{i-1}$.

Set $s_{-1} := 0$ and $s_{r+1} := 1$, and denote by π_i the path defined by

$$\pi_i(t) := \pi((s_{i-1} + t(s_i - s_{i-1})) - \pi(s_{i-1}), \quad i = 0, \ldots, r+1.$$

It is clear that $\pi = \pi_0 * \pi_1 * \cdots * \pi_{r+1}$.

Definition. If $m_\alpha > -1$, then $e_\alpha \pi := 0$. Otherwise,

$$e_\alpha \pi := \pi_0 * \eta_1 * \eta_2 * \cdots * \eta_r * \pi_{r+1},$$

where $\eta_i = \pi_i$ if the function h_α behaves on $[s_{i-1}, s_i]$ as in (1), and $\eta_i = s_\alpha(\pi_i)$ if the function h_α behaves on $[s_{i-1}, s_i]$ as in (2).

The definition of the operator f_α is similar. Let $t_0 \in [0,1]$ be maximal such that $h_\alpha(t_0) = m_\alpha$. If $h_\alpha(1) - m_\alpha \geq 1$, then fix $t_1 \in [t_0, 1]$ minimal such that $h_\alpha(t) \geq m_\alpha + 1$ for $t \in [t_1, 1]$.

Choose $t_0 = s_0 < s_1 < \cdots < s_r = t_1$ such that either

(1) $h_\alpha(s_i) = h_\alpha(s_{i-1})$ and $h_\alpha(t) \geq h_\alpha(s_{i-1})$ for $t \in [s_{i-1}, s_i]$;

(2) or h_α is strictly increasing on $[s_{i-1}, s_i]$ and $h_\alpha(t) \geq h_\alpha(s_i)$ for $t \geq s_i$.

Set $s_{-1} := 0$ and $s_{r+1} := 1$, and denote by π_i the path defined by

$$\pi_i(t) := \pi((s_{i-1} + t(s_i - s_{i-1})) - \pi(s_{i-1}), \quad i = 0, \ldots, r+1.$$

It is clear that $\pi = \pi_0 * \pi_1 * \cdots * \pi_{r+1}$.

Definition. If $h_\alpha(1) - m_\alpha < 1$, then $f_\alpha \pi := 0$. Otherwise,

$$f_\alpha \pi := \pi_0 * \eta_1 * \eta_2 * \cdots * \eta_r * \pi_{r+1},$$

where $\eta_i = \pi_i$ if the function h_α behaves on $[s_{i-1}, s_i]$ as in (1), and $\eta_i = s_\alpha(\pi_i)$ if the function h_α behaves on $[s_{i-1}, s_i]$ as in (2).

Example. Suppose $\mathfrak{g} = \mathfrak{sl}_3$ and μ is the highest root. The eight paths obtained from $\pi_\mu : t \mapsto t\mu$ by applying the operators f_α, e_α are the paths $\pi_\beta(t) := t\beta$, where β is an arbitrary root; for α simple one gets in addition:

$$\pi(t) := \begin{cases} -t\alpha, & \text{for } 0 \leq t \leq 1/2; \\ (t-1)\alpha, & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

2. Some simple properties of the operators

Denote by $\mathcal{A}$ the subalgebra of $\mathrm{End}_{\mathbb{Z}}\, \mathbb{Z}\Pi$ generated by the root operators. For $\pi \in \Pi$ let $m_\alpha := \min\{h_\alpha(t) \mid t \in [0,1]\}$ be the minimal value attained by the function h_α and denote by $\pi^*(t) := \pi(1-t) - \pi(1)$ the *dual* path of π. The following properties are obvious by the definition of the root operators:

LEMMA 2.1. a) *If $e_\alpha \pi \neq 0$, then $e_\alpha \pi(1) = \pi(1) + \alpha$, and if $f_\alpha \pi \neq 0$, then $f_\alpha \pi(1) = \pi(1) - \alpha$.*

b) *If* $e_\alpha\pi \neq 0$, *then* $f_\alpha e_\alpha \pi = \pi$, *and if* $f_\alpha\pi \neq 0$ *then* $e_\alpha f_\alpha \pi = \pi$.

c) $e_\alpha^n \pi = 0$ *if and only if* $n > |m_\alpha|$, *and* $f_\alpha^n\pi = 0$ *if and only if* $n > \langle \pi(1), \alpha^\vee\rangle - m_\alpha$.

d) *The* $\mathcal{A}$*-module* $\mathcal{A}\pi \subset \mathbb{Z}\Pi$ *generated by* π *has as basis the set of all paths* $\eta \in \Pi$ *contained in* $\mathcal{A}\pi$.

e) $(f_\alpha\pi)^* = e_\alpha\pi^*$ *and* $(e_\alpha\pi)^* = f_\alpha\pi^*$.

Let $\mathbb{Z}\Pi_{\text{int}}$ be the submodule of $\mathbb{Z}\Pi$ spanned by the paths ending in an integral weight. Clearly, $\mathbb{Z}\Pi_{\text{int}}$ is stable under the root operators. Choose $\rho \in X$ such that $\langle \rho, \alpha^\vee\rangle = 1$ for all simple roots. The following is an easy consequence of Lemma 2.1.

LEMMA 2.2. a) *For* $\pi \in \Pi_{\text{int}}$ *let* n_1, n_2 *be maximal such that* $e_\alpha^{n_1}\pi \neq 0$ *and* $f_\alpha^{n_2}\pi \neq 0$. *Then* $\langle \pi(1), \alpha^\vee\rangle = n_2 - n_1$.

b) $e_\alpha \pi = 0$ *for all simple roots if and only if the shifted path* $\rho + \pi$ *is completely contained in the interior of the dominant Weyl chamber.*

Let $\nu \in X$ be an integral weight and denote by $\Pi_{\text{int}}(\nu)$ the set of elements π in Π_{int} such that $\pi(1) = \nu$. Fix a simple root α and let $\varphi_j : \Pi_{\text{int}}(\nu) \to \Pi_{\text{int}}(\nu - j\alpha) \cup \{0\}$ be the map defined by $\pi \mapsto f_\alpha^j \pi$ for $j \geq 0$ and $\pi \mapsto e_\alpha^j\pi$ for $j \leq 0$. By Lemma 2.2 we have:

LEMMA 2.3. *Set* $N := \langle \nu, \alpha^\vee\rangle$. *The map* φ_j *is injective for* $0 \leq j \leq N$ *if* $N \geq 0$ *and for* $N \leq j \leq 0$ *if* $N \leq 0$.

For $n \in \mathbb{N}$ and $\pi \in \Pi$ denote by $n\pi$ the path $(n\pi)(t) := n\pi(t)$. The definition for the operators e_α and f_α given here has the advantage (compared with [8]) that it is obviously compatible with the "stretching" of paths:

LEMMA 2.4. a) $n(f_\alpha \pi) = f_\alpha^n(n\pi)$.

b) $n(e_\alpha\pi) = e_\alpha^n(n\pi)$.

Let $\mathcal{G}$ be the colored, oriented graph associated to Π_{int}: The points of $\mathcal{G}$ are the elements of Π_{int}, and we put an arrow colored by a simple root $\pi \xrightarrow{\alpha} \pi'$ between two elements if $f_\alpha \pi = \pi'$, or equivalently $e_\alpha \pi' = \pi$. For $\pi \in \Pi_{\text{int}}$ let $\mathcal{G}(\pi)$ be the connected component of $\mathcal{G}$ containing π. The set of points of $\mathcal{G}(\pi)$ is then just B_π, the set of paths in $\mathcal{A}\pi$. Note that $\mathcal{G}(\pi)$ determines completely the $\mathcal{A}$-module structure of $\mathcal{A}\pi$.

An isomorphism $\phi : \mathcal{G}(\pi_1) \to \mathcal{G}(\pi_2)$ of such graphs is a map which is a bijection on the set of points of the graphs, and which in addition has the property that $\phi(f_\alpha\pi) = f_\alpha\phi(\pi)$ for all simple roots and all points π of $\mathcal{G}(\pi_1)$.

LEMMA 2.5. *For* $\pi, \pi_1, \pi_2 \in \Pi_{\text{int}}$ *let* $\mathcal{G}(\pi), \mathcal{G}(\pi_1)$ *and* $\mathcal{G}(\pi_2)$ *be the associated graphs.*

a) *The injection* $j: B_\pi \mapsto B_{n\pi}$, $\pi' \mapsto n\pi'$, *satisfies* $j(f_\alpha\pi') = f_\alpha^n j(\pi')$.

b) *If* $\phi_n:\mathcal{G}(n\pi_1) \to \mathcal{G}(n\pi_2)$ *is an isomorphism for some* $n \in \mathbb{N}$ *such that* $\phi_n(n\pi_1) = n\pi_2$, *then there exists an isomorphism* $\phi:\mathcal{G}(\pi_1) \to \mathcal{G}(\pi_2)$ *such that* $\phi(\pi_1) = \pi_2$.

Proof. Part a) is just a reformulation of Lemma 2.4. To prove b) note that the image of $j_1 : B_{\pi_1} \mapsto B_{n\pi_1}$ is just the set of paths obtained from $n\pi_1$ by applying the operators e_α^n and f_α^n. Since the same is true for j_2, we see that ϕ_n induces a bijection $\mathrm{Im}(j_1) \to \mathrm{Im}(j_2)$ and hence a bijection $\phi : B_{\pi_1} \mapsto B_{\pi_2}$ such that $\phi(\pi_1) = \pi_2$. Since ϕ_n is a graph isomorphism, ϕ induces in fact an isomorphism $\phi : \mathcal{G}(\pi_1) \to \mathcal{G}(\pi_2)$. □

2.6. *Concatenation of modules.* Let $M \subset \mathbb{Z}\Pi_{\mathrm{int}}$ be an $\mathcal{A}$-stable submodule having as a basis the set of paths $B := M \cap \Pi_{\mathrm{int}}$. We say that B has the *integrality property* if for all $\pi \in B$ and all simple roots the minimum attained by the function $h_\alpha(t) := \langle \pi(t), \alpha^\vee \rangle$ is an integer. In the following we set $\pi * 0 = 0 * \pi := 0$ for $\pi \in \Pi$.

Suppose M_1 and M_2 are two $\mathcal{A}$-submodules of $\mathbb{Z}\Pi_{\mathrm{int}}$ having $B_1, B_2 \subset \Pi_{\mathrm{int}}$ as bases. Assume further that both have the integrality property. For $\pi \in B_1$ and $\eta \in B_2$ let $\pi * \eta$ be the concatenation of the two paths.

Denote by m_1 the minimum of the function h_α for π and by m_2 the minimum for η. Since $\pi(1)$ is an integral weight, we get:

$$f_\alpha(\pi * \eta) = \begin{cases} (f_\alpha \pi) * \eta, & \text{if } m_1 < \langle \pi(1), \alpha^\vee \rangle + m_2; \\ \pi * (f_\alpha \eta); & \text{otherwise.} \end{cases}$$

By Lemma 2.2 one can describe the action of f_α and e_α on $\pi * \eta$ as follows:

LEMMA 2.7. *Let* M_1,$M_2 \subset \mathbb{Z}\Pi_{\mathrm{int}}$ *be* $\mathcal{A}$*-submodules having* B_1,$B_2 \subset \Pi_{\mathrm{int}}$ *as bases, and suppose that* B_1,B_2 *have the integrality property. For* $\pi \in B_1$ *and* $\eta \in B_2$,

$$f_\alpha(\pi * \eta) = \begin{cases} (f_\alpha \pi) * \eta, & \textit{if there exists } n \geq 1 \textit{ such that } f_\alpha^n \pi \neq 0 \textit{ but } e_\alpha^n \eta = 0; \\ \pi * (f_\alpha \eta), & \textit{otherwise.} \end{cases}$$

Similarly, $e_\alpha(\pi * \eta) = \pi * (e_\alpha \eta)$ *if there exists* $n \geq 1$ *such that* $e_\alpha^n \eta \neq 0$ *but* $f_\alpha^n \pi = 0$, *and* $e_\alpha(\pi * \eta) = (e_\alpha \pi) * \eta$ *otherwise.*

In particular, if we denote by $M_1 * M_2$ *the* $\mathbb{Z}$*-span of the concatenations*

$$B_1 * B_2 := \{\pi * \eta \mid \pi \in B_1, \eta \in B_2\},$$

then $M_1 * M_2 \subset \mathbb{Z}\Pi_{\mathrm{int}}$ *is an* $\mathcal{A}$*-submodule.*

Remark 2.8. For $\pi \in B_1 * B_2$ the minimum of the function h_α is an integer for all simple roots, so $B_1 * B_2$ has again the integrality property.

Note that the module structure on $M_1 * M_2$ depends only on the module structure of M_1 and M_2 and not on the paths: Let N_1, N_2 be $\mathcal{A}$-submodules of $\mathbb{Z}\Pi_{\text{int}}$ having as bases the subsets $P_1, P_2 \subset \Pi_{\text{int}}$ of paths and suppose that P_1, P_2 have the integrality property. The following is obvious:

LEMMA 2.9. *If $\phi_i{:}N_i \to M_i$, $i = 1{,}2$, are $\mathcal{A}$-module isomorphisms such that $\phi_i(P_i) = B_i$, then the induced maps*

$$\phi_1 * \text{id}\colon N_1 * M_2 \longrightarrow M_1 * M_2, \quad \pi * \eta \mapsto \phi_1(\pi) * \eta$$

and

$$\text{id} * \phi_2\colon M_1 * N_2 \longrightarrow M_1 * M_2, \pi * \eta \mapsto \pi * \phi_2(\eta)$$

are isomorphisms of $\mathcal{A}$-modules.

2.10. *Some $\mathfrak{sl}_2$-theory.* The results in 2.1–2.3 show a certain resemblance with standard results in the representation theory of the Lie algebra $\mathfrak{sl}_2$. We conclude this section with a few remarks that make this resemblance more explicit. Denote by $\mathcal{B}$ the subalgebra of $\text{End}_{\mathbb{Z}}\,\mathbb{Z}\Pi_{\text{int}}$ generated by the restriction of the root operators to $\mathbb{Z}\Pi_{\text{int}}$, and let $\hat{\mathcal{B}}$ be the subalgebra of $\text{End}_{\mathbb{Z}}\,\mathbb{Z}\Pi_{\text{int}}$ consisting of all endomorphisms that can locally be approximated by elements of $\mathcal{B}$. Since the root operators are locally nilpotent, the operators

$$x_\alpha := \sum_{i\geq 1} e_\alpha^i f_\alpha^{i-1}, \quad y_\alpha := \sum_{i\geq 1} f_\alpha^i e_\alpha^{i-1}, \quad h_\alpha := \sum_{i\geq 1} (e_\alpha^i f_\alpha^i - f_\alpha^i e_\alpha^i)$$

are examples for elements of $\hat{\mathcal{B}}$. The following proposition follows easily from Lemma 2.1 and 2.2 by applying the operators to an element in Π_{int}:

PROPOSITION 2.11. *If π is an element of Π_{int}, then $h_\alpha \pi = \langle \pi(1), \alpha^\vee \rangle \pi$. Further,*

$$[x_\alpha, y_\alpha] = h_\alpha, \quad [h_\alpha, x_\alpha] = 2x_\alpha, \quad [h_\alpha, y_\alpha] = -2y_\alpha,$$

so the elements x_α, y_α and h_α span a Lie subalgebra of $\text{End}_{\mathbb{Z}}\,\mathbb{Z}\Pi_{\text{int}}$ isomorphic to $\mathfrak{sl}_2(\mathbb{Z})$.

Remark 2.12. The x_α respectively y_α do not satisfy the Serre relations, but the h_α commute. Let $\mathfrak{h}$ be the subalgebra of $\text{End}_{\mathbb{Z}}\,\mathbb{Z}\Pi_{\text{int}}$ spanned by the h_α. The "character" of M_π considered in the introduction can hence be viewed as the (usual) character of M_π as an $\mathfrak{h}$-module.

The results above can be easily extended to the q-analogue of $\mathfrak{sl}_2$. We define the corresponding operators on $\mathbb{Z}\Pi_{\text{int}} \otimes_{\mathbb{Z}} \mathbb{Z}[q, q^{-1}]$. Set $K_\alpha := q^{h_\alpha}$, so that $K_\alpha \pi := q^{\langle \nu, \alpha^\vee \rangle} \pi$ for $\pi \in \Pi_{\text{int}}(\nu)$. Let $[j]$ denote the Laurent polynomial $(q^j - q^{-j})/(q - q^{-1})$. We set

$$E_\alpha := \sum_{i\geq 1} ([i] - [i-1]) e_\alpha^i f_\alpha^{i-1}, \quad F_\alpha := \sum_{i\geq 1} ([i] - [i-1]) f_\alpha^i e_\alpha^{i-1}$$

and

$$H_\alpha := (K_\alpha - K_\alpha^{-1})/(q - q^{-1}).$$

PROPOSITION 2.13. *$H_\alpha\pi = [\langle\pi(1),\alpha^\vee\rangle]\pi$ for $\pi\in\Pi_{\rm int}$. Further,*

$$[E_\alpha,F_\alpha] = H_\alpha,\quad K_\alpha E_\alpha K_\alpha^{-1} = q^2X_\alpha \quad\text{and}\quad K_\alpha Y_\alpha K^{-1} = q^{-2}F_\alpha,$$

so the elements K_α,E_α and F_α satisfy the relations of the generators of the q-analogue $U_q(\mathfrak{sl}_2)$ of the enveloping algebra of $\mathfrak{sl}_2(\mathbb{Z})$.

Remark 2.14. The paths form naturally a basis of the crystal lattice in $\mathbb{Z}\Pi_{\rm int}\otimes_{\mathbb{Z}}\mathbb{Q}(q)$ for the action of $U_q(\mathfrak{sl}_2)$ ([5], [9]). Note that the operators $\tilde{f}_\alpha$ and $\tilde{e}_\alpha$ associated in [5] to the operators F_α and E_α are here just again the root operators f_α and e_α.

3. Continuity

Compared to the definition given in [8], the main advantage of the definition of the root operators given here is that the action is "continuous". For $\pi_1,\pi_2\in\Pi$, fix a parameterization. With respect to this parameterization we set:

$$d(\pi_1,\pi_2) := \max\{|\langle\pi_1(t)-\pi_2(t),\alpha^\vee\rangle| \mid \alpha \text{ simple},\, t\in[0,1]\}.$$

Denote by $\mathfrak{c}$ the maximum $\max\{|\langle\alpha,\gamma^\vee\rangle| \mid \alpha,\gamma \text{ simple roots }\}$.

PROPOSITION 3.1. a) *Let π_1,$\pi_2\in\Pi_{\rm int}$ be such that $d(\pi_1,\pi_2)<\varepsilon<1$ and $\min\{\langle\pi_j(t),\alpha^\vee\rangle \mid t\in[0,1]\}\in\mathbb{Z}$ for $j=1,2$. Then $f_\alpha^n\pi_1\neq 0$ (respectively $e_\alpha^n\pi_1\neq 0$) if and only if $f_\alpha^n\pi_2\neq 0$ (respectively $e_\alpha^n\pi_2\neq 0$) for all $n\geq 1$.*

b) *Suppose π_1,$\pi_2\in\Pi$ are paths such that $d(\pi_1,\pi_2)<\varepsilon$ and $f_\alpha\pi_1$,$f_\alpha\pi_2\neq 0$. Then $d(f_\alpha\pi_1,f_\alpha\pi_2)<3\mathfrak{c}\varepsilon$.*

c) *Suppose π_1,$\pi_2\in\Pi$ are paths such that $d(\pi_1,\pi_2)<\varepsilon$ and $e_\alpha\pi_1$,$e_\alpha\pi_2\neq 0$. Then $d(e_\alpha\pi_1,e_\alpha\pi_2)<3\mathfrak{c}\varepsilon$.*

Proof. If $d(\pi_1,\pi_2)<1$ and the minima are integers, then we have necessarily

$$m=\min\{\langle\pi_1(t),\alpha^\vee\rangle \mid t\in[0,1]\} = \min\{\langle\pi_2(t),\alpha^\vee\rangle \mid t\in[0,1]\}\in\mathbb{Z}$$

and $\langle\pi_1(1),\alpha^\vee\rangle = \langle\pi_2(1),\alpha^\vee\rangle$, which proves part a) by Lemma 2.1.

To prove b), let φ_1,φ_2 be nondecreasing functions such that $f_\alpha\pi_1(t) = \pi_1(t)-\varphi_1(t)\alpha$ and $f_\alpha\pi_2(t)=\pi_2(t)-\varphi_2(t)\alpha$. Then

$$\begin{aligned} d(f_\alpha\pi_1,f_\alpha\pi_2) &= d(\pi_1-\varphi_1\alpha,\pi_2-\varphi_2\alpha)\\ &\leq d(\pi_1,\pi_2)+\mathfrak{c}\max\{|\varphi_1(t)-\varphi_2(t)| \mid t\in[0,1]\}\\ &< \varepsilon+\mathfrak{c}\max\{|\varphi_1(t)-\varphi_2(t)| \mid t\in[0,1]\}.\end{aligned}$$

CLAIM. $\max\{|\varphi_1(t)-\varphi_2(t)| \mid t\in[0,1]\}<2\varepsilon$.

Note that the claim implies the proposition: $d(f_\alpha\pi_1, f_\alpha\pi_2) < \varepsilon + 2c\varepsilon \leq 3c\varepsilon$.

Proof of the claim. Set $m_i := \min\{\langle\pi_i(t), \alpha^\vee\rangle \mid t \in [0,1]\}$, $i = 1, 2$. Note that $|m_1 - m_2| < \varepsilon$. Suppose first $t \in [0,1]$ is such that neither φ_1 nor φ_2 is constant on an arbitrary small neighborhood of t. Since

$$\varphi_1(t) = \langle\pi_1(t), \alpha^\vee\rangle - m_1, \quad \varphi_2(t) = \langle\pi_2(t), \alpha^\vee\rangle - m_2,$$

we get $|\varphi_1(t) - \varphi_2(t)| \leq \varepsilon + |m_1 - m_2| < 2\varepsilon$.

Next suppose $p, q \in [0,1]$ are such that $p < q$ and φ_2 is constant on $[p, q]$, but φ_2 is not constant on an arbitrary small neighborhood of p and q, or $p = 0$. In addition we assume that $|\varphi_1(p) - \varphi_2(p)| < 2\varepsilon$. We prove now that $|\varphi_1(t) - \varphi_2(t)| < 2\varepsilon$ for all $t \in [p, q]$:

Since φ_2 is constant and φ_1 is nondecreasing, it suffices to prove that $|\varphi_1(q) - \varphi_2(q)| < 2\varepsilon$. The assumption that φ_2 is not locally constant at q implies $\varphi_2(q) = \langle\pi_2(q), \alpha^\vee\rangle - m_2$. If φ_1 is constant on $[p, q]$ too, then there is nothing to prove. If $\varphi_1(q) < \varphi_2(q)$, then we have ($\varphi_1$ is nondecreasing) $|\varphi_2(q) - \varphi_1(q)| \leq |\varphi_2(p) - \varphi_1(p)| < 2\varepsilon$.

So suppose that $\varphi_1(q) \geq \varphi_2(q)$ and fix now $q_0 \leq q$ maximal such that φ_1 is not locally constant at q_0. Then $\varphi_1(q) = \varphi_1(q_0) = \langle\pi_1(q_0), \alpha^\vee\rangle - m_1 \leq \langle\pi_1(q), \alpha^\vee\rangle - m_1$ by the definition of φ_1. Since we assume that $\varphi_1(q) \geq \varphi_2(q)$, we get

$$|\varphi_1(q) - \varphi_2(q)| \leq |\langle\pi_1(q), \alpha^\vee\rangle - m_1 - (\langle\pi_2(q), \alpha^\vee\rangle - m_2)| < 2\varepsilon.$$

Let x be such that $\varphi_1(t) = 1$ for $t \geq x$ and $\varphi_1(t) < 1$ for $t < x$. Without loss of generality we assume that $\varphi_2(t) < 1$ for $t < x$ too. Then every point $t \in [0, x]$ is contained in some interval $[p, q]$, $p < q$, such that either φ_1 and φ_2 are nowhere locally constant on $[p, q]$, or either φ_1 or φ_2 is constant on the interval and the function is not locally constant at p (except $p = 0$) and q. Since $|\varphi_1(0) - \varphi_2(0)| = 0$, this implies by the considerations above $|\varphi_1(t) - \varphi_2(t)| < 2\varepsilon$ for $t \in [0, x]$.

Since φ_1 is constant, $\varphi_1(t) \geq \varphi_2(t)$ for $t \geq x$ and φ_2 is nondecreasing, $|\varphi_1(x) - \varphi_2(x)| < 2\varepsilon$ implies $|\varphi_1(t) - \varphi_2(t)| < 2\varepsilon$ for $t \geq x$, which finishes the proof of the claim and hence the proof of b).

The proof of c) is similar. □

4. Lakshmibai-Seshadri paths

First let λ be a dominant integral weight. In [8], the $\mathcal{A}$-module $\mathcal{A}\pi_\lambda$ generated by the path $t \mapsto t\lambda$ is described as the module spanned by the Lakshmibai-Seshadri paths (L-S paths) of shape λ.

In this section, we introduce the notion of an L-S path of class λ, where λ is now an *arbitrary* integral weight (and not necessarily an element of the Tits cone!). The two notions coincide for dominant weights. As in the case of dominant weights, the L-S paths of class λ have the integrality property and they are stable under the action of the root operators. But if λ is not in the Tits cone, then in general the module $\mathcal{A}\pi_\lambda$ is a proper submodule of the $\mathcal{A}$-module spanned by the L-S paths of class λ.

An important notion for the definition of L-S paths is the distance function $\mathrm{dist}(\mu,\nu)$ on Weyl group orbits, which has been proposed by M. Kashiwara to the author as a replacement for the length function on W used in [8]. The use of dist simplified many proofs given in a previous version of this article.

For $\lambda \in X$ and $\nu,\mu \in W\lambda$ write $\nu \geq \mu$ if there exist sequences of weights $\nu = \nu_0, \nu_1, \ldots, \nu_s = \mu$ and positive real roots $\beta_1, \ldots, \beta_s$ such that

$$\nu_i = s_{\beta_i}(\nu_{i-1}) \quad \text{and} \quad \langle \nu_{i-1}, \beta_i^\vee \rangle < 0 \quad \text{for all } i = 1, \ldots, s.$$

If $\nu \geq \mu$, then denote by $\mathrm{dist}(\nu,\mu)$ the maximal length s of all possible such sequences. Clearly, $\mathrm{dist}(\mu_1,\mu_2) + \mathrm{dist}(\mu_2,\mu_3) \leq \mathrm{dist}(\mu_1,\mu_3)$ if $\mu_1 \geq \mu_2 \geq \mu_3$.

LEMMA 4.1. a) *If $\mu \geq \nu$ and α is a simple root such that $\langle \mu,\alpha^\vee \rangle < 0$ but $\langle \nu,\alpha^\vee \rangle \geq 0$, then $s_\alpha(\mu) \geq \nu$ and* $\mathrm{dist}(s_\alpha(\mu),\nu) < \mathrm{dist}(\mu,\nu)$.

b) *If $\mu \geq \nu$ and α is a simple root such that $\langle \mu,\alpha^\vee \rangle \leq 0$ but $\langle \nu,\alpha^\vee \rangle > 0$, then $\mu \geq s_\alpha(\nu)$ and* $\mathrm{dist}(\mu,s_\alpha(\nu)) < \mathrm{dist}(\mu,\nu)$.

c) *If $\mu \geq \nu$ and α is a simple root such that $\langle \mu,\alpha^\vee \rangle,\langle \nu,\alpha^\vee \rangle > 0$ (respectively $\langle \mu,\alpha^\vee \rangle,\langle \nu,\alpha^\vee \rangle < 0$), then* $\mathrm{dist}(\mu,\nu) = \mathrm{dist}(s_\alpha(\mu),s_\alpha(\nu))$.

COROLLARY 1. *Suppose $\mu \geq \nu$ is such that $\mathrm{dist}(\mu,\nu) = 1$ and β is a positive real root such that $s_\beta(\mu) = \nu$. If α is a simple root such that $\langle \mu,\alpha^\vee \rangle \leq 0$ and $\langle \nu,\alpha^\vee \rangle > 0$ (or $\langle \mu,\alpha^\vee \rangle < 0$ but $\langle \nu,\alpha^\vee \rangle \geq 0$), then $\alpha = \beta$.*

Remark 4.2. Suppose λ is a dominant weight, and for $\mu,\nu \in W\lambda$ fix $\tau,\kappa \in W/W_\lambda$ such that $\tau(\lambda) = \mu$ and $\kappa(\lambda) = \nu$. Then $\mu \geq \nu$ if and only if $\tau \geq \kappa$ in the Bruhat order, and $\mathrm{dist}(\mu,\nu) = l(\tau) - l(\kappa)$.

Proof of the lemma. Let $\mu = \nu_0, \nu_1, \ldots, \nu_s = \nu$ be a sequence of weights of maximal length and let $\beta_1, \ldots, \beta_s$ be the corresponding positive real roots. Fix i minimal such that $\langle \nu_i, \alpha^\vee \rangle < 0$ but $\langle \nu_{i+1}, \alpha^\vee \rangle \geq 0$.

The sequence $s_\alpha(\mu) = s_\alpha(\nu_0), s_\alpha(\nu_1), \ldots, s_\alpha(\nu_i)$ has the property that

$$s_{s_\alpha(\beta_j)}(s_\alpha(\nu_{j-1})) = s_\alpha(\nu_j) \quad \text{and} \quad \langle s_\alpha(\nu_{j-1}), s_\alpha(\beta_j^\vee) \rangle < 0.$$

So if we prove that $s_\alpha(\nu_i) = \nu_{i+1}$, then it follows that $s_\alpha(\mu) \geq \nu$. Further, since any such sequence between $s_\alpha(\mu)$ and $s_\alpha(\nu_i) = \nu_{i+1}$ can be extended to a sequence between μ and $s_\alpha(\nu_i)$ by adding μ to the sequence of weights and α to the sequence of positive real roots ($\langle \mu, \alpha^\vee \rangle < 0$!), the maximality of the length of the sequence we started with implies that $\mathrm{dist}(s_\alpha(\mu), \nu) = \mathrm{dist}(\mu, \nu) - 1$.

It remains to prove that $s_\alpha(\nu_i) = \nu_{i+1}$. So for simplicity we may assume that $d(\mu,\nu) = 1$, β is a positive real root such that $s_\beta(\mu) = \nu$ and α is a simple root such that $\langle \mu, \alpha^\vee \rangle < 0$ and $\langle \nu, \alpha^\vee \rangle \geq 0$. Suppose that $\alpha \neq \beta$ and consider the sequence $\nu_0 := \mu$, $\nu_1 := s_\alpha(\mu)$, $\nu_2 := s_\alpha(\nu)$ and $\nu_3 := \nu$. Then $s_\alpha(\nu_0) = \nu_1$ and $\langle \nu_0, \alpha^\vee \rangle < 0$, and $s_\alpha(\nu_2) = \nu_3$ and $\langle \nu_2, \alpha^\vee \rangle \leq 0$. Since

$$s_{s_\alpha(\beta)}(\nu_1) = \nu_2, \quad \text{and} \quad \langle \nu_1, s_\alpha(\beta^\vee) \rangle = \langle \mu, \beta^\vee \rangle < 0,$$

one obtains $\operatorname{dist}(\mu,\nu) \geq 3$ (respectively $\operatorname{dist}(\mu,\nu) \geq 2$ if $\langle \nu_2, \alpha^\vee \rangle = 0$), in contradiction to the assumption $\operatorname{dist}(\mu,\nu) = 1$.

The proofs of b) and c) are similar. □

Definition. A rational path $\pi = (\underline{\nu}, \underline{a})$ of class λ is a pair of sequences where $\underline{\nu} : \nu_1 > \cdots > \nu_s$ is a linearly ordered sequence of weights in $W\lambda$, $\underline{a} : a_0 = 0 < a_1 < \cdots < a_r = 1$ is a sequence of rational numbers. We identify π with the path

$$\pi(t) := \sum_{i=1}^{j-1} (a_i - a_{i-1})\nu_i + (t - a_{j-1})\nu_j \quad \text{for } a_{j-1} \leq t \leq a_j.$$

To ensure that $\pi(1)$ is an integral weight, we introduce now the a-chain (see [7], [8]). Let $0 < a < 1$ be a rational number and $\mu, \nu \in W\lambda$:

Definition. An *a-chain* for (μ,ν) is a sequence $\mu = \lambda_0 > \lambda_1 > \cdots > \lambda_s = \nu$ of weights in $W\lambda$ such that either $s = 0$ and $\mu = \lambda_0 = \nu$, or $\lambda_i = s_{\beta_i}(\lambda_{i-1})$ for some positive real roots $\beta_1, \ldots, \beta_s$, and $\operatorname{dist}(\lambda_{i-1}, \lambda_i) = 1$ and $a\langle \lambda_{i-1}, \beta_i^\vee \rangle \in \mathbb{Z}$ for all $i = 1, \ldots, s$.

The "integrality" condition implies that $a(\mu - \nu) = \sum_{i=1}^{s} a(\lambda_{i-1} - \lambda_i) = \sum_{i=0}^{s} a\langle \lambda_{i-1}, \beta_i^\vee \rangle \beta_i$ is a sum of positive roots.

LEMMA 4.3. *Let $\mu = \lambda_0 > \lambda_1 > \cdots > \lambda_s = \nu$ be an a-chain for (μ,ν) and fix a simple root α.*

a) *If $\langle \mu,\alpha^\vee \rangle < 0$ and $\langle \lambda_i,\alpha^\vee \rangle \geq 0$ for some i, then there exists an a-chain for $(s_\alpha(\mu),\nu)$.*

b) *If $\langle \nu,\alpha^\vee \rangle > 0$ and $\langle \lambda_i,\alpha^\vee \rangle \leq 0$ for some i, then there exists an a-chain for $(\mu,s_\alpha(\nu))$.*

Proof. Assume first that $\langle \mu, \alpha^\vee \rangle < 0$, and let i be minimal with the property that $\langle \lambda_{i+1}, \alpha^\vee \rangle \geq 0$. Further, let $\beta_1, \ldots, \beta_s$ be the positive real roots corresponding to the a-chain. Since $\langle \lambda_j, \beta_j^\vee \rangle = \langle s_\alpha(\lambda_j), s_\alpha(\beta_j^\vee) \rangle$, one sees as in the proof of Lemma 4.1 that $s_\alpha(\mu) = s_\alpha(\lambda_0) > \cdots > s_\alpha(\lambda_i) = \lambda_{i+1} > \cdots > \lambda_s = \nu$ is an a-chain for $(s_\alpha\mu, \nu)$. The proof of b) is similar. □

Definition. A rational path $\pi = (\underline{\nu}; \underline{a})$ of class $\lambda \in X$ is called an L-S *path of class* λ if for all $i = 1, \ldots, s-1$ there exists an a_i-chain for (ν_i, ν_{i+1}).

Remark 4.4. a) If $\pi = (\underline{\nu};\underline{a})$ is an L-S path of class λ, then it is an L-S path of class $w(\lambda)$ for all $w \in W$.

b) See [8]: If λ is a dominant weight, then $\pi = (\underline{\nu};\underline{a})$ is an L-S path of class λ if and only if $(\tau_1, \dots, \tau_s; a_0, \dots, a_s)$ is an L-S path of shape λ, where the $\tau_i \in W/W_\lambda$ are such that $\tau_i(\lambda) = \nu_i$.

We say that a function h attains on $[0,1]$ a local minimum at $t = t_0$ if either h is constant, or if there exists an $\varepsilon > 0$ such that $h(t) \geq h(t_0)$ for $|t - t_0| < \varepsilon$ and $h(t) > h(t_0)$ for either $t_0 < t < t_0 + \varepsilon$ or $t_0 - \varepsilon < t < t_0$.

LEMMA 4.5. a) *If π is an* L-S *path of class λ, then $\pi \in \Pi_{\mathrm{int}}$.*

b) *If $\pi = (\underline{\nu};\underline{a})$ is an* L-S *path, then $\pi' = (\nu_i, \dots, \nu_j; 0, a_i \dots, a_{j-1}, 1)$ is an* L-S *path for all $1 \leq i \leq j \leq s$.*

c) *If π is an* L-S *path and $a_{i-1} \leq x \leq a_i$ is such that $\langle \pi(x), \alpha^\vee \rangle \in \mathbb{Z}$ for some simple root α, then $x\langle \nu_i, \alpha^\vee \rangle \in \mathbb{Z}$.*

d) *Let $\pi = (\underline{\nu};\underline{a})$ be an* L-S *path and fix a simple root α. If the function $h_\alpha(t) := \langle \pi(t), \alpha^\vee \rangle$ attains at $t = t_0$ a local minimum, then $h_\alpha(t_0) \in \mathbb{Z}$.*

In particular, the L-S *paths have the integrality property.*

Proof. The chain condition implies $a_j(\nu_j - \nu_{j+1})$ is a sum of roots, so

$$\pi(1) = \sum_{j=1}^{s} (a_j - a_{j-1})\nu_j = \nu_s + \sum_{j=1}^{s-1} a_j(\nu_j - \nu_{j+1}) \in X,$$

proving a). Similarly, one has for c): $\pi(x) = x\nu_i + \sum_{j=1}^{i-1} a_j(\nu_j - \nu_{j+1})$, which implies that $\langle \pi(x), \alpha^\vee \rangle \in \mathbb{Z}$ if and only if $x\langle \nu_i, \alpha^\vee \rangle \in \mathbb{Z}$. The proof of b) is obvious; it remains to prove d).

We may assume $t_0 = a_i$ for some i. To prove that $h_\alpha(a_i)$ is an integer, by b) one can assume that $i = s-1$. So $h_\alpha(a_{s-1}) = \langle \pi(1), \alpha^\vee \rangle - (1 - a_{s-1})\langle \nu_s, \alpha^\vee \rangle$. Hence it is sufficient to prove that $(1 - a_{s-1})\langle \nu_s, \alpha^\vee \rangle \in \mathbb{Z}$. This is obvious if $\langle \nu_s, \alpha^\vee \rangle = 0$. Since $h_\alpha(t)$ attains at a_{s-1} a local minimum, one has otherwise $\langle \nu_s, \alpha^\vee \rangle > 0$ and $\langle \nu_{s-1}, \alpha^\vee \rangle \leq 0$.

By Lemma 4.3 this implies that $\pi' = (\dots, \nu_{s-1}, s_\alpha(\nu_s); \dots, a_{s-1}, a_s)$ is an L-S path. Now by the chain condition one knows that $\nu_s - \pi(1)$ as well as $s_\alpha(\nu_s) - \pi'(1)$ are elements of the root lattice; so, also, $\pi(1) - \pi'(1)$ is in the root lattice. But $\pi(1) - \pi'(1) = (1 - a_{s-1})\langle \nu_s, \alpha^\vee \rangle \alpha$ is in the root lattice only if $(1 - a_{s-1})\langle \nu_s, \alpha^\vee \rangle \in \mathbb{Z}$. □

Remark 4.6. The same arguments prove the following: For an L-S path $\pi = (\underline{\nu};\underline{a})$ let $\nu_i = \mu_0 > \mu_1 > \cdots > \mu_r = \nu_{i+1}$ be an a_i-chain for (ν_i, ν_{i+1}). If $\langle \nu_i, \alpha^\vee \rangle < 0$ for a simple root α and $\langle \mu_j, \alpha^\vee \rangle \geq 0$ for some j, or $\langle \nu_{i+1}, \alpha^\vee \rangle > 0$ and $\langle \mu_j, \alpha^\vee \rangle \leq 0$ for some j, then $h_\alpha(a_i) = \langle \pi(a_i), \alpha^\vee \rangle \in \mathbb{Z}$.

PROPOSITION 4.7. *Let $\eta = (\underline{\nu};\underline{a})$ be an* L-S *path and assume that the function $h_\alpha(t) := \langle \eta(t), \alpha^\vee \rangle$ attains at $t = a_i$ a local minimum.*

a) *Suppose there exists a* $y > a_i$ *such that* $h_\alpha(y) = h_\alpha(a_i) + 1$ *and* $h_\alpha(t) \geq h_\alpha(a_i)$ *for all* $a_i \leq t \leq y$. *Then there exist* $a_i \leq a_j < x \leq y$ *such that*

$$h_\alpha(a_i) = h_\alpha(a_j) < h_\alpha(t) < h_\alpha(x) = h_\alpha(y)$$

for $a_j < t < x$, *and the function* h_α *is strictly increasing on* $[a_j,x]$. *Further,* η' *is an* L-S *path, where:*

$$\eta' := (\nu_1, \ldots, \nu_j, s_\alpha(\nu_{j+1}), \ldots, s_\alpha(\nu_l), \nu_l, \ldots, \nu_r; a_0, \ldots, a_{l-1}, x, a_l, \ldots, a_r).$$

b) *Suppose there exists an* $x < a_i$ *such that* $h_\alpha(a_i) + 1 = h_\alpha(x)$ *and* $h_\alpha(t) \geq h_\alpha(a_i)$ *for all* $x \leq t \leq a_i$. *Then there exist* $x \leq y < a_k \leq a_i$ *such that*

$$h_\alpha(x) = h_\alpha(y) > h_\alpha(t) > h_\alpha(a_k) = h_\alpha(a_i)$$

for $y < t < a_k$ *and the function* h_α *is strictly decreasing on* $[y,a_k]$. *Further,* η' *is an* L-S *path, where:*

$$\eta' := (\nu_1, \ldots, \nu_l, s_\alpha(\nu_l), \ldots, s_\alpha(\nu_k), \nu_{k+1}, \ldots, \nu_r; a_0, \ldots, a_{l-1}, y, a_l, \ldots, a_r).$$

Remark 4.8. If $s_\alpha(\nu_{j+1}) = \nu_j$ or $x = a_l$ etc., then the corresponding entries are not listed twice.

COROLLARY 2. a) *The* $\mathbb{Z}$*-module* $L_\lambda \subset \mathbb{Z}\Pi_{\text{int}}$ *generated by all* L-S *paths of class* λ *is an* $\mathcal{A}$*-submodule.*

b) *On the set of* L-S *paths the root operators defined in Section* 1 *coincide with the root operators defined in* [8].

COROLLARY 3. *If* λ *is a dominant weight, then* π_λ *is the only* L-S *path* π *of class* λ *such that* $e_\alpha\pi = 0$ *for all simple roots. Further, any* L-S *path* π *of class* λ *is of the form* $\pi = f_{\alpha_1} \cdots f_{\alpha_r}\pi_\lambda$ *for some simple roots* $\alpha_1, \ldots, \alpha_r$.

Remark 4.9. If λ is not in the Tits cone, then $\mathcal{A}\pi_\lambda$ can be a proper submodule of L_λ. For example, in the rank two case, suppose that λ is not in the Tits cone. Consider the L-S paths $\pi = (\underline{\nu}, \underline{a})$ of class λ such that for all i there exists a simple root such that $\nu_{i-1} = s_\alpha(\nu_i)$. It is easy to see that these paths span a proper $\mathcal{A}$-stable submodule of L_λ.

Proof of the corollaries. Assume that h_α attains at $t_0 = a_i$ its minimum for the last time, and $t_1 > a_i$ is the first time such that h_α attains the value $h_\alpha(a_i) + 1$. Since by the integrality property one has $h_\alpha(t) \geq h_\alpha(a_i) + 1$ for $t \geq t_1$, one sees that η' in a) above is $f_\alpha\eta$. Similarly, if h_α attains at $t_1 = a_i$ its minimum for the first time and $t_0 < a_i$ is the last time such that h_α attains the value $h_\alpha(a_i) + 1$, then η' in b) above is equal to $e_\alpha\eta$.

Further, since h_α is always strictly increasing on $[t_0, t_1]$ (respectively decreasing), on the set of L-S paths the root operators defined in Section 1 coincide with the root operators defined in [8].

Suppose now λ is a dominant weight. If $\pi = (\underline{\nu}, \underline{a})$ is an L-S path of class λ such that $\nu_1 \neq \lambda$, then there exists a simple root α such that $\langle \nu_1, \alpha^\vee \rangle < 0$. By the integrality property and Lemma 2.1 this implies $e_\alpha \pi \neq 0$. So there exist some simple roots such that $\pi' = (\underline{\nu}', \underline{a}') = e_{\alpha_1} \dots e_{\alpha_r} \pi$ is such that $\nu_1' = \lambda$, and hence $\pi' = \pi_\lambda$. □

Proof of the proposition. The proofs of a) and b) are similar, so only the proof of a) is given. Let $a_i \leq a_j < y$ be maximal such that $h_\alpha(a_i) = h_\alpha(a_j)$, and let $a_j < x \leq y$ be minimal such that $h_\alpha(x) = h_\alpha(y) = h_\alpha(a_i) + 1$. By Lemma 4.5 it follows that the function h_α is strictly increasing on $[a_j, x]$.

It remains to prove that η' is an L-S path of class λ. Now h_α attains at $t = a_j$ a local minimum, so $h_\alpha(a_j) \in \mathbb{Z}$, and by the choice of j one has $\langle \nu_j, \alpha^\vee \rangle \leq 0$ and $\langle \nu_{j+1}, \alpha^\vee \rangle > 0$. So by Lemma 4.3 there exists an a_j-chain for $(\nu_j, s_\alpha(\nu_{j+1}))$. Further, since $h_\alpha(t) \notin \mathbb{Z}$ for $a_j < t < x$, it follows by Remark 4.6 that for all $k = j+1, \dots, l-1$: If $\nu_k = \mu_0 > \dots > \mu_r = \nu_{k+1}$ is an a_k-chain for (ν_k, ν_{k+1}), then $s_\alpha(\nu_k) > \dots > s_\alpha(\mu_r)$ is an a_k-chain for $(s_\alpha(\nu_k), s_\alpha(\nu_{k+1}))$. Eventually, by Lemma 4.5 c), $s_\alpha(\nu_l) > \nu_l$ is an x-chain for $(s_\alpha(\nu_l), \nu_l)$, and hence η' is an L-S path of class λ. □

5. Gluing L-S paths

The next step towards a proof of the isomorphism theorem will be to investigate modules of the form $\mathcal{A}(\pi_\lambda * \pi_\mu)$, where λ, μ are rational weights and $\lambda + \mu$ is an integral weight.

For a path $\pi \in \Pi$ and $s, s' \in [0,1]$, $s \leq s'$, let π^s, $\pi_s^{s'}$ and $\pi_{s'}$ be the paths

$$\pi^s : [0,s] \to X_\mathbb{Q},\ t \mapsto \pi(t), \quad \pi_s^{s'} : [s,s'] \to X_\mathbb{Q},\ t \mapsto \pi(t),$$

and $\pi_{s'} : [s',1] \to X_\mathbb{Q}$, $t \mapsto \pi(t)$. If π, η, σ are paths, then let $\pi^s \circ \eta_s^{s'} \circ \sigma_{s'}$ be the path obtained by "gluing" the paths π^s, $\eta_s^{s'}$ and $\sigma_{s'}$, i.e.:

$$\pi^s \circ \eta_s^{s'} \circ \sigma_{s'}(t) := \begin{cases} \pi(t), & \text{for } t \leq s; \\ \eta(t) + [\pi(s) - \eta(s)], & \text{for } s \leq t \leq s'; \\ \sigma(t) + [\pi(s) - \eta(s) + \eta(s') - \sigma(s')], & \text{for } s' \leq t; \end{cases}$$

For $\lambda, \mu \in X$ let π_λ and π_μ be the paths $t \mapsto t\lambda$ respectively $t \mapsto t\mu$. Denote by θ the trivial path $t \mapsto 0$ for all $t \in [0,1]$. To simplify the notation we write also θ for $\theta_s^{s'}$. Next we investigate the $\mathcal{A}$-module $\mathcal{A}\pi$ generated by $\pi = \pi_\lambda^s \circ \theta \circ \pi_{\mu,s'}$.

Remark 5.1. Let λ, μ be rational weights such that $\nu = \lambda + \nu$ is an integral weight. The path $\pi_\lambda * \pi_\mu$ can also be described in the form above: Fix $n \geq 2$ such that $n\lambda, n\mu \in X$ are integral weights. Then:

$$\pi_\lambda * \pi_\mu = \pi_{n\lambda}^{\frac{1}{n}} \circ \theta \circ \pi_{n\mu, 1-\frac{1}{n}}$$

up to reparametrization. The advantage of the somewhat heavy looking notion on the right side is that $\pi_{n\lambda}$ and $\pi_{n\mu}$ are L-S paths.

We introduce now the "gluing pair" which can be viewed as a variation of the defining chain for Young tableaux introduced by Lakshmibai, Musili and Seshadri (see for example [7]). For two rational weights ν, μ we write

$$\nu \triangleright \mu \quad \text{if for all positive real roots } \beta: \quad \langle \nu, \beta^\vee \rangle < 0 \Rightarrow \langle \mu, \beta^\vee \rangle \leq 0.$$

Note that if ν is a dominant rational weight, then obviously $\nu \triangleright \mu$ for any μ. The notion $\nu \triangleright \mu$ is due Kashiwara [4].

LEMMA 5.2. a) *If $\nu \triangleright \mu$ and α is a simple root such that $\langle \nu,\alpha^\vee \rangle < 0$, then $s_\alpha(\nu) \triangleright s_\alpha(\mu)$.*

b) *If $\nu \triangleright \mu$ and α is a simple root such that $\langle \nu,\alpha^\vee \rangle > 0$ and $\langle \mu,\alpha^\vee \rangle \geq 0$, then $s_\alpha(\nu) \triangleright s_\alpha(\mu)$.*

Proof. For any positive real root $\beta \neq \alpha$ we have:

$$\begin{aligned} \langle s_\alpha(\nu), \beta^\vee \rangle &< 0 \Leftrightarrow \langle \nu, s_\alpha(\beta^\vee) \rangle \\ &< 0 \Rightarrow \langle \mu, s_\alpha(\beta^\vee) \rangle \leq 0 \Leftrightarrow \langle s_\alpha(\mu), \beta^\vee \rangle \leq 0. \end{aligned}$$ □

5.3. Let $\sigma = (\lambda_1, \ldots, \lambda_r; a_0, \ldots, a_r)$ be an L-S path of class λ and let $\delta = (\mu_1, \ldots, \mu_t; b_0, b_1, \ldots)$ be an L-S path of class μ. Suppose now that $0 < s \leq s' < 1$ are such that $a_{r-1} < s$ and $s' < b_1$, and $\eta = \sigma^s \circ \theta \circ \delta_{s'} \in \Pi_{\text{int}}$.

Definition. A pair (λ_{r+1}, μ_0), $\lambda_{r+1} \in W\lambda$ and $\mu_0 \in W\mu$, of weights is called a *gluing pair* for η if $\lambda_{r+1} \triangleright \mu_0$, and if there exists an s-chain for $(\lambda_r, \lambda_{r+1})$ and an s'-chain for (μ_0, μ_1).

Remark 5.4. If $\lambda_r \neq \lambda_{r+1}$, then the condition on λ_{r+1} implies that $\sigma' = (\ldots, \lambda_r, \lambda_{r+1}; \ldots, a_{r-1}, s, a_r)$ is an L-S path. Similarly, if $\mu_0 \neq \mu_1$, then the condition on μ_0 implies that $\delta' = (\mu_0, \mu_1, \ldots; b_0, s', b_1, \ldots)$ is an L-S path.

Example. Let λ, μ be rational weights such that $\nu = \lambda + \mu$ is an integral weight. If $\lambda \triangleright \mu$ (for example if λ is dominant!), then by Remark 5.1 one sees that $\pi_\lambda * \pi_\mu$ is as in 5.3 with gluing pair $(n\lambda, n\mu)$.

LEMMA 5.5. *Let $\eta \in \Pi_{\text{int}}$ be as in* 5.3. *If there exists a gluing pair for η, then for all simple roots α the local minima of the function $h_\alpha(t)\colon= \langle \eta(t),\alpha^\vee \rangle$ are integers.*

Proof. If the minimum is attained at $t = t_0$ and $t_0 < s$ or $t_0 > s'$, then the claim follows from the corresponding property for L-S paths (Lemma 4.5) since $\eta(1) \in X$. Suppose now h_α attains a local minimum at $t_0 = s$ (or $t_0 = s'$; recall that h_α is constant on $[s, s']$), and this minimum is only attained on $[s, s']$. We may hence assume that $\langle \lambda_r, \alpha^\vee \rangle < 0$ and $\langle \mu_1, \alpha^\vee \rangle > 0$.

If $\langle\lambda_{r+1},\alpha^\vee\rangle \geq 0$, then $h_\alpha(s) \in \mathbb{Z}$ since $\sigma' = (\ldots,\tau_r,\tau_{r+1};\ldots,a_{r-1},s,1)$ is an L-S path by assumption, and $h_\alpha(s) = \langle\eta(s),\alpha^\vee\rangle = \langle\sigma'(s),\alpha^\vee\rangle \in \mathbb{Z}$ by Lemma 4.5. So we may assume that $\langle\lambda_{r+1},\alpha^\vee\rangle < 0$ and hence $\langle\mu_0,\alpha^\vee\rangle \leq 0$. Since $\delta' = (\mu_0,\mu_1,\ldots;b_0,s',b_1,\ldots)$ is an L-S path and $\langle\mu_1,\alpha^\vee\rangle > 0$, it follows by Lemma 4.5 that $\langle\delta'(s'),\alpha^\vee\rangle \in \mathbb{Z}$. Since $\eta(1) - \delta'(1) = \eta(s') - \delta'(s')$ is an integral weight, it follows that $h_\alpha(s') = h_\alpha(s) \in \mathbb{Z}$. □

PROPOSITION 5.6. *Let σ be an* L-S *path of class λ and let δ be an* L-S *path of class μ, and suppose $\eta = \sigma^s \circ \theta \circ \delta_{s'} \in \Pi_{\text{int}}$ is as in* 5.3 *with gluing pair* (λ_{r+1},μ_0). *Then the $\mathcal{A}$-module $\mathcal{A}\eta$ has the integrality property.*

Further, for a path $\eta' \in \mathcal{A}\eta$ there exist an L-S *path σ' of class λ and an* L-S *path δ' of class μ such that $\eta' = \sigma^s \circ \theta \circ \delta_{s'}$ is as in* 5.3. *Also there exists a $w \in W$ such that $(w(\lambda_{r+1}),w(\mu_0))$ is a gluing pair for η'.*

Proof. By Lemma 5.5, the first part of the proposition follows from the second part. To prove the second part, it is sufficient to consider the case $\eta' = f_\alpha\eta$ or $\eta' = e_\alpha\eta$. Fix a simple root α, and for a root operator, let $t_0 < t_1$ be as in Section 1. If $t_0 > s'$ or $t_1 < s$, then it follows from Proposition 4.7 that one can write $f_\alpha\eta$, respectively $e_\alpha\eta$, again as $\eta' = \sigma'^s \circ \theta \circ \delta'_{s'}$ as in 5.3, and one can take (λ_{r+1},μ_0) as a gluing pair.

For f_α assume that $t_1 = s$, so that $\langle\lambda_r,\alpha^\vee\rangle > 0$. Set $n := \langle\sigma(1)-\sigma(t_0),\alpha^\vee\rangle$; then $f_\alpha\eta = (f_\alpha^n\sigma)^s \circ \theta \circ \delta_{s'}$. And since $h_\alpha(t_1) = \langle\sigma(t_1),\alpha^\vee\rangle \in \mathbb{Z}$, there exists an s-chain also for $(s_\alpha(\lambda_r),\lambda_{r+1})$ (Lemma 4.5 c)), so (λ_{r+1},μ_0) is a gluing pair for $f_\alpha\eta$. The same arguments prove for e_α that if $t_0 = s'$ (and hence $\langle\mu_1,\alpha^\vee\rangle < 0$), then $e_\alpha\eta = \sigma^s \circ \theta \circ (e_\alpha^m\delta)_{s'}$ with gluing pair (λ_{r+1},μ_0), where $m = -\langle\delta(t_1),\alpha^\vee\rangle$.

Similarly, if we assume for f_α that $t_0 = s'$ and $\langle\mu_0,\alpha^\vee\rangle \leq 0$, then $f_\alpha\eta = \sigma^s \circ \theta \circ (f_\alpha^m\delta)_{s'}$ with gluing pair (λ_{r+1},μ_0), where $m = \langle\delta(t_1),\alpha^\vee\rangle$. And if $t_1 = s$ and $\langle\lambda_{r+1},\alpha^\vee\rangle \geq 0$, then $e_\alpha\eta = (e_\alpha^m\sigma)^s \circ \theta \circ \delta_{s'}$ with gluing pair (λ_{r+1},μ_0), where $m = \langle\sigma(t_0) - \sigma(1),\alpha^\vee\rangle$.

For f_α assume now that $t_0 = s'$ and $\langle\mu_0,\alpha^\vee\rangle > 0$. Note that this implies that $\langle\lambda_{r+1},\alpha\rangle \geq 0$. Further, since $t_0 = s'$, one knows that $\langle\lambda_r,\alpha\rangle \leq 0$, so in any case there exists an s-chain also for $(\lambda_r, s_\alpha(\lambda_{r+1}))$ by Lemma 4.3. Also, $h_\alpha(s') \in \mathbb{Z}$ implies $\langle\delta(s'),\alpha\rangle \in \mathbb{Z}$, and hence there exists also an s'-chain for $(s_\alpha(\mu_0), s_\alpha(\mu_1))$. Eventually, by Lemma 5.2 one knows that $s_\alpha(\lambda_{r+1}) \triangleright s_\alpha(\mu_0)$. So if one sets $n := \langle\delta(s'),\alpha\rangle + 1$, then $f_\alpha\eta = \sigma^s \circ \theta \circ (f_\alpha^n\delta)_{s'}$ with gluing pair $(s_\alpha(\lambda_{r+1}), s_\alpha(\mu_0))$.

Similarly, if $t_1 = s$ and $\langle\lambda_{r+1},\alpha^\vee\rangle < 0$, then $e_\alpha\eta = (e_\alpha^m\sigma)^s \circ \theta \circ \delta_{s'}$ with gluing pair $(s_\alpha(\lambda_{r+1}), s_\alpha(\mu_0))$, where $m = \langle\sigma(t_0) - \sigma(1),\alpha^\vee\rangle$.

Suppose now $t_0 < s \leq s' < t_1$. In the following we consider only the operator f_α since the proof for e_α is similar. By Lemma 5.5 (and the fact $h_\alpha(s) = h_\alpha(s') \notin \mathbb{Z}$) one has $\langle\lambda_r,\alpha^\vee\rangle > 0$ and $\langle\mu_1,\alpha^\vee\rangle > 0$. Set $n = \langle\sigma(1) -$

$\sigma(t_0), \alpha^\vee\rangle$ and $m = \langle\delta(t_1), \alpha^\vee\rangle$ (these are integers!), then $f_\alpha\eta = (f_\alpha^n\sigma)^s \circ \theta \circ (f_\alpha^m\delta)_{s'}$.

If $\lambda_r \neq \lambda_{r+1}$, by Remark 5.4, $\sigma' = (\ldots, \lambda_r, \lambda_{r+1}; \ldots, s, 1)$ is an L-S path of class λ. Since $\langle\sigma'(s), \alpha^\vee\rangle = \langle\eta(s), \alpha^\vee\rangle \notin \mathbb{Z}$, it follows by Lemma 4.5 that $\langle\lambda_{r+1}, \alpha^\vee\rangle > 0$ and, as in the proof of Proposition 4.7, there exists an s-chain for $(s_\alpha(\lambda_r), s_\alpha(\lambda_{r+1}))$. If $\lambda = \lambda_{r+1}$, such a chain trivially exists.

Note that $\langle\mu_0, \alpha^\vee\rangle > 0$; otherwise $\delta' = (\mu_0, \mu_1 \ldots; b_0, s', b_1, \ldots)$ would be an L-S path with the property: $\langle\delta'(s'), \alpha^\vee\rangle \in \mathbb{Z}$. Since $\delta'(s')$ and $\eta(s')$ differ only by an integral weight, this would contradict the assumption $\langle\eta(s'), \alpha^\vee\rangle = \langle\eta(s), \alpha^\vee\rangle \notin \mathbb{Z}$. Now the same arguments as for λ_{r+1} prove that there exists an s'-chain for $(s_\alpha(\mu_0), s_\alpha(\mu_1))$. Since $s_\alpha(\lambda_{r+1}) \triangleright s_\alpha(\mu_0)$ by Lemma 5.2, this proves that $(s_\alpha(\lambda_{r+1}), s_\alpha(\mu_0))$ is a gluing pair for $f_\alpha\eta$. □

PROPOSITION 5.7. *Let λ,μ be rational weights such that λ is dominant and $\lambda + \mu = \nu$ is an integral dominant weight, and set $\pi = \pi_\lambda * \pi_\mu$. The module $\mathcal{A}\pi$ has the integrality property, and π is the only path in $\mathcal{A}\pi$ such that $\pi(1) = \nu$ and $e_\alpha\pi = 0$ for all simple roots.*

Proof. Fix $n \geq 2$ and s, s' as in Remark 5.1 and Example 5.4 such that $\pi = \pi_{n\lambda}^s \circ \theta \circ \pi_{n\mu,s'}$. Since $(n\lambda, n\mu)$ is a gluing pair for π, the first claim follows from Proposition 5.6. Suppose now $\pi' = \pi_1^s \circ \theta \circ \pi_{2,s'} \in \mathcal{A}\pi$ is such that $\pi'(1) = \nu$ and $e_\alpha\pi' = 0$ for all simple roots. Then $e_\alpha\pi_1 = 0$ for all simple roots, so $\pi_1 = \pi_{n\lambda}$. Now by Proposition 5.6 one can choose $(n\lambda, w(n\mu))$ as a gluing pair for π' for some $w \in W_\lambda$.

Since $\pi = \pi_\lambda * \pi_\mu$ is in $\mathcal{P}^+$, one knows that $\langle\mu, \alpha^\vee\rangle \geq 0$ for α simple such that $\langle\lambda, \alpha^\vee\rangle = 0$. In particular, if $\langle w(\mu), \alpha^\vee\rangle < 0$, then $s_\alpha w < w$. But if $\langle w(\mu), \alpha^\vee\rangle < 0$ and $\pi_2 = (\underline{\nu}', \underline{a}')$, then $\langle\nu_1', \alpha^\vee\rangle \geq 0$ since $\pi' \in \mathcal{P}^+$. Hence by Lemma 4.3, there exists an a_1'-chain for $(s_\alpha w(n\mu), n\mu)$. Since $n\lambda$ is dominant we have $n\lambda \triangleright s_\alpha w(n\mu)$, so that $(n\lambda, s_\alpha w(n\mu))$ is also a gluing pair for π'. Thus in the following we may take $(n\lambda, n\mu)$ as a gluing pair for π'. But since $\mu \geq \nu_1'$, one gets $\pi'(1) = \lambda + (\pi_2(1) - \pi_2(s')) = \lambda + \mu = \nu$ if and only if $\pi_2 = \pi_{n\mu}$, and hence $\pi = \pi'$. □

6. Linking

Let $\mathfrak{c}$ be the constant introduced in section 3. To use the "continuity" of the root operators, we introduce now the notion of *linking*. Two paths $\eta, \eta' \in \Pi_{\text{int}}$ such that $\eta(1) = \eta'(1)$ are called *linked of level L* ($\eta \overset{L}{\sim} \eta'$), if there exist paths $\eta = \pi_0, \ldots, \pi_t = \eta'$ such that: $\eta(1) = \pi_i(1)$ for all $0 \leq i \leq t$, the modules $\mathcal{A}\pi_i$ have the integrality property for all $0 \leq i \leq t$, and there exist parametrizations of the paths such that $d(\pi_i, \pi_{i+1}) < 3^{-L}\mathfrak{c}^{-L}$ for all $0 \leq i \leq t$. Such a sequence of paths is called a linking chain.

LEMMA 6.1. *If $\eta \overset{L}{\sim} \eta'$ and $n_1+n_2+\cdots \leq L$, then $f_{\alpha_1}^{n_1}e_{\alpha_2}^{n_2}\ldots\eta = 0$ if and only if $f_{\alpha_1}^{n_1}e_{\alpha_2}^{n_2}\ldots\eta' = 0$.*

Proof. By the definition of linking chain it is sufficient to prove the lemma for η, η' such that $d(\eta, \eta') \leq 3^{-L}\mathfrak{c}^{-L}$. But then the lemma follows immediately from Proposition 3.1. □

Example. Let λ, μ be rational weights such that $\nu = \lambda + \mu$ is an integral weight, and assume that $\lambda \triangleright \mu$ (for example if λ is dominant). For $x \in [0,1]$, consider the paths $\pi_x := \pi_{x\lambda} * \pi_{\mu+(1-x)\lambda}$. Then $\pi_0 = \pi_\nu$ is an L-S path of class ν, and $\pi_1 = \pi_\lambda * \pi_\mu$. If $x > 0$, then for appropriate choices of n, s, s' one gets (modulo reparametrization, see Example 5.4):

$$\pi_x = \pi_{nx\lambda}^{s} \circ \theta \circ \pi_{s',n(\mu+(1-x)\lambda)},$$

where $n \geq 2$ is chosen such that $nx\lambda, n(\mu+(1-x)\lambda)$ are integral weights. Since $\lambda \triangleright \mu$ implies $x\lambda \triangleright \mu + (1-x)\lambda$, $(nx\lambda, n(\mu + (1-x)\lambda))$ is a gluing pair for π_x. In particular, $\mathcal{A}\pi_x$ is integral for all $x \in [0,1]$. Further, since $\pi_x(t) - \pi_y(t) = 2t(x-y)\lambda$ for $t \leq 1/2$ and $\pi_x(t) - \pi_y(t) = 2(1-t)(x-y)\lambda$ for $t \geq 1/2$, one can choose, for any given L, $x_0 = 0, \ldots, x_N = 1$ such that $d(\pi_{x_i}, \pi_{x_{i+1}}) < 3^{-L}\mathfrak{c}^{-L}$ for $i = 0, \ldots, N$. Hence: $\pi_\nu \overset{L}{\sim} \pi_\lambda * \pi_\mu$ for arbitrary L.

As a first application one can extend the result of Proposition 5.7:

PROPOSITION 6.2. *Let λ,μ be rational weights such that λ is dominant and $\nu = \lambda + \mu$ is an integral dominant weight. Then $\pi = \pi_\lambda * \pi_\mu$ is the only path in $\mathcal{A}\pi$ ending in $\nu = \pi(1)$.*

Proof. By the example above one knows that $\pi_\nu \overset{L}{\sim} \pi_\lambda * \pi_\mu$ for arbitrary L. Let now $D = f_{\alpha_1}^{n_1}e_{\alpha_2}^{n_2}\ldots f_{\alpha_t}^{n_t}$ be a monomial in the root operators and suppose that $D\pi(1) = \nu$. By Lemma 6.1 it follows that $D\pi_\nu \neq 0$, and since $D\pi_\nu(1) = \nu$, one has in fact $D\pi_\nu = \pi_\nu$ by Corollary 3. Since $e_\alpha \pi_\nu = 0$ for all simple roots, it follows in turn from Lemma 6.1 that $e_\alpha D\pi = 0$ for all simple roots, and now Proposition 5.7 implies that $D\pi = \pi$. □

THEOREM 6.3. *Let λ,μ be rational weights such that λ is dominant and $\nu = \lambda + \mu$ is an integral dominant weight. The map $\pi_\lambda * \pi_\mu \mapsto \pi_\nu$ extends to an isomorphism $\Phi: \mathcal{A}(\pi_\lambda * \pi_\mu) \xrightarrow{\sim} \mathcal{A}\pi_\nu$ of $\mathcal{A}$-modules.*

Proof. Let $D = f_{\alpha_1}^{n_1}e_{\alpha_2}^{n_2}\ldots f_{\alpha_r}^{n_r}$ be a monomial of root operators. By Lemma 6.1 and the example above, one knows that $D\pi_\nu = 0$ if and only if $D(\pi_\lambda * \pi_\mu) = 0$. To prove that the map $\Phi : a(\pi_\lambda * \pi_\mu) \mapsto a(\pi_\nu)$ is well defined, one has to show that if $D' = f_{\gamma_1}^{m_1}e_{\gamma_2}^{m_2}\ldots f_{\gamma_s}^{m_s}$ and $D\pi_\nu, D'\pi_\nu \neq 0$, then

$$D\pi_\nu = D'\pi_\nu \Leftrightarrow D(\pi_\lambda * \pi_\gamma) = D'(\pi_\lambda * \pi_\gamma). \tag{6.1}$$

Set $D'' = e_{\alpha_r}^{n_r} \dots f_{\alpha_2}^{n_2} e_{\alpha_1}^{n_1} D'$; then 6.1 is equivalent to

$$\pi_\nu = D''\pi_\nu \Leftrightarrow \pi_\lambda * \pi_\gamma = D''(\pi_\lambda * \pi_\gamma). \tag{6.2}$$

If one of the equalities in 6.2 holds, then $D''\pi_\nu(1) = D''(\pi_\lambda * \pi_\gamma)(1) = \nu$, so (6.2) follows from Proposition 6.2. Both modules have the paths as a basis, and the morphism maps paths to paths. So $\Phi(a_1\pi_1 + \cdots + a_r\pi_r) = 0$ only if some of the paths with $a_i \neq 0$ have the same image. But this is excluded by (6.1), so Φ is injective. Since Φ is clearly surjective, this proves the theorem. □

7. The Isomorphism Theorem for $\mathcal{P}^+$

For a path $\pi \in \mathcal{P}^+$ let $M_\pi := \mathcal{A}\pi$ be the module generated by π and denote by B_π the basis of M_π consisting of the set of paths contained in M_π. For $\lambda := \pi(1)$ let π_λ be the path $t \mapsto t\lambda$, set $M_\lambda := \mathcal{A}\pi_\lambda$ and denote by B_λ the basis of M_λ of L-S paths.

THEOREM 7.1. *The map $\pi_\lambda \mapsto \pi$ extends to an isomorphism $M_\lambda \to M_\pi$ of $\mathcal{A}$-modules.*

COROLLARY 1. a) (*Integrality property*) *For any $\eta \in B_\pi$ and any simple root α the minimum attained by the function h_α is an integer.*

b) *π is the only path in B_π such that $e_\alpha\pi = 0$ for all simple roots.*

c) *Every element $\eta \in B_\pi$ is of the form $\eta = f_{\alpha_1} f_{\alpha_2} \dots f_{\alpha_s}\pi$.*

Proof. Parts b) and c) follow from the isomorphism theorem and the corresponding properties for the set of L-S paths B_λ (Corollary 3). To prove a), fix a simple root α and $\eta \in B_\pi$. Let $\eta' \in B_\lambda$ be the path corresponding to η under the isomorphism $M_\lambda \to M_\pi$. Since η' has the integrality property, we know that if $n, m \in \mathbb{N}$ are maximal such that $f_\alpha^n\eta' \neq 0$, respectively $e^m\eta' \neq 0$, then pn and pm are maximal such that $f_\alpha^{pn}(p\eta') \neq 0$, respectively $e^{pm}(p\eta') \neq 0$. By the isomorphism theorem this is also true for η. For the minimum q attained by h_α for the path η we know $m \leq |q|$. Let $p \in \mathbb{N}$ be such that $p|q| \in \mathbb{Z}$. Now pm is maximal such that $e_\alpha^{pm}(p\eta) \neq 0$, but $p|q| \geq pm$ and $e_\alpha^{p|q|}(p\eta) \neq 0$. This implies $p|q| = pm$ and hence $q = m \in \mathbb{Z}$. □

Proof of Theorem 7.1. By Lemma 2.5, it is sufficient to consider the case where $\pi = \pi_{\nu_1} * \cdots * \pi_{\nu_s}$ and $\nu_1, \dots, \nu_s$ are integral weights. We proceed by induction on s. If $s = 1$, then there is nothing to prove; the case $s = 2$ has been proved in Theorem 6.3. Suppose now $s \geq 3$ and $\pi = \pi_{\nu_1} * \cdots * \pi_{\nu_s}$. Set $\pi_1 := \pi_{\nu_1} * \cdots * \pi_{\nu_{s-1}}$ and $\lambda_1 := \pi_1(1)$. By induction, the map $\pi_{\lambda_1} \to \pi_1$ extends to an isomorphism of $\mathcal{A}$-modules $\mathcal{A}\pi_{\lambda_1} \to \mathcal{A}\pi_1$, and by Lemma 2.9, this isomorphism induces an isomorphism $\psi : \mathcal{A}\pi_{\lambda_1} * \mathcal{A}\pi_{\nu_s} \to \mathcal{A}\pi_1 * \mathcal{A}\pi_{\nu_s}$

of $\mathcal{A}$-modules such that $\psi(\pi_{\lambda_1} * \pi_{\nu_s}) = \pi_{\nu_1} * \cdots * \pi_{\nu_{s-1}} * \pi_{\nu_s}$. So we get an isomorphism of $\mathcal{A}$-modules $\mathcal{A}(\pi_{\lambda_1} * \pi_{\nu_s}) \to \mathcal{A}(\pi_{\nu_1} * \cdots * \pi_{\nu_s}) = \mathcal{A}\pi$.

Now by Theorem 6.3 we have for $\lambda := \lambda_1 + \nu_s = \pi(1)$ an isomorphism $\mathcal{A}\pi_\lambda \to \mathcal{A}(\pi_{\lambda_1} * \pi_{\nu_s})$ such that $\pi_\lambda \mapsto \pi_{\lambda_1} * \pi_{\nu_s}$, so the composition of these two gives the desired isomorphism $\mathcal{A}\pi_\lambda \to \mathcal{A}\pi$ such that $\pi_\lambda \mapsto \pi$. □

8. The action of the Weyl group

The $\mathfrak{sl}_2(\mathbb{Z})$-action constructed in subsection 2.10 suggests the following operators on Π_{int}:

$$\tilde{s}_\alpha(\pi) := \begin{cases} f_\alpha^n \pi; & \text{if } n := \langle \pi(1), \alpha^\vee \rangle \geq 0, \\ e_\alpha^{-n} \pi; & \text{if } n := \langle \pi(1), \alpha^\vee \rangle < 0. \end{cases}$$

Note that $\tilde{s}_\alpha^2 = 1$ and $\tilde{s}_\alpha(\pi)(1) = s_\alpha(\pi(1))$. In fact:

THEOREM 8.1. *The map $s_\alpha \mapsto \tilde{s}_\alpha$ on the simple reflections in W extends to a representation $W \to \operatorname{End}_{\mathbb{Z}} \Pi_{\text{int}}$ such that $w(\pi)(1) = w(\pi(1))$ for $\pi \in \Pi_{\text{int}}$ and $w \in W$.*

Proof. It remains to prove that the braid relations are satisfied in the rank two case for $\mathfrak{g}$ finite-dimensional. Without loss of generality we may assume that $\pi \in \Pi_{\text{int}}$ is such that $\pi(1)$ is a dominant weight. Let $w_0 = s_\alpha s_\gamma \ldots = s_\gamma s_\alpha \ldots$ be the two different decompositions of the longest word w_0 in the Weyl group. We have to prove that $\tilde{s}_\alpha \tilde{s}_\gamma \ldots (\pi) = \tilde{s}_\gamma \tilde{s}_\alpha \ldots (\pi)$. This is obvious if $\lambda := \pi(1)$ is not regular, so we may assume in the following that λ is regular. Replacing π by $m\pi$ for some $m \in \mathbb{N}$, by Lemma 2.4 we may assume that $\pi = \pi_\lambda * \pi_\mu * \cdots * \pi_\nu$, where $\lambda, \mu, \ldots, \nu$ are integral weights, so that π is a concatenation of L-S paths. Further, if $\pi \in \mathcal{P}^+$, then $\tilde{s}_\alpha \tilde{s}_\gamma \ldots (\pi) = \tilde{s}_\gamma \tilde{s}_\alpha \ldots (\pi)$ is the unique path in $\mathcal{A}\pi$ ending in $w_0(\lambda)$. So we may assume $\pi \notin \mathcal{P}^+$.

Denote by π^n the n-fold concatenation: $\pi * \cdots * \pi$ and set $\langle \pi(1), \alpha^\vee \rangle = k > 0$. Then $f_\alpha^m(\pi * \pi) = \tilde{s}_\alpha(\pi) * f_\alpha^{m-k}\pi$ for $m \geq k$ (Lemma 2.7). Let η be a concatenation of L-S paths. If p is maximal such that $e_\alpha^p \eta \neq 0$, then choose $N < n$ such that $\langle \pi^{n-N}(1), \alpha^\vee \rangle \geq p$. We get by Lemma 2.7 for $m \geq kN$:

$$f_\alpha^m(\pi^n * \eta) = (\tilde{s}_\alpha \pi)^N * f_\alpha^{m-kN}(\pi^{n-N} * \eta).$$

Let $\rho \in X$ be such that $\langle \rho, \alpha^\vee \rangle = \langle \rho, \gamma^\vee \rangle = 1$. For $n \in \mathbb{N}$ choose $q \in \mathbb{N}$ such that $\pi_{q\rho} * \pi^n \in \mathcal{P}^+$, so that $\tilde{s}_\alpha \tilde{s}_\gamma \ldots (\pi_{q\rho} * \pi^n) = \tilde{s}_\gamma \tilde{s}_\alpha \ldots (\pi_{q\rho} * \pi^n)$. The arguments above show that for $n \gg 0$ there exist $\pi_1 \in B_{q\rho}$ and $\pi_2 \in \mathcal{A}\pi^{n-1}$ such that

$$\tilde{s}_\alpha \tilde{s}_\gamma \ldots (\pi_{q\rho} * \pi^n) = \pi_1 * \tilde{s}_\alpha \tilde{s}_\gamma \ldots (\pi) * \pi_2.$$

Similarly, $\tilde{s}_\gamma \tilde{s}_\alpha \ldots (\pi_{q\rho} * \pi^n) = \pi_1 * \tilde{s}_\gamma \tilde{s}_\alpha \ldots \ldots (\pi) * \pi_2$, where $\pi_1 \in B_{q\rho}$ and $\pi_2 \in \mathcal{A}\pi^{n-1}$. But this implies $\tilde{s}_\gamma \tilde{s}_\alpha \ldots \ldots (\pi) = \tilde{s}_\alpha \tilde{s}_\gamma \ldots (\pi)$. □

9. Weyl's character formula

Fix ρ in the weight lattice X such that $\langle \rho, \alpha^\vee \rangle = 1$ for all simple roots. For $\pi \in \mathcal{P}^+$ let $M_\pi := \mathcal{A}\pi$ be the $\mathcal{A}$-module generated by π and let $B_\pi := M_\pi \cap \Pi$ be the $\mathbb{Z}$-basis of M_π consisting of the paths contained in M_π. Denote by $\operatorname{Char} M_\pi := \sum_{\eta \in B_\pi} e^{\eta(1)}$ the character of M_π.

THEOREM 9.1. (*Weyl's character formula*).

$$\sum_{\sigma \in W} \operatorname{sgn}(\sigma) e^{\sigma(\rho)} \operatorname{Char} M_\pi = \sum_{\sigma \in W} \operatorname{sgn}(\sigma) e^{\sigma(\rho+\lambda)}.$$

In particular, $\operatorname{Char} M_\pi$ *is equal to the character of the irreducible, integrable* $\mathfrak{g}$*-module* V_λ *of highest weight* $\lambda := \pi(1)$.

Proof. Set $\Omega(\mu) := \{(\eta, \sigma) \mid \eta \in B_\pi, \sigma \in W, \sigma(\rho) + \eta(1) = \mu\}$ for $\mu \in X$. Since $\Omega(\tau(\mu)) = \{(\tau(\eta), \tau\sigma) \mid (\eta, \sigma) \in \Omega(\mu)\}$, we may assume that μ is dominant. Further, $\sigma(\rho) \prec \rho$ for $\sigma \neq 1$, and $\eta = f_{\alpha_1}^{n_1} \ldots f_{\alpha_r}^{n_r} \pi$, so that $\eta(1) \prec \pi(1) = \lambda$ for $\eta \neq \pi$. Hence $\Omega(\lambda + \rho) = \{(\pi, 1)\}$ and

$$\sum_{(\eta,\sigma) \in \Omega(\lambda+\rho)} \operatorname{sgn}(\sigma) e^{\sigma(\rho)+\eta(1)} = e^{\lambda+\rho}.$$

Let $\mu \neq \rho + \lambda$ be dominant such that $\Omega = \Omega(\mu) \neq \emptyset$. It remains to show:

$$\sum_{(\sigma,\eta) \in \Omega(\mu)} \operatorname{sgn}(\sigma) e^{\sigma(\rho)+\eta(1)} = 0. \tag{9.1}$$

Fix $(\eta_0, \sigma_0) \in \Omega$, and choose $t_0 \in [0,1]$ maximal such that $\sigma_0(\rho) + \eta_0(t_0)$ is dominant but not regular. If such a t_0 does not exist, then necessarily $\sigma_0 = 1$ and $\langle \rho + \eta_0(t), \alpha^\vee \rangle > 0$ for all $t \in [0,1]$. By the integrality property of the paths this implies $\langle \eta_0(t), \alpha^\vee \rangle \geq 0$ for all $t \in [0,1]$ and hence $\eta_0 = \pi$, in contradiction to the assumption $\mu \neq \rho + \lambda$.

Fix a simple root α such that $\langle \sigma_0(\rho) + \eta_0(t_0), \alpha^\vee \rangle = 0$ and consider

$$\Omega_0 := \{(\eta, \sigma) \in \Omega \mid \sigma(\rho) + \eta(t) = \sigma_0(\rho) + \eta_0(t) \text{ for all } t \in [t_0, 1]\}.$$

We define an involution i_α on Ω_0 such that $i_\alpha((\eta, \sigma)) = (\eta', s_\alpha \sigma)$. Note that the existence of such an involution implies

$$\sum_{(\eta,\sigma) \in \Omega_0} \operatorname{sgn}(\sigma) e^{\sigma(\rho)+\eta(1)} = 0.$$

Since $\Omega = \Omega_0 \cup \cdots \cup \Omega_r$ is a disjoint union for some $\eta_0, \ldots, \eta_r \in \Omega$, this implies 9.1. (Recall that $\Omega = \Omega(\mu)$ is a finite set by Corollary 1). To construct i_α let (η, σ) first be such that $\langle \sigma(\rho), \alpha^\vee \rangle < 0$. Since $\langle \sigma(\rho) + \eta(t), \alpha^\vee \rangle > 0$ for $t > t_0$, for $m := |\langle \sigma(\rho), \alpha^\vee \rangle|$ we get $f_\alpha^m \eta \neq 0$ and $s_\alpha \sigma(\rho) + f_\alpha^m \eta(t) = \sigma(\rho) + \eta(t)$ for $t \geq t_0$. In particular, $(f_\alpha^m \eta, s_\alpha \sigma) \in \Omega_0$. We set $i_\alpha(\eta, \sigma) := (f_\alpha^m \eta, s_\alpha \sigma)$.

Similarly, if $\langle\sigma(\rho),\alpha^\vee\rangle = m > 0$, then $i_\alpha(\eta,\sigma) := (e_\alpha^m\eta, s_\alpha\sigma) \in \Omega_0$. It is now easy to see that $i_\alpha^2 = \mathrm{id}$, so that i_α is an involution. □

10. The decomposition rules

The decomposition rules stated in the introduction are immediate consequences of the character formula (Theorem 9.1). For $\pi \in \mathcal{P}^+$ let $M_\pi := \mathcal{A}\pi$ be the module generated by π and let $B_\pi = \Pi \cap M_\pi$ be its basis.

For $\pi_1, \pi_2 \in \mathcal{P}^+$ one knows by Corollary 1 that if $\eta \in B_{\pi_1} * B_{\pi_2}$, then its weight $\eta(1)$ can be written as $\pi_1(1)+\pi_2(1)-\sum_i a_i\beta_i$, where the β_i are positive real roots and $a_i \geq 0$. So by weight considerations there exists for η a sequence $n_1,\dots,n_p$ such that $\pi := e_{\alpha_1}^{n_1}\dots e_{\alpha_p}^{n_p}\eta$ has the property $e_\alpha\pi = 0$ for all simple roots. Since $B_{\pi_1} * B_{\pi_2}$ has the integrality property this implies $\pi \in \mathcal{P}^+$. Since π is the only path in $\mathcal{A}\pi$ such that $e_\alpha\pi = 0$ for all simple roots we get:

$$M_{\pi_1} * M_{\pi_2} = \bigoplus_\pi M_\pi,$$

where π runs over all $\pi \in B_{\pi_1} * B_{\pi_2}$ such that $\pi \in \mathcal{P}^+$. To see that the elements $\pi \in B_{\pi_1} * B_{\pi_2} \cap \mathcal{P}^+$ are in fact of the form $\pi_1 * \pi'$ note that if $\pi = \eta * \pi'$ is such that $e_\alpha\eta \neq 0$, then $e_\alpha\pi \neq 0$ by Lemma 2.7 and hence $\pi \notin \mathcal{P}^+$. The proof of the restriction formula is similar. By the integrality property and Corollary 1, there exists for $\eta \in B_\pi$ a sequence $n_1, n_2, \dots$ and simple roots in $\mathfrak{l}$ such that $\sigma := e_{\alpha_1}^{n_1}e_{\alpha_2}^{n_2}\dots\eta \in \mathcal{P}_\mathfrak{l}^+$. Since σ is the only path in $\mathcal{A}_\mathfrak{l}\sigma$ such that $e_\alpha\sigma = 0$ for all simple roots in $\mathfrak{l}$, we get the following sum over all paths in B_π contained in $\mathcal{P}_\mathfrak{l}^+$: $M_\pi = \bigoplus_\eta \mathcal{A}_\mathfrak{l}\pi_\eta$.

11. The rank 2 case

We conclude with a description of B_π, $\pi \in \mathcal{P}^+$, in the rank 2 case. Let α, γ be the simple roots and set $a := |\langle\alpha,\gamma^\vee\rangle|$, $b := |\langle\gamma,\alpha^\vee\rangle|$ and $x := ab$. We assume in addition that $x > 0$. Consider the sequence $\{y_i\}_{i\in\mathbb{N}}$ defined by $y_0 = 1$, and

$$y_i := 1 - \frac{1}{xy_{i-1}} \quad \text{if} \quad y_{i-1} \neq 0 \quad \text{and} \quad y_i := 0 \quad \text{otherwise.}$$

A small calculation shows (compare also [3]):

Lemma 11.1. a) *If $x = 1$, then $y_0 = 1$ and $y_i = 0$ for $i \geq 1$.*
b) *If $x = 2$, then $y_0 = 1$,$y_1 = 1/2$ and $y_i = 0$ for $i \geq 2$.*
c) *If $x = 3$, then $y_0 = 1$,$y_1 = 2/3$,$y_2 = 1/2$,$y_3 = 1/3$ and $y_i = 0$ for $i \geq 4$.*
d) *If $x \geq 4$, then $y_i \geq 1/2 + \sqrt{1/4 - 1/x}$ for all $i \geq 0$ and the sequence $\{y_i\}_{i\in\mathbb{N}}$ is strictly decreasing.*

Remark 11.2. If $y_i \neq 0$, then $xy_i \geq 1$.

Set $Y_i := y_0 y_1 \dots y_i$, and for a sequence $n_1, m_1, n_2, \dots \geq 0$ of integers set

$$M_\gamma^i := x^{i-1}(bn_i y_{2i-2} - m_i)Y_{2i-3}, \quad M_\alpha^i := x^{i-1}b(am_i y_{2i-1} - n_{i+1})Y_{2i-2}.$$

THEOREM 11.3. *Let $\pi_0 \in \mathcal{P}^+$ be such that $\pi_0(1) = \lambda$. For every element $\pi \in B_{\pi_0}$ there exists a unique sequence of integers $n_1, m_1, n_2, m_2, \dots$ such that $\pi := f_\gamma^{n_1} f_\alpha^{m_1} f_\gamma^{n_2} \dots \pi_0$. This sequence satisfies the following inequalities: $am_1 y_0 \geq n_2$, $bn_2 y_1 \geq m_2$, $am_2 y_2 \geq n_3$, … and*

$$\begin{array}{lllll} 0 & \leq & n_1 & \leq & \langle\lambda,\gamma^\vee\rangle + a(m_1+m_2+\cdots) - 2(n_2+n_3+\cdots), \\ 1 & \leq & m_1 & \leq & \langle\lambda,\alpha^\vee\rangle + b(n_2+n_3+\cdots) - 2(m_2+m_3+\cdots), \\ 1 & \leq & n_2 & \leq & \langle\lambda,\gamma^\vee\rangle + a(m_2+m_3+\cdots) - 2(n_3+n_4+\cdots), \\ & & \cdots & & \cdots \end{array}$$

Further, if a sequence satisfies these inequalities, then $\pi := f_\gamma^{n_1} f_\alpha^{m_1} f_\gamma^{n_2} \dots \pi_0 \neq 0$, and $e_\gamma f_\alpha^{m_1} f_\gamma^{n_2} f_\alpha^{m_2} \dots \pi_0 = 0$, $e_\alpha f_\gamma^{n_2} f_\alpha^{m_2} \dots \pi_0 = 0$, $e_\gamma f_\alpha^{m_2} \dots \pi_0 = 0, \dots$ and $m := \max\{0, -M_\gamma^1, -M_\alpha^1, -M_\gamma^2, -M_\alpha^2, \dots\}$ is maximal such that $e_\alpha^m \pi \neq 0$ and n_1 is maximal such that $e_\gamma^{n_1}\pi \neq 0$.

Example. Suppose $\mathfrak{g}$ is of type $\mathtt{A}_2$ and $\lambda = k\omega_\gamma + l\omega_\alpha$ (where $\omega_\gamma, \omega_\alpha$ are the fundamental weights such that $\omega_\gamma(\alpha) = 0$ and $\omega_\alpha(\gamma) = 0$). Then

$$\begin{aligned} B_{\pi_\lambda} = {} & \{f_\gamma^{n_1}\pi_\lambda \mid 0 \leq n_1 \leq k\} \cup \{f_\gamma^{n_1} f_\alpha^{m_1}\pi_\lambda \mid 0 \leq n_1 \leq k + m_1, 1 \leq m_1 \leq l\} \\ & \cup \{f_\gamma^{n_1} f_\alpha^{m_1} f_\gamma^{n_2}\pi_\lambda \mid 0 \leq n_1 \leq k + m_1 - 2n_2, 1 \leq m_1 \leq l + n_2, \\ & \quad 1 \leq n_2 \leq k, m_1 \geq n_2\}. \end{aligned}$$

If $\pi \in \mathcal{A}\pi_\lambda$ is of the first type, then $e_\alpha \pi = 0$; if π is of the second type, then $e_\alpha^m \pi = 0$ for $m > m_1 - n_1$; if π is of the third type, then $e_\alpha^m \pi = 0$ for $m > \max\{n_2, m_1 - n_1\}$.

To prove the theorem by induction, we need the following

LEMMA 11.4. *If $\pi = f_\gamma^{n_1} f_\alpha^{m_1} f_\gamma^{n_2} \dots \pi_0 \neq 0$ is such that*

$$am_1 y_0 - n_2 \geq 0, \quad bn_2 y_1 - m_2 \geq 0, \quad am_2 y_2 - n_3 \geq 0, \ \dots \tag{11.1}$$

then $m := \max\{m \in \mathbb{N} \mid e_\alpha^m \pi \neq 0\} = \max\{0, -M_\gamma^1, -M_\alpha^1, -M_\gamma^2, \dots\}$.

Proof of the theorem. We show first that the lemma implies the theorem. To have $m = 0$, we need $M_\alpha^i, M_\gamma^i \geq 0$ for all i, which is equivalent to

$$bn_1 y_0 - m_1 \geq 0, \quad am_1 y_1 - n_2 \geq 0, \quad bn_2 y_2 - m_2 \geq 0, \ \dots.$$

Since the sequence $\{y_i\}$ is not increasing, this proves inductively the equivalence of (11.1) and $e_\gamma f_\alpha^{m_1} f_\gamma^{n_2} \dots \pi_0 = 0$, $e_\alpha f_\gamma^{n_2} \dots \pi_0 = 0$, etc. The second set of inequalities is just to ensure that $\pi \neq 0$:

If $e_\gamma f_\alpha^{m_i} f_\gamma^{n_{i+1}} \dots \pi_0 = 0$, then $f_\gamma^n f_\alpha^{m_i} f_\gamma^{n_{i+1}} \dots \pi_0 = 0$ if and only if

$$n > \langle f_\alpha^{m_i} f_\gamma^{n_{i+1}} \dots \pi_0(1), \gamma^\vee \rangle = \langle \lambda, \gamma^\vee \rangle + a(m_i + m_{i+1} + \cdots) - 2(n_{i+1} + \cdots).$$

To prove that the sequence is unique, we construct the sequence $n_1, m_1, n_2, \dots$ as follows: Choose n_1 maximal such that $e_\gamma^{n_1}\pi \neq 0$, choose m_1 maximal such that $e_\alpha^{m_1} e_\gamma^{n_1}\pi \neq 0$, etc. We have seen that the sequence $m_1, n_2, \dots$ satisfies the inequalities, and the inequality for n_1 is also clearly satisfied. Since the $m_1, n_2, \dots$ are positive, the construction shows that the sequence is unique. Clearly, n_1 is maximal such that $e_\gamma^{n_1}\pi \neq 0$, and the statement about the maximal $m \in \mathbb{N}$ such that $e_\alpha^m \pi \neq 0$ follows by the lemma. □

Proof of the lemma. We proceed by induction on the length of the sequence. So we may assume that (11.1) is equivalent to

$$e_\gamma f_\alpha^{m_1} f_\gamma^{n_2} \dots \pi_0 = 0, \quad e_\alpha f_\gamma^{n_2} \dots \pi_0 = 0, \dots.$$

Let φ_α^i and φ_γ^i be the increasing functions on $[0,1]$ defined by

$$f_\gamma^{n_i} f_\alpha^{m_i} f_\gamma^{n_{i+1}} \dots \pi_0(t) = f_\alpha^{m_i} f_\gamma^{n_{i+1}} \dots \pi_0(t) - \varphi_\gamma^i(t)\gamma,$$

and $f_\alpha^{m_i} \dots \pi_0(t) = f_\gamma^{n_{i+1}} \dots \pi_0(t) - \varphi_\alpha^i(t)\alpha$. If $e_\gamma(f_\alpha^{m_i} \dots \pi_0) = 0$, then

$$\varphi_\gamma^i(t) \leq \langle f_\alpha^{m_i} f_\gamma^{n_{i+1}} \dots \pi_0(t), \gamma^\vee \rangle \tag{11.2}$$

for all $t \in [0,1]$, and we have equality if φ_γ^i is not constant on an arbitrary small neighborhood of t. Now in the situation of the lemma we have

$$h_\alpha(t) = \langle \pi(t), \alpha^\vee \rangle = \langle f_\gamma^{n_2} f_\alpha^{m_2} \dots \pi_0(t), \alpha^\vee \rangle + b\varphi_\gamma^1(t) - 2\varphi_\alpha^1(t). \tag{11.3}$$

By assumption (and 11.2) we know that $\langle f_\gamma^{n_2} f_\alpha^{m_2} \dots \pi(t), \alpha^\vee \rangle - \varphi_\alpha^1(t) \geq 0$. Since φ_γ^1 is not decreasing, we know that if the function $h_\alpha(t)$ attains its minimum for the first time at $t = t_0$, then φ_α^1 is not constant near t_0 and hence

$$\langle f_\gamma^{n_2} f_\alpha^{m_2} \dots \pi(t_0), \alpha^\vee \rangle - \varphi_\alpha^1(t_0) = 0 \tag{11.4}$$

and $-m = \min\{h_\alpha(t) \mid t \in [0,1]\} = \min\{b\varphi_\gamma^1(t) - \varphi_\alpha^1(t) \mid t \in [0,1]\}$. Set

$$p_i := \min_{t\in[0,1]} \{by_{2i-2}\varphi_\gamma^i(t) - \varphi_\alpha^i(t)\}, \quad q_i := \min_{t\in[0,1]} \{ay_{2i-1}\varphi_\alpha^i(t) - \varphi_\gamma^{i+1}(t)\}.$$

SUBLEMMA 11.5. a) *Let* $p{:} = p_i x^{i-1} Y_{2i-3}$ *and set* $q{:} = q_i b x^{i-1} Y_{2i-2}$. *Then* $p \leq M_\gamma^i$ *and* $p \leq q$, *and if* $p < M_\gamma^i$ *then* $p = q$.

b) *Let* $q{:} = q_i b x^{i-1} Y_{2i-2}$ *and set* $p{:} = p_{i+1} x^i Y_{2i-1}$. *Then* $q \leq M_\alpha^i$ *and* $q \leq p$, *and if* $q < M_\alpha^i$ *then* $q = p$.

Proof of the sublemma. Obviously for a):

$$p \leq x^{i-1} Y_{2i-3}(b\varphi_\gamma^i(1) y_{2i-2} - \varphi_\alpha^i(1)) = M_\gamma^i.$$

By (11.2), $\langle f_\alpha^{m_i} f_\gamma^{n_{i+1}} \dots \pi_0(t), \gamma^\vee\rangle \geq \varphi_\gamma^i(t)$, and hence

$$p \leq x^{i-1} Y_{2i-3} \min_{t\in[0,1]} \{b\langle f_\alpha^{m_i} f_\gamma^{n_{i+1}} \dots \pi_0(t), \gamma^\vee\rangle y_{2i-2} - \varphi_\alpha^i(t)\}. \tag{11.5}$$

The function in (11.5) is equal to

$$b y_{2i-2}(\langle f_\alpha^{m_{i+1}}(t) \dots \pi_0(t), \gamma^\vee\rangle - \varphi_\gamma^{i+1}(t)) + \varphi_\alpha^i(t)(x y_{2i-2} - 1) - b\varphi_\gamma^{i+1}(t) y_{2i-2}.$$

By assumption (see 11.2) the first part is nonnegative, and it is zero at $t = t_0$ if φ_γ^{i+1} is not constant on an arbitrary small neighborhood of t_0. So as in (11.4), the minimum is equal to the minimum of the second part. It follows by (11.5):

$$\begin{aligned} p &\leq x^{i-1} Y_{2i-3} \min_{t\in[0,1]} \{\varphi_\alpha^i(t)(x y_{2i-2} - 1) - b y_{2i-2}\varphi_\gamma^{i+1}(t)\} \\ &= b x^{i-1} Y_{2i-2} \min_{t\in[0,1]} \{a y_{2i-1}\varphi_\alpha^i(t) - \varphi_\gamma^{i+1}(t)\} = q. \end{aligned} \tag{11.6}$$

It remains to prove that $p = q$ if $p < M_\gamma^i$. Let $c_0 \in [0,1]$ be minimal such that φ_γ^i is constant for $t \geq c_0$. If $p < M_\gamma^i$, then p is attained for some $t_0 \leq c_0$, and in addition we may assume that φ_γ^i is not constant in a small neighborhood of t_0. Hence we have $\langle f_\alpha^{m_i} \dots \pi_0(t_0), \gamma^\vee\rangle = \varphi_\gamma^i(t_0)$ (see 11.2) and equality for $t = t_0$ in (11.5) and (11.6). The proof of b) is similar. □

End of the proof of the lemma. We have proved already that

$$-m = \min_{t\in[0,1]} \{b\varphi_\gamma^1(t) - \varphi_\alpha^1(t)\}.$$

By Lemma 11.5 this implies $-m \leq M_\alpha^i, M_\gamma^i$ for all i. If $-m < M_\alpha^i, M_\gamma^i$ for all i, then we obtain by induction and the equality in (11.5) for $\pi = f_\gamma^{n_1} \dots f_\gamma^{n_s} f_\alpha^{m_s}\pi_0$:

$$\begin{aligned} -m &= c \min_{t\in[0,1]} \{b y_{2s-2}\varphi_\gamma^s(t) - \varphi_\alpha^s(t)\} \\ &= c \min_{t\in[0,1]} \{b y_{2s-2}\langle f_\alpha^{m_s}\pi_0(t), \gamma^\vee\rangle - \varphi_\alpha^s(t)\} \\ &= c \min_{t\in[0,1]} \{b y_{2s-2}\langle \pi_0(t), \gamma^\vee\rangle + \varphi_\alpha^s(t)(x y_{2s-2} - 1)\} = 0, \end{aligned}$$

since $(x y_{2s-2} - 1) \geq 0$ (Remark 11.2) and $\langle \pi_0(t), \gamma^\vee\rangle \geq 0$. The same arguments show that if $\pi = f_\gamma^{n_1} f_\alpha^{m_1} f_\gamma^{n_2} \dots f_\alpha^{m_s} f_\gamma^{n_{s+1}}\pi_0$, then $m = 0$, which finishes the proof of the lemma.

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(Received September 1, 1993)