Elliptic Functions and related topics

Problem sheet 1

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Exercise 1. (4 points)
Let $a, b \in \mathbb{C}$ with $0 < |a| < |b|$ and $f : \mathbb{C} \setminus \{a, b\} \to \mathbb{C}$ be defined by

$$f(z) = \frac{1}{z^2 - (a+b)z + ab}.$$

Find the Laurent expansions of f which converge for

(a) $|z| < |a|$,

(b) $|a| < |z| < |b|$,

(c) $|z| > |b|$.

Exercise 2. (4 points)
Denote the open unit disk in $\mathbb{C}$ by D. Prove that the series

$$\sum_{n=0}^{\infty} \frac{z^{2^n}}{1 - z^{2 \cdot 2^n}}$$

defines a holomorphic function on D as well as on $\mathbb{C} \setminus \overline{D}$.

Exercise 3. (4 points)

(a) We write $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ for the upper half-plane in $\mathbb{C}$. Prove that every entire function $f : \mathbb{C} \to \mathbb{H}$ is constant.

(b) Use Liouville's theorem to prove the Fundamental Theorem of Algebra.

Exercise 4. (4 points)
Let $G \subseteq \mathbb{C}$ be a domain (i.e. an open, connected subset of $\mathbb{C}$) and $U \subset G$ be a bounded, simply connected subdomain of G with $\overline{U} \subset G$ and a smooth boundary. For a meromorphic function $f : G \to \widehat{\mathbb{C}}$ show the equality

$$\frac{1}{2\pi i} \int_{\partial U} \frac{f'}{f}(z)dz = \sum_{a \in U} \operatorname{ord}_a(f),$$

where $\operatorname{ord}_a(f)$ counts the order of a zero (resp. pole) of f in a if it is positive (resp. negative) and is zero otherwise.

Deadline: Tuesday, Oct. 14, 2014 at the beginning of the lecture.