Elliptic Functions and related topics

Problem sheet 11

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Exercise 1. (4 points)
Let f be an arithmetic function (i.e., $f : \mathbb{N} \to \mathbb{C}$). We call f *multiplicative*, if $f(mn) = f(m) \cdot f(n)$ whenever m and n are coprime, and *completely multiplicative*, if the above is true for all $m, n \in \mathbb{N}$. For $s \in \mathbb{C}$ with sufficiently large real part we define the *Dirichlet series* associated to f by

$$D_f(s) := \sum_{n=1}^{\infty} f(n) n^{-s}.$$

(We assume that for $\mathrm{Re}(s) > \sigma_a \in \mathbb{R}$, the above series is absolutely convergent).

(a) Show that if f is multiplicative, then it holds that

$$D_f(s) = \prod_p \left(\sum_{\ell=0}^{\infty} f(p^\ell) p^{-\ell s} \right),$$

where the product runs over all primes and $\mathrm{Re}(s) > \sigma_a$. We call this identity an *Euler product*.

Hint: Consider the product over all primes $p \leq N$ for some $N \in \mathbb{N}$ and show that the difference of the left and right hand side goes to 0 for $N \to \infty$.

(b) Show that if f is completely multiplicative, then the above Euler product simplifies to

$$D_f(s) = \prod_p (1 - f(p) p^{-s})^{-1}.$$

(c) Suppose that f is multiplicative and satisfies the recurrence relation

$$f(p^{r+1}) = f(p) f(p^r) - p f(p^{r-1})$$

for all primes p and $r \in \mathbb{N}$. Show that in this case, the Euler product in (a) simplifies to

$$D_f(s) = \prod_p (1 - f(p) p^{-s} + p^{1-2s})^{-1}.$$

Hint: First prove that for any multiplicative arithmetic function we have $f(1) = 1$.

Exercise 2. (4 points)
Let $f : \mathbb{H} \to \mathbb{C}$ be a holomorphic function which is defined by the Fourier series

$$f(\tau) := \sum_{n=1}^{\infty} a(n) e^{2\pi i n \tau}$$

and satisfies the functional equation $f(-\frac{1}{\tau}) = \tau^k f(\tau)$ for some even number $k \geq 4$ (we call such a function a *cusp form of weight* k). One can show that $a(n) \leq C_f n^{\frac{k}{2}}$ for some positive constant C_f which depends on f. Thus the Dirichlet series $D_a(s)$ is absolutely convergent for $\mathrm{Re}(s) > \frac{k}{2}$. Show that the function

$$\mathbb{D}_f(s) := (2\pi)^{-s}\Gamma(s) D_a(s)$$

has an analytic continuation to $\mathbb{C}$ and satisfies the functional equation

$$\mathbb{D}_f(s)(k-s) = i^k \mathbb{D}_f(s).$$

Hint: Substitute $t = 2\pi m y$ in the integral representation of the Γ-function to obtain $\mathbb{D}_f(s) = \int_0^\infty f(iy) y^{s-1} dy$ and then use the functional equation to show that $\mathbb{D}_f(s) = \int_1^\infty f(iy)[y^s + i^k y^{k-s}]\frac{dy}{y}$.

Exercise 3. (4 points)
Let χ be a primitive Dirichlet character $\pmod N$ and define for $t > 0$

$$\theta(\chi, t) := \begin{cases} \sum\limits_{n=1}^{\infty} \chi(n) e^{-\pi t n^2} & \text{if } \chi \text{ is even,} \\ \sum\limits_{n=1}^{\infty} n\chi(n) e^{-\pi t n^2} & \text{if } \chi \text{ is odd.} \end{cases}$$

Prove the functional equations $\theta(\chi, t) = \frac{g(\chi)}{\sqrt{N^2 t}} \theta(\overline{\chi}, \frac{1}{N^2 t})$ for even χ and $\theta(\chi, t) = -iN^{-2} t^{-\frac{3}{2}} g(\chi) \theta(\overline{\chi}, \frac{1}{N^2 t})$ for odd χ.

Exercise 4. (4 points)
Show that the Mellin transform of $\theta(\chi, t)$ from Exercise 3 converges for all $s \in \mathbb{C}$ and that it equals $\pi^{-s}\Gamma(s) L(\chi, 2s - \delta_\chi)$, where $\delta_\chi = 0$ if χ is even and $\delta_\chi = 1$ if χ is odd, for $\mathrm{Re}(s) > \frac{1}{2}$.

Deadline: Tuesday, Jan. 13, 2014 at the beginning of the lecture.