Elliptic Functions and related topics

Problem sheet 12

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Exercise 1. (4 points)
As you have seen in the lecture, the group $\mathrm{SL}_2(\mathbb{Z})$ is generated by the matrices $T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

(a) Describe an algorithm that decomposes a given matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ into a product of S and T.

(b) Write the following matrices as a product of T and S,

$$A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \; B = \begin{pmatrix} -3 & -5 \\ 5 & 8 \end{pmatrix}, \; C = \begin{pmatrix} 2 & 7 \\ 1 & 4 \end{pmatrix}.$$

Exercise 2. (4 points)
For $N \in \mathbb{N}$ define the sets

$$\begin{aligned} \Gamma_0(N) &:= \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \; : \; c \equiv 0 \pmod{N} \} \\ \Gamma_1(N) &:= \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N) \; : \; a \equiv d \equiv 1 \pmod{N} \} \\ \Gamma(N) &:= \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(N) \; : \; b \equiv 0 \pmod{N} \}. \end{aligned}$$

(a) Show that all the above sets are finite-index subgroups of $\mathrm{SL}_2(\mathbb{Z})$ and that moreover, $\Gamma(N)$ is a normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

(b) For a prime p, determine $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(p)]$.

Exercise 3. (4 points)

(a) Show that the group $\mathrm{SL}_2(\mathbb{Z})$ (and therefore all its subgroups) acts from the left on the projective line over $\mathbb{Q}$, $\mathbb{P}_1(\mathbb{Q}) := \mathbb{Q} \cup \{\infty\}$ via Möbius transformations with the usual settings for the operations involving ∞ (e.g., $\frac{a}{\infty} := 0$ etc.).

(b) For p, q distinct primes, determine the number of orbits as well as a system of orbit representatives for the action of $\Gamma_0(p), \Gamma_0(p^2)$, and $\Gamma_0(pq)$ on $\mathbb{P}_1(\mathbb{Q})$.

Remark: These orbits are called *cusps* of the respective group.

Exercise 4. (4 points)
By adding the cusps introduced in Exercise 3, the quotient space $\Gamma_0(N) \setminus \mathbb{H}$ can be turned into a compact Riemann surface $X_0(N)$ (you don't need to prove this). Use this to prove (without appeal to the valence formula), that $\dim_{\mathbb{C}}(S_2(\Gamma_0(N)) =$

$genus(X_0(N))$, where $S_k(\Gamma)$ denotes the space of cusp forms of weight k for the group Γ.
Hint: Show that every cusp form of weight 2 defines a holomorphic differential on $X_0(N)$ and then use that the dimension of the space of holomorphic differentials on a compact Riemann surface equals the surface's genus.

Deadline: Tuesday, Jan. 20, 2014 at the beginning of the lecture.