Elliptic Functions and related topics

Problem sheet 12

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Exercise 1. (4 points)
Let $D := \frac{d}{d\tau} = q\frac{d}{dq}$ be the renormalized derivative and $E_2 = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n$ be the Eisenstein series of weight 2. Prove that for a modular form f of weight $k \in \mathbb{Z}$ for $\mathrm{SL}_2(\mathbb{Z})$ the so-called *Serre derivative*

$$\vartheta f := Df - \frac{k}{12}E_2 f$$

is a modular form of weight $k+2$ for $\mathrm{SL}_2(\mathbb{Z})$.

Exercise 2. (4 points)
Let $f : \mathbb{H} \to \mathbb{C}$ be a smooth function and define the operator R_k for $k \in \mathbb{R}$ by

$$R_k f(\tau) := 2i\frac{\partial f}{\partial \tau}(\tau) + k\,\mathrm{Im}(\tau)^{-1} f(\tau).$$

Show that

$$R_k(f|_k\gamma)(\tau) = (R_k f)|_{k+2}\gamma(\tau)$$

for all $\gamma \in \mathrm{SL}_2(\mathbb{R})$.

Exercise 3. (4 points)

(a) Let $d \in \mathbb{N}$. Show that the function $E_2(\tau) - dE_2(d\tau)$ is a modular form of weight 2 for $\Gamma_0(d)$.

(b) Let $f(\tau) = \sum_{n=0}^{\infty} a_n q^n$ be a modular form of weight k for some $\Gamma_0(N)$. Show that the function $g(\tau) = \sum_{n=0}^{\infty} a_{2n+1}q^{2n+1}$ is a modular form for $\Gamma_0(4N)$.

Exercise 4. (4 points)
Using the theory of modular forms, reprove the congruence

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691},$$

where $\Delta(\tau) = \sum_{n=1} \tau(n)q^n$. In Exercise 2 on Problem sheet 6 we had proven this using the theory of elliptic functions.

Deadline: Tuesday, Jan. 27, 2014 at the beginning of the lecture.