Elliptic Functions and related topics

Problem sheet 14

Dr. L. Rolen, Dr. M. H. Mertens

Exercise 1. (4 points)
Recall the Maaß raising operator

$$R_k := 2i\frac{\partial}{\partial\tau} + k\,\mathrm{Im}(\tau)^{-1}$$

from the previous problem sheet, which maps (possibly non-holomorphic) modular forms of weight k to (non-holomorphic) modular forms of weight $k+2$.

(a) Define the iterated Maaß raising operator R_k^n by $R_k^0 := \mathrm{id}$ and

$$R_k^n := R_{k+2(n-1)} \circ \ldots \circ R_{k+2} \circ R_k.$$

Prove the explicit formula

$$R_k^n = \sum_{m=0}^{n} \binom{n}{m} \frac{\Gamma(k+n)}{\Gamma(k+n-m)}\,\mathrm{Im}(\tau)^{-m}(2i)^{n-m}\frac{\partial^{n-m}}{\partial\tau^{n-m}},$$

where Γ denotes the usual Gamma function.

(b) Let $k<0$ be an integer and $f:\mathbb{H}\to\mathbb{C}$ a weakly holomorphic modular form of weight k for some group Γ (i.e. f is holomorphic on $\mathbb{H}$ and has at most a pole at the cusps of Γ). Show that $\frac{d^{1-k}}{d\tau^{1-k}}f$ is a weakly holomorphic modular form of weight $2-k$.

Exercise 2. (4 points)
Let f and g be modular forms of weights k and ℓ for $\mathrm{SL}_2(\mathbb{Z})$ and define

$$[f,g](\tau) := (kf(\tau)g'(\tau) - \ell f'(\tau)g(\tau)),$$

where $f' := \frac{1}{2\pi i}\frac{d}{d\tau}f$.

(a) Show that $[f,g]$ is a cusp form of weight $k+\ell+2$ for $\mathrm{SL}_2(\mathbb{Z})$.

(b) Prove that the graded $\mathbb{C}$-vector space $M_* := \bigoplus_{k>0} M_k(\mathrm{SL}_2(\mathbb{Z}))$ has a Lie algebra structure by means of the above bracket, i.e. that the above bracket

   (i) is well-defined on M_*,

   (ii) is $\mathbb{C}$-bilinear,

   (iii) satisfies the Jacobi identity $[f,[g,h]]+[g,[h,f]]+[h,[f,g]]=0$ for all $f,g,h\in M_*$,

(iv) satisfies $[f, f] = 0$ for all $f \in M_*$.

(c) Find an expression for the Ramanujan τ function in terms of the divisor sums σ_3 and σ_5 which does not involve convolutions of one of these with itself, i.e. for example expressions of the form $\sum_{r+s=n} \sigma_5(r)\sigma_5(s)$.

Exercise 3. (4 points)
Let $f(\tau) =: \sum\limits_{n=0}^{\infty} a(n)^n$ be a modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$. Find the Fourier coefficients of $T_n(f)$.

Exercise 4. (4 points)
Let f be a weight k cusp form for $\mathrm{SL}_2(\mathbb{Z})$, which is a simultaneous eigenform for all Hecke operators T_n and has leading coefficient 1. Show that the Fourier coefficients of f are algebraic integers of degree $\leq \dim S_k$.

Deadline: Tuesday, Feb 03, 2015 at the beginning of the lecture.