Elliptic Functions and related topics

Problem sheet 2

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Exercise 1. (4 points)
Let $p(X) = 4(X-a)(X-b)(X-c)$ be a polynomial where $a, b, c \in \mathbb{C}$ are all distinct and let G be a function satisfying the differential equation $G'^2 = p(G)$. For some $\gamma \in \mathbb{C} \setminus \{0\}$ define the function $H = \sqrt{\gamma(G-a)}$.

(a) Show that H satisfies a differential equation of the form

$$H'^2 = AH^4 + BH^2 + C,$$

where $A, B, C \in \mathbb{C}$ and $AC \neq 0$. Give the coefficients A, B, C explicitly in terms of the parameter γ and the zeros of p.

(b) Determine the parameter γ in terms of a, b, c so that the differential equation for H can be given in Legendre form

$$H'^2 = C(1-H^2)(1-k^2H^2).$$

Also show the equation

$$k^2 = \frac{a-b}{a-c}$$

for one possible chice of γ.

Exercise 2. (4 points)

(a) Let $p(X) = a_0X^3 + a_1X^2 + a_2X + a_3 \in \mathbb{C}[X]$ be a polynomial of degree 3. Find a linear substitution $Y := \alpha X + \beta$ such that $p(Y)$ is in Weierstrass form, i.e.

$$p(Y) = 4X^3 - c_1X - c_2.$$

(b) Let p be a polynomial of degree 3 in Weierstrass form. Show that the roots of p are all distinct if and only if $\Delta := c_1^3 - 27c_2^2 \neq 0$.
Hint: Show the equality $\Delta = 16(a-b)^2(b-c)^2(c-a)^2$ where a, b, c are the complex roots of p.

Exercise 3. (4 points)

(a) For $0 \leq x \leq 1$ let

$$F(x) := \int_0^x \frac{1}{\sqrt{1-t^4}} dt$$

and let G be the inverse function of F. As you have seen in the lecture, F satisfies the addition formula

$$F(x) + F(y) = F\left(\frac{x\sqrt{1-y^4} + y\sqrt{1-x^4}}{1+x^2y^2} \right)$$

for all $x, y \geq 0$ which are small enough. Deduce from there the addition formula

$$G(u+v) = \frac{G(u)G'(v) + G(v)G'(u)}{1 + G^2(u)G^2(v)}$$

for all $0 \leq u, v$ sufficiently small.

(b) Now let G be a meromorphic function with poles in a set P_G satisfying the differential equation $G'^2 = 1 - G^4$ with the initial conditions $G(0) = 0$ and $G'(0) = 1$. Further let $v \in \mathbb{C}$ with $G'(v) = 1$. Prove that for all $m, n \in \mathbb{Z}$ and $u \in \mathbb{C} \setminus P_G$ we have $G(u + (m + in)v) = G(u)$.

Exercise 4. (4 points)
Use Fagnano's theorem to show that the duplication of the arc of a lemniscate is constructible with ruler and compass.

Deadline: Tuesday, Oct. 21, 2014 at the beginning of the lecture.