# Elliptic Functions and related topics

## Problem sheet 3

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**Exercise 1.** (4 points)
Let $\Omega = \mathbb{Z}\omega_1 \oplus \cdots \oplus \mathbb{Z}\omega_n$ be a full lattice in $\mathbb{R}^n$ endowed with the standard scalar product. By the definition of a lattice, the $n$-tuple $(\omega_1, \ldots, \omega_n)$ forms a basis of $\mathbb{R}^n$, so let $(\omega_1', \ldots, \omega_n')$ denote the corresponding Gram-Schmidt-orthogonalized basis (i.e. $\omega_j' = \omega_j + \sum_{i<j} \mu_{i,j}\omega_i$ for certain $\mu_{i,j}$ such that $\langle \omega_i', \omega_j' \rangle = 0$ for $i \neq j$). Show that for every $\omega \in \Omega \setminus \{0\}$ we have the inequality

$$\|\omega\| \geq \min\{\|\omega_1'\|, \ldots, \|\omega_n'\|\}.$$

**Exercise 2.** (4 points)
Let $\omega_1 := 5 + 2i\sqrt{7}$ and $\omega_2 := \frac{7}{2} + i\frac{3\sqrt{7}}{2}$ and define the lattice $\Omega := \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$.

(a) Show that $\Omega = \mathbb{Z} \oplus \left(\frac{1+i\sqrt{7}}{2}\right)\mathbb{Z}$.

(b) Using Exercise 1, determine

$$\min(\Omega) := \min\{|\omega| \ : \ \omega \in \Omega \setminus \{0\}\}$$

and

$$S(\Omega) := \{\omega \in \Omega \setminus \{0\} \ : \ |\omega| = \min(\Omega)\}.$$

**Exercise 3.** (4 points)
Find integers $m_1, m_2, m_3$, not all zero, such that

$$(*) \qquad \left| m_1\sqrt{2} + m_2\left(\sqrt{3} + i\sqrt{5}\right) + m_3 i\sqrt{7} \right| < 1$$

and that $m_1^2 + m_2^2 + m_3^2$ is minimal under the condition $(*)$.

**Exercise 4.** (4 points)
Let $\Omega = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ be a full lattice in $\mathbb{C}$ and let $\omega = m_1\omega_1 + m_2\omega_2 \in \Omega$. Show the equivalence of the following statements.

*(i)* There exists an $\omega' \in \Omega$ such that $\Omega = \mathbb{Z}\omega \oplus \mathbb{Z}\omega'$.

*(ii)* $m_1$ and $m_2$ are coprime.

*(iii)* If $n \in \mathbb{Z}$, $n > 1$, then $\frac{1}{n}\omega \notin \Omega$.

**Deadline:** Tuesday, Oct. 28, 2014 at the beginning of the lecture.