Elliptic Functions and related topics

Problem sheet 4

Dr. L. Rolen, Dr. M. H. Mertens

Throughout, let $L = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ denote a full lattice in $\mathbb{C}$.

Exercise 1. (4 points)
Let f be entire function. Show that

(a) if for every $\omega \in L$ there is a polynomial P_ω such that $f(z+\omega) = f(z) + P_\omega(z)$, then f is itself a polynomial,

(b) if for every $\omega \in L$ there exists a constant $c(\omega) \in \mathbb{C}$ such that $f(z+\omega) = c(\omega)f(z)$, then there are $a, b \in \mathbb{C}$ such that $f(z) = a \cdot \exp(bz)$ for all $z \in \mathbb{C}$.

Exercise 2. (4 points)
Classify all elliptic functions that have simple poles in the points $\frac{\omega}{2}$, $\omega \in L \setminus 2L$, and are holomorphic otherwise.

Exercise 3. (4 points)
Let $\wp = \wp_L$ denote the Weierstraß $\wp$-function of the lattice L.

(a) Show the differential equation

$$\wp''(z) = 6\wp(z)^2 + 2(e_1e_2 + e_2e_3 + e_3e_1).$$

(b) Find a non-trivial polynomial $P(X,Y)$ such that $P(\wp', \wp'') = 0$.

Hint: For part (b) it is OK to use a computer algebra system like Magma, Sage, Maple... to multiply out the polynomials.

Exercise 4. (4 points)
Let $G_k = G_k(L)$, $k \geq 4$ even, denote the weight k Eisenstein series of the lattice L.

(a) Show that that $G_k(\mathbb{Z} \oplus i\mathbb{Z}) = 0$ for $k \not\equiv 0 \pmod 4$ and $G_k\left(\mathbb{Z} \oplus \frac{1+i\sqrt{3}}{2}\mathbb{Z}\right) = 0$ for $k \not\equiv 0 \pmod 6$.

(b) Let $\tau \in \mathbb{H}$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Prove the transformation law

$$G_k(\mathbb{Z} \oplus \tau\mathbb{Z}) = (c\tau + d)^{-k} G_k\left(\mathbb{Z} \oplus \frac{a\tau + b}{c\tau + d}\mathbb{Z}\right).$$

Deadline: Tuesday, Nov. 04, 2014 at the beginning of the lecture.