Elliptic Functions and related topics

Problem sheet 5

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Exercise 1. (4 points)

(a) Find polynomials $P, Q \in \mathbb{Q}[X,Y]$ such that $G_{12} = P(G_4, G_6)$ and $G_{14} = Q(G_4, G_6)$.

(b) For $\tau \in \mathbb{H} = \{\tau \in \mathbb{C} \,:\, \operatorname{Im}\tau > 0\}$ and $k \geq 2$ an integer show the formula

$$\sum_{n\in\mathbb{Z}} \frac{1}{(\tau+n)^k} = \frac{(-2\pi i)^k}{(k-1)!}\sum_{r=1}^{\infty} r^{k-1}e^{2\pi i r\tau}.$$

Hint: Use the partial fraction decomposition of the contangent function.

Exercise 2. (4 points)

(a) Let L be a lattice that is stable under conjugation, i.e. $\overline{L} = L$. Show that $\overline{\wp_L(z)} = \wp_L(\overline{z})$ and $\overline{\wp_L'(z)} = \wp_L'(\overline{z})$, so that in particular, $\wp(z)$ is real for all $z \in \mathbb{C}\setminus L$ which are real or purely imaginary, and $\wp(z)$ is real resp. purely imaginary if $z \in \mathbb{C}\setminus L$ is so as well.

(b) Show that the following statements are all equivalent:

(i) $g_2(L)$ and $g_3(L)$ are both real.

(ii) For all $k \geq 4$, $G_k(L)$ is real.

(iii) Either all of the values e_1, e_2, e_3 are real or exactly one of them is real and the other two are complex conjugates of each other.

(iv) L is stable under conjugation.

Exercise 3. (4 points)
Let $L = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ be a rectangular lattice, i.e. such that $\frac{1}{i}\omega_1 > 0$ and $\omega_2 > 0$ and let $z \in \mathbb{C}\setminus L$. Show that $\wp(z)$ is real if and only if there is an $\omega \in L$ such that

$$z \in \frac{\omega}{2} + \mathbb{R} \quad \text{or} \quad z \in \frac{\omega}{2} + i\mathbb{R}.$$

Exercise 4. (4 points)
Let $L = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ be a rectangular lattice as in Exercise 3. Show that $\wp$ maps the open rectangle V with vertices $0, \frac{\omega_1}{2}, \frac{\omega_3}{2}, \frac{\omega_2}{2}$, biholomorphically to the lower half plane $\mathbb{H}^- := \{z \in \mathbb{C} \,:\, \operatorname{Im} z < 0\}$ by proving the following assertions.

(a) $\wp(V) \subseteq \mathbb{H}^-$.

(b) $\wp\left(\frac{\omega_3}{2} + V\right) = \wp(V) \subseteq \mathbb{H}^-$ and $\wp\left(\frac{\omega_1}{2} + V\right) = \wp\left(\frac{\omega_2}{2} + V\right) \subseteq \mathbb{H}$.

(c) $\wp : V \to \mathbb{H}^-$ is biholomorphic.

Deadline: Tuesday, Nov. 04, 2014 at the beginning of the lecture.