Elliptic Functions and related topics

Problem sheet 6

Dr. L. Rolen, Dr. M. H. Mertens

Exercise 1. (4 points)
Recall the definition of the Bernoulli numbers,

$$\frac{t}{e^t-1} =: \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n, \qquad (|t| < 2\pi).$$

Prove that

(a) $B_n \in \mathbb{Q}$ for all $n \in \mathbb{N}_0$,

(b) $B_1 = -\frac{1}{2}$ and $B_{2n+1} = 0$ for all $n \in \mathbb{N}$,

(c) Calculate B_{2n} for $n \in \{0, ..., 6\}$.

Exercise 2. (4 points)
Let Δ be the normalized discriminant function, i.e.

$$\Delta(\tau) = \sum_{n=1}^{\infty} \tau(n) q^n = q - 24q^2 + 252q^3 - 1472q^4 + O(q^5), \qquad q := e^{2\pi i \tau}.$$

(a) Use the results from Exercise 1 of Problem sheet 5 about the relations among the Eisenstein series to show that

$$\Delta(\tau) = \frac{1}{1728}(E_4(\tau)^3 - E_6(\tau)^2) = \frac{691}{762048}(E_{12}(\tau) - E_6(\tau)^2) = \frac{691}{432000}(E_4(\tau)^3 - E_{12}(\tau)).$$

(b) Show the congruence $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$ for all $n \in \mathbb{N}$.

Hint: Using a computer to check the constants is acceptable.

Exercise 3. (4 points)
Prove the following identity,

$$\int_1^{\infty} \frac{1}{\sqrt{4x^3 - 4x}} dx = \int_0^1 \frac{1}{\sqrt{1-t^4}} dt = \frac{\Gamma\left(\frac{1}{4}\right)^2}{4\sqrt{2\pi}},$$

where $\Gamma(s) := \int_0^{\infty} t^{s-1} e^{-t} dt$ denotes the gamma function.
Hint: The first equality is just a simple substitution, for the second, write $\Gamma\left(\frac{1}{4}\right)$ as a double integral and use polar coordinates to compute in another way.

Exercise 4. (4 points)
Let $L = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ be a full lattice in $\mathbb{C}$ and $\wp = \wp_L$ the corresponding Weierstraß $\wp$-function.

(a) For all $z, w \in \mathbb{C}$ such that $z, w, z \pm w \notin L$ prove that

$$\wp(z+w) - \wp(z-w) = -\frac{\wp'(z)\wp'(w)}{(\wp(z) - \wp(w))^2}.$$

(b) Use the Addition theorem and the differential equation for the $\wp$-function to find a polynomial $0 \neq P \in R[X, Y, Z]$, where $R = \mathbb{Z}[g_2, g_3]$ such that $P(\wp(z), \wp(w), \wp(z+w)) = 0$ for all $z, w \in \mathbb{C}$ such that $z, w, z + w \notin L$.

Hint: It is advisable to use a computer to find that polynomial.

Deadline: Tuesday, Nov. 18, 2014 at the beginning of the lecture.