Elliptic Functions and related topics

Problem sheet 9

Dr. L. Rolen, Dr. M. H. Mertens

Exercise 1. (4 points)
Let G, H be abelian groups and $\phi : G \to H$ and $\psi : H \to G$ be homomorphisms such that $[H : \phi(G)]$ and $[G : \psi(H)]$ are both finite. Suppose that there is an integer $m \geq 2$ such that

$$\psi \circ \phi(g) = m \cdot g \qquad \text{for all } g \in G,$$
$$\phi \circ \psi(h) = m \cdot h \qquad \text{for all } h \in H.$$

Prove that then $[G : mG]$ is finite and, more specifically, that

$$[G : mG] \leq [G : \psi(H)][H : \phi(G)].$$

Exercise 2. (4 points)
For a sequence of natural numbers N_r, its zeta-function is defined by

$$Z(T) := \exp\left(\sum_{r=1}^{\infty} N_r \frac{T^r}{r}\right).$$

Let a be a natural number.

(a) Show that for

$$N_r^{(a)} := \begin{cases} a & r \text{ even}, \\ 0 & r \text{ odd}, \end{cases},$$

the zeta function is a rational function in T if and only if a is even.

(b) Find a polynomial equation whose number of solutions over $\mathbb{F}_{p^r}$ equals $N_r^{(2)}$.

Exercise 3. (4 points)
Let q be a prime power and $\psi : \mathbb{F}_q \to \mathbb{C}^*$ be a fixed additive character. For multiplicative characters $\chi, \chi_1, \chi_2 : \mathbb{F}_q^* \mapsto \mathbb{C}^*$, recall the definitions of the Gauß sum $g(\chi) := \sum_{x \in \mathbb{F}_q} \chi(x)\psi(x)$ and the Jacobi sum $J(\chi_1, \chi_2) := \sum_{x \in \mathbb{F}_q} \chi_1(x)\chi_2(1-x)$. Prove the following assertions supposing that χ, χ_1, χ_2 are non-trivial, ε denotes the trivial character of $\mathbb{F}_q^*$, and $\overline{\chi}$ denotes the conjugate character.

(a) $g(\varepsilon) = -1$; $J(\varepsilon, \varepsilon) = q - 2$; $J(\varepsilon, \chi) = -1$; $J(\chi, \overline{\chi}) = -\chi(-1)$;
$J(\chi_1, \chi_2) = J(\chi_2, \chi_1)$;

(b) $g(\chi)g(\overline{\chi}) = \chi(-1)q$; $|g(\chi)| = \sqrt{q}$;

(c) $J(\chi_1, \chi_2) = \frac{g(\chi_1)g(\chi_2)}{g(\chi_1\chi_2}$ if $\chi_1 \neq \chi_2$.

Exercise 4. (4 points)
Let $p \equiv 1 \pmod 4$ be a prime and let $\chi_4 : \mathbb{F}_p^* \to \mathbb{C}^*$ be a multiplicative character of exact order 4. Show that $\chi(4) = \chi(-1) = 1$ if $p \equiv 1 \pmod 8$ and that $\chi(-1) = \chi(4) = -1$ if $p \equiv 5 \pmod 8$. Conclude that $\chi(-4) = 1$ in all cases.

Deadline: Tuesday, Dec. 9, 2014 at the beginning of the lecture.