ON THE CATEGORY OF FINITE-DIMENSIONAL REPRESENTATIONS OF $\mathrm{OSp}(r|2n)$: PART I

MICHAEL EHRIG AND CATHARINA STROPPEL

Abstract. We study the combinatorics of the category $\mathcal{F}$ of finite-dimensional integrable modules for the orthosymplectic Lie supergroup $\mathrm{OSp}(r\,|\,2n)$. In particular we present a positive counting formula for the dimension of the space of homomorphism between two projective modules. This refines earlier results of Gruson and Serganova. Moreover we construct an algebra $A_{\mathcal{B}}$ whose module category shares the combinatorics with $\mathcal{F}$. This algebra arises as a subquotient of a suitable limit of type D Khovanov algebras. It will turn out that A is isomorphic to the endomorphism algebra of a minimal projective generator of $\mathcal{F}$. In this way we provide a direct link from $\mathcal{F}$ to the geometry of isotropic Grassmannians and Springer fibres of type B/D, and to parabolic categories $\mathcal{O}$ of type B/D, with maximal parabolic of type A. We also indicate why the category $\mathcal{F}$ is not highest weight in general.

Contents

1. Introduction

Fix as ground field the complex numbers $\mathbb{C}$. This is the first part of a series of three papers, where we describe the category $\mathcal{C}$ of *finite-dimensional representations* of the orthosymplectic Lie supergroup $G = \mathrm{OSp}(r|2n)$ respectively the finite-dimensional *integrable representations* of the orthosymplectic Lie superalgebra $\mathfrak{g} = \mathfrak{osp}(r|2n)$. In particular we are interested in the combinatorics and the structure of the locally finite endomorphism ring of a projective generator of this category. (To be more precise: a projective generator only exists as a pro-object, but we still call it a projective generator and refer to [BD16, Theorem 2.4] for a detailed treatment.)

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Our main result is an *explicit description of the endomorphism ring of a minimal projective generator* for any block $\mathcal{B}$ in $\mathcal{C}$. We first describe in detail the underlying vector space in Theorem A, and then formulate the endomorphism theorem in Theorem B. As a consequence we deduce that the endomorphism algebra can be equipped with a $\mathbb{Z}$-grading. The definitions and results are illustrated by several examples. Theorem A provides an elementary way to compute dimensions of homomorphism spaces between projective objects, and Theorem B allows a concrete description of the corresponding categories. In small examples, we provide a description of the category $\mathcal{C}$ in terms of a quiver with relations.

The proof of Theorem B will appear in Part II, but we explain here the main ideas of the proof and the important and new phenomena which appear on the way. We believe that they are interesting on their own and also provide a conceptual explanation for the lack of desired properties of the category $\mathcal{C}$ (in comparison to the type A case). The (rather long) technical arguments required for the complete proof of Theorem B will appear in Part II, together with several applications to the representation theory. We also defer to part II the proof of Lemma 4.16, which is an easy observation as soon as the theory of Jucys-Murphys elements for Brauer algebras is available (which will be the case in Part II).

Understanding the representation theory of algebraic supergroups and in particular their category $\mathcal{C}$ of finite-dimensional representations is an interesting and difficult task with several major developments in recent years. We refer to the articles [Ser14], [Bru14], [MW14] for a nice description and overview of the state of art. Despite these remarkable results, in particular for the general linear case, but also for the category $\mathcal{O}$ for classical Lie superalgebras, there is still an amazingly poor understanding of the category $\mathcal{C}$ outside of type A.

At least for the orthosymplectic case we can provide here some new insights into the structure of these categories by giving a construction of endomorphism algebras of projective objects.

Our results are in spirit analogous to [BS12b] and many of the applications deduced there for the general linear Lie algebra can be deduced here as well (investigated in detail in Parts II and III). The orthosymplectic case however requires new arguments and a *totally new line of proof.* There are several subtle differences which make the case treated here substantially harder, the proofs more involved and conceptually different. The categories are much less well behaved than in type A. To prove the main Theorem B we first need to develop the basic underlying combinatorics for the orthosymplectic case, make it accessible for explicit calculations and also for categorification methods, then use non-trivial results from the representation theory of Brauer algebras and the Schur-Weyl duality for orthosymplectic Lie supergroups, and finally connect both with the theory of Khovanov algebras of type D. On the way we explain why (and to which extent) these categories are not highest weight, but we still manage to describe their combinatorics in terms of certain maximal parabolic Kazhdan-Lusztig polynomials of type B (or equivalently D by [ES13a, 9.7]).

The main results and the idea of the proof. To explain our results in more detail, fix $r, n \in \mathbb{Z}_{\geq 0}$ and consider a *vector superspace*, that is a $\mathbb{Z}_2$-graded vector space, $V = V_0 \oplus V_1$ of superdimension $(r|2n)$ with its Lie superalgebra $\mathfrak{gl}(V)$ of endomorphisms, see Section 3 for a precise definition. Then $\mathfrak{g} = \mathfrak{osp}(r|2n)$ is the Lie super subalgebra of $\mathfrak{gl}(V)$ which leaves invariant a fixed non-degenerate super-symmetric bilinear form β on V (that is a form of degree zero, symmetric on V_0 and antisymmetric on V_1), and $G = \mathrm{OSp}(r|2n)$ is the corresponding supergroup of automorphisms preserving this form. In particular, the extremal cases $r = 0$ respectively $n = 0$ give the classical simple Lie algebras $\mathfrak{so}(r)$ respectively $\mathfrak{sp}(2n)$ with the corresponding orthogonal and symplectic groups.

For simplicity, we restrict ourselves in this introduction to the case where $r = 2m + 1$ *is odd.*

Now consider the category $\mathcal{C}'$ of finite-dimensional representations of the supergroup $G' = \mathrm{SOSp}(r|2n)$, that is finite-dimensional representations for its Lie algebra $\mathfrak{g}$ in the sense of [Ser11], [Ser14] Like in the ordinary semisimple Lie algebra case, simple objects in $\mathcal{C}'$ are, up to a parity shift π, the highest weight modules $L^{\mathfrak{g}}(\lambda)$ which arise as quotients of Verma modules whose highest weights λ are integral and dominant. Hence for each such λ we have two irreducible representations, $L^{\mathfrak{g}}(\lambda)$ and $\pi L^{\mathfrak{g}}(\lambda)$ in the category $\mathcal{C}'$. More precisely $\mathcal{C}'$ decomposes into a sum of two equivalent categories $\mathcal{C}' = \mathcal{F}' \oplus \pi(\mathcal{F}')$, such that the simple objects in $\mathcal{F}'$ are labelled by integral dominant weights. In particular, it suffices to study the category $\mathcal{F}'$. Similarly we obtain the categories $\mathcal{C}$ and $\mathcal{F}$ if we work $G = \mathrm{OSp}(r|2n)$. Under our assumption an object in $\mathcal{F}$ is just an object in $\mathcal{F}'$ together with an action of the nontrivial element $\sigma \in G$ not contained in G' by multiplication by $\pm$ (In which case we leave out the decoration $\mathfrak{g}$ in the notation).

In contrast to the ordinary semsimple Lie algebra case, finite-dimensional representations of $\mathfrak{g}$ are in general *not completely reducible*. Already the tensor products $V^{\otimes d}$ of the natural representations V need not be.[1] One goal of our series of papers is to understand possible extensions between simple modules.

The category $\mathcal{F}$ is an interesting abelian tensor category with enough projective and injective modules (which in fact coincide, [BKN11, Proposition 2.2.2]). We have therefore a non-semisimple Calabi-Yau category which has additionally a monoidal structure.

The indecomposable projective modules are precisely the projective covers $P(\lambda)$ of the simple objects $L(\lambda)$. Given a block $\mathcal{B}$ of $\mathcal{C}$ there is the notion of *atypicality* or *defect*, $\mathrm{def}(\mathcal{B})$, which measures the non-semisimplicity of the block. In case the atypicality is zero, the block is semisimple. In general our Theorem B implies that the Loewy length of any projective module in $\mathcal{B}$ equals $2\,\mathrm{def}(\mathcal{B}) + 1$. Up to equivalence, the block $\mathcal{B}$ is determined by its atypicality, [GS10, Theorem 2], see also Remark 7.7.

[1] They are in fact semsimple in case of the general linear Lie superalgebra by the Schur-Weyl duality theorem of Sergeev [Ser84] and Berele-Regev [BR87], see [BS12a, Theorem 7.5], but in general not semisimple for $\mathfrak{osp}(r|2n)$, see [ES14, (1.1), Remark 3.3].

Remark. Our assumption on r comes into the picture here, since usually people (including also the above cited references) would consider the category of finite-dimensional representations for the group $G' = \mathrm{SOSp}(r|2n)$ instead of the group $G = \mathrm{OSp}(r|2n)$. In case r is odd, this makes no difference, since via the isomorphism of groups (1.4), the representation theory does not change in the sense that any block for G' gives rise to two equivalent blocks for G each of which is equivalent to the original block for G', see Section 4.2.1. In the even case the interplay is more involved. We however prefer to work with $\mathrm{OSp}(r|2n)$ instead of $\mathrm{SOSp}(r|2n)$, for instance because it allows us to make the connection to Deligne categories [Del96], [CH15] and Brauer algebras [Bra37].

For the rest of the introduction we identify now blocks for G' with blocks for G where the non-trivial element from $\mathbb{Z}/2\mathbb{Z}$ in (1.4) *acts as the identity.*

Dimension formula. Let λ, μ be dominant integral weights for $\mathfrak{g}$. To access the dimension of $\mathrm{Hom}_{\mathcal{F}}(P(\lambda), P(\mu))$ we encode the highest weights λ and μ in terms of *diagrammatic weights* λ and μ in the spirit of [BS11a], see Definition 6.6. Such a diagrammatic weight is a certain infinite sequence of symbols from $\{\times, \circ, \wedge, \vee\}$, with the property that two weights λ and μ are in the same block (abbreviating that $L(\lambda)$ and $L(\mu)$ are in the same block), if and only if the *core symbols* $\times$ and $\circ$ of the associated diagrammatic weights are at the same positions and the parity of the number of $\wedge$'s agree, see also Proposition 7.5 for a more precise statement. From Proposition 7.5 it also follows that the set $\Lambda(\mathcal{B})$ of diagrammatic weights attached to a block $\mathcal{B}$ is contained in a diagrammatic *block* Λ in the sense of [ES13a, 2.2].

Following [ES13a] we attach to the diagrammatic weights λ and μ via Definition 5.9 a pair of *cup diagrams* $\underline{\lambda}$, $\underline{\mu}$. If they have the same core symbols one can put the second on top of the first to obtain a *circle diagram* $\underline{\lambda}\overline{\mu}$. Our main combinatorial result (Theorem 7.1, Theorem B) is a *counting formula* for the dimensions:

Theorem A. *The dimension of* $\mathrm{Hom}_{\mathcal{F}}(P(\lambda), P(\mu))$ *equals the number of orientations* $\underline{\lambda}\nu\overline{\mu}$ *of* $\underline{\lambda}\overline{\mu}$ *if the circle diagram* $\underline{\lambda}\overline{\mu}$ *is defined and contains no non-propagating line, and the dimension is zero otherwise.*

By an *orientation* we mean another diagrammatic weight ν from the same block which, when putting it into the middle of the circle diagram, makes it oriented in the sense of Definition 5.17. In other words, we factorize the symmetric Cartan matrix C (see Theorem 4.18) into a product $C = AA^T$ with positive integral entries.

In [ES13a, 6.1] it was explained how to introduce an algebra structure $\mathbb{D}_\Lambda$ on the vector space with basis all oriented circle diagrams $\underline{\lambda}\nu\overline{\mu}$, where $\lambda, \mu, \nu \in \Lambda$. This algebra is called the *Khovanov algebra of type*[2] D attached to the (diagrammatic) block Λ. By [ES13a, Theorem 6.2] it restricts to an algebra structure on the vector space $\mathbb{D}_{\Lambda(\mathcal{B})}$ spanned by all circle diagrams $\underline{\lambda}\nu\overline{\mu}$ with $\lambda, \mu \in \Lambda(\mathcal{B})$ via the obvious idempotent truncation. Let $\mathbb{1}_\mathcal{B}$ be the

[2]Some readers might prefer to see here Khovanov algebras of type B appearing, but as shown in [ES13a, 9.7], this is just a matter of perspective: a Khovanov algebra of type B_n is isomorphic to one of type D_{n+1}.

corresponding idempotent projecting onto this subalgebra and consider the idempotent truncation $\mathbb{1}_{\mathcal{B}}\mathbb{D}_\Lambda\mathbb{1}_{\mathcal{B}}$. To make the connection with the combinatorics of Theorem A, we prove in Proposition 7.3 that its oriented circle diagrams which contain at least one non-propagating line, span an ideal $\mathbb{I}$ in $\mathbb{1}_{\mathcal{B}}\mathbb{D}_\Lambda\mathbb{1}_{\mathcal{B}}$. We call this the *nuclear ideal* and its elements *nuclear morphisms.*

Now our main theorem is the following, where $P = \oplus_{\lambda\in\Lambda(\mathcal{B})}P(\lambda)$ is a minimal projective generator of the chosen block $\mathcal{B}$.

Theorem B. *There is an isomorphism of algebras*

$$\mathbb{1}_{\mathcal{B}}\mathbb{D}_\Lambda\mathbb{1}_{\mathcal{B}}/\mathbb{I} \;\;\cong\;\; \operatorname{End}^{\mathrm{fin}}_{\mathcal{F}}(P).$$

Here, $\operatorname{End}^{\mathrm{fin}}_{\mathcal{F}}(P) = \oplus_{\lambda\in\Lambda(\mathcal{B})}\operatorname{Hom}_{\mathcal{F}}(P(\lambda), P)$ denotes the locally finite endomorphism ring of P. This locally finiteness adjustment is necessary, since the labelling set $\Lambda(\mathcal{B})$ of the indecomposable projective modules in $\mathcal{B}$ is infinite, and so we have to work with infinite blocks of diagrammatic weights. But we like to stress that for any chosen finite sum $\oplus_{\lambda\in J\subset\Lambda(\mathcal{B})}P(\lambda)$, the corresponding (ordinary) endomorphism ring is automatically finite dimensional. In practise, the endomorphism ring can then be computed in a quotient of an appropriate Khovanov algebra (of type B/D) attached to a finite diagrammatic block.

Since $\mathbb{D}_\Lambda$ is by construction a (non-negatively) $\mathbb{Z}$-graded algebra, and $\mathbb{I}$ is a graded ideal, we deduce that

Corollary C. $\mathbb{1}_{\mathcal{B}}\mathbb{D}_\Lambda\mathbb{1}_{\mathcal{B}}/\mathbb{I} \cong \operatorname{End}^{\mathrm{fin}}_{\mathcal{F}}(P)$ *is a graded algebra.*

In analogy to the general linear supergroup case, [BS12b], it is natural to expect that this grading is in fact a Koszul grading in the sense of [MOS09] which is a version of [BGS96] for locally finite algebras with infinitely many idempotents. This expectation is easy to verify for OSp(3|2) using the explicit description in Section 2.5, but it *fails to be true* in general, see Section 9.4.

The Khovanov algebras $\mathbb{D}_\Lambda$ of type D for *finite* diagrammatic blocks arose originally from classical highest weight Lie theory, since they describe blocks of parabolic category $\mathcal{O}$ of type B or equivalently of type D with maximal parabolic of type A, see [ES13a, Theorem 9.1 and Theorem 9.22], and hence describe the category of perverse sheaves on isotropic Grassmannians. They also have an interpretation in the context of the geometry of the Springer fibers of type D or C for nilpotent elements corresponding to two-row partitions, [ES12], [Wil15].

Our infinite diagrammatic weights Λ can be interpreted as elements in an appropriate limit of a sequence of finite diagrammatic weights. As in [BS11a] the resulting algebras $\mathbb{D}_\Lambda$ could then also be viewed as limit algebras $\mathbb{D}_{\Lambda_n}$ for certain finite blocks Λ_n. Hence, up to the ideal $\mathbb{I}$, our main theorem connects the category $\mathcal{F}$ to classical (that means *non-super*) infinite-dimensional highest weight Lie theory and classical (i.e. non-super) geometry in an appropriate limit. This is similar to the result for the general linear supergroups. [BS12b, Theorem 1.2]. It is also a shadow of the socalled super duality conjectures [CLW11], but in a subtle variation, since we deal here with finite-dimensional representations instead of the highest weight category $\mathcal{O}$. Moreover, taking this limit for type D Khovanov algebras is technically slightly more difficult than in type A, since the (naive) parallel construction mimicking the type A case would produce infinite weights with infinite defect. To circumvent this problem we apply a rather brutal diagrammatic trick and introduce so-called *frozen vertices* which force our infinite cup diagramms to have a finite number of cups, which means the defect stays finite. This procedure crucially depends on r and n. We expect that this diagrammatic trick also provides the passage between the limit categories introduced by Serganova in [Ser14] and the category $\mathcal{F}$.

Gruson-Serganova combinatorics. The proof of Theorem A is heavily based on the main combinatorial results of Gruson and Serganova, [GS10] and [GS13], who also introduced a version of cup diagram combinatorics for $\mathrm{SOSp}(r|2n)$ very similar to ours. An explicit translation between the two set-ups is given below in (7.55). There are however some small, but important differences in our approaches:

- Gruson and Serganova work with certain natural, but *virtual* modules in the Grothendieck group (the Euler characteristics $\mathcal{E}(\lambda)$), whereas our combinatorics relies on *actual filtrations* of the projective modules with the subquotients being shadows of cell modules for the Brauer algebra.
- Gruson and Serganova's formulas are *alternating* summation formulas, whereas ours are *positive counting formulas.*
- Gruson and Serganova work with the *special* orthosymplectic group, whereas we work with the orthosymplectic group, which is better adapted to the diagram combinatorics and connects directly to the representation theory of Brauer algebras via [Ser14, Theorem 3.4], [LZ15, Theorem 5.6].
- Gruson and Serganova's cup diagram combinatorics unfortunately does not give a direct connection to the theory of Hecke algebras and Kazhdan-Lusztig polynomials, whereas our Khovanov algebra of type D is built from the Kazhdan-Lusztig combinatorics of the hermitian symmetric pair $(\mathrm{D_n}, \mathrm{A_{n-1}})$, see [LS12], [ES13a].

Comparing Theorem A with [BS11b, (5.15)] and [BS12b, Theorem 2.1], our formulas indicate that one could expect some highest weight structure or at least some cellularity of each block $\mathcal{B}$ of $\mathcal{F}$ explaining our positive counting formulas and appearance of Kazhdan-Lusztig polynomials. But blocks of $\mathcal{F}$ are not highest weight and not even cellular in general, as the example from Section 2 illustrates, and there are no obvious candidates for cell modules. This is a huge difference to the case of $\mathfrak{gl}(m|n)$, where parabolic induction of a finite-dimensional representation of the Levi subalgebra $\mathfrak{gl}(m|n)_0 = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$ produces a finite-dimensional *Kac module.* These modules are the standard modules for the highest weight structure of the category of integrable finite-dimensional representations in that case, see [Bru03, Theorem 4.47] or [BS12b, Theorem 1.1]. Such a parabolic subalgebra, and hence such a class of modules is however not available for

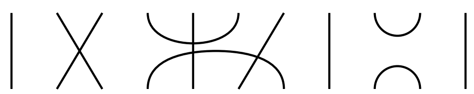
$\mathfrak{g} = \mathfrak{osp}(r|2n)$ if $r \geq 2, n \geq 1$. Nevertheless, we claim that our counting formula arises from some natural filtrations on projective objects, whose origin we like to explain now.

Tensor spaces and Brauer algebras. The tensor spaces $V^{\otimes d}$ for $d \geq 0$ from above already contain in some sense the complete information about the category $\mathcal{F}$. Namely, each indecomposable projective $P(\lambda)$ occurs in $V^{\otimes d}$ for some large enough d, see e.g. [CH15, Lemma 7.5]. By weight considerations and the action of σ one can easily check that $\mathrm{Hom}_G(V^{\otimes d}, V^{\otimes d'}) = \{0\}$ if d and d' have different parity (see also Remark 1). Hence to understand the spaces of morphisms between projective modules in a fixed block $\mathcal{B}$ of $\mathcal{F}$, it suffices to consider the tensor spaces for each parity of d separately. Moreover, since the trivial representation appears as a quotient of $V \otimes V$ (via the pairing given by β), we have a surjection $P \otimes V \otimes V \twoheadrightarrow P \otimes \mathbb{C} = P$ which splits if P is projective. Thus we obtain

Lemma D. *Let $J \subset \Lambda(\mathcal{B})$ be a finite subset of weights such that all $P(\lambda)$ are in the same block $\mathcal{B}$ of $\mathcal{F}$. Then $P' = \oplus_{\lambda \in J} P(\lambda)$ appears as a direct summand of $V^{\otimes d}$ for some large enough d.*

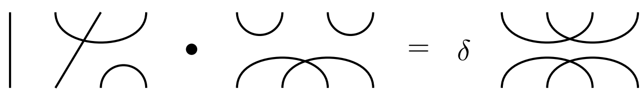
To achieve our goal (to determine the endomorphism ring of all such P') we first consider endomorphisms of these tensor spaces $V^{\otimes d}$. For this we use a super analogue of a result from classical invariant theory of the semisimple orthogonal and symplectic Lie algebras studied by Brauer in [Bra37].

For fixed $d \in \mathbb{Z}_{\geq 0}$ and $\delta \in \mathbb{C}$, the *Brauer algebra* $\mathrm{Br}_d(\delta)$ is an algebra structure on the vector space with basis all equivalence classes of Brauer diagrams for d. A *Brauer diagram for* d is a partitioning of the set $\{\pm 1, \pm 2, \ldots, \pm d\}$ into two element subsets. One can display this by identifying $\pm j$ with the point $(j, \pm 1)$ in the plane and connect two points in the same subset by an arc inside the rectangle $[1, d] \times [-1, 1]$. Here is an example of a Brauer diagram for $d = 11$:



(1.1)

Given two Brauer diagrams D_1 and D_2 we can stack D_2 on top of D_1. The result is again a Brauer diagram D after we removed possible internal loops and the process is independent of the chosen visualization. Setting $D_1 D_2 = \delta^c D$, where c is the number of internal loops removed, defines the associative algebra structure $\mathrm{Br}_d(\delta)$ on the vector space with basis given by Brauer diagrams. Here is an example of the product of two basis vectors:



(1.2)

We use the following important result.

Proposition E ([Ser14, Theorem 3.4], [LZ14a, Theorem 5.6]). *Let $\delta = r - 2n$. Then the canonical algebra homomorphism*

$$\mathrm{Br}_d(\delta) \twoheadrightarrow \mathrm{End}_{\mathrm{OSp}(r|2n)}(V^{\otimes d}). \tag{1.3}$$

is surjective.

Hereby a Brauer diagram D acts on a tensor product $v_1 \otimes v_2 \otimes \cdots \otimes v_d$ as follows: We identify the d tensor factors with the bottom points of the diagrams. Whenever there is a cap (connecting horizontally two bottom points) we pair the corresponding vectors using β and obtain a scalar multiple w of the vector $v_{i_1} \otimes \cdots \otimes v_{i_t}$, where t equals d minus the number of caps and $v_{i_j} = v_{i_k}$ if the jth top point not connected to another top point (by a cup) is connected with the kth point at the bottom not connected by a cap. Finally we insert for each cup a pair of new factors arising as the image of 1 under counit map $\mathbb{C} \mapsto V \otimes V$, see e.g. [Ser14, (3.3)] or [ES14] for details.

We like to stress that the map (1.3) fails to be surjective in general if we work with $G = \mathrm{SOSp}(2m|2n)$ or its Lie algebra $\mathfrak{osp}(2m|2n)$, see e.g. [LZ15] and [ES14, Remark 5.8].

To be able to control the category $\mathcal{C}$ by the action of the Brauer algebra on tensor spaces we prefer to work with $\mathrm{OSp}(r|2n)$ instead of the more commonly studied semsisimple supergroup $\mathrm{SOSp}(r|2n)$ or with its Lie algebra $\mathfrak{g}$. This requires then however a translation and adaption of the results from the literature (including [GS10], [GS13]) to $\mathrm{OSp}(r|2n)$. In case $r = 2m + 1$ is odd this is an easy task, since we have

$$\mathrm{OSp}(2m+1|2n) \cong \mathrm{SOSp}(2m+1|2n) \times \mathbb{Z}/2\mathbb{Z}, \tag{1.4}$$

where the generator of the cyclic group is minus the identity. If $r = 2m$ is even, we only have a semidirect product

$$\mathrm{OSp}(2m|2n) \cong \mathrm{SOSp}(2m|2n) \rtimes \mathbb{Z}/2\mathbb{Z}, \tag{1.5}$$

and the situation is rather involved. A larger part of the present paper is devoted to this problem. We believe that in contrast to the case of $\mathrm{SOSp}(r|2n)$, the blocks for $\mathrm{OSp}(r|2n)$ are completely determined by their atypicality, see Remark 7.7.

Remark. Instead of considering only single tensor product spaces $V^{\otimes d}$ as in (1.3), one might prefer to work with the tensor subcategory $(V, \otimes)$ of $\mathcal{F}(\mathrm{OSp}(r|2n))$ generated by V (for any fixed nonnegative integers r, n). Then the surjection (1.3) can in fact be extended to a full monoidal functor from the Brauer *category* $\mathrm{Br}(\delta)$ to $(V, \otimes)$, see e.g. [CW12]. An object in the Brauer category (which is just a natural number d) is sent to $V^{\otimes d}$ and a basis morphism (that is a Brauer diagram as in (1.1) but not necessarily with the same number of bottom and top points) is sent to the corresponding intertwiner. Hence the Brauer category controls all intertwiners. Again, this statement is not true for the special orthosymplectic groups, not even for the odd cases $\mathrm{SOSp}(2m+1|2n)$, since one can find some integer d with a nontrivial morphism from $V^{\otimes d}$ to $V^{\otimes (d+1)}$, see Remark 9.1. Such a morphism however can not come from a morphism in the Brauer category, since for a diagram in the Brauer category the number of dots on the top and on the number of dots on the bottom of the diagram have the same parity. The

Brauer category can also be identified with Deligne's universal symmetric category Rep(O_δ), [Del96], as used e.g. in [CH15], [Ser14].

We therefore chose to work with $G = \mathrm{OSp}(r|2n)$ instead of the more commonly studied supergroup SOSp($r|2n$).

As a direct consequence of (1.3) and Lemma D, we can find an idempotent $e = e_{d,\delta}$ in $\mathrm{Br}_d(\delta)$ such that the following holds

Proposition F. *Let $I, P' = \oplus_{\lambda\in J} P(\lambda)$ be as in Lemma D. There is a surjective algebra homomorphism*

$$\Phi = \Phi_{d,\delta} : \quad e\,\mathrm{Br}_d(\delta)e \quad \twoheadrightarrow \quad \mathrm{End}_{\mathcal{F}}(P') \tag{1.6}$$

identifying the primitive idempotents in both algebras.

Comes and Heidersdorf obtain in [CH15, Theorem 7.3] a classification of the indecomposable summands in $V^{\otimes d}$ in terms of idempotents of the Brauer algebra and our (yet another) cup diagram combinatorics for the Brauer algebra developed in [ES13b]. They moreover prove in [CH15, Lemma 7.15] that the indecomposable *projective* summands $P(\lambda)$ correspond to cup diagrams with maximal possible number, namely $\min\{m, n\}$, of cups. Unfortunately their theorem provides no way to read off the weight λ from the cup diagram. In part II we will show that a diagrammatic trick as in [BS12a, Lemma 8.18] for the walled Brauer algebra can be applied in our set-up as well (with roughly the same proof) and provides a correspondence that allows to read off the highest weights of the projective summands.

More precisely, let c be the cup diagram corresponding to a projective summand $P^{\mathfrak{g}}(\lambda)$ in $V^{\otimes d}$ via [CH15, Lemma 7.15]. Let ν be the corresponding diagrammatic weight, that is the unique diagrammatic weight ν such that $c = \underline{\nu}$, see Remark 5.12. Now given such a diagrammatic weight ν let $\nu^\dagger$ be the diagrammatic weight obtained by changing all $\wedge$'s into $\vee$'s and all $\vee$'s into $\wedge$'s. Then the correspondence is given by the following:

Proposition G. *In the set-up from above we have $\nu^\dagger = \lambda^\infty$, with λ^∞ the infinite diagrammatic weight attached to λ via* (5.41).

For Examples see Section 9. Note that for P' as in Lemma D, Proposition G provides a description of the idempotent e in (1.6).

The shadow of a quasi-hereditary or cellular structure. Fortunately, the representation theory of $\mathrm{Br}_d(\delta)$ for arbitrary $\delta \in \mathbb{Z}$ is by now reasonably well understood thanks to the results in [Mar09], [CDVM09], [CDV11], [ES13b] and [ES15]. In particular it is known that $\mathrm{Br}_d(\delta)$ is a quasi-hereditary algebra if $\delta \neq 0$ and still cellular in case $\delta = 0$, [Mar09], see also [ES13b]. For the sake of simplicity let us assume for the next paragraph that $\delta \neq 0$. Let $\mathcal{P}_d$ be the usual labelling set of simple modules for $\mathrm{Br}_d(\delta)$ by partitions, see [Mar09], [CDV11], and denote by $L_d(\alpha)$, $P_d(\alpha)$, and $\Delta_d(\alpha)$ the simple module, its projective cover and the corresponding standard module respectively attached to $\alpha \in \mathcal{P}_d$. Then standard properties for quasi-hereditary algebras, the BGG-reciprocity, see [Don98, A2.2 (iv)], and the existence of a duality preserving the simple objects, give us that

$$\begin{aligned}\dim \mathrm{Hom}_{\mathrm{Br}_d(\delta)}(P_d(\alpha), P_d(\beta)) &= [P_d(\beta) : L(\alpha)] \\ &= \sum_{\eta \in \mathcal{P}_d} [\Delta_d(\eta) : L(\alpha)](P_d(\beta) : \Delta(\eta)) \\ &= \sum_{\eta \in \mathcal{P}_d} (P_d(\alpha) : \Delta(\eta))(P_d(\beta) : \Delta(\eta)) \quad (1.7)\end{aligned}$$

where $[M : L]$ denotes the multiplicity of a simple module L in a Jordan-Hölder series of M and $(P : \Delta)$ denotes the multiplicity of Δ appearing as a subquotient in a standard filtration of P. As first observed in [Mar09], see also [CDV11], all the occurring multiplicities are either 0 or 1 and given by some parabolic Kazhdan-Lusztig polynomial (which is in fact monomial) evaluated at 1.

Now since $\mathrm{Br}_d(\delta)$ is quasi-hereditary with standard modules $\Delta_d(\alpha)$, the idempotent truncation $e\,\mathrm{Br}_d(\delta)e$ is cellular, with cell modules $\Delta_d(\alpha)e$, see [KX98, Proposition 4.3]. Hence the endomorphism algebra in question is by Proposition F *a quotient* of a cellular algebra. Unfortunately, we have the following:

Quotients of cellular algebras need not be cellular.

However, there is still some extra structure. Given a projective $e\,\mathrm{Br}_d(\delta)e$-module $P_d(\lambda)$ with $\lambda^\dagger \in J$ and e as in Proposition F, and a fixed filtration with subquotients certain cell modules $\Delta^{e\,\mathrm{Br}_d(\delta)e}(\nu^\dagger)$, then this filtration induces a filtration[3] of the projective module $P(\lambda^\dagger) \in \mathcal{F}$ via the algebra homomorphism $\Phi_{d,\delta}$.

The shape of the successive subquotients, $\Delta^{\mathcal{F}}(\lambda^\dagger, \nu^\dagger) = \Delta(\lambda^\dagger, \nu^\dagger)$ do however in general not only depend on $\nu^\dagger$, but also on $\lambda^\dagger$, that means on the projective module we chose. (A priori, in case of higher multiplicities, two subquotients might even differ although they arise from isomorphic cell modules in $P_d(\lambda)$. But this turns out to be irrelevant for our counting and so we can ignore it.) It is the multiplicities of these quotients of the cell modules which we count in our main Theorem A. In particular we still have a well-defined *positive counting formula* for the multiplicities for each given pair (λ, ν), although we do not have standard or cell modules we have still some control.

The failure of quasi-heriditarity and cellularity of the category $\mathcal{F}$ is encoded in the kernel of the maps $\Phi_{d,\delta}$.

We need now to connect this information with Theorem A and describe the kernel.

Graded version $\mathrm{Br}_d^{\mathrm{gr}}(\delta)$ of the Brauer algebra $\mathrm{Br}_d(\delta)$. To determine the number $(P_d(\alpha) : \Delta_d(\eta))$ one can, as in [CDV11] or [ES13b], first assign to the partition α and η a diagrammatic weight, denoted by the same letter and compute the corresponding cup diagram $\underline{\alpha}$ using the rules in Definition 5.9.

[3]More generally given a finite-dimensional algebra A and a quotient algebra A/I with surjection $\Psi : A \to A/I$, any A-module filtration of Ae_λ for an idempotent e_λ induces a filtration on $A/I\Psi(e_\lambda)$ by taking just the image.

Then the multiplicity in question is non-zero (and therefore equal to 1) if and only if $\underline{\alpha}\eta$ is oriented in the sense of (5.48), see [ES13a, (8.64)].

Now consider the endomorphism ring $\mathrm{B}_d(\delta) := \mathrm{End}_{\mathrm{Br}_d(\delta)}(\oplus_{\alpha\in\mathcal{P}_d} P_d(\alpha))$ of a minimal projective generator of $\mathrm{Br}_d(\delta)$. That is $\mathrm{B}_d(\delta)$ is the basic algebra underlying $\mathrm{Br}_d(\delta)$. Then a basis of $\mathrm{B}_d(\delta)$ can be labelled by pairs of oriented cup diagrams of the form $(\underline{\alpha}\eta, \underline{\beta}\eta)$ or equivalently by oriented circle diagrams $\underline{\alpha}\eta\overline{\beta}$, where $\alpha, \eta, \beta \in \mathcal{P}_d$.

By [ES13a, Section 6.2], there is an algebra structure $\mathrm{B}_d^{\mathrm{gr}}(\delta)$ on the vector space spanned by such circle diagrams using the multiplication rules of the type D Khovanov algebras from [ES13a]. Using the degree function on circle diagrams from (5.48), this turns $\mathrm{B}_d(\delta)$ into a $\mathbb{Z}$-graded algebra $\mathrm{B}_d^{\mathrm{gr}}(\delta)$. By [ES13b] together with [ES15, Theorem A], this gives a new realization of the basic Brauer algebra, namely a graded lift of our basic algebra $\mathrm{B}_d(\delta)$:

Theorem H. *The algebra $\mathrm{B}_d^{\mathrm{gr}}(\delta)$ is isomorphic to the basic Brauer algebra $\mathrm{B}_d(\delta)$ as ungraded algebras.*

In fact this grading can also be extended to provide a grading on $\mathrm{Br}_d(\delta)$, but for our purposes it suffices to work with the basic algebra $\mathrm{B}_d(\delta)$.

Explicit endomorphism algebra. Given the diagrammatic description $\mathrm{B}_d^{\mathrm{gr}}(\delta)$ of $\mathrm{B}_d(\delta)$, the idempotent truncation $e\mathrm{Br}_d^{\mathrm{gr}}(\delta)e$ (which is by definition a subalgebra) is easily described by only allowing certain cup diagrams depending on e, in fact precisely the $\underline{\lambda}$ corresponding to elements in J. However, the description of the kernel of $\Phi_{d,\delta}$ is more tricky. For (1.3) this kernel was described in [LZ14b], but their description is not very suitable for our purposes. Instead we obtain a similar result as in [BS12a, Theorem 8.1 and Corollary 8.2] (although the proof is quite different), which will be explained in Part II. It implies that the kernel is controlled by the ideal $\mathbb{I}$ of nuclear endomorphisms.

To summarize: for any choice of block $\mathcal{B}$ and set of weights J as in Lemma D and $\Phi_{d,\delta}$ as in Proposition F, we will map in Part II the circle diagrams from $\mathrm{Br}_d^{\mathrm{gr}}(\delta)$, picked out by $e\mathrm{Br}_d^{\mathrm{gr}}(\delta)e$, to the corresponding basis element of some Khovanov algebra $\mathbb{D}_\Delta$ using the identification from Proposition G and the identification from Definition 6.6 of integral highest weights with diagrammatic weights. We will show that under this assignment the kernel of $\Phi_{d,\delta}$ restricted to $e\,\mathrm{Br}_d(\delta)e$ is mapped to the ideal $\mathbb{I}$ of nuclear circle diagrams. As a result we deduce then finally Theorem B.

We like to stress that although our results are similar to the results from [BS12b], [BS12a], the line of arguments is in some sense opposite. Whereas in [BS12a] the known input was the finite-dimensional representation theory of $\mathfrak{gl}(m|n)$ from which the existence of the graded walled Brauer algebra was deduced, the known input now is the graded Brauer algebra from [ES15], from which then the information about the representation theory of $\mathrm{OSp}(r|2n)$ is deduced. In particular, our arguments here rely on a good understanding of the (graded) Brauer algebra. Moreover, the combinatorics needed here does not (yet) have a conceptual description in terms of crystal bases and categorifications, although first steps in this direction can be found for the Brauer algebra in [ES13b] and for the category $\mathcal{O}$ for $\mathfrak{g}$ in [BW13].

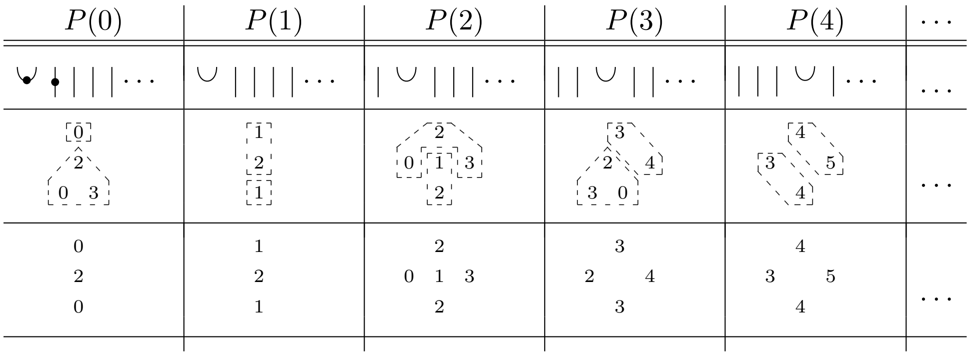


FIGURE 1. Indecomposable projectives in $\mathbb{1}_{\mathcal{B}}\mathbb{D}_{\Lambda}\mathbb{1}_{\mathcal{B}}$ versus $\mathbb{1}_{\mathcal{B}}\mathbb{D}_{\Lambda}\mathbb{1}_{\mathcal{B}}/\mathbb{I}$.

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2. An illustrating example: $\mathcal{F}(\mathrm{SOSp}(3|2))$

Before we start we describe blocks of $\mathcal{F}(\mathrm{SOSp}(3|2))$ in terms of a quiver with relations using Theorems A and B, see also Section 4.2.1 for the precise passage to $\mathcal{F}(\mathrm{OSp}(3|2))$.

In this case $m = n = 1$ and $\delta = 1$. By [GS10, Lemma 7 (ii)], all blocks are semisimple or equivalent to the principal block $\mathcal{B}$ (of atypicality 1) containing the trivial representation. Hence we restrict ourselves to this block. The explicit description of this category is not new, but was obtained already by Germoni in [Ger00, Theorem 2.1.1]. We reproduce the result here using our diagram algebras.

2.1. **The indecomposable projectives and the algebra.** By Definition 4.2, the block $\mathcal{B}$ contains the simple modules $L^{\mathfrak{g}}(\lambda)$ of (with our choice of Borel) highest weight λ, where $\lambda \in \{\lambda_a \mid a \geq 0\}$ with $\lambda_0 = (0\,|\,0)$ in the standard basis and $\lambda_a = (a\,|\,a-1)$ if $a > 0$. We abbreviate the corresponding module by $L(a)$ and let $P(a)$ be its projective cover. We assign to $P(a)$ via Definitions 6.6 and 5.9 the cup diagram $\underline{\lambda_a}$ as shown in the second line of Figure 1 (with infinitely many rays to the right), see also Section 9.2.

The oriented circle diagrams built from the given cup diagrams are displayed in Figure 2. They are obtained by putting one of the cup diagrams upside down on top of another one and then equip the result with an orientation as in (5.48). These diagrams then form a basis of the algebra $\mathbb{1}_{\mathcal{B}}\mathbb{D}_{\Lambda}\mathbb{1}_{\mathcal{B}}$. The multiplication is given by the rules from [ES13a, Section 6.2].

The last two (framed) oriented circle diagrams in Figure 2 are exactly those which contain at least one non-propagating line. They span the nuclear ideal $\mathbb{I}$, in $\mathbb{1}_{\mathcal{B}}\mathbb{D}_{\Lambda}\mathbb{1}_{\mathcal{B}}$, see Lemma 7.3.

The algebra $\mathbb{1}_{\mathcal{B}}\mathbb{D}_{\Lambda}\mathbb{1}_{\mathcal{B}}$ can be equipped with a positive $\mathbb{Z}$-grading, via (5.48), such that the basis vectors are homogeneous of degree as displayed in Figure 2 with homogeneous ideal $\mathbb{I}$. It descends to a grading on $\mathbb{1}_{\mathcal{B}}\mathbb{D}_{\Lambda}\mathbb{1}_{\mathcal{B}}/\mathbb{I}$, and hence gives a grading on the category $\mathcal{B}$.

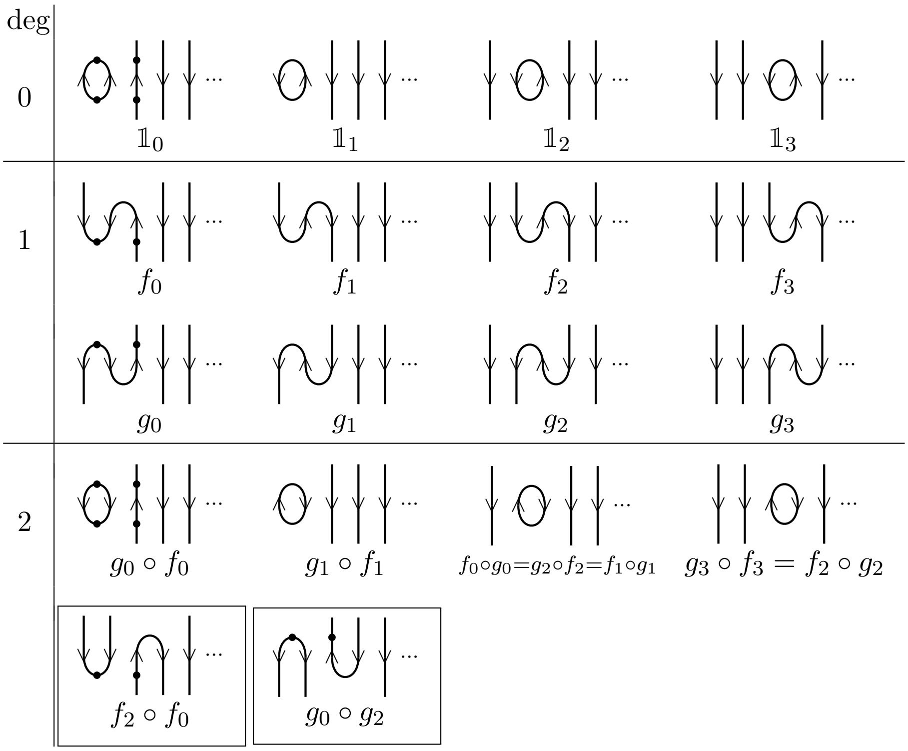
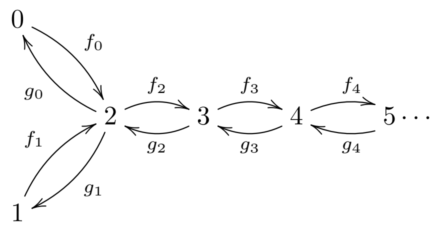


FIGURE 2. The homogeneous basis vectors of $\mathbb{1}_{\mathcal{B}}\mathbb{D}_\Lambda\mathbb{1}_{\mathcal{B}}$.

2.2. **The block $\mathcal{B}$ in terms of a quiver with relations.** From the definition of the multiplication, see [ES13a], we directly deduce, using Theorem B, an explicit description of the locally finite endomorphism ring $\mathrm{End}^{\mathrm{fin}}_{\mathcal{F}}(P)$:

Theorem A. *The algebra $\mathbb{1}_{\mathcal{B}}\mathbb{D}_\Lambda\mathbb{1}_{\mathcal{B}}/\mathbb{I}$ is isomorphic (as graded algebras) to the path algebra of the following infinite quiver (with grading given by putting all arrows in degree 1)*

(2.8)



modulo the (homogeneous) ideal generated by (the homogeneous relations) $f_{i+1} \circ f_i = 0 = g_i \circ g_{i+1}$, $g_{i+1} \circ f_{i+1} = f_i \circ g_i$ for $i \geq 0$ and $g_0 \circ f_1 = 0 = g_1 \circ f_0$, $f_0 \circ g_0 = g_1 \circ f_1 = g_2 \circ f_2$ and $f_2 \circ f_0 = 0 = g_0 \circ g_2$. Here, the last two relations are the relations from $\mathbb{I}$. In particular, the category of finite-dimensional modules of this algebra is equivalent to the principal block $\mathcal{B}$ of $\mathcal{F}(\mathrm{SOSp}(3|2))$.

The structure of the indecomposable projective modules for $\mathbb{1}_{\mathcal{B}}\mathbb{D}_\Lambda\mathbb{1}_{\mathcal{B}}$ is displayed in the third line of Figure 1, where each number stands for the corresponding simple module. The height where the number of a simple module occurs, indicates the degree it is concentrated in, when we consider it as a module for the graded algebra. We displayed the grading filtration which in this case however agrees with the radical and the socle filtration. In comparison, the fourth line shows the structure of the indecomposable projective modules for $\mathbb{1}_{\mathcal{B}}\mathbb{D}_\Lambda\mathbb{1}_{\mathcal{B}}/\mathbb{I}$.

The description of the category $\mathcal{F}(\mathrm{SOSp}(3|2))$ in Theorem A reproduces Germoni's result. The algebra $\mathbb{1}_{\mathcal{B}}\mathbb{D}_\Lambda\mathbb{1}_{\mathcal{B}}/\mathbb{I}$ in this example also occurs under the name *zigzag algebra* (of type D_∞) in the literature, [CL10, 2.3]. In contrast to the general case of $\mathcal{F}(\mathrm{OSp}(r|2n))$, it is representation finite as shown in [Ger00].

2.3. **The failure of quasi-hereditarity and cellularity.** By [ES13a, Section 6], $\mathbb{D}_\Lambda$ is quasi-hereditary, and so $\mathbb{1}_{\mathcal{B}}\mathbb{D}_\Lambda\mathbb{1}_{\mathcal{B}}$ is a cellular algebra, [KX98, Proposition 4.3]. Hence we have cell modules $\Delta(\lambda) = \Delta_{\mathbb{1}_{\mathcal{B}}\mathbb{D}_\Lambda\mathbb{1}_{\mathcal{B}}}(\lambda)$, indexed by some labelling set (in fact certain weights $\lambda \in \Lambda$, but we ignore this here). We indicate in Figure 1 (by grouping the composition factors) these cell modules. Note that there are two cell modules with simple head labelled by 1, since the truncation of our quasi-hereditary algebra $\mathbb{D}$ is not compatible with the quasi-hereditary ordering. Hence although $\mathbb{D}_\Lambda$ is quasi-hereditary, the truncation $\mathbb{1}_{\mathcal{B}}\mathbb{D}_\Lambda\mathbb{1}_{\mathcal{B}}$ is only cellular. Factoring out the ideal $\mathbb{I}$ of nuclear morphisms means we kill some of the simple composition factors. The result is displayed then in the last line in Figure 1. One can also see there for instance that the cell module $\Delta(2)$ gives rise to a different subquotient in $P(0)$ than in $P(3)$, namely in the notation from Section 1 we have

$$\Delta(2) = \begin{array}{ccc} & 2 & \\ 0 & & 3 \end{array} \quad \rightsquigarrow \quad \Delta(0,2) \;=\; \begin{array}{c} 2 \\ 0 \end{array} \quad \text{and} \quad \Delta(3,2) \;=\; \begin{array}{c} 2 \\ 3 \end{array} \tag{2.9}$$

We leave it to the reader to show that this algebra is not cellular.

2.4. **The Calabi-Yau property.** We observe that the resulting projective modules for $\mathbb{1}_{\mathcal{B}}\mathbb{D}_\Lambda\mathbb{1}_{\mathcal{B}}/\mathbb{I}$ are self-dual and they are in fact the maximal self-dual quotients of the indecomposable projective modules for $\mathbb{1}_{\mathcal{B}}\mathbb{D}_\Lambda\mathbb{1}_{\mathcal{B}}$. Hence the projective modules become injective, a property which is well-known to hold in $\mathcal{B}$, see [BKN11]. More conceptually let $\tilde{\tau} : \mathbb{1}_{\mathcal{B}}\mathbb{D}_\Lambda\mathbb{1}_{\mathcal{B}} \to \mathbb{C}$ be the linear (trace) map defined on basis vectors b from Figure 2 by

$$\tilde{\tau}(b) \quad = \quad \begin{cases} 1 & \text{if } b \text{ is of the form } \underline{\lambda}\nu\overline{\lambda} \text{ (i.e. it has reflection symmetry} \\ & \text{in the horizontal reflection line), and } \deg(b) = 2, \\ 0 & \text{otherwise.} \end{cases}$$

and consider the corresponding bilinear map τ defined on basis vectors as

$$\tau : \quad \mathbb{1}_{\mathcal{B}}\mathbb{D}_\Lambda\mathbb{1}_{\mathcal{B}} \times \mathbb{1}_{\mathcal{B}}\mathbb{D}_\Lambda\mathbb{1}_{\mathcal{B}} \;\to\; \mathbb{C} \tag{2.10}$$

$$\tau(b_1, b_2) \;=\; \tilde{\tau}(b_1 b_2), \tag{2.11}$$

This is by definition a symmetric form, which is however degenerate with radical $\mathrm{rad}(\tau)$ spanned by the nuclear morphisms $f_2 \circ f_0$, $g_0 \circ g_2$. In particular $\mathbb{1}_{\mathcal{B}}\mathbb{D}_\Lambda\mathbb{1}_{\mathcal{B}}/\mathbb{I} = \mathbb{1}_{\mathcal{B}}\mathbb{D}_\Lambda\mathbb{1}_{\mathcal{B}}/\mathrm{rad}(\tau)$ is a noncommutative (symmetric) Frobenius algebra. Hence we have the following

The block $\mathcal{B}$ is the maximal Calabi-Yau quotient (with respect to τ) of the category of finite-dimensional $\mathbb{1}_{\mathcal{B}}\mathbb{D}_\Lambda\mathbb{1}_{\mathcal{B}}$-modules.

A corresponding characterisation holds for arbitrary blocks and arbitrary m, n and will be studied in detail in a subsequent paper.

2.5. **Koszulity.** By constructing an explicit (infinite) linear projective resolutions for each simple module one can check in this special example, that the algebra $\mathbb{1}_{\mathcal{B}}\mathbb{D}_{\Lambda}\mathbb{1}_{\mathcal{B}}/\mathbb{I}$ here is a locally finite Koszul algebra in the sense of [MOS09]. In general, the algebras from Theorem B are however not Koszul, see Section 9.4.

3. The orthosymplectic supergroup and its Lie algebra

For the general theory of Lie superalgebras we refer to [Mus12].

3.1. **Lie superalgebras.** By a *(vector) superspace* we always mean a finite-dimensional $\mathbb{Z}/2\mathbb{Z}$ -graded vector space $V = V_0 \oplus V_1$. For any homogeneous element $v \in V$ we denote by $|v| \in \{0,1\}$ its parity. The integer $\dim V_0 - \dim V_1$ is called the *supertrace* of V, and we denote by $\mathrm{sdim} V = \dim V_0 \,|\, \dim V_1$ its *superdimension.* Given a superspace V let $\mathfrak{gl}(V)$ be the corresponding *general Lie superalgebra*, i.e. the superspace $\mathrm{End}_{\mathbb{C}}(V)$ of all endomorphism with the superbracket defined on homogeneous elements by

$$[X,Y] = X \circ Y - (-1)^{|X|\cdot|Y|} Y \circ X. \tag{3.12}$$

If V has superdimension $a \,|\, b$ then $\mathfrak{gl}(V)$ is also denoted by $\mathfrak{gl}(a \,|\, b)$. It can be realized as the space of $(a+b) \times (a+b)$-matrices viewed as superspace with the matrix units on the block diagonals being even, and the other matrix units being odd elements, and the bracket given by the supercommutator (3.12).

We fix now $r, n \in \mathbb{Z}_{\geq 0}$ and a superspace $V = V_0 \oplus V_1$ of superdimension $r \,|\, 2n$ equipped with a non-degenerate supersymmetric bilinear form $\langle -, - \rangle$, i.e. a bilinear form $V \times V \to \mathbb{C}$ which is symmetric when restricted to $V_0 \times V_0$, skew-symmetric on $V_1 \times V_1$ and zero on mixed products. From now on we fix also $m \in \mathbb{Z}_{\geq 0}$ such that $r = 2m$ or $r = 2m+1$. We denote by $\delta = r - 2n$, the supertrace of the natural representation.

Definition 3.1. The *orthosymplectic Lie superalgebra* $\mathfrak{g} = \mathfrak{osp}(V)$ is the Lie supersubalgebra of $\mathfrak{gl}(V)$ consisting of all endomorphisms which respect a fixed supersymmetric bilinear form. Explicitly, a homogeneous element $X \in \mathfrak{osp}(V)$ has to satisfy for any homogeneous $v \in V$

$$\langle Xv, w \rangle + (-1)^{|X|\cdot|v|} \langle v, Xw \rangle = 0. \tag{3.13}$$

In case one prefers a concrete realization in terms of endomorphism of a superspace one could choose a homogeneous basis v_i of V and consider the supersymmetric bilinear form given by the (skew)symmetric matrices

$$J^{\mathrm{sym}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \mathbf{1}_m \\ 0 & \mathbf{1}_m & 0 \end{pmatrix} \qquad \text{and} \qquad J^{\mathrm{skew}} = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix}$$

where $\mathbf{1}_k$ denotes the respective identity matrix and r is either equal to $2m+1$ or equal to $2m$, in the latter case the first column and row of J^{sym}

are removed. Then $\mathfrak{g}$ can explicitly be realized as the Lie super subalgebra of matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $\mathfrak{gl}(r|2n)$ where

$$A^t J^{\text{sym}} + J^{\text{sym}} A = B^t J^{\text{sym}} - J^{\text{skew}} C = D^t J^{\text{skew}} + J^{\text{skew}} D = 0.$$

Then $\mathfrak{g}_0$ (resp. $\mathfrak{g}_1$) is the subset of all such matrices with $B = C = 0$ (resp. $A = D = 0$). In particular, $\mathfrak{g}_0 \cong \mathfrak{so}(r) \oplus \mathfrak{sp}(2n)$ with its standard Cartan $\mathfrak{h} = \mathfrak{h}_0$ of all diagonal matrices. We denote therefore $\mathfrak{g}$ also by $\mathfrak{osp}(r|2n)$.

Let

$$X = X(\mathfrak{g}) = \bigoplus_{i=1}^{m} \mathbb{Z}\varepsilon_i \oplus \bigoplus_{j=1}^{n} \mathbb{Z}\delta_j. \tag{3.14}$$

be the integral weight lattice. Here the ε_i's and δ_j's are the standard basis vectors of $\mathfrak{h}^*$ picking out the i-th respectively $(r+j)$-th diagonal matrix entry. We fix on $\mathfrak{h}^*$ the standard symmetric bilinear form $(\varepsilon_i, \varepsilon_j) = \delta_{i,j}$, $(\varepsilon_i, \delta_j) = 0$, $(\delta_i, \delta_j) = -\delta_{i,j}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. We define the parity (an element in $\mathbb{Z}/2\mathbb{Z}$) of the ε's to be 0 and the parity of the δ's to be 1 and extend linearly to the whole weight lattice. In the following by a *weight* we always mean an *integral weight*. We will often denote weights as $(m+n)$-tuples $(a_1, a_2, \ldots, a_m \mid b_1, b_2, \ldots, b_n)$, with the coefficients of the ε's to the left and those of the δ's on the right of the vertical line.

Then $\mathfrak{g}$ decomposes into *root spaces* that is into weight spaces with respect to the adjoint action of $\mathfrak{h}$,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

One can check that $\mathfrak{g}_\alpha$ is either even or odd. Hence we can talk about *even roots* and *odd roots*. Explicitly, the roots for $\mathfrak{osp}(2m|2n)$ respectively $\mathfrak{osp}(2m+1|2n)$ are the following, with $1 \leq i \leq r$, $1 \leq j \leq n$,

$$\begin{aligned} \Delta(2m|2n) &= \{\pm\varepsilon_i \pm \varepsilon_{i'}, \pm\delta_j \pm \delta_{j'} \mid i \neq i'\} \cup \{\pm\varepsilon_i \pm \delta_j\}, \qquad (3.15) \\ \Delta(2m+1|2n) &= \{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_{i'}, \pm\delta_j \pm \delta_{j'} \mid i \neq i'\} \cup \{\pm\delta_j, \pm\varepsilon_i \pm \delta_j\}, \end{aligned}$$

where all signs can be chosen independently, and the indices are such that the expressions exist. (In each case the first set contains the even and the second the odd roots).

3.2. **Supergroups and super Harish-Chandra pairs.** Let $\mathrm{G}(r|2n)$ be the affine algebraic supergroup $\mathrm{OSp}(r|2n)$ over $\mathbb{C}$. Using scheme-theoretic language, $\mathrm{G}(r|2n)$ can be regarded as a functor G from the category of commutative superalgebras over $\mathbb{C}$ to the category of groups, mapping a commutative superalgebra $A = A_0 \oplus A_{\overline{1}}$ to the group $\mathrm{G}(A)$ of all invertible $(r+2n) \times (r+2n)$ orthosymplectic matrices over A, see [Ser11, Section 3]. This functor is representable by *an affine super Hopf algebra* (i.e. a finitely generated supercommutative super Hopf algebra) $R = \mathbb{C}[G]$, and there is a contravariant equivalence of categories between the categories of algebraic supergroups and of affine super Hopf algebras extending the situation of algebraic groups in the obvious way, see e.g. [Fio03], [Mas13]. By restricting the functor G to commutative algebras defines an (ordinary) algebraic group G_0 represented by $R/I = \mathbb{C}[G]/I$, where I is the ideal generated by the odd part of R. In case of $\mathrm{G}(r|2n)$ this algebraic group is just $\mathrm{O}(r) \times \mathrm{Sp}(2n)$. Similarly, we also have the affine algebraic supergroup $G' = \mathrm{SOSp}(r|2n)$ over $\mathbb{C}$ with algebraic group $\mathrm{SO}(r) \times \mathrm{Sp}(2n)$. They both have $\mathfrak{osp}(r|2n)$ as the associated Lie superalgebra. We refer to [Ser11, Section 3] for more details on these constructions.

We are interested in the category $\mathcal{C}(r|2n)$ of finite-dimensional G-modules or equivalently the category of *integrable* $\mathfrak{g}$-modules, that is *Harish-Chandra modules* for the super Harish-Chandra pair $(\mathfrak{g}, G, \mathrm{a})$, where a is the adjoint action, see [Vis11]. To make this more precise we recall some facts.

Definition 3.2. A *super Harish-Chandra pair* is a triple $(\mathfrak{g}, G_0, \mathrm{a})$ where $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie superalgebra, G_0 is an algebraic group with Lie algebra $\mathfrak{g}_0$, and a is a G_0-module structure on $\mathfrak{g}$ whose differential is the adjoint action of $\mathfrak{g}_0$. A *Harish-Chandra module* for such a triple or shorter a $(\mathfrak{g}, G_0, \mathrm{a})$*-module* is then a $\mathfrak{g}$-module M with a compatible G_0-module structure (that means the derivative of the G_0-action agrees with the action of $\mathfrak{g}_0$). We denote by $(\mathfrak{g}, G_0, \mathrm{a}) - \mathrm{mod}$ the category of finite-dimensional $(\mathfrak{g}, G_0, \mathrm{a})$-modules.

Given any super Harish-Chandra pair $(\mathfrak{g}, G_0, \mathrm{a})$ one can construct a Hopf superalgebra $R = \mathbb{C}[G]$ such that $\mathfrak{g}$ is the Lie algebra of the supergroup G and $R/I = \mathbb{C}[G_0]$. Namely $R = \mathrm{Hom}_{U(\mathfrak{g}_0)}(U(\mathfrak{g}), \mathbb{C}[G_0])$, where $U(\mathfrak{h})$ denotes the universal enveloping (super)algebra of a Lie superalgebra $\mathfrak{h}$, and where $U(\mathfrak{g}_0)$ acts by left invariant derivations on $\mathbb{C}[G_0]$, see [Ser11, (3.1)] for precise formulas and the description of the Hopf algebra structure - with the dependence on the action a. This assignment $\Phi : (\mathfrak{g}, G_0, \mathrm{a}) \mapsto G$ for any super Harish-Chandra pair can be extended in fact to the following equivalence of categories, see [Vis11], [Bal11] for the super case, but the arguments are very much parallel to the classical case from [Kos77].

Proposition 3.3. *The assignment* $\Phi : (\mathfrak{g}, G_0, \mathrm{a}) \mapsto G$ *induces the following:*

1. *The category of super Harish-Chandra pairs is equivalent to the category of algebraic supergroups.*
2. *Moreover the category of finite-dimensional* $(\mathfrak{g}, G_0, \mathrm{a})$*-modules, denoted by* $(\mathfrak{g}, G_0, \mathrm{a}) - \mathrm{mod}$*, is equivalent to the category* $G - \mathrm{mod}$ *of finite-dimensional* G*-modules.*

The category $\mathcal{C}(G)$ of finite-dimensional G-modules has enough projectives and enough injectives, [Ser11, Lemma 9.1], in fact projective and injective modules agree, [BKN11, Proposition 2.2.2].

4. Finite-dimensional representations

We are interested in the category of Harish-Chandra modules for the particular super Harish-Chandra pairs arising from the (special) orthosymplectic supergroups. Since the action a in this cases is always the adjoint action, we will usually omit it in the notation. From now on we fix $r, n \in \mathbb{Z}_{\geq 0}$ and use the following abbreviations:

$$\mathfrak{g} = \mathfrak{osp}(r|2n) \qquad G = \mathrm{OSp}(r|2n), \qquad G' = \mathrm{SOSp}(r|2n),$$
$$\mathcal{C} = \mathcal{C}(\mathrm{OSp}(r|2n)) \qquad \mathcal{C}' = \mathcal{C}(\mathrm{SOSp}(r|2n))$$

The simple objects in $\mathcal{C}'$ are (viewed as Harish-Chandra modules) highest weight modules, and every simple object is up to isomorphism and parity shift uniquely determined by its highest weight, see e.g. [Ser11, Theorem 9.9]. More precisely, the category $\mathcal{C}'$ decomposes into a direct sum of two equivalent subcategories

$$\mathcal{C}' = \mathcal{F}(\mathrm{SOSp}(r|2n)) \oplus \Pi\mathcal{F}(\mathrm{SOSp}(r|2n))$$

namely $\mathcal{F}(\mathrm{SOSp}(r|2n))$ and its parity shift $\Pi\mathcal{F}(\mathrm{SOSp}(r|2n))$, where the category $\mathcal{F}(\mathrm{SOSp}(r|2n))$ contains all objects such that the parity of any weight space agrees with the parity of the corresponding weight. Similarly, the categories $\mathcal{C}$ decomposes into $\mathcal{F} = \mathcal{F}(\mathrm{OSp}(r|2n))$ and its parity shift, where $\mathcal{F}$ consists of those modules that lie in $\mathcal{F}(\mathrm{SOSp}(r|2n))$ when restricted to $\mathrm{SOSp}(r|2n)$. Therefore it suffices to restrict ourselves to study the summands

$$\mathcal{F}' = \mathcal{F}(\mathrm{SOSp}(r|2n)) \qquad \text{respectively} \qquad \mathcal{F} = \mathcal{F}(\mathrm{OSp}(r|2n)),$$

which we will consider now in more detail.

4.1. **Finite-dimensional representations of** $\mathrm{SOSp}(r|2n)$**.** We first consider the case of the *special* orthosymplectic group. With a fixed Borel subalgebra in G', every irreducible module in $\mathcal{F}'$ (viewed as integrable module for $\mathfrak{g}$) is a quotient of a Verma module, in particular a highest weight module $L(\lambda)$ for some highest weight λ, see [Ser11, Theorem 9.9]. The occurring highest weights are precisely the *dominant* weights. The explicit dominance condition on the coefficients of λ in our chosen basis (3.14) depends on the choice of Borel we made, since in the orthosymplectic case Borels are not always pairwise conjugate. We follow now closely [GS13] and fix the slightly unusual choice of Borel with maximal possible number of odd simple roots, see [GS10], with the simple roots given as follows:

For $\mathfrak{osp}(2m+1|2n)$:

$$\text{if } m \geq n: \begin{cases} \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{m-n} - \varepsilon_{m-n+1}, \\ \varepsilon_{m-n+1} - \delta_1, \delta_1 - \varepsilon_{m-n+2}, \varepsilon_{m-n+2} - \delta_2, \ldots, \varepsilon_m - \delta_n, \delta_n. \end{cases}$$

$$\text{if } m < n: \begin{cases} \delta_1 - \delta_2, \delta_2 - \delta_3, \ldots, \delta_{n-m-1} - \delta_{n-m}, \\ \delta_{n-m} - \varepsilon_1, \varepsilon_1 - \delta_{n-m+1}, \delta_{n-m+1} - \varepsilon_2, \ldots, \varepsilon_m - \delta_n, \delta_n. \end{cases}$$

For $\mathfrak{osp}(2m|2n)$:

$$\text{if } m > n: \begin{cases} \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{m-n-1} - \varepsilon_{m-n}, \\ \varepsilon_{m-n} - \delta_1, \delta_1 - \varepsilon_{m-n+1}, \varepsilon_{m-n+1} - \delta_2, \ldots, \delta_n - \varepsilon_m, \\ \delta_n + \varepsilon_m. \end{cases}$$

$$\text{if } m \leq n: \begin{cases} \delta_1 - \delta_2, \delta_2 - \delta_3, \ldots, \delta_{n-m} - \delta_{n-m+1}, \\ \delta_{n-m+1} - \varepsilon_1, \varepsilon_1 - \delta_{n-m+2}, \delta_{n-m+2} - \varepsilon_2, \ldots, \delta_n - \varepsilon_m, \\ \delta_n + \varepsilon_m. \end{cases}$$

This choice of Borel has the advantage that the dominance conditions look similar to the ordinary ones for semisimple Lie algebras and moreover is best adapted to our diagrammatics. To formulate it, let ρ be half of the sum of positive even roots minus the sum of positive odd roots, explicitly given as follows.

For $\mathfrak{g} = \mathfrak{osp}(2m+1|2n)$: In this case $\frac{\delta}{2} = m - n + \frac{1}{2}$ and

$$\rho = \begin{cases} \left(\frac{\delta}{2}-1, -\frac{\delta}{2}-2, \dots, \frac{1}{2}, -\frac{1}{2}, \dots, -\frac{1}{2} \,\middle|\, \frac{1}{2}, \dots, \frac{1}{2}\right) & \text{if } m \geq n, \\ \left(-\frac{1}{2}, \dots, -\frac{1}{2} \,\middle|\, -\frac{\delta}{2}, -\frac{\delta}{2}-1, \dots, \frac{1}{2}, \dots, \frac{1}{2}\right) & \text{if } m < n. \end{cases}$$

For $\mathfrak{g} = \mathfrak{osp}(2m|2n)$: In this case $\frac{\delta}{2} = m - n$ and

$$\rho = \begin{cases} \left(\frac{\delta}{2}-1, \frac{\delta}{2}-2, \dots, 1, 0, \dots, 0 \,\middle|\, 0, \dots, 0\right) & \text{if } m > n, \\ \left(0, \dots, 0 \,\middle|\, -\frac{\delta}{2}, -\frac{\delta}{2}-1, \dots, 1, 0, \dots, 0\right) & \text{if } m \leq n. \end{cases}$$

Remark 4.1. Note that $n = 0$ gives $\rho = (m-1, m-2, \dots, 0)$ in the even orthogonal case and $\rho = (m - \frac{1}{2}, m - \frac{3}{2}, \dots, \frac{1}{2})$ in the odd orthogonal case; and $\rho = (n, n-1, \dots, 1)$ in case $m = 0$. These are precisely the values for ρ for the semisimple Lie algebras of type $\mathbf{D}_m$, $\mathbf{B}_m$, $\mathbf{C}_n$.

Definition 4.2. For our choice of Borel, a weight $\lambda \in X(\mathfrak{g})$ is *dominant* if

$$\lambda + \rho = \sum_{i=1}^{m} a_i \varepsilon_i + \sum_{j=1}^{n} b_j \delta_j \tag{4.16}$$

satisfies the following dominance condition, see [GS10].

For $\mathfrak{g} = \mathfrak{osp}(2m+1|2n)$:

(i) either $a_1 > a_2 > \cdots > a_m \geq \frac{1}{2}$ and $b_1 > b_2 > \cdots > b_n \geq \frac{1}{2}$,
(ii) or $a_1 > a_2 > \cdots > a_{m-l-1} > a_{m-l} = \cdots = a_m = -\frac{1}{2}$ and
$b_1 > b_2 > \cdots > b_{n-l-1} \geq b_{n-l} = \cdots = b_n = \frac{1}{2}$,

For $\mathfrak{g} = \mathfrak{osp}(2m|2n)$:

(i) either $a_1 > a_2 > \cdots > a_{m-1} > |a_m|$ and $b_1 > b_2 > \cdots > b_n > 0$,
(ii) or $a_1 > a_2 > \cdots > a_{m-l-1} \geq a_{m-l} = \cdots = a_m = 0$ and
$b_1 > b_2 > \cdots > b_{n-l-1} > b_{n-l} = \cdots = b_n = 0$.

The set of dominant weights is denoted $X^+(\mathfrak{g})$.

Note $X^+(\mathfrak{osp}(2m+1|2n)) \subset \frac{1}{2} + \mathbb{Z}^{m+n}$ and $X^+(\mathfrak{osp}(2m|2n)) \subset \mathbb{Z}^{m+n}$.

Definition 4.3. Assume $r = 2m$. If $\lambda \in X^+(\mathfrak{g})$, written in the form 4.16, satisfies $a_m \neq 0$, then we write $\lambda = \lambda_+$ if $a_m > 0$, and we write $\lambda = \lambda_-$ if $a_m < 0$.

Definition 4.4. Weights satisfying (i) are called *tailless* and the number $l+1$ from Definition 4.2 is the *tail length,* tail(λ)*, of* λ.

Example 4.5. The zero weight is always dominant with maximal possible tail length, namely tail$(0) = \min\{m, n\}$.

For $\lambda \in X^+(\mathfrak{g})$ let $P^{\mathfrak{g}}(\lambda)$ be the projective cover of $L^{\mathfrak{g}}(\lambda)$, see [BKN11] for a construction, and $I^{\mathfrak{g}}(\lambda)$ its injective hull. Then the $P^{\mathfrak{g}}(\lambda)$ (respectively $I^{\mathfrak{g}}(\lambda)$), with $\lambda \in X^+(\mathfrak{g})$, form a complete non-redundant set of representatives for the isomorphism classes of indecomposable projective (respectively injective) objects in $\mathcal{F}'$.

4.2. **Finite-dimensional representations of** $\mathrm{OSp}(r|2n)$**.** We recall the classification of the simple finite-dimensional representations of G using the one for G'.

For this let $\sigma \in \mathbb{Z}/2\mathbb{Z}$ be the non-unit element. Via (1.5) it corresponds to an element in $\mathrm{OSp}(2m|2n)$, also called σ, which acts as an involution on $\mathrm{SOSp}(2m|2n)$ preserving the maximal torus. On weights it acts as $\sigma(\varepsilon_m) = -\varepsilon_m$ and $\sigma(\varepsilon_i) = \varepsilon_i$, $\sigma(\delta_i) = \delta_i$ for $1 \leq i \leq m-1$, $1 \leq j \leq n$. We have $\mathrm{OSp}(2m|2n) = \mathrm{SOSp}(2m|2n) \cup \sigma\,\mathrm{SOSp}(2m|2n)$.

To construct the irreducible representations we use a very special case of Harish-Chandra induction which we recall now. Let $(\mathfrak{g}, H, a)$ be a super Harish-Chandra pair and H' a subgroup of H such that $(\mathfrak{g}, H', a' = a_{|H'})$ is also a super-Harish Chandra pair. Then there is a *(Harish-Chandra) induction functor*

$$\mathrm{Ind}_{\mathfrak{g},H'}^{\mathfrak{g},H} : \quad (\mathfrak{g}, H', a') - \mathrm{mod} \quad \longrightarrow \quad (\mathfrak{g}, H, \mathrm{a}) - \mathrm{mod}, \tag{4.17}$$

where $\mathrm{Ind}_{\mathfrak{g},H'}^{\mathfrak{g},H} N = \{f : H \to N \mid f(xh) = xf(h), h \in H, x \in H'\}$ is the usual induction for algebraic groups, [Jan03, 3.3]. The H-action is given by the right regular action and the $\mathfrak{g}$-action is just the $\mathfrak{g}$-action on N. This functor $\mathrm{Ind}_{\mathfrak{g},H'}^{\mathfrak{g},H}$ is left exact. It sends injective objects to injective objects, [Jan03, Proposition 3.9], and it is right adjoint to the restriction functor $\mathrm{Res}_{\mathfrak{g},H}^{\mathfrak{g},H'}$, [Jan03, Proposition 3.4].

We apply this to the two super Harish-Chandra pairs $(\mathfrak{g}, G')$ and $(\mathfrak{g}, G)$.

4.2.1. *The odd case:* $\mathrm{SOSp}(2m+1|2n)$. In the odd case, the element σ is central and thanks to (1.4) we can describe the simple objects in $\mathcal{F}$:

Proposition 4.6. *For* $G = \mathrm{OSp}(2m+1|2n)$ *the set*

$$X^+(G) \quad = \quad X^+(\mathfrak{g}) \times \mathbb{Z}/2\mathbb{Z} = \{(\lambda, \epsilon) \mid \lambda \in X^+(\mathfrak{g}), \epsilon \in \{\pm\}\}$$

is a labelling set for the isomorphism classes of irreducible G*-modules in* $\mathcal{F}$*. The simple module* $L(\lambda, \pm)$ *is hereby just the simple* G'*-module* $L^{\mathfrak{g}}(\lambda)$ *extended to a module for* G *by letting* σ *act by* ± 1*.*

Note that $\mathrm{Ind}_{\mathfrak{g},G'}^{\mathfrak{g},G} L^{\mathfrak{g}}(\lambda) \cong L(\lambda, +) \oplus L(\lambda, -)$. By construction, the category $\mathcal{F}$ decomposes as $\mathcal{F}^+ \oplus \mathcal{F}^-$, where $\mathcal{F}^\pm$ is the full subcategory of $\mathcal{F}$ containing all representations with composition factors only of the form $L(\lambda, \pm)$, and moreover $\mathcal{F}^\pm \cong \mathcal{F}'$.

Remark 4.7. In particular we have for $\lambda, \mu \in X^+(\mathfrak{g})$

$$\mathrm{Hom}_{\mathcal{F}}(I(\lambda, +), I(\mu, -)) \;=\; \{0\} \;=\; \mathrm{Hom}_{\mathcal{F}}(I(\lambda, -), I(\mu, +)), \tag{4.18}$$
$$\mathrm{Hom}_{\mathcal{F}}(P(\lambda, +), P(\mu, -)) \;=\; \{0\} \;=\; \mathrm{Hom}_{\mathcal{F}}(P(\lambda, -), P(\mu, +)), \tag{4.19}$$

and the nonzero morphism spaces are controlled by those for $\mathfrak{g}$:

$$\mathrm{Hom}_{\mathcal{F}}(P(\lambda, \pm), P(\mu, \pm)) \;=\; \mathrm{Hom}_{\mathcal{F}'}(P^{\mathfrak{g}}(\lambda), P^{\mathfrak{g}}(\mu)). \tag{4.20}$$

Corollary 4.8. *Given* (λ, ϵ) *and* (μ, ϵ') *in* $X^+(G)$ *it holds:* $L(\lambda, \epsilon)$ *and* $L(\mu, \epsilon')$ *are in the same block of* $\mathcal{F}$ *if and only if* $\epsilon = \epsilon'$ *and* $L^{\mathfrak{g}}(\lambda)$ *and* $L^{\mathfrak{g}}(\mu)$ *are in the same block of* $\mathcal{F}'$*.*

4.2.2. *The even case:* OSp($2m|2n$). In the even case the situation is slightly more involved, since σ is not central. We first construct the irreducible representations using Harish-Chandra induction.

Remark 4.9. Note that the natural vector representation $V = \mathbb{C}^{2m+1|2n}$ can be identified with $L(\epsilon, -1)$. In particular, σ acts on a d-fold tensor product $V^{\otimes d}$ by -1 iff d is odd and by 1 iff d is even. This implies that there is no G-equivariant morphism from $V^{\otimes d}$ to $V^{\otimes d'}$ in case $d \not\equiv d' \bmod 2$.

Definition 4.10. For $G = \mathrm{OSp}(2m|2n)$ we introduce the following set:

$$X^+(G) \quad = \quad \{(\lambda, \epsilon) \mid \lambda \in X^+(\mathfrak{g})/\sigma \text{ and } \epsilon \in \mathrm{Stab}_\sigma(\lambda)\},$$

where Stab_σ denotes the stabilizer of λ under the group generated by σ.

Notation 4.11. To avoid overloading of notation we usually just write λ instead of (λ, ϵ) if the representatives of λ have trivial stabilizer. Otherwise the orbit has a unique element. In this case the stabilizer has two elements and we often write $(\lambda, +)$ for (λ, e) and $(\lambda, -)$ for (λ, σ). In addition we write λ^G for the σ-orbit of $\lambda \in X^+(\mathfrak{g})$. We will omit this superscript if the orbit consists of a single element.

Proposition 4.12. *Consider* $\mathfrak{g} = \mathfrak{osp}(2m|2n)$, $G = \mathrm{OSp}(2m|2n)$, *and* $G' = \mathrm{SOSp}(2m|2n)$. *Assume*

$$\lambda = \sum_{i=1}^{m} a_i \varepsilon_i + \sum_{j=1}^{n} b_j \delta_j - \rho \in X^+(\mathfrak{g}) \tag{4.21}$$

and let $L^{\mathfrak{g}}(\lambda) \in \mathcal{F}'$ *be the corresponding irreducible highest weight representation of* G' *with injective cover* $I^{\mathfrak{g}}(\lambda)$. *Then with* $\mathrm{Ind}_{\mathfrak{g},G'}^{\mathfrak{g},G}$ *from* (4.17) *the following holds:*

(1) for induced irreducible representations:
 (a) If $a_m \neq 0$ *then the* $(\mathfrak{osp}(2m|2n), \mathrm{OSp}(2m|2n))$*-module*
 $$L(\lambda^G) = L(\lambda^G, e) \quad := \quad \mathrm{Ind}_{\mathfrak{g},G'}^{\mathfrak{g},G} L^{\mathfrak{g}}(\lambda) \tag{4.22}$$
 is irreducible. Moreover,
 $$\mathrm{Ind}_{\mathfrak{g},G'}^{\mathfrak{g},G} L^{\mathfrak{g}}(\lambda) \quad \cong \quad \mathrm{Ind}_{\mathfrak{g},G'}^{\mathfrak{g},G} L^{\mathfrak{g}}(\sigma(\lambda)). \tag{4.23}$$
 (b) If $a_m = 0$ *then*
 $$\mathrm{Ind}_{\mathfrak{g},G'}^{\mathfrak{g},G} L^{\mathfrak{g}}(\lambda) \quad =: \quad L(\lambda, +) \oplus L(\lambda, -) \tag{4.24}$$
 is a direct sum of $L(\lambda, +)$, *and* $L(\lambda, -)$, *two non-isomorphic irreducible* $(\mathfrak{osp}(2m|2n), \mathrm{OSp}(2m|2n))$*-modules. As* G'*-modules they are isomorphic to* $L^{\mathfrak{g}}(\lambda)$.

(2) for induced injective representations:
 (a) If $a_m \neq 0$ *then* $I(\lambda^G) := \mathrm{Ind}_{\mathfrak{g},G'}^{\mathfrak{g},G} I^{\mathfrak{g}}(\lambda)$ *is the indecomposable injective hull of* $L(\lambda^G)$.
 (b) If $a_m = 0$ *then* $\mathrm{Ind}_{\mathfrak{g},G'}^{\mathfrak{g},G} I^{\mathfrak{g}}(\lambda) \cong I(\lambda, +) \oplus I(\lambda, -)$, *where* $I(\lambda, \pm)$ *denotes the injective hull of* $L(\lambda, \pm)$.

(3) The same formulas hold for the indecomposable projective objects.

As a consequence we obtain the following:

Proposition 4.13. *The set* $\{L(\lambda,\epsilon) \mid (\lambda,\epsilon) \in X^+(G)\}$ *is a complete non-redundant set of representatives for the isomorphism classes of irreducible* G*-modules in* $\mathcal{F}$.

Proof of Propositions 4.12 and 4.13. The arguments for part (1) of Proposition 4.12 and the classification of irreducible representations in Proposition 4.13 are precisely as in the classical case, see e.g. [GW09, 5.5.5]. By construction and the proof there,

$$\operatorname{Res}^{\mathfrak{g},G'}_{\mathfrak{g},G} L(\lambda,\pm) \cong L^{\mathfrak{g}}(\lambda) \text{ and } \operatorname{Res}^{\mathfrak{g},G'}_{\mathfrak{g},G} L(\lambda^G) \cong L^{\mathfrak{g}}(\lambda) \oplus L^{\mathfrak{g}}(\sigma(\lambda)) \tag{4.25}$$

where $(\lambda,\pm)$ is as in (1)(b) respectively λ as in (1)(a). More precisely, it is proved that the modules $L(\lambda,\pm)$ are isomorphic to $L^{\mathfrak{g}}(\lambda)$ as G'-modules; with the action extended to G such that σ acts on the highest weight vector by multiplication with the scalar 1 or -1 (but see also Lemma 4.16 and Remark 4.7).

Since the functor $\operatorname{Ind}^{\mathfrak{g},G}_{\mathfrak{g},G'}$ sends injective objects to injective objects, and is right adjoint to the restriction functor, [Jan03, Propositions 3.4. and 3.9], the statements (2) of Proposition 4.12 can be deduced as follows. Let $\mu \in X^+(\mathfrak{g})$ with

$$\mu = \sum_{i=1}^{m} a'_i\varepsilon_i + \sum_{j=1}^{n} b'_j\delta_j - \rho. \tag{4.26}$$

If $a_m \neq 0$ in (4.21) we obtain for any simple G-module $L \in \mathcal{F}$

$$\begin{aligned}&\operatorname{Hom}_{\mathcal{F}}(L, \operatorname{Ind}^{\mathfrak{g},G}_{\mathfrak{g},G'} I^{\mathfrak{g}}(\lambda))\\ \cong &\begin{cases}\operatorname{Hom}_{\mathcal{F}'}(L^{\mathfrak{g}}(\mu)\oplus L^{\mathfrak{g}}(\sigma(\mu)), I^{\mathfrak{g}}(\lambda)) & \text{if } L = L(\mu^G), \text{ i.e. } a'_m \neq 0,\\ \operatorname{Hom}_{\mathcal{F}'}(L^{\mathfrak{g}}(\mu), I^{\mathfrak{g}}(\lambda)) = \{0\}, & \text{if } L = L(\mu,\pm), \text{ i.e. } a'_m = 0,\end{cases}\end{aligned}$$

where the isomorphism holds by adjunction and the first paragraph of the proof. Whereas if $a_m = 0$, we have

$$\begin{aligned}&\operatorname{Hom}_{\mathcal{F}}(L, \operatorname{Ind}^{\mathfrak{g},G}_{\mathfrak{g},G'} I^{\mathfrak{g}}(\lambda))\\ \cong &\begin{cases}\operatorname{Hom}_{\mathcal{F}'}(L^{\mathfrak{g}}(\mu)\oplus L^{\mathfrak{g}}(\sigma(\mu)), I^{\mathfrak{g}}(\lambda)) = \{0\}, & \text{if } L = L(\mu^G), \text{ i.e. } a'_m \neq 0,\\ \operatorname{Hom}_{\mathcal{F}'}(L^{\mathfrak{g}}(\mu), I^{\mathfrak{g}}(\lambda)), & \text{if } L = L(\mu,\pm), \text{ i.e. } a'_m = 0,\end{cases}\end{aligned}$$

where the first homomorphism space vanishes since $I^{\mathfrak{g}}(\lambda)$ is the injective hull of $L^{\mathfrak{g}}(\lambda)$. This proves part (2) in Proposition 4.12.

To prove the analogous statements (3) for indecomposable projective modules, recall that any indecomposable projective is also injective, [BKN11, Proposition 2.2.2]. Hence $P(\lambda^G) \cong I(\Phi(\lambda^G))$ and $P(\lambda,\pm) \cong I(\Phi(\lambda,\pm))$ for some function $\Phi : X^+(\mathfrak{g}) \to X^+(\mathfrak{g})$. By [BKN11, Proposition 2.2.1], the function Φ can be computed as follows: let $N = \dim \mathfrak{g}_1 = 8mn$ and consider the 1-dimensional $\mathfrak{g}_0$-module $\bigwedge^N \mathfrak{g}_1$ of weight ν. Then there is an isomorphism of G'-modules $P^{\mathfrak{g}}(\lambda) \cong I^{\mathfrak{g}}(\lambda+\nu)$. Set $\mu = \lambda+\nu$. Then, using the explicit description (3.15) of the odd roots in $\mathfrak{osp}(2m|2n)$, one can easily check that in ν the coefficient for ε_m vanishes, and therefore $a_m \neq 0$ if and only if $a'_m \neq 0$ in the notation of (4.21) and (4.26). Hence, Φ preserves the condition $a_m \neq 0$. Therefore, the formulas for induced projective modules agree with the formulas for the induced injective modules. □

We have restriction formulas for the projective-injective modules:

Lemma 4.14. *Let $\lambda \in X^+(\mathfrak{g})$. There are isomorphisms of G'-modules*

$$\begin{aligned} \operatorname{Res}^{\mathfrak{g},G'}_{\mathfrak{g},G} I(\lambda^G) &\cong I^{\mathfrak{g}}(\lambda) \oplus I^{\mathfrak{g}}(\sigma(\lambda)) \text{ if } \lambda \neq \sigma(\lambda) \text{ and} \\ \operatorname{Res}^{\mathfrak{g},G'}_{\mathfrak{g},G} I(\lambda,\pm) &\cong I^{\mathfrak{g}}(\lambda) \text{ otherwise.} \end{aligned}$$

Similarly for the indecomposable projective objects.

Proof. Let $P \in \mathcal{F}$ be indecomposable projective. Then $\operatorname{Hom}_G(P,\_\,)$ is exact. For our special case, the induction functor $\operatorname{Ind}^{\mathfrak{g},G}_{\mathfrak{g},G'}$ is exact as well, [Jan03, 3.8.(3), see also 4.9] using (1.5), and right adjoint to the restriction functor, hence we obtain that $\operatorname{Res}^{\mathfrak{g},G'}_{\mathfrak{g},G} P$ is projective. The restriction formulas for projective modules follow then using adjunction from (4.22) and (4.24). Using the identification with indecomposable injective objects (as in the last part of the proof of Proposition 4.13), the claims follow also for these. □

Lemma 4.15. *Assume $\lambda \in X^+(\mathfrak{g})$ with $a_m = 0$ in the notation from* (4.21)*. Then as G'-modules $I(\lambda,+) \cong I(\lambda,-)$, similarly for $P(\lambda,\pm)$.*

Proof. We first claim that our Harish-Chandra induction commutes with Lie algebra induction in the following sense. Let M be a finite-dimensional Harish-Chandra module for $(\mathfrak{g}_0, G')$. Then there is a natural isomorphism of Harish-Chandra modules for $(\mathfrak{g}, G)$ as follows

$$\begin{aligned} U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} (\operatorname{Ind}^{\mathfrak{g}_0,G}_{\mathfrak{g}_0,G'} M) &\cong \operatorname{Ind}^{\mathfrak{g},G}_{\mathfrak{g},G'}(U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} M). \\ u \otimes f &\mapsto f_u, \end{aligned} \tag{4.27}$$

where $f_u(g) = u \otimes f(g)$ for any $g \in G$. The map is obviously well-defined and injective, and therefore also an isomorphism by a dimension count using that $U(\mathfrak{g})$ is free over $U(\mathfrak{g}_0)$ of finite rank.

By Proposition 4.12 (2) (b) we obtain $P(\lambda,+) \oplus P(\lambda,-) \cong \operatorname{Ind}^{\mathfrak{g},G}_{\mathfrak{g},G'} P^{\mathfrak{g}}(\lambda)$. On the other hand, by [BKN11, Proof of Proposition 2.2.2], the indecomposable $(\mathfrak{g}, G')$-module $P^{\mathfrak{g}}(\lambda)$ is a summand of $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} L_0(\lambda)$, where $L_0(\lambda)$ is the irreducible $(\mathfrak{g}_0, G')$-Harish-Chandra module of highest weight λ.

Together with (4.27) and Proposition 4.12 we get that $P(\lambda,\pm)$ is a summand of

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} (\operatorname{Ind}^{\mathfrak{g}_0,G}_{\mathfrak{g}_0,G'} L_0(\lambda)) \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} (L_0(\lambda,+) \oplus L_0(\lambda,-))$$

By carefully following the highest weight vectors through the isomorphism we obtain that $P(\lambda,\pm)$ is in fact a summand of $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} L_0(\lambda,\pm)$. By Proposition 4.12 (1) (b) the action of σ on $L_0(\lambda,+)$ is given by a scalar, hence it acts by the same scalar on the highest weight vector of $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} L_0(\lambda,+)$, and thus also on the highest weight vector of $P(\lambda,\pm)$. The analogous statements hold then for $I(\lambda,\pm)$ as well (again via the identification Φ from the proof of Proposition 4.13). □

The following separates the indecomposable projective modules $P(\lambda,\pm)$ into two groups, depending on the sign:

Lemma 4.16. *Let $G = \mathrm{OSp}(2m|2n)$. Consider the set $X^+(\mathfrak{g})_{\mathrm{sign}} = \{\lambda \in X^+(\mathfrak{g}) \mid a_m = 0\}$ in the notation from* (4.21)*. Then the sign in the labelling* *of the irreducible modules from* (4.24) *can be consistently chosen such that the following holds for any* $\lambda, \mu \in X^+(\mathfrak{g})_{\text{sign}}$*:*

$$\operatorname{Hom}_{\mathcal{F}}(I(\lambda,+), I(\mu,-)) = \{0\} = \operatorname{Hom}_{\mathcal{F}}(I(\lambda,-), I(\mu,+)), \quad (4.28)$$
$$\operatorname{Hom}_{\mathcal{F}}(P(\lambda,+), P(\mu,-)) = \{0\} = \operatorname{Hom}_{\mathcal{F}}(P(\lambda,-), P(\mu,+)). \quad (4.29)$$

Proof. The proof of this Lemma will be given in Part II of this series. It is a consequence of the action of the Jucys-Murphy elements of the Brauer algebra. The proof is a rather easy induction argument, but requires a few combinatorial facts about the action of Jucys-Murphy elements. □

We deduce now a few dimension formulas for homomorphism spaces.

Proposition 4.17. *Let* $\lambda, \mu \in X^+(\mathfrak{g})$*. With the notations from Proposition 4.12, in particular* (4.21) *and* (4.26)*, we have the following.*

(1) If $a_m = 0 \neq a'_m$ *then*

$$\begin{aligned} \dim \operatorname{Hom}_{\mathcal{F}}(I(\lambda,\pm), I(\mu^G)) &= \dim \operatorname{Hom}_{\mathcal{F}'}(I^{\mathfrak{g}}(\lambda), I^{\mathfrak{g}}(\mu)) && (4.30)\\ &= \dim \operatorname{Hom}_{\mathcal{F}'}(I^{\mathfrak{g}}(\lambda), I^{\mathfrak{g}}(\sigma(\mu)) && (4.31)\\ \dim \operatorname{Hom}_{\mathcal{F}}(I(\mu^G), I(\lambda,\pm)) &= \dim \operatorname{Hom}_{\mathcal{F}'}(I^{\mathfrak{g}}(\mu), I^{\mathfrak{g}}(\lambda)) && (4.32)\\ &= \dim \operatorname{Hom}_{\mathcal{F}'}(I^{\mathfrak{g}}(\sigma(\mu)), I^{\mathfrak{g}}(\lambda)) && (4.33)\end{aligned}$$

(2) If $a_m > 0 < a'_m$ *or* $a_m < 0 > a'_m$ *then*

$$\begin{aligned} \dim \operatorname{Hom}_{\mathcal{F}}(I(\lambda^G), I(\mu^G)) &= \dim \operatorname{Hom}_{\mathcal{F}'}(I^{\mathfrak{g}}(\lambda), I^{\mathfrak{g}}(\mu)) && (4.34)\\ &= \dim \operatorname{Hom}_{\mathcal{F}'}(I^{\mathfrak{g}}(\sigma(\lambda)), I^{\mathfrak{g}}(\sigma(\mu)) && (4.35)\end{aligned}$$

(3) If $a_m = 0 = a'_m$ *then*

$$\begin{aligned} \dim \operatorname{Hom}_{\mathcal{F}}(I(\lambda,\pm), I(\mu,\pm)) &= \dim \operatorname{Hom}_{\mathcal{F}'}(I^{\mathfrak{g}}(\lambda), I^{\mathfrak{g}}(\mu)), && (4.36)\\ \dim \operatorname{Hom}_{\mathcal{F}}(I(\lambda,\pm), I(\mu,\mp)) &= \{0\}, && (4.37)\end{aligned}$$

The analogous formulas hold for indecomposable projective objects.

Proof. For the first statement (4.30) we calculate using Proposition 4.12, adjunction of restriction and induction and Lemma 4.14

$$\begin{aligned} \dim \operatorname{Hom}_{\mathcal{F}}(I(\lambda,\pm), I(\mu^G)) &= \dim \operatorname{Hom}_{\mathcal{F}}(I(\lambda,\pm), \operatorname{Ind}_{\mathfrak{g},G'}^{\mathfrak{g},G} I^{\mathfrak{g}}(\mu))\\ &= \dim \operatorname{Hom}_{\mathcal{F}'}(\operatorname{Res}_{\mathfrak{g},G}^{\mathfrak{g},G'} I(\lambda,\pm), I^{\mathfrak{g}}(\mu))\\ &= \dim \operatorname{Hom}_{\mathcal{F}'}(I^{\mathfrak{g}}(\lambda), I^{\mathfrak{g}}(\mu)).\end{aligned}$$

Similarly, (4.31) holds. Again, the same formulas hold for projective objects. On the categories $\mathcal{F}$ and $\mathcal{F}'$ there is the usual duality $\mathbf{d}$, [Mus12, 13.7.1], given by taking the sum of the vector space dual of the weight spaces with the action of $\mathfrak{g}$, G, G' twisted by the Chevalley automorphism. This duality sends simple objects to simple objects and their injective hulls to the projective covers. Applying $\mathbf{d}$ to (4.30) resp.(4.31) gives (4.32) and (4.33).

For the statement (4.34) we calculate

$$\begin{aligned} \dim \operatorname{Hom}_{\mathcal{F}}(I(\lambda^G), I(\mu^G)) &= \dim \operatorname{Hom}_{\mathcal{F}}(I(\lambda^G), \operatorname{Ind}_{\mathfrak{g},G'}^{\mathfrak{g},G} I^{\mathfrak{g}}(\mu))\\ &= \dim \operatorname{Hom}_{\mathcal{F}'}(\operatorname{Res}_{\mathfrak{g},G}^{\mathfrak{g},G'} I(\lambda^G), I^{\mathfrak{g}}(\mu))\\ &= \dim \operatorname{Hom}_{\mathcal{F}'}(I^{\mathfrak{g}}(\lambda) \oplus I^{\mathfrak{g}}(\sigma(\lambda)), I^{\mathfrak{g}}(\mu))\\ &= \dim \operatorname{Hom}_{\mathcal{F}'}(I^{\mathfrak{g}}(\lambda), I^{\mathfrak{g}}(\mu)),\end{aligned}$$

again using Proposition 4.12, adjunction and Lemma 4.14 for the first to third equalities. The last one follows from the Gruson-Serganova combinatorics, [GS13], see Proposition 8.1 (1). Hence, (4.34) and similarly (4.35) follow.

The second part of (3) is exactly (4.28). For the first part we calculate

$$\begin{aligned}\dim \mathrm{Hom}_{\mathcal{F}'}(I^{\mathfrak{g}}(\lambda), I^{\mathfrak{g}}(\mu)) \;&=\; \dim \mathrm{Hom}_{\mathcal{F}'}(\mathrm{Res}^{\mathfrak{g},G'}_{\mathfrak{g},G}\, I(\lambda,\pm), I^{\mathfrak{g}}(\mu))\\ =\; \dim \mathrm{Hom}_{\mathcal{F}'}(I(\lambda,\pm), \mathrm{Ind}^{\mathfrak{g},G}_{\mathfrak{g},G'}\, I^{\mathfrak{g}}(\mu)) &= \dim \mathrm{Hom}_{\mathcal{F}}(I(\lambda,\pm), I(\mu,\pm)),\end{aligned}$$

where we used again Lemma 4.14, adjunction, Proposition 4.12 and finally (4.28). Again, the analogous formulas for the projectives hold as well. □

4.3. **The Cartan matix.** We apply the results so far to deduce the symmetry of the Cartan matrix:

Proposition 4.18. *Consider $G = \mathrm{OSp}(r|2n)$ for fixed m, n. The Cartan matrix of $\mathcal{F}$ is symmetric, i.e. for any $\lambda, \mu \in X^+(G)$ we have an equality of multiplicities of irreducible modules in a Jordan-Hölder series:*

$$[P(\lambda) : L(\mu)] = [P(\mu) : L(\lambda)], \tag{4.38}$$

and therefore $\dim \mathrm{Hom}_{\mathcal{F}}(P(\lambda), P(\mu)) = \dim \mathrm{Hom}_{\mathcal{F}}(P(\mu), P(\lambda))$.

Proof. We first claim the analogous formulas for $\mathcal{F}'$. So given $\lambda, \mu \in X^+(\mathfrak{g})$, the multiplicity $[P^{\mathfrak{g}}(\lambda) : L^{\mathfrak{g}}(\mu)]$ is the coefficient of the class of $L^{\mathfrak{g}}(\mu)$ when we express the class of $[P^{\mathfrak{g}}(\lambda)]$ in terms of the classes of the irreducible modules of $\mathcal{F}'$ in the Grothendieck group of $\mathcal{F}'$. Now by [GS13] we have another class of linearly independent elements in the Grothendieck group, namely the Euler-characteristics $\mathcal{E}^{\mathfrak{g}}(\nu)$, where ν runs through all tailless elements in $X^+(\mathfrak{g})$ and the classes $[P^{\mathfrak{g}}(\lambda)]$ are all in the $\mathbb{Z}$-lattice spanned by these, with coefficients denoted by $(P^{\mathfrak{g}}(\lambda) : \mathcal{E}^{\mathfrak{g}}(\nu))$, see [GS13, Theorem 1]. Hence

$$\begin{aligned}[P^{\mathfrak{g}}(\lambda) : L^{\mathfrak{g}}(\mu)] \;&=\; \sum_{\nu} (P^{\mathfrak{g}}(\lambda) : \mathcal{E}^{\mathfrak{g}}(\nu))[\mathcal{E}^{\mathfrak{g}}(\nu) : L^{\mathfrak{g}}(\mu)]\\ &=\; \sum_{\nu} [\mathcal{E}^{\mathfrak{g}}(\nu) : L^{\mathfrak{g}}(\lambda)][\mathcal{E}^{\mathfrak{g}}(\nu) : L^{\mathfrak{g}}(\mu)]\\ &=\; [P^{\mathfrak{g}}(\mu) : L^{\mathfrak{g}}(\lambda)],\end{aligned}$$

where the second equality is the BGG-reciprocity, [GS13, Theorem 1], and the third equality holds then by symmetry. Hence the analogue of (4.38) for $\mathcal{F}'$ holds. Now $\dim \mathrm{Hom}_{\mathcal{F}'}(P^{\mathfrak{g}}(\lambda), L^{\mathfrak{g}}(\lambda)) = 1$, since $L^{\mathfrak{g}}(\lambda)$ is a highest weight module, and therefore $\dim \mathrm{Hom}_{\mathcal{F}'}(P^{\mathfrak{g}}(\lambda), P^{\mathfrak{g}}(\mu)) = [P^{\mathfrak{g}}(\mu) : L^{\mathfrak{g}}(\lambda)]$. Hence the proposition holds for $\mathcal{F}'$. (Alternatively one could use that $P^{\mathfrak{g}}(\lambda) \cong I^{\mathfrak{g}}(\lambda)$ and apply the usual simple preserving duality on $\mathcal{F}'$). Proposition 4.12 implies that $\dim \mathrm{End}_{\mathcal{F}}(L) = 1$ for any irreducible object in $\mathcal{F}$. Then the statement from the proposition follows directly from the statement for $\mathcal{F}'$ and the formulas for the dimensions of homomorphism spaces (Lemma 4.16 and Proposition 4.17). □

4.4. **Hook partitions.** Let still $G = \mathrm{OSp}(r|2n)$ for $r = 2m+1$ or $r = 2m$ and recall (from Propositions 4.6 and 4.10 and (4.2)) the labelling sets $X^+(G)$ respectively $X^+(\mathfrak{g})$ for the isomorphism classes of irreducible objects in $\mathcal{F}$ and $\mathcal{F}'$ respectively.

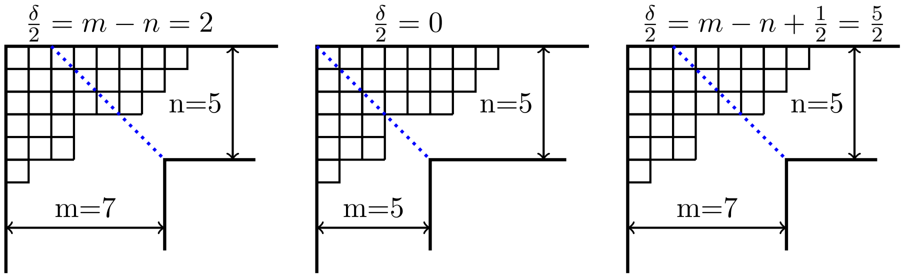


FIGURE 3. The translation between weights and hook partitions.

A different common labelling of the simple modules in $\mathcal{F}'$ is given by hook partitions, see e.g. [CW12]. To make the connection, recall that a *partition*, denoted[4] by $^\ulcorner\gamma$, is a weakly decreasing sequence of non-negative integers, $^\ulcorner\gamma = (^\ulcorner\gamma_1 \geq {}^\ulcorner\gamma_2 \geq \cdots)$. We denote by $^\ulcorner\gamma^t$ its transpose partition, i.e. $^\ulcorner\gamma^t_i = |\{k \mid \lambda_k \geq i\}|$. A partition $^\ulcorner\gamma$ is called (n,m)-*hook* if $^\ulcorner\gamma_{n+1} \leq m$. The partition $^\ulcorner\gamma = (8,7,6,3,3,1)$ is for instance $(5,7)$-hook and $(5,5)$-hook, but not $(2,5)$-hook, see Figure 3. Note that the empty partition $\varnothing$ is (n,m)-hook for any $n, m \geq 0$. and corresponds to the zero weight via the following.

Definition 4.19. Given an (n,m)-hook partition $^\ulcorner\gamma$ we associate *weights*

$$\mathrm{wt}(^\ulcorner\gamma) \in X^+(\mathfrak{osp}(2m+1|2n)), \quad \text{respectively} \quad \mathrm{wt}(^\ulcorner\gamma) \in X^+(\mathfrak{osp}(2m|2n))$$

defined, via (4.21), as follows, (with $1 \leq i \leq m$, $1 \leq j \leq n$):

- in the odd case $\mathrm{wt}(^\ulcorner\gamma) = (a_1, a_2, \ldots, a_m \mid b_1, b_2, \ldots, b_n) - \rho$, where

$$b_j = \max\left\{^\ulcorner\gamma_j - j - \frac{\delta}{2} + 1, \frac{1}{2}\right\} \quad \text{and} \quad a_i = \max\left\{^\ulcorner\gamma^t_i - i + \frac{\delta}{2}, -\frac{1}{2}\right\},$$

- in the even case $\mathrm{wt}(^\ulcorner\gamma) = (a_1, a_2, \ldots, a_m \mid b_1, b_2, \ldots, b_n) - \rho$, where

$$b_j = \max\left\{^\ulcorner\gamma_j - j - \frac{\delta}{2} + 1, 0\right\} \quad \text{and} \quad a_i = \max\left\{^\ulcorner\gamma^t_i - i + \frac{\delta}{2}, 0\right\}.$$

The a_i and b_j give a different way to describe (n,m)-hook partitions by encoding the number of boxes below and to the right of the $\lfloor \frac{\delta}{2} \rfloor$-shifted diagonal (which we just call *diagonal*). For example let $\lambda = (8,7,6,3,3,1)$. Consider it as a hook partition, for instance as $(5,7)$-hook respectively $(5,5)$-hook, and mark the diagonal which intersects the inflexion point of the hook (it is exactly the diagonal, where the boxes have content $\frac{\delta}{2} + \frac{1}{2}$ respectively $\frac{\delta}{2} + 1$, where the content is the row minus the column number of the box).

- In the even case on the other hand a_i counts the number of boxes in column i strictly below the diagonal. While b_j counts the number of boxes in row j on and to the right of this diagonal. In the first two cases of Figure 3 we get $\mathbf{a} = (7,5,4,1,0,0,0)$, respectively $\mathbf{a} = (5,3,2,0,0)$, and on the other hand $\mathbf{b} = (6,4,2,0,0)$, respectively $\mathbf{b} = (8,6,4,0,0)$.
- In the odd case this implies that a_i counts the number of boxes in column i on and below the diagonal minus $\frac{1}{2}$. On the other hand b_j

[4] We chose this notation to distinguish partitions from integral weights

counts the number of boxes in row j strictly to the right of this diagonal minus $\frac{1}{2}$ and takes the absolute value of this expression. In the third case above in Figure 3 this gives $\mathbf{a} = (\frac{15}{2}, \frac{11}{2}, \frac{9}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$ and $\mathbf{b} = (\frac{11}{2}, \frac{7}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2})$.

(Note that we also count the numbers of boxes which can be put in the region between the marked diagonal and the partition, i.e. above or to the left of the diagram depending on the given diagonal.)

Definition 4.20. A *signed* (n, m)*-hook partition* is an (n, m)-hook partition $^\ulcorner\gamma$ with $^\ulcorner\gamma_{n+1} \geq m$ or a pair $(^\ulcorner\gamma, \epsilon)$ of an (n, m)-hook partition with $^\ulcorner\gamma_{n+1} < m$ and a sign $\epsilon \in \{\pm\}$.

The following easy identification allows us to work with hook partitions plus signs (in the odd case) respectively with signed hook partitions (in the even case) instead of dominant weights:

Lemma 4.21. *The assignments* $^\ulcorner\gamma \mapsto \mathrm{wt}(^\ulcorner\gamma)$ *defines a bijection*

$$\begin{array}{rcl}\Psi = \Psi_{2m+1,2n}:\ \{(n,m)-\textit{hook partitions}\} \times \mathbb{Z}/2\mathbb{Z} & \overset{1:1}{\leftrightarrow} & X^+(G) \\ (^\ulcorner\gamma, \pm) & \mapsto & (\mathrm{wt}(^\ulcorner\gamma), \pm)\end{array} \tag{4.39}$$

in case $G = \mathrm{OSp}(2m+1|2n)$*, and in case* $G = \mathrm{OSp}(2m|2n)$ *a bijection*

$$\begin{array}{rcl}\Psi = \Psi_{2m,2n}:\ \{(n,m)-\textit{signed hook partitions}\} & \overset{1:1}{\leftrightarrow} & X^+(G) \\ ^\ulcorner\gamma & \mapsto & \mathrm{wt}(^\ulcorner\gamma) \\ (^\ulcorner\gamma, \pm) & \mapsto & (\mathrm{wt}(^\ulcorner\gamma), \pm)).\end{array}$$

In either case: given $\lambda \in X^+(G)$*, we denote by* $^\ulcorner\lambda$ *the unique hook partition such that* $^\ulcorner\lambda$ *respectively* $(^\ulcorner\lambda, \pm)$ *is the preimage of* λ *under* Ψ *and call it the underlying hook partition.*

Proof. Take an (n, m)-hook partition $^\ulcorner\gamma$. To see that the maps are well-defined it suffices to show that $\mathrm{wt}(^\ulcorner\gamma)$ us a dominant weight for $\mathfrak{g}$ (since we can clearly ignore the signs).

Let us first consider the case $\Psi_{2m,2n}$. Since $^\ulcorner\gamma$ is a partition we have $a_{i+1} < a_i$ and $b_{j+1} < b_j$ whenever they are defined and non-zero. For the map to be well-defined it remains to show that the number of zero a's is equal or one larger than the number of zero b's.

Claim: For $s \leq \min\{m, n\}$ we have $a_{m-s} > 0$ implies $b_{n-s} > 0$. If $b_{n-s} = 0$ then $^\ulcorner\gamma_{n-s} - m + n - n + s + 1 \leq 0$, hence $^\ulcorner\gamma_{n-s} \leq m - s - 1$ and so $^\ulcorner\gamma$ has at most $n - s - 1$ rows of length $m - s$. This means $^\ulcorner\gamma^t_{m-s} \leq n - s - 1$ and thus $a_{m-s} = {^\ulcorner\gamma^t_{m-s}} - n + m - m + s \leq n - s - 1 - n + s = -1$ which is a contradiction and the claim follows. This shows that there are at least as many zero a's as b's. It suffices now to show that $a_{m-r} = 0$ forces $b_{n-r+1} = 0$. So assume $a_{m-r} = 0$. Since $a_{m-n-1} = {^\ulcorner\gamma^t_{m-n-1}} - n + m - m + n - 1 = {^\ulcorner\gamma^t_{m-n-1}} + 1 > 0$ we see that $a_{m-r} = 0$ implies $r \leq n$ and so b_{n-r+1} must exist. If $b_{n-r+1} = {^\ulcorner\gamma_{n-r+1}} - m + n - n + r - 1 + 1 > 0$,then $^\ulcorner\gamma_{n-r+1} > m - r$ which implies $^\ulcorner\gamma^t_{m-r} \leq n - r + 1$ and therefore $a_{m-r} = {^\ulcorner\gamma^t_{m-r}} - n + m - m + r \geq n - r + 1 - n + r \geq 1$ which is a contradiction. Hence the map is well-defined and obviously injective.

Clearly the weights in the image satisfy $a_m \geq 0$. For the description of the image it suffices to show that if $(\mathbf{a} \,|\, \mathbf{b}) \in X^+(\mathfrak{g})$ satisfies the dominance condition from Definition 4.2 with $a_m > 0$ in case (i) then it comes from a hook partition. It is enough to see it defines a partition, since b_j is only defined for $1 \leq j \leq n$ and a_i for $1 \leq i \leq m$, hence if it is a partition it must be (n, m)-hook. For that it suffices to see that $a_i \neq 0$ with $i = \frac{\delta}{2} + k$ for some k implies $b_k \geq 1$. Write $i = m - s$ then this is equivalent to ($a_{m-s} \neq 0$ implies $b_{n-s} \geq 1$), since $k = i - \frac{d}{2} = i - m + n = n - s$. But this was exactly the claim above. The arguments for $\Psi_{2m+1,2n}$ are analogous, but this last step is even easier. □

Definition 4.22. The tail length $\mathrm{tail}(\lambda)$ of $\lambda \in X^+(G)$ or equivalently of the underlying hook partition, is equal to $\min\{m, n\} - d$, where d is the number of boxes on the diagonal of the hook partition.

It is easy to check that this notion agrees with the notion of tail from Definition 4.4. Note that $\mathrm{tail}(\lambda)$ counts the number of missing boxes on the diagonal of the hook partition, in particular, it is maximal possible for the empty partition, i.e. the zero weight.

We will present now a new (and more convenient) way of encoding dominant weights and the labeling set of irreducible finite-dimensional representations of G in terms of *diagrammatic weights*. This is in the spirit of [BS12b] built on the combinatorics introduced in [ES13a].

5. Diagrammatics

We attach now a certain diagrammatic weight to each simple object in $\mathcal{F}(G)$. This will allow us to develop a diagrammatic description of the morphism spaces between indecomposable projective objects in the corresponding categories $\mathcal{F}(G)$.

5.1. Diagrammatic weights attached to $X^+(G)$.

To establish the combinatorics consider the non-negative number line $\mathcal{L}$ and call its integral points *vertices*.

Definition 5.1. An *(infinite) diagrammatic weight* or just a *diagrammatic weight* λ is a diagram obtained by labelling each of the vertices by exactly one of the symbols $\times$ (cross), $\circ$ (nought), $\vee$ (down), $\wedge$ (up); for the position zero we do not distinguish the labels $\wedge$ and $\vee$ and use instead the label $\Diamond$. The vertices labelled $\circ$ or $\times$ are called *core symbols* and the diagram obtained from λ by removing all symbols $\wedge$, $\vee$ and $\diamond$ is called its *core diagram*.

For a diagrammatic weight λ we denote by $\# \times (\lambda)$, $\# \times (\lambda)$, $\# \wedge (\lambda)$, $\#\vee(\lambda)$ the number of crosses, noughts, downs and ups respectively occurring in λ.

Definition 5.2. A diagrammatic weight λ is called

- *finite* if $\# \vee (\lambda) + \# \wedge (\lambda) + \# \times (\lambda) < \infty$, and
- *of partition type* if $\# \vee (\lambda) + \# \circ (\lambda) + \# \times (\lambda) < \infty$, and
- *super* or *of finite type* if $\# \wedge (\lambda) + \# \circ (\lambda) + \# \times (\lambda) < \infty$.

Hence a finite weight has only noughts far to the right, and a weight of finite type has only $\vee$'s far to the right, and a weight of partition type has only $\wedge$'s far to the right. For instance, consider the diagrammatic weights

$$\vee \ \circ \ \times \ \wedge \ \times \ \wedge \ \wedge \ \wedge \ \vee \ \wedge \ \wedge \ \wedge \ \vee \ \vee \ \wedge \ \vee \wedge ? \ ? \ ? \ ? \cdots \tag{5.40}$$

where the ?'s and the dots indicate either only $\circ$'s, only $\vee$'s or only $\wedge$'s respectively. Then the resulting three weights λ_{fin}, λ_{fint}, and λ_{part} are finite, finite type or partition type respectively.

Definition 5.3. Two diagrammatic weights λ and μ are *linked* or *in the same block* if their core diagrams coincide, and in addition the parities of the total number of $\wedge$'s agree, in formulas

$$\# \wedge (\lambda) \equiv \# \wedge (\mu) \mod 2.$$

in case there is no $\diamond$.

We now assign to each (n, m)-hook partition γ a diagrammatic weight .

Definition 5.4. For any partition $^{\ulcorner}\gamma$ and $\delta = r - 2n$ set

$$\mathcal{S}(^{\ulcorner}\gamma) \quad = \quad \left(\frac{\delta}{2} + i - {^{\ulcorner}\gamma_i} - 1 \right)_{i \geq 1}. \tag{5.41}$$

This is a strictly increasing sequence of half-integers (i.e. from $\mathbb{Z} + \frac{1}{2}$) if r is odd, and of integers in case r is even. In case r is odd we identify the vertices of $\mathcal{L}$ order-preserving with $\mathbb{Z}_{\geq 0} + \frac{1}{2}$. That means we have then vertices $\frac{1}{2}, \frac{3}{2}, \ldots$. In case r is even, we identify the vertices of $\mathcal{L}$ order-preserving with $\mathbb{Z}_{\geq 0}$.

Definition 5.5. To the sequence $\mathcal{S}(^{\ulcorner}\gamma)$ we then assign an infinite diagrammatic weight $^{\ulcorner}\gamma^{\infty}$ by attaching to the vertex p the label

$$\begin{cases} \circ & \text{if } p \text{ nor } -p \text{ occurs in } \mathcal{S}(\gamma), \\ \vee & \text{if } -p, \text{ but not } p, \text{ occurs in } \mathcal{S}(\gamma), \\ \wedge & \text{if } p, \text{ but not } -p, \text{ occurs in } \mathcal{S}(\gamma), \\ \times & \text{if both, } -p \neq p \text{ occur in } \mathcal{S}(\gamma), \\ \diamond & \text{if } p = 0 \text{ occurs in } \mathcal{S}(\gamma). \end{cases} \tag{5.42}$$

Note that there are only finitely many labels different from $\wedge$, hence these resulting diagrammatic weights are all of partition type. Moreover, in this one can only get $\circ$ or $\wedge$ at position zero.

The empty partition gives in case of odd r the following diagrams

$$\begin{cases} \underbrace{\overset{\frac{1}{2}}{\circ}\ \cdots\ \circ}_{m-n}\ \ \underbrace{\overset{\frac{\delta}{2}}{\wedge}\ \ \wedge\ \cdots\ \wedge\ \ \wedge}_{2n}\ \ \textcircled{$\wedge$}\ \ \textcircled{$\wedge$}\ \ \cdots & \text{if } \delta > 0, \\ \underbrace{\overset{\frac{1}{2}}{\times}\ \cdots\ \overset{-\frac{\delta}{2}}{\times}}_{n-m}\ \ \underbrace{\wedge\ \ \wedge\ \cdots\ \wedge\ \ \wedge}_{2m}\ \ \textcircled{$\wedge$}\ \ \textcircled{$\wedge$}\ \ \cdots & \text{if } \delta < 0. \end{cases} \tag{5.43}$$

(The circles around the $\wedge$ in $\textcircled{$\wedge$}$ should be ignored for the moment. They will play an important role later). In case r is even, the empty partition gives the following diagrammatic weights

$$\begin{cases} \overset{0}{\circ}\ \underbrace{\cdots \ \circ}_{m-n} \ \underbrace{\overset{\frac{\delta}{2}}{\wedge} \ \wedge \ \cdots \ \wedge}_{2n} \ \circledwedge \ \cdots & \text{if } \delta \geq 0, \\ \overset{0}{\diamond} \ \underbrace{\times \ \cdots \ \overset{-\frac{\delta}{2}}{\times}}_{n-m} \ \underbrace{\wedge \ \cdots \ \wedge}_{2m-1} \ \circledwedge \ \cdots & \text{if } \delta < 0, \end{cases} \tag{5.44}$$

Again, the circle around the $\wedge$ in $\circledwedge$ should be ignored for the moment. We refer to Section 9.2 for more examples.

Lemma 5.6. *Let $\lambda \in X^+(G)$. We have $\mathcal{S}(\lambda)_i < 0$ (respectively $\mathcal{S}(\lambda)_i \geq 0$) in* (5.41) *iff the i-th row in the underlying Young diagram ends above or on (respectively strictly below) the $\frac{\delta}{2}$-shifted diagonal.*

Proof. Note that $\mathcal{S}(\lambda)_i < 0$ iff $\frac{\delta}{2} + i - {}^\ulcorner\lambda_i - 1 < 0$ or equivalently ${}^\ulcorner\lambda_i > i + \frac{\delta}{2} - 1$. □

The tail length of $\lambda \in X^+(G)$ can be expressed again combinatorially.

Corollary 5.7. *The length of the tail of λ equals* $\mathrm{tail}(\lambda) = n - s$ *where* $s = \# \vee (\lambda^\infty) + \# \times (\lambda^\infty)$.

Proof. If $m \geq n$ then there is a box on the diagonal in row i iff $\mathcal{S}(\lambda)_i < 0$. This implies that there are exactly s boxes on the shifted diagonal, hence $\mathrm{tail}(\lambda) = n - s$. If on the other hand $m < n$ then again s is the number of rows that end above or on the shifter diagonal, but we have to subtract the first $n - m$ rows, thus there are $s - (n - m)$ boxes on the diagonal, hence $\mathrm{tail}(\lambda) = m - s + (n - m) = n - s$. □

The following characterizes the weights with non-zero tail in the even case.

Corollary 5.8. *Assume $r = 2m$ and let $\lambda \in X^+(\mathfrak{g})$ with the notation from* (4.21)*. Consider the underlying (n, m)-hook partition ${}^\ulcorner\lambda$ and the diagrammatic weight ${}^\ulcorner\lambda^\infty$ given by $\mathcal{S}({}^\ulcorner\lambda)$. Then the following are equivalent:*

$$a_m > 0 \Leftrightarrow {}^\ulcorner\lambda_{n+1} = m \Leftrightarrow \mathrm{Ind}_{\mathfrak{g},G'}^{\mathfrak{g},G} L^{\mathfrak{g}}(\lambda) \textit{ is irreducible} \Leftrightarrow \mathcal{S}({}^\ulcorner\lambda)_{n+1} = 0.$$

Moreover, in this case the associated diagrammatic weight ${}^\ulcorner\lambda^\infty$ has label $\wedge$ at position zero, and $\mathrm{tail}(\lambda) = 0$.

Proof. Obviously $a_m > 0$ is equivalent to ${}^\ulcorner\lambda_{n+1} = m$ by Definition 4.19, and hence to $\mathrm{tail}(\lambda) = 0$ by definition. It is moreover equivalent to $\mathrm{Ind}_{\mathfrak{g},G'}^{\mathfrak{g},G} L^{\mathfrak{g}}(\lambda)$ being irreducible by Proposition 4.12. On the other hand ${}^\ulcorner\lambda_{n+1} = m$ if and only if $\mathcal{S}({}^\ulcorner\lambda)_{n+1} = m - n + n + 1 - {}^\ulcorner\lambda_{n+1} - 1 = m - {}^\ulcorner\lambda_{n+1} = 0$ (which then obviously causes an $\wedge$ at position zero). □

5.2. **Cup diagrams.** Given a diagrammatic weight λ which is finite, of finite type or of partition type, we like to assign a unique cup diagram. For this we say that two vertices in a diagrammatic weight are *neighboured* if they are only separated by vertices with labels $\circ$'s and $\times$'s.

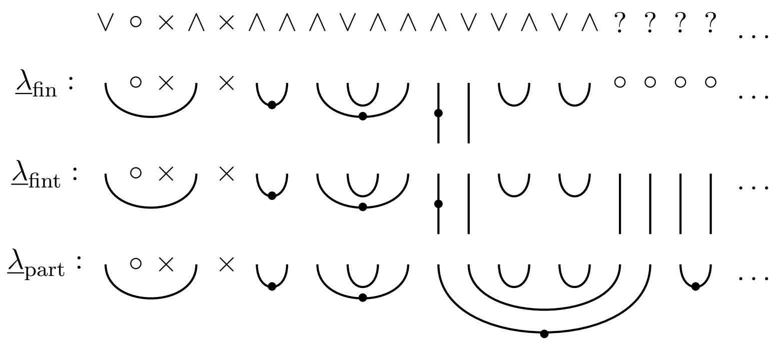
Definition 5.9. The *infinite decorated cup diagram* or just *cup diagram*, denoted $\underline{\lambda}$, associated with a diagrammatic weight λ (finite, finite type or partition type) is obtained by applying the following steps in order.

(Cup 1) First connect neighboured vertices labelled $\vee\wedge$ successively by a cup (ignoring already joint vertices) as long as possible. (The result is independent of the order in which the connections are made).
(Cup 2) Attach to each remaining $\vee$ a vertical ray.
(Cup 3) Connect from left to right pairs of two neighboured $\wedge$'s by cups (viewing $\diamond$ as an $\wedge$).
(Cup 4) If a single $\wedge$ remains, attach a vertical ray.
(Cup 5) Finally put a decoration $\bullet$ on each cup created in step (Cup 3) and each ray created in (Cup 4).

The arcs for the connections should always be drawn without intersections.

Remark 5.10. Observe that the conditions finite, of finite type and of partition type make sure that the algorithm producing the cup diagram is well-defined. In case of a partition type diagrammatic weight the steps (Cup 1) and (Cup 4) can be removed and the diagram will never have dotted or undotted rays, but infinitely many dotted cups. In case the diagram is of finite type it will have only rays far to the right.

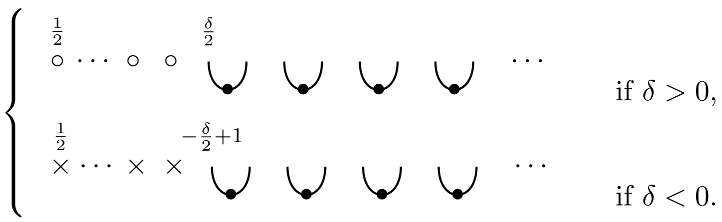
Examples 5.11. The three diagrammatic weights λ_{fin}, λ_{fint}, and λ_{part} from (5.40) give the following three cup diagrams.



(5.45)

The empty partition gives always an infinite cup diagram (for the diagrammatic weight see (5.43) and (5.44)):

In the case of $G = \mathrm{OSp}(2m+1|2n)$:



(5.46)

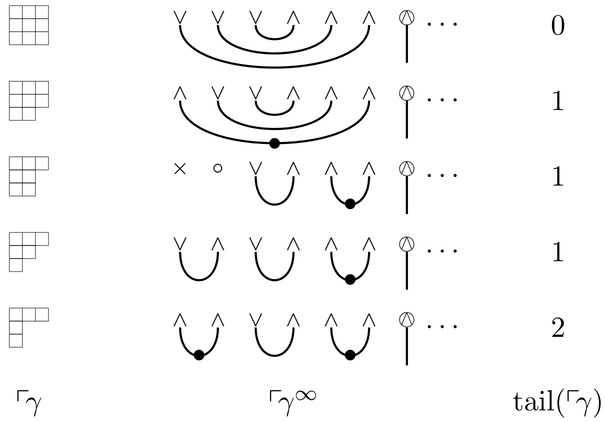
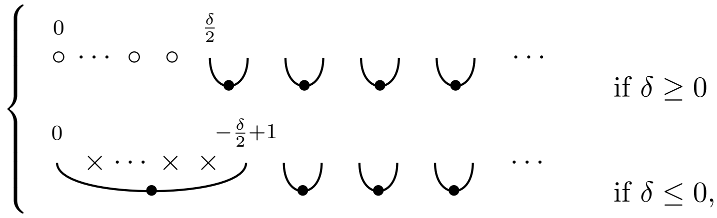


FIGURE 4. Super cup diagrams $^\ulcorner\gamma^\infty$ associated to hook partitions in the case SOSp(7|6).

In the case of $G = \mathrm{OSp}(2m|2n)$:



(5.47)

Remark 5.12. Note that, by construction, there might be cups nested inside each other, but such cups cannot be dotted. By construction, there is also never a $\bullet$ to the right of a ray. Given any such cup diagram c there is a unique diagrammatic weight λ such that $\underline{\lambda} = c$. Namely λ is the unique diagrammatic weight such that, when put on top of c, the core symbols match and all cups and rays are oriented in the unique degree zero way as displayed in (5.48).

Definition 5.13. We call cups or rays with a decoration $\bullet$ *dotted* and those without decorations *undotted*. The total number (possibly infinite) of undotted plus dotted cups in a cup diagram c is called its *defect* or *atypicality* and denoted $\mathrm{def}(c)$ and define the defect $\mathrm{def}(\lambda)$ of a diagrammatic weight of finite type to be the defect $\mathrm{def}(\underline{\lambda})$ of its associated cup diagram.

5.3. **(Nuclear) circle diagrams.** A pair of compatible cup diagrams can be combined to a circle diagram:

Definition 5.14. Given λ, μ, ν diagrammatic weights. We call the ordered pair (λ, μ) a *circle diagram* if λ and μ have the same core diagrams. We usually denote this circle diagrams by $\underline{\lambda}\overline{\mu}$ and think of it as a diagram obtained from putting the cup diagram $\underline{\mu}$ upside down on top of the cup diagram $\underline{\lambda}$.

For examples we refer to Figure 2, where the diagrammatic weight in the middle of each circle diagram should be ignored. The connected components in a circle diagram are (ignoring dots) either *lines* or *circles*.

Remark 5.15. It is easy to check that the total number x of cups and caps in a connected component of a circle diagram is always even if the component is a circle or a propagating line, and is odd if it is a non-propagating line. (The statement is obvious for lines and circles where $x \leq 2$. If $x \leq 2$ then one can find a kink built out of one cup and one cap, can remove this from the circle diagram and then argue by induction).

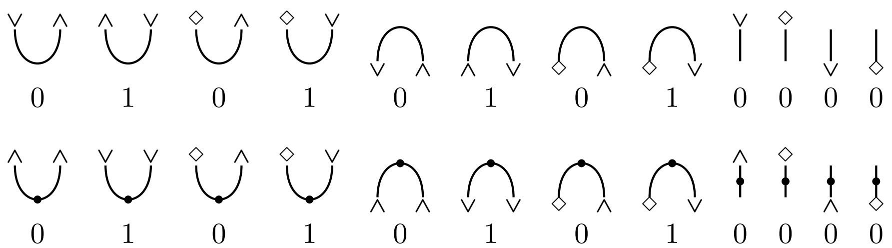
We now introduce the following important set of nuclear circle diagrams

Definition 5.16. Given two diagrammatic weights λ, μ we call the circle diagram $\underline{\lambda}\overline{\mu}$ *nuclear* if it contains at least one line which is not propagating.

In Figure 2, the last two circle diagrams (again ignoring the diagrammatic weight in the middle) are nuclear, the others not.

5.4. **Orientations and degree.** Assume λ is a diagrammatic weight and $\underline{\lambda}$ its associated decorated cup diagram. An *orientation* of $\underline{\lambda}$ is a diagrammatic weight ν such that λ and ν have the same core diagram and if we put ν on top of $\underline{\lambda}$ (identifying along the corresponding vertices), then all cups and rays in the resulting diagram are 'oriented' in one of the ways displayed in (5.48). An oriented infinite decorated cup diagram is such a pair $(\underline{\lambda}, \nu)$.

We usually just draw the cup diagram with the orientation on top and think of it in a topological way. The dots on cups and caps could be thought of as orientation reversing points justifying why we have to put two $\wedge$'s or two $\vee$'s at the endpoints of a dotted cup.



(5.48)

For instance, the cup diagram in (5.45) together with the weight in (5.45) is an oriented cup diagram. In fact $\underline{\lambda}\lambda$ is always an oriented cup diagram for any diagrammatic weight λ. Note that (5.45) has 2^6 possible orientations, namely precisely given by those weights which we obtain by taking any subset of the cups in (5.45) and change the corresponding label in ν from $\vee$ to $\wedge$ respectively $\wedge$ to $\vee$. In general a cup diagram c has precisely $2^{\mathrm{def}(c)}$ number of orientations.

Definition 5.17. A triple (λ, ν, μ) of diagrammatic weights is an *oriented circle diagram* if $\underline{\lambda}\overline{\mu}$ is a circle diagram and ν is an orientation of both $\underline{\lambda}$ and $\underline{\mu}$. We usually write such a triple as $\underline{\lambda}\nu\overline{\mu}$ and display it as the diagram $\underline{\lambda}\overline{\mu}$ with some labelling in the middle turning it into an oriented diagram in the sense that locally every arc looks like one of the form (5.48).

We refer to Figure 2 for all possible orientations on circle diagrams obtained from the cup diagrams in Figure 1.

Lemma 5.18. *If a circle diagram can be oriented, then there are precisely 2^x possible orientations, where x is the number of circles in the diagram.*

Remark 5.19. Note that not every circle diagram can be oriented. As shown in [ES13b, Lemma 4.8] to be orientable one needs at least that each circle in $\underline{\lambda}\overline{\mu}$ has an even number of $\bullet$'s. By [ES13b, Lemma 4.8] a circle diagram which is not nuclear and does not contain a ray at position zero can be oriented if and only if each component (circle or line) has an even number of dots. One easily checks that if it contains a ray at position zero then it is orientable if and only if each other component (circle or line) has an even number of dots (but the line passing through zero need not have an even number of dots).

Definition 5.20. The *degree* of an oriented cup diagram $\underline{\lambda}\nu$ or an oriented circle diagram $\underline{\lambda}\nu\overline{\mu}$ is the sum of the degrees of its components of the form (5.48), where the degree of each component is listed below each picture.

It follows directly from the definitions that $\underline{\lambda}\lambda$ is the unique orientation of $\underline{\lambda}$ of degree zero and all other orientations have positive degrees. In [ES13b] we called cups or caps of degree 0 *anticlockwise* and those of degree 1 *clockwise*. Then the degree is just the number of clockwise cups plus clockwise caps. For examples we refer again to Figure 2.

6. Diagrammatic weights

The goal of this section is to assign to each irreducible finite-dimensional $\mathrm{OSp}(r|2n)$-module in $\mathcal{F}$ a certain cup diagram which then allows us to make the connection with Khovanov's algebra and to formulate and prove the main theorem (Theorem 7.1).

6.1. **Fake cups.** Our infinite diagrammatic weights λ^∞ attached to $\ulcorner\gamma \in X^+(\mathfrak{g})$ via (5.5) and their decorated cup diagrams $\underline{\gamma^\infty}$ are slightly more general than those allowed in [ES13b] in the sense that they might have infinite defect. Diagrammatic weights with infinite defect were carefully avoided however in [BS11a] and in [ES13b], since the associated Khovanov algebra would not be well-defined. Note moreover that $\underline{\gamma}^\infty$ only depends on γ and $\frac{\delta}{2}$, but not on m, n itself. We will next introduce a dependence on m, n which also also has the effect of giving a certain finiteness condition which allows us to avoid working with infinite defects. This will finally put us into the framework from [ES13b] and will be enable us to talk about the Khovanov algebra associated to a block of $\mathrm{OSp}(r|2n)$. The defect will then correspond to the usual notion of atypicality of weights in the context of Lie superalgebras.

We start by incorporating the dependence on m and n.

Definition 6.1. Given $\ulcorner\lambda \in X^+(\mathfrak{g})$ with associated infinite cup diagram $\underline{\ulcorner\gamma^\infty}$, a cup C is a *fake cup* if C is dotted and there are at least $\mathrm{tail}(\ulcorner\gamma)$ dotted cups to the left of C. The vertices attached to fake cups are called *frozen vertices*. We indicate the frozen vertices by $\oslash$.

Remark 6.2. Note that Corollary 5.7 gives a formula to compute the tail length $\mathrm{tail}(\ulcorner\gamma)$. For instance, $\mathrm{tail}(\varnothing)$ equals $\min\{m, n\}$ and there are infinitely many frozen vertices to the right of the vertices indicated by $\oslash$ in

(5.43) and (5.44). By definition, fake cups are never nested insider another cup, since dotted cups are never nested. Moreover, all dotted cups to the right of a fake cup are obviously also fake cups.

For the empty partition the frozen vertices are indicated in (5.43) and (5.44), where the dependence on m and n is also illustrated; see also Section 9.2 for more examples.

6.2. **The super diagrammatic weight $\lambda^{\oslash}$.** Recall the infinite diagrammatic weight $\ulcorner\gamma^{\infty}$ attached to a hook partition $\ulcorner\gamma$. Given $\lambda \in X^{+}(\mathfrak{g})$ we define the super diagrammatic weight $\ulcorner\lambda^{\oslash}$ as the one obtained from $\ulcorner\lambda^{\infty}$ by replacing all the frozen labels by $\vee$'s except of the leftmost one, which we leave as an $\wedge$ (respectively $\diamond$).

Example 6.3. For instance consider $G = \mathrm{OSp}(6|4)$, that is $m = 3$, $n = 2$. First consider $\lambda \in X^{+}(\mathfrak{g})$ with corresponding hook partition $\ulcorner\lambda = (4, 2, 1)$. Then we have

$$\ulcorner\lambda^{\infty} : \diamond \circ \wedge \vee \wedge \underline{\oslash\oslash\oslash\oslash}\cdots \quad \rightsquigarrow \quad \ulcorner\lambda^{\oslash} : \diamond \circ \wedge \vee \wedge \underline{\wedge \vee \vee \vee}\cdots \tag{6.49}$$

where we indicated the relevant positions by a horizontal line. For the hook partition $\ulcorner\lambda = (4, 1, 1)$ we obtain

$$\ulcorner\lambda^{\infty} : \circ \wedge \wedge \vee \wedge \underline{\oslash\oslash\oslash\oslash}\cdots \quad \rightsquigarrow \quad \ulcorner\lambda^{\oslash} : \circ \wedge \wedge \vee \wedge \underline{\wedge \vee \vee \vee}\cdots \tag{6.50}$$

Note that in case $G = \mathrm{OSp}(7|4)$, the weights $\lambda \in X^{+}(\mathfrak{g})$ with hook partitions $\ulcorner\lambda = (5, 2, 1)$ respectively $(5, 1, 1)$ give rise to the same four diagrams as above, but placed on the positive half-integer line instead of the positive integer line.

If $\ulcorner\gamma^{\infty}$ has a $\circ$ at position zero or if $\ulcorner\gamma^{\infty}$ is supported on half-integers, then we more precisely write $(\ulcorner\lambda^{\oslash}, \epsilon)$ instead of just $\ulcorner\lambda^{\oslash}$, where $\epsilon \in \{1, 2\}$ is the parity of the number of $\wedge$'s, that is $\#\wedge(\ulcorner\gamma) + \#\times(\ulcorner\gamma) \equiv \epsilon \bmod 2$. If ϵ' is the opposite parity to ϵ, then denote by $(\ulcorner\lambda^{\oslash}, \epsilon')$ the diagram obtained from $\ulcorner\lambda^{\oslash}$ by replacing all labels at frozen vertices with $\vee$'s. By definition, the cup diagrams for $(\ulcorner\lambda^{\oslash}, \epsilon)$ and $(\ulcorner\lambda^{\oslash}, \epsilon')$ differ precisely by a dot on the leftmost ray. (Note that this ray is not attached to the zero vertex by assumption.)

Example 6.4. As above let $G = \mathrm{OSp}(6|4)$ and assume the situation (6.50).

$$\ulcorner\lambda^{\infty} : \circ \wedge \wedge \vee \wedge \underline{\oslash\oslash\oslash\oslash}\cdots \quad \rightsquigarrow \quad \begin{array}{l} (\ulcorner\lambda^{\oslash}, \overline{1}) : \circ \wedge \wedge \vee \wedge \underline{\vee \vee \vee \vee}\cdots \\ (\ulcorner\lambda^{\oslash}, \overline{2}) : \circ \wedge \wedge \vee \wedge \underline{\wedge \vee \vee \vee}\cdots \end{array}$$

If $\ulcorner\lambda^{\infty}$ has an $\wedge$ at position zero, but does not satisfy one of the equivalent conditions from Corollary 5.8, then write again $(\ulcorner\lambda^{\oslash}, \epsilon)$ instead of just $\ulcorner\gamma^{\oslash}$, where $\epsilon \in \{1, 2\}$ such that $\#\wedge(\ulcorner\lambda) + \#\times(\ulcorner\gamma) \equiv \epsilon \bmod 2$. If ϵ' is the parity opposite to ϵ, then denote by $(\ulcorner\gamma^{\oslash}, \epsilon')$ the diagram obtained from $\ulcorner\lambda^{\oslash}$ by replacing all labels at frozen vertices with $\vee$'s and also turn the $\wedge$ at position zero into a $\vee$ (this one is not frozen by assumption).

Example 6.5. Let $G = \mathrm{OSp}(6|4)$ and assume the situation (6.49). Then

$$\ulcorner\lambda^{\infty} : \diamond \circ \wedge \vee \wedge \underline{\oslash\oslash\oslash\oslash}\cdots \quad \rightsquigarrow \quad \begin{array}{l} (\ulcorner\lambda^{\oslash}, \overline{1}) : \diamond \circ \wedge \vee \wedge \underline{\vee \vee \vee \vee}\cdots \\ (\ulcorner\lambda^{\oslash}, \overline{2}) : \diamond \circ \wedge \vee \wedge \underline{\wedge \vee \vee \vee}\cdots \end{array}$$

6.3. **The diagrammatic weight associated to irreducible modules.** We now can assign to each dominant weight a diagram weight.

Definition 6.6. Consider $G = \mathrm{SOSp}(r|2n)$. Given $\lambda \in X^+(G)$ with underlying hook partition $^\ulcorner\lambda$, we define the *(super) diagrammatic weight attached to* λ, and also denoted by λ, as follows

$$\lambda = \begin{cases} {}^\ulcorner\lambda^{\oslash} & \text{if } \lambda = \Psi({}^\ulcorner\lambda), & (6.51a) \\ ({}^\ulcorner\lambda^{\oslash}, \bar{1}) & \text{if } \lambda = \Psi(({}^\ulcorner\lambda, +)), & (6.51b) \\ ({}^\ulcorner\lambda^{\oslash}, \bar{2}) & \text{if } \lambda = \Psi(({}^\ulcorner\lambda, -)), & (6.51c) \end{cases}$$

where we use the identifications from Lemma 4.21.

Remark 6.7. As a result we have attached to any $\lambda \in X^+(G)$ a cup diagram $\underline{\lambda}$ which has an infinite number of undotted rays, tail length many dotted cups, and at most one dotted ray. Observe that $\underline{\lambda}$ coincides with $\underline{\lambda^\infty}$, except that each fake cup is replaced by two vertical rays (with the leftmost ray possibly dotted). In other words, we keep the undotted cups, but force the diagram to have exactly as many dotted cups as the length of the tail by taking the first $\mathrm{tail}(\lambda)$ dotted cups. Note also that the core diagram of the diagrammatic weight λ is the same as the core diagram of λ^∞ from Definition 5.5.

For more examples we refer to Section 9.1

Remark 6.8. The weight diagrams attached to the pair $(\lambda, \pm)$ can be viewed as super analogues of the notion of *associated partitions* which was used by Weyl to label pairs of irreducible representations for $\mathrm{O}(r)$ which restrict to isomorphic representations for $\mathrm{SO}(r)$, see Section 9.1 for more details.

Remark 6.9. One can show that (for fixed G) the assignment which sends $\lambda \in X^+(G)$ to the diagrammatic weight λ is injective.

Proposition 6.10. *Assume $r = 2m$ and let $\lambda \in X^+(\mathfrak{g})$. Then there are the following two cases:*

(1) $\mathrm{Ind}_{\mathfrak{g},G'}^{\mathfrak{g},G} L^{\mathfrak{g}}(\lambda) = L(\lambda)$ is irreducible. Then, at position zero, the attached diagrammatic weight λ has a $\diamond$ and the cup diagram $\underline{\lambda}$ has a ray.

(2) $\mathrm{Ind}_{\mathfrak{g},G'}^{\mathfrak{g},G} L^{\mathfrak{g}}(\lambda) \cong L(\lambda,+) \oplus L(\lambda,-)$. Then the diagrammatic weight λ has a $\circ$ or a $\diamond$ at position zero. In the latter case $\underline{\lambda}$ has a dotted cup starting at the zero position.

Proof. In the situation (1) Corollary 5.8 implies that there is a $\diamond$ at position zero and the tail is zero. Hence the dotted cup attached to the zero position in λ^∞ is fake, and thus gives a ray in $\underline{\lambda}$. Situation (2) is equivalent to ${}^\ulcorner\lambda_{n+1} < m$ and $\circ$ or $\diamond$ can occur at position zero. Assume first $\mathrm{tail}({}^\ulcorner\lambda) = 0$, this means ${}^\ulcorner\lambda_n \geq m$. Then $\mathcal{S}({}^\ulcorner\lambda)_n = \frac{\delta}{2} + n - 1 - {}^\ulcorner\gamma_n = m - n + n - 1 - {}^\ulcorner\gamma_n < 0$, and $\mathcal{S}({}^\ulcorner\lambda)_{n+1} = \frac{\delta}{2} + n + 1 - 1 - {}^\ulcorner\gamma_{n+1} = m - n + n - {}^\ulcorner\gamma_{n+1} > 0$. Since the sequence $\mathcal{S}({}^\ulcorner\lambda)$ is strictly increasing, the value zero does not occur and therefore we have $\circ$ at position zero. in particular, if $\diamond$ occurs than we must have $\mathrm{tail}({}^\ulcorner\lambda) > 0$, in which case the dotted cup attached to zero in ${}^\ulcorner\lambda^\infty$ is not fake and so $\underline{\lambda}$ has a dotted cup starting at the zero position. □

Corollary 6.11. *Assume* $\lambda = (\lambda^{\mathfrak{g}}, \pm)$ *and* $\mu = (\mu^{\mathfrak{g}}, \pm) \in X^+(G)$. *Then* $\underline{\lambda}\overline{\mu}$ *is orientable if and only if each component contains an even number of dots. Moreover the number of possible orientations equals* 2^c*, where* c *is the number of closed components in* $\underline{\lambda}\overline{\mu}$*.*

Proof. By Remark 5.19 the "if" statement holds as well as the claim about the number of orientations. On the other hand, again by Remark 5.19, orientability implies that each closed component and each line not passing through zero must have an even number of dots. Now by Proposition 6.10, both, $\underline{\lambda}$ and $\underline{\mu}$, have a dotted cup attached to zero and so to be orientable, the line passing through zero also needs to have an even number of dots. □

6.4. **Blocks and diagrammatic linkage.**

Definition 6.12. We say that two elements $\lambda, \mu \in X^+(G)$ are *diagrammatically linked* if their attached super diagrammatic weights λ and μ are in the same block in the sense of Definition 5.3.

Lemma 6.13. *Given* $\lambda \in X^+(G)$ *then* $\operatorname{def}(\lambda) = n - \#\times(\lambda)$ *with the notation from Definition 5.13. In particular, if* λ *and* μ *give rise to the same core diagram then* $\operatorname{def}(\lambda) = \operatorname{def}(\mu)$*.*

Proof. Note that passing from $\ulcorner\lambda^{\oslash}$ to λ does not change the total number of cups in the corresponding cup diagram. Now, the number of undotted cups in $\underline{\ulcorner\lambda^{\oslash}}$ equals $\# \vee (\lambda^\infty)$, whereas the number of dotted cups is by construction equal to $\operatorname{tail}(\lambda) = n - s$, where $s = \# \vee (\lambda^\infty) + \# \times (\lambda^\infty)$. The claim follows. □

Lemma 6.14. *Two diagrammatically linked elements* $\lambda, \mu \in X^+(G)$ *have the same defect.*

Proof. Since they have by definition the same core diagram this follows directly from Lemma 6.13. □

Proposition 6.15. *Let* $G = \mathrm{OSp}(r|2n)$*. Assume* $\lambda, \mu \in X^+(G)$ *such that the circle diagram* $\underline{\lambda}\overline{\mu}$ *is not nuclear. If* λ *belongs to case* (6.51b) *and* μ *belongs to* (6.51c) *(or vice versa) then this circle diagram is not orientable. Moreover* λ *and* μ *are only diagrammatically linked if they both have a* $\diamond$ *at position zero and the same core diagram.*

Proof. Suppose first that we are in the odd case or λ and μ have $\circ$ at position zero. Then λ and μ are not diagrammatically linked, since the parity of $\#\wedge(\lambda)$ and of $\#\wedge(\mu)$ are different by assumption. By (5.48) every orientation ν of $\underline{\lambda}$ satisfies $\# \wedge (\nu) \equiv \# \wedge (\lambda) \bmod 2$. Hence any orientation $\underline{\lambda}\nu\overline{\mu}$ implies $\# \wedge (\lambda) \equiv \# \wedge (\mu) \bmod 2$ which is a contradiction.

Otherwise suppose we are in the even case and λ and μ have a $\diamond$ at position zero. Then they are by Definition 5.3 diagrammatically linked if they both have the same core diagram. Now assume $\underline{\lambda}\nu\overline{\mu}$ is oriented, then we can count $\diamond$ as an $\wedge$ or as a $\vee$ such that the diagram locally looks like a diagram of the form (5.48) (noting that by Proposition 6.10 both λ and μ have no ray at position zero). In either case it implies that $\# \wedge (\lambda) \equiv \# \wedge (\mu) \bmod 2$, which is a contradiction. □

The following is only applicable in case $G = \mathrm{OSp}(2m|2n)$, since otherwise $(\lambda, +), (\lambda, -)$ are in different diagrammatic blocks.

Proposition 6.16. *Consider elements $\mu, (\lambda, +), (\lambda, -) \in X^+(G)$ and the corresponding diagrammatic weights which we denote by the same notation. (they cover exactly the three cases in Definition 6.6). Assume that these weights are in the same diagrammatic block.*

(1) Then $\underline{\mu}\overline{(\lambda, +)}$ is not nuclear if and only if $\underline{\mu}\overline{(\lambda, -)}$ is not nuclear.
(2) In this case we have moreover that the number of possible orientations of $\underline{\mu}\overline{(\lambda, +)}$ equals the number of possible orientations of $\underline{\mu}\overline{(\lambda, -)}$.
(3) This number of possible orientations is non-zero if and only if every closed component in $\underline{\mu}\overline{(\lambda, \pm)}$ and each line not passing through zero contains an even number of dots in which case it equals 2^c where c is the number of closed components.

Proof. The first statement is obvious, since $(\lambda, +)$ differs from $(\lambda, -)$ only by a dot. By Proposition 6.10, the cup diagrams $\underline{(\lambda, +)}$ and $\underline{(\lambda, -)}$ have both a dotted cup at the position zero, whereas $\underline{\mu}$ has a ray at position zero. This implies that $\underline{\mu}\overline{(\lambda, \pm)}$ have both a propagating line through zero in case they are not nuclear. By construction, this propagating line contains the (leftmost) ray in which $\underline{(\lambda, +)}$ differs from $\underline{(\lambda, -)}$. But since $\diamond$ can be considered as an $\wedge$ or as a $\vee$ with respect to orientations, every orientation of $\underline{\mu}\overline{(\lambda, +)}$ is also one of $\underline{\mu}\overline{(\lambda, -)}$ and vice versa. This shows (2). For the third one note that a closed component is orientable precisely if and only it if has an even number of dots. $\square$

7. Homomorphisms between projectives via Khovanov algebras

Our main theorem gives now a description of the underlying vector space of $\mathrm{Hom}_{\mathcal{F}}(P(\lambda), P(\mu))$ for any $\lambda, \mu \in X^+(G)$, which in particular includes an explicit counting formula for the dimension of the spaces of morphisms between two indecomposable projective objects. In the special case $G = \mathrm{OSp}(2m + 1|2n)$ this gives Theorem A from the introduction.

7.1. **The main theorem.** Recall the vector space $\mathbb{I}$ from Definition 5.16.

Theorem 7.1. *Consider $G = \mathrm{OSp}(r|2n)$ for fixed m, n. For any $\lambda, \mu \in X^+(G)$ we have an isomorphisms of vector spaces*

$$\mathrm{Hom}_{\mathcal{F}}(P(\lambda), P(\mu)) \quad \cong \quad \mathbb{B}(\lambda, \mu)/\mathbb{I}_{\lambda,\mu}. \tag{7.52}$$

Here, $\mathbb{B}(\lambda, \mu)$ is the vector space with basis all oriented circle diagrams of the form $\underline{\lambda}\nu\overline{\mu}$ for some diagrammatic weights ν, and $\mathbb{I}_{\lambda,\mu}$ is the vector subspace spanned by its set of nuclear diagrams, hence

$$\mathbb{B}(\lambda, \mu)/\mathbb{I}_{\lambda,\mu} = \{\underline{\lambda}\nu\overline{\mu} \mid \underline{\lambda}\nu\overline{\mu} \in \mathbb{B} \text{ and } \underline{\lambda}\overline{\mu} \notin \mathbb{I}\}\,. \tag{7.53}$$

Proof. Theorem 7.1 follows directly from the Dimension Formula (Theorem 8.4) . $\square$

The following is a shadow of the duality explained in [MW14, 5.5]:

Corollary 7.2. *Let $G = \mathrm{OSp}(2m+1|2n)$ and $G^t = \mathrm{OSp}(2n+1|2m)$. Let $\lambda, \mu \in X^+(G)$ and $\lambda^t, \mu^t \in X^+(G^t)$ be the corresponding element with the same sign, but transposed partition. Then*

$$\mathrm{Hom}_{\mathcal{F}(G)}(P(\lambda), P(\mu)) \quad \cong \quad \mathrm{Hom}_{\mathcal{F}(G^t)}(P(\lambda^t), P(\mu^t)).$$

Proof. This follows directly from the theorem, since the associated diagrammatic weight for λ^t is obtained from that of λ by swapping $\circ$ with $\times$ and $\times$ with $\circ$. This swapping is however irrelevant for the dimension counting, since it does (up to core symbols) not change the corresponding cup diagram. and possible orientations. □

Before we prove the Dimension Formula in Theorem 8.4 (and hence Theorem 7.1) we first explain how to put an algebra structure on the space $\bigoplus_{\lambda,\mu} \mathbb{B}(\lambda, \mu)/\mathbb{I}_{\lambda,\mu}$ as required in Theorem B.

7.2. **The algebra structure and the nuclear ideal.** Let $G = \mathrm{OSp}(r|2n)$ and consider a fixed block $\mathcal{B}$ of $\mathcal{F}$. Let $P = \oplus_\lambda P(\lambda)$ be a minimal projective generator, that is the direct sum runs over all $\lambda \in X^+(G)$ such that $P(\lambda) \in \mathcal{B}$. By Proposition 7.5 below, the corresponding set $\Lambda(\mathcal{B})$ of diagrammatic weights is contained in a block Λ in the sense of Definition 5.3. Let $\mathbb{D}_\Lambda$ be the Khovanov algebra of type D attached to Λ as defined in [ES13a]. Let $\mathbb{1}_\mathcal{B} = \sum_{\lambda \in \Lambda(\mathcal{B})}$ be the idempotent in $\mathbb{D}_\Lambda$ corresponding to $\Lambda(\mathcal{B})$, see [ES13a, Theorem 6.2]. We consider now the algebra $\mathbb{1}_\mathcal{B}\mathbb{D}_\Lambda\mathbb{1}_\mathcal{B}$. By definition it has a basis given by all oriented circle diagrams $\underline{\lambda}\nu\overline{\mu}$, where $\lambda, \mu \in \Lambda(\mathcal{B})$. We first observe the following crucial fact:

Lemma 7.3. *The subspace $\mathbb{I}_\mathcal{B}$ of $\mathbb{1}_\mathcal{B}\mathbb{D}_\Lambda\mathbb{1}_\mathcal{B}$ spanned by all nuclear basis vectors is an ideal.*

Proof. Let $x \in \mathbb{I}_\mathcal{B}$ be a basis vector. Hence we can find $\lambda, \mu \in \Lambda'$ such that $x \in \mathbb{B}(\lambda, \mu) \cap \mathbb{I}_\mathcal{B}$ and x contains at least one non-propagating line. It is enough to show that $cx, xc \in \mathbb{I}$ for any basis element c of $\mathbb{1}_\mathcal{B}\mathbb{D}_\Lambda\mathbb{1}_\mathcal{B}$. The algebra $\mathbb{D}_\Lambda$ has an anti-automorphism which sends a basis element $a\nu b$ to $b^*\nu a^*$ in the notation from Definition 5.17, see [ES13a, Corollary 6.4]. Obviously this descends to an anti-automorphism of $\mathbb{1}_\mathcal{B}\mathbb{D}_\Lambda\mathbb{1}_\mathcal{B}$ which preserve $\mathbb{I}_\mathcal{B}$. Therefore, it is enough to show $bc \in \mathbb{I}_\mathcal{B}$.

Consider the non-propagating lines in b. Then the number of those ending at the top equals the number of those ending at the bottom since the weights in Λ' are linked and have the same defect by Lemma 6.14. Hence assume there is at least one such line L ending at the bottom.

From the surgery procedure defining the algebra structure we see directly that any surgery involving such a line and a circle either preserves this property ([ES13a, first two cases in Remark 5.13] and [ES13a, Remark 5.15]), or produces zero ([ES13a, last two cases in Remark 5.13] and [ES13a, Reconnect in 5.2.3]). Hence the claim follows. □

Then thanks to Theorem 7.1, there is a canonical isomorphism of vector spaces $\mathrm{End}^{\mathrm{fin}}_\mathcal{F}(P) \cong \mathbb{1}_\mathcal{B}\mathbb{D}_\Lambda\mathbb{1}_\mathcal{B}/\mathbb{I}_\mathcal{B}$, sending a basis vector to the corresponding basis vector of $\mathbb{1}_\mathcal{B}\mathbb{D}_\Lambda\mathbb{1}_\mathcal{B}$ denoted in the same way. In particular, $\mathrm{End}^{\mathrm{fin}}_\mathcal{F}(P)$

inherits an algebra structure from the Khovanov algebra $\mathbb{D}_\Lambda$ via this identification. In part II of this series we show (a more general version of Theorem B that the two algebras are isomorphic.

7.3. **Dictionary to GS-weights.** To prove Proposition 7.1 we have to connect the diagram calculus developed in [GS13] to our calculus. For later reference and to make a precise connection to [GS13] we give an explicit dictionary, although we could prove the result more directly. The GS-diagrammatic weight $\mathrm{GS}(\lambda)$ associated with $\lambda \in X^+(\mathfrak{g})$ is a certain labelling $\mathcal{L}$ with the symbols $<, >, \times, \circ, \otimes$ with almost all vertices labelled $\circ$. Gruson and Serganova obtain this labelling as the image of a composite of two maps

$$\begin{aligned}\mathrm{GS}: \quad X^+(\mathfrak{g}) &\longrightarrow \{\mathrm{GS-diagrams\ with\ tail}\}\\ &\longrightarrow \{\mathrm{coloured\ \ GS-diagrams\ without\ tail}\} \quad (7.54)\end{aligned}$$

We refer to [GS13] for details, but will briefly recall the construction in Section 7.6 below. (The additional signs appearing in [GS13] and in the weights for $X^+(\mathfrak{g})$ do not play any role for us thanks to (4.23) and therefore we can ignore them.)

For convenience we list now the explicit map T which translates from GS-weights $\mathrm{GS}(\lambda)$ to our diagrammatic weights $\lambda^\infty = \mathrm{T}(\mathrm{GS}(\lambda))$ and vice versa. The following dictionary gives us the translation, with the first line containing the label in $\mathrm{GS}(\lambda)$ and the second line the corresponding label in the diagrammatic weight $T(\mathrm{GS}(\lambda))$:

$\mathrm{GS}(\lambda)$	$<$	$>$	$\times$	$\circ$	$\otimes$	at 0: $\otimes$	$>$	$\circ$
$\mathrm{T}(\mathrm{GS}(\lambda))$	$\times$	$\circ$	$\vee$	$\wedge$	$\wedge$	$\diamond$	$\circ$	$\diamond$

(7.55)

Even though the vertex $\frac{1}{2}$ will play a special role in the proofs to come, only the vertex 0 in the even case has a special assignment rule.

7.4. **Comparison of the two cup diagram combinatorics.** Gruson and Serganova assigned to any $\mathrm{GS}(\lambda)$-weight also some cup diagram (without any decorations). We claim that our combinatorics refines their combinatorics in the following sense (with the felicitous consequence that the assignment from $X^+(G)$ to cup diagrams is injective):

Proposition 7.4. *Let $\lambda \in X^+(\mathfrak{g})$ with associated hook diagram $\ulcorner\lambda$.*

(1) The assignment T*, from* (7.55) *satisfies*

$$\mathrm{T}(\mathrm{GS}(\lambda)) \quad = \quad \ulcorner\lambda^\infty. \qquad (7.56)$$

(2) Moreover, the cup diagram attached to $\mathrm{GS}(\lambda^{\mathfrak{g}})$ *in the sense of* [GS13] *agrees with our cup diagram* $\underline{\ulcorner\lambda^\infty}$ *when forgetting the decorations and fake cups, and with* $\underline{\ulcorner\lambda^{\oslash}}$ *when forgetting the decoration and all rays.*

(3) Under this correspondence the cups attached to $\otimes$'s correspond precisely to the dotted, non-fake cups in $\underline{\ulcorner\lambda^\infty}$*, and to the dotted cups in* $\underline{\ulcorner\lambda^{\oslash}}$*.*

Proof. It suffices to prove the statements involving $\ulcorner\lambda^\infty$, since the others follow then directly from the definition of $\underline{\ulcorner\lambda^{\oslash}}$. The proof is given in the next section. ∎

7.5. **Blocks in terms of diagrammatic blocks.** Before we prove this result, we deduce some important consequence:

Proposition 7.5. *Let $\lambda, \mu \in X^+(G)$. Then $P(\lambda)$ and $P(\mu)$ (and hence then also $L(\lambda)$ and $L(\mu)$) are in the same block if and only if λ^∞ and μ^∞ have the same core diagrams in the sense of Definition 5.1 and additionally $\# \wedge (\ulcorner\lambda^{\oslash}) \equiv \# \wedge (\ulcorner\mu^{\oslash}) \bmod 2$ in case no $\diamondsuit$ occurs.*

Proof. Observe that the assignment T sends core symbols in the sense of [GS13] to the core symbols in the sense of Definition 5.1.

Let us first assume $G = \mathrm{OSp}(2m+1|2n)$. Then by Definition 4.6 we have $\lambda = (\lambda^{\mathfrak{g}}, \epsilon)$ and $\mu = (\mu^{\mathfrak{g}}, \epsilon')$ for some $\lambda^{\mathfrak{g}}, \mu^{\mathfrak{g}} \in X^+(\mathfrak{g})$. Now by Corollary 4.8, $P^{\mathfrak{g}}(\lambda)$ and $P^{\mathfrak{g}}(\mu)$ are in the same block if and only if $\epsilon = \epsilon'$ (that means σ acts by the same scalar) and $P(\lambda^{\mathfrak{g}})$ and $P(\mu^{\mathfrak{g}})$ are in the same block for $\mathcal{F}'$. By [GS13], the latter holds precisely if the associated weight diagrams $\mathrm{GS}(\lambda^{\mathfrak{g}})$ and $\mathrm{GS}(\mu^{\mathfrak{g}})$ have the same core diagram in the sense of [GS13], and hence by Proposition 7.4 λ^∞ and μ^∞ have the same core diagrams in the sense of Definition 5.1. Therefore $P(\lambda)$ and $P(\mu)$ are in the same block if and only if λ^∞ and λ^∞ have the same core diagrams and $\# \wedge (\ulcorner\lambda^{\oslash}) \equiv \# \wedge (\ulcorner\mu^{\oslash}) \bmod 2$, since this parity is given by ϵ.

Let now $G = \mathrm{OSp}(2m|2n)$. Assume $\mathrm{Hom}_{\mathcal{F}}(P(\lambda), P(\mu)) \neq \{0\}$ then by Lemma 4.17 and Proposition 6.15 $\mathrm{Hom}_{\mathcal{F}'}(\mathrm{res}\, P(\lambda), \mathrm{res}\, P(\mu)) \neq \{0\}$. By Lemma 4.14 $\mathrm{res}\, P(\lambda)$ and $\mathrm{res}\, P(\mu)$ give rise to weight diagrams in the sense of [GS13] which have the some core diagrams, hence λ^∞ and μ^∞ have the same core diagrams thanks to Proposition 7.4. If the core diagram contains a $\circ$ at zero, then the corresponding weights $\ulcorner\gamma^{\oslash}$ with this same core diagram fall into two classes, the ones where $\# \wedge (\ulcorner\gamma^{\oslash})$ is even (and so it corresponds to some $(\gamma, +) \in X^+(G)$) and the ones where $\# \wedge (\ulcorner\gamma^{\oslash})$ is odd (and so it corresponds to some $(\gamma, -) \in X^+(G)$). By Proposition 4.17 (3) both sets in fact give rise to a block, since they form a block if we consider the corresponding weights in $\gamma \in X^+(\mathfrak{g})$. Hence the claim follows in this case. For $\lambda \in X^+(G)$ let $P^{\mathfrak{g}}(\lambda^{\mathfrak{g}})$ by the unique summand in $\mathrm{res}\, P(\lambda)$ such that $\lambda^{\mathfrak{g}} \in X^+(\mathfrak{g})$ satisfies $a_m \geq 0$ in the notation (4.21). Now, if the core diagram contains a $\diamond$ at zero, then the same argument as above gives that $\mathrm{Hom}_{\mathcal{F}}(P(\lambda), P(\mu)) \neq \{0\}$ implies that the core diagrams must be the same. On the other hand if the core diagrams are the same, $P^{\mathfrak{g}}(\lambda^{\mathfrak{g}})$ and $P^{\mathfrak{g}}(\mu^{\mathfrak{g}})$ are in the same block by [GS13] and hence are connected by a sequence of homomorphisms between projective modules which can be chosen of the form $P^{\mathfrak{g}}(\nu^{\mathfrak{g}})$ for some ν, but this gives then by Proposition 4.17 a sequence of non-zero morphisms connecting $P(\lambda)$ and $P(\mu)$. The claim follows. □

Corollary 7.6. *Let $\lambda, \mu \in X^+(G)$. If $P(\lambda)$ and $P(\mu)$ are in the same block $\mathcal{B}$ of $\mathcal{F}$ then $\mathrm{def}(\lambda) = \mathrm{def}(\mu)$. In particular, we can talk about the defect of a block $\mathcal{B}$ of $\mathcal{F}$.*

Remark 7.7. Using the Dictionary to [GS13] which we will develop in (7.55), one can show that the defect is precisely the *atypicality* of the block in the sense of Lie superalgebras. We expect that, in contrast to the SOSp-case treated in [GS10, Theorem 2], the blocks depend up to equivalence of categories only on the atypicality, see Section 9 for examples.

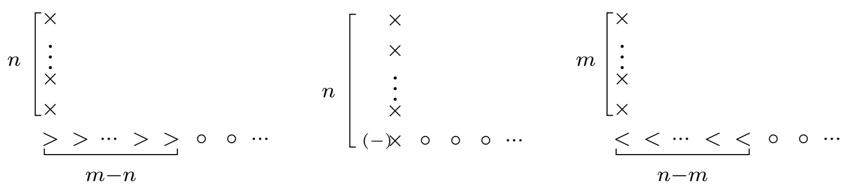
To prove Proposition 7.4 we first need to recall some of the construction from [GS13].

7.6. **The Gruson-Serganova combinatorics.** We start by recalling the construction of the map GS from [GS13]. Recall the notion of vertices on $\mathcal{L}$ as in Section 5.1. The first map in (7.54) takes a weight $\lambda \in X^+(\mathfrak{g})$ writes $\lambda + \rho$ in the form (4.16) and puts at the vertex p of $\mathcal{L}$ then α_p symbols $>$ and β_p symbols $<$, where

$$\alpha_p = |\{1 \leq j \leq m \mid a_j = \pm p\}| \text{ and } \beta_p = |\{1 \leq i \leq n \mid b_i = \pm p\}|$$

and a symbol $\circ$ if $\alpha_p = \beta_p = 0$. We use the abbreviation $\times$ for a pair $>$ and $<$ at a common vertex. We call the resulting diagram a GS-*diagram with tail*.

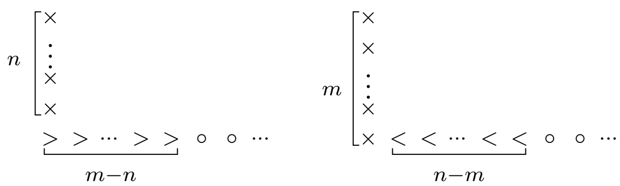
Case: $\mathfrak{osp}(2m+1|2n)$*:* In this case the the dominance condition is equivalent to the statement that there is at most one symbol, $>$, $<$, $\times$ or $\circ$ at each vertex $p > \frac{1}{2}$ and at $\frac{1}{2}$ at most one $<$ or $>$, but possibly many $\times$. If there are only $\times$'s at $\frac{1}{2}$ we have to put an indicator which is $(+)$ if $a_j = \frac{1}{2}$ for some j and $(-)$ otherwise. For instance, the diagram for the trivial weight are the following for $n > m$, $m = n$, $m > n$ respectively.



(7.57)

The tail length is the number of $\times$ at the leftmost vertex, subtracting one if the indicator is $(+)$ and the tail are all symbols $\times$ at position $\frac{1}{2}$ except for one if the indicator is $(+)$.

Case: $\mathfrak{osp}(2m|2n)$*:* The dominance condition in this case is equivalent to the statement that there is at most one symbol, $>$, $<$, $\times$ or $\circ$ at each vertex $p > 0$ and at 0 either $\circ$ or at most one $>$ but possibly many $\times$. If there is a $\circ$ at 0 one has to remember a sign to distinguish $a_m > 0$ and $a_m < 0$ (denoted by $[\pm]$ in [GS13]). For instance the trivial weight corresponds to the following for $m > n$, $n \geq m$ respectively.



(7.58)

Note that the tail length in this case is the number of $\times$ at the leftmost vertex.

For the second map (7.54) we have to turn the diagram with tail into a coloured weight diagram. In case of $\mathfrak{osp}(2m+1|2n)$ proceed as follows: First remove the tail of the diagram, but remember the number $l = \operatorname{tail}(\lambda)$, of symbols removed (note that in case of an indicator this can mean that one symbol $\times$ at position $\frac{1}{2}$ is kept). Ignoring the core symbols $<$ and $>$, connect neighboured pairs $\times$ $\circ$ (in this order) successively by a cup. Then number the vertices not connected to a cup and not containing $<$ or $>$ from the left by $1, 2, 3, \ldots$. Then relabel those positions with number $1, 3, 5, \ldots, 2l-1$ etc. by $\otimes$. (The symbol $\otimes$ should indicate that at least apart from the special case of the left most vertex a $\times$ was actually moved and placed on top of a $\circ$). Finally connect neighboured pairs $\otimes$ and $\circ$ successively by a cup.

The resulting diagram with all labels at cups removed is the GS-*cup diagram* attached to λ. In [GS13] these new labels $\otimes$ are called *coloured* and we call the attached cups *coloured*; note they are by construction never nested inside other cups. The resulting labelling of $\mathcal{L}$ (after all cups are removed) is the *coloured* GS-*diagram without tail* attached to λ.

In case of $\mathfrak{osp}(2m+1|2n)$ proceed in the same way but viewing the vertex 0 as the vertex $\frac{1}{2}$ and always using the rule that if there are only $\times$ at position zero we use the rule that the indicator is $(+)$. Note that whether a_m is strictly larger or smaller than 0 does not play a role in the construction of the diagram.

The following gives an easy example:

Lemma 7.8. *With the assignment* T*, from* (7.55)*, we have* $\mathrm{T}(\mathrm{GS}(0)) = 0^\infty$*, and Proposition 7.4 holds for* $\lambda = 0$*.*

Proof. ▷ ***Case*** $\mathfrak{osp}(2m+1|2n)$*:* The weights from the diagrams (7.57) with tail are transferred into the cup diagram with m, respectively n in the last case, coloured cups placed next to each other starting at position $-\frac{\delta}{2}+1$, 0, and $\frac{\delta}{2}$ respectively. On the other hand, our diagrammatics assigns to the empty partition the diagrammatic weights (5.43) and hence produce a cup diagrams with n, respectively m in the last case, dotted cups placed next to each other starting at position $-\frac{\delta}{2}+1$, 0 and $\frac{\delta}{2}$ respectively, see (5.46). The corresponding coloured weight diagram contains the $>$'s and $<$'s as and only $\circ$ and $\otimes$ at the positions of the cups. Applying T this translates into the diagrammatic weights (5.43). Hence the claim is true in case $\mathfrak{osp}(2m+1|2n)$:
▷ ***Case*** $\mathfrak{osp}(2m|2n)$*:* In this case the diagrams (7.58) with tail are transformed into a cup diagram with n, respectively $m-1$ in the second case, coloured cups placed next to each other starting at positions $\frac{\delta}{2}$, respectively $-\frac{\delta}{2}+2$. In the latter case it also contains one uncoloured cup connecting position zero and $-\frac{\delta}{2}+1$. Using our diagrammatics will produce the weights diagrams in (5.44), which in turn produce cup diagrams with n, respectively $m-1$ dotted cups placed next to each other starting at positions $\frac{\delta}{2}$, respectively $-\frac{\delta}{2}+2$, see (5.47). □

7.7. **The proof of Proposition 7.4.** The proof proceeds by induction on the number of boxes in the corresponding hook partition (where we are allowed to ignore the sign).

Proof of Proposition 7.4. In case of the empty partition the claim follows from Lemma 7.8 above.

▶ *Adding a box:* In the situation that the partition for λ is obtained from the one for μ by adding a box, we first summarize a few general results on what kind of configurations are not possible in the two combinatorics.

- If for the μ it holds, that $b_i > 0$ and we can add a box in row i. Then $b_j > b_i + 1$ for all $j < i$. This implies that in the GS weight there is no symbol $<$ or $\times$ at positions $b_i + 1$, i.e. immediately to the right of b_i.

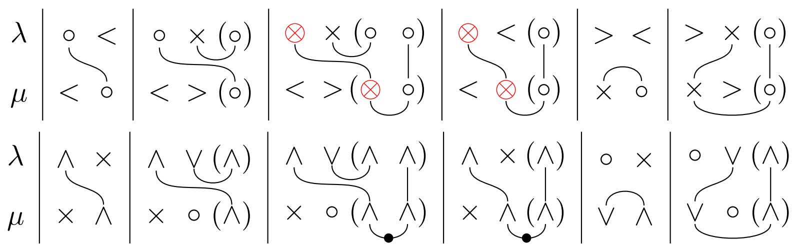
- If for the μ it holds, that $a_i > 0$ and we can add a box in column i. Then $a_j > a_i + 1$ for all $j < i$. This implies that in the GS weight there is no symbol $>$ or $\times$ at positions $a_i + 1$.

Note that there can be a symbol $\otimes$ at the position $a_i + 1$ respectively $b_i + 1$ from the tail.

We start with the cases that the box is added far away from the diagonal, i.e. not on the diagonal or next to it. In these cases the odd and even case behave exactly the same.

The additional box is added far above the diagonal. We add the box in position (j_0, i_0) and the box in question is not immediately above the diagonal. In both the even and odd case $b_{j_0} > \frac{1}{2}$ is increased by 1 and all other a's and b's are preserved. This means a symbol $<$ is moved to the right from position b_{j_0} to $b_{j_0} + 1$. Note that if the symbol $<$ is part of a $\times$ there cannot be a symbol $\otimes$ at position $b_{j_0} + 1$ by construction of the coloured GS-diagram.

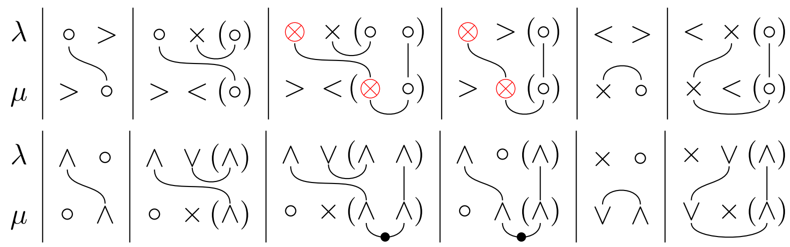
The table below lists the possible configurations at b_{j_0} and $b_{j_0} + 1$ (with corresponding translations, obtained via T, displayed in the second row and the symbols in brackets are added to indicate the shape of the cup diagram).



For the second row we argue as follows. Since $b_{j_0} > 0$ with $b_{j_0} = \mu_{j_0} - j_0 - \frac{\delta}{2} + 1$. This implies that $\mathcal{S}(\mu)_{j_0} = -b_{j_0}$, which in turn will be decreased by 1 by assumption. Having either the symbol $\vee$ or $\times$ at position b_{j_0} with the symbol $\vee$ being moved to the right. Furthermore $\mathcal{S}(\mu)_j < \mathcal{S}(\mu)_{j_0} - 1$ since $\mu_j > \mu_{j_0}$ for $j < j_0$, which in turn implies that at position $b_{j_0} + 1$ there is the symbol $\wedge$ or $\circ$.

In all of the listed cases neither the tail length nor the number of dotted cups is changed, thus all fake cups are unchanged, and if an $\wedge$ in μ^∞ is frozen and moved, it is still frozen in λ^∞. Hence the claim follows in this case.

The box is added far below the diagonal. We add the box in position (j_0, i_0) and not adjacent to the diagonal. In this case $a_{i_0} > \frac{1}{2}$ is increased by one and all other a_i's and b_i's are left unchanged. Thus we move a symbol $>$ to the right. As before there cannot be the symbol $\otimes$ at position $a_{i_0} + 1$ if there is the symbol $\times$ at position a_{i_0}. In total this gives us the configurations in the first row below (showing positions a_{i_0} and $a_{i_0}+1$)



For the second row note that since $a_{i_0} > 0$ we have $a_{i_0} = \mu^t_{i_0} - i_0 + \frac{\delta}{2} = j_0 - 1 - i_0 + \frac{\delta}{2}$. Which implies $\mathcal{S}(\mu)_{j_0} = \frac{\delta}{2} + j_0 - \mu_{i_0} - 1 = a_{i_0} + 1 > 0$, which will be decreased by 1 (since μ_{i_0} is increased by 1). Thus we have the symbol $\wedge$ or $\times$ at position $a_{i_0}+1$ with $\wedge$ moved to the left. Furthermore $\mathcal{S}(\mu)_j < \mathcal{S}(\mu)_{j_0} - 1$ since $\mu_j > \mu_{j_0}$ for $j < j_0$, which in turn implies that at position a_{i_0} there is either the symbol $\vee$ or $\circ$.

Again, in all cases neither the tail length nor the number of dotted cups change, thus all fake cups are unchanged. Additionally if an $\wedge$ in μ^∞ is frozen and moved in λ^∞ it will be frozen in λ^∞ as well.

To add a box on or adjacent to the diagonal we have to distinguish the even and odd case.

▷ ***Case*** $\mathfrak{osp}(2m+1|2n)$*:* We distinguish three possibilities: adding the box exactly above the shifted diagonal, adding the box exactly below the shifted diagonal, and adding the box on the diagonal.

The additional box is added exactly above the diagonal. We add the box in position (j_0, i_0) and thus it holds $i_0 - j_0 = \frac{\delta}{2} + \frac{1}{2}$. In addition $b_{j_0} = \frac{1}{2}$ and increased by 1, while all other a's and b's are left unchanged. Thus a symbol $>$ is moved from position $\frac{1}{2}$ to position $\frac{3}{2}$. If this symbol is not part of a symbol $\times$ then the arguments are the same as for adding a box far above the diagonal and we refer to that case. If on the other hand it is part of a $\times$ this implies that the indicator is $(+)$ since $a_{i_0-1} = \frac{1}{2}$. Thus the $\times$ at position $\frac{1}{2}$ is not coloured and one obtains the left block below.

To obtain the right block, note that $\mathcal{S}(\mu)_{j_0} = -\frac{1}{2}$ which is decreased to $-\frac{3}{2}$. Obtaining the other possible entries is done as before. Again neither tail length nor number of dotted cups changes.

The additional box is added exactly below the diagonal. We add the box in position (j_0, i_0) and thus it holds $i_0 - j_0 = \frac{\delta}{2} - \frac{3}{2}$. In addition $a_{i_0} = \frac{1}{2}$ and increased by 1, while all other a's and b's are left unchanged. Thus a symbol $<$ is moved from position $\frac{1}{2}$ to position $\frac{3}{2}$. If this symbol is not part of a symbol $\times$ then the arguments are the same as for adding a box far below the diagonal and we refer to that case. As in the case above this implies that the indicator is $(+)$ since $a_{i_0} = \frac{1}{2}$ and we obtain the left block below.

For the right block note that $\mathcal{S}(\mu)_{j_0} = \frac{3}{2}$ which is decreased to $\frac{1}{2}$. The rest of the arguments is the same as before.

The additional box is added exactly on the diagonal. We add the box in position (j_0, i_0) and thus it holds $i_0 - j_0 = \frac{\delta}{2} - \frac{1}{2}$. In addition $a_{i_0} = -\frac{1}{2}$ and increased by 1, while all other a's and b's are left unchanged. The $\frac{1}{2}$ position for μ contains only the symbol $\times$ and the indicator is $(-)$ since $a_{i_0} = -\frac{1}{2}$. Thus in the GS combinatorics adding the box on the diagonal does not change the diagrammatic weight itself but the indicator from $(-)$ to $(+)$. Which decreases the tail length by 1.

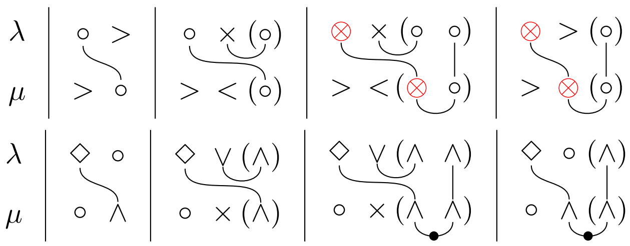
Since in this case $\mathcal{S}(\mu)_{j_0} = \frac{1}{2}$ and decreased to $-\frac{1}{2}$ thus changing the first cup from a dotted cup to an undotted cup and preserving all frozen variables.

▷ ***Case*** $\mathfrak{osp}(2m|2n)$*:* Again we distinguish three possibilities: adding the box exactly above the shifted diagonal, adding the box exactly below the shifted diagonal, and adding the box on the diagonal.

The additional box is added exactly above the diagonal. We add the box in position (j_0, i_0) and thus it holds $i_0 - j_0 = \frac{\delta}{2} + 1$. In addition $b_{j_0} = 1$ and increased by 1, while all other a's and b's are left unchanged. This is done in the same way as adding a box far above the diagonal.

The additional box is added exactly below the diagonal. We add the box in position (j_0, i_0) and thus it holds $i_0 - j_0 = \frac{\delta}{2} - 1$. In addition $a_{i_0} = 0$ and increased by 1, while all other a's and b's are left unchanged.

Note that $b_{j_0-1} > 0$. Furthermore the rest of the diagonal to the lower right is empty, implying $a_i = 0$ for $i > i_0$ and $b_j = 0$ for $j > j_0 - 1$, which implies that in the tail we have exactly once the symbol $>$ and possibly some $\times$. The $\times$ are distributed onto the coloured diagram, leaving $>$ at position zero. This leads to the following configurations (at positions zero and 1, the rest is unchanged) in the first row:



The second line is obtained as follows. It holds $\mathcal{S}(\mu)_{j_0} = 1$ which implies that at position 1 there is either an $\wedge$ or a $\times$. In addition, since $\mu_j > \mu_{j_0}$ for $j < j_0$ it holds that $\mathcal{S}(\mu)_j \leq -1$ for $j < j_0$ (the case $= -1$ giving us the symbol $\times$ at position 1) and since $\mu_j \leq \mu_{j_0}$ for $j > j_0$ it holds $\mathcal{S}(\mu)_j \geq 2$ for $j > j_0$.

Again, neither tail length nor number of dotted cups changes.

The additional box is added exactly on the diagonal. We add the box in position (j_0, i_0) and thus it holds $i_0 - j_0 = \frac{\delta}{2}$. In addition $b_{j_0} = 0$ and increased by 1, while all other a's and b's are left unchanged. It holds $a_{i_0} = 0$ as are all a_i for $i > i_0$ and all b_j for $j > j_0$. Which means that the diagram with a tail has only the symbol $\times$ at position zero (possibly multiple times), with all but one being distributed when forming the diagram without tail. Since we add a box on the diagonal all $b_j > 1$ for $j < j_0$ and all $a_i \geq 1$ for $i < i_0$. This implies that we have the symbol $\circ$ or $>$ at position 1, giving us the following configurations in the left block below (showing positions zero and 1)

For the right block, note that $\mu_{j_0} = j_0 + \frac{\delta}{2} - 1$, hence $\mathcal{S}(\mu)_{j_0} = 0$, which will be decreased by 1. Furthermore $\mathcal{S}(\mu)_j < -1$ for $j < j_0$ since $\mu_j > \mu_{j_0}$ for $j < j_0$ and $\mathcal{S}(\mu)_j \geq 1$ for $j > j_0$. Giving us that we have the symbol $\diamond$ at position zero and the either $\wedge$ or $\circ$ at position 1.

Note that in both cases the tail length decreases by 1 each, but we also loose the decoration on the first dotted cup from the left or loose the dotted cup altogether, thus all fake cups and corresponding frozen vertices remain unchanged.

Since the cup diagrams agree, their leftmost label determines if they are coloured (in the sense of [GS13] or dotted in our sense, hence the statement follows from the definition of (7.55). $\square$

Definition 7.9. Assume $\lambda \in X^+(\mathfrak{g})$ and let D be the cup diagram associated with $\ulcorner\lambda^\infty$ in the sense of Gruson and Serganova. Then a *consistent labelling* of D is a labelling of the vertices with $>$, $<$, $\times$, $\circ$ such that the core symbols match with the core symbols of $\ulcorner\lambda^\infty$ and each cup is labelled by precisely one $\times$ and one $\circ$.

Definition 7.10. For a tailless weight $\nu \in X^+(\mathfrak{g})$ let $A(\lambda, \nu) = 1$ if $\mathrm{GS}(\nu)$ is a consistent labelling of D and $A(\lambda, \nu) = 0$ otherwise. In case of a consistent labelling we let $c(\lambda, \nu)$ be the number of coloured cups plus the number of coloured cups labelled $\circ$ and $\times$ in this order and we set β to be the number of cups with left vertex at position zero labelled with $\times$ via ν. Here $o = \frac{1}{2}$ in the odd case and $o = 0$ in the even case.

8. Counting formulas

We present now several dimension formulas for homomorphism spaces.

8.1. **Dimensions of homomorphism spaces: alternating formula.** We start with the following dimension formula deduced from the results, [GS13, Theorems 1 to 4], of Gruson and Serganova:[5]

Proposition 8.1. *Let* $\lambda, \mu \in X^+(\mathfrak{g})$. *Then*

(1) *in the even case* ($r = 2m$) *we have* $\mathrm{Hom}_{\mathcal{F}'}(P^{\mathfrak{g}}(\lambda), P^{\mathfrak{g}}(\mu)) = \{0\}$ *if* $a_m > 0 > a_{m'}$ *or* $a_m < 0 < a_{m'}$ *in the notation,* (4.21) *and* (4.26)

[5]It might help to say that the first condition is only implicitly contained in [GS13].

(2) and otherwise

$$\dim \operatorname{Hom}_{\mathcal{F}'}(P^{\mathfrak{g}}(\lambda), P^{\mathfrak{g}}(\mu)) = \sum a(\lambda,\nu)a(\mu,\nu) \tag{8.59}$$

where the sum runs through all tailless dominant weights ν *and*

$$a(\eta,\nu) \quad = \quad (-1)^{\beta}(-1)^{c(\eta,\nu)}A(\lambda,\mu) \tag{8.60}$$

for any $\eta \in X^+(\mathfrak{g})$.

Remark 8.2. Note that $a(\eta,\nu) \in \{-1,0,1\}$. In particular, the numbers are not always non-negative, and the above sum might have some (non-trivial) cancellations. In the framework of Gruson and Serganova, the $a(\eta,\nu)$ are coefficients expressing the so-called *Euler classes* $\mathcal{E}^{\mathfrak{g}}(\nu)$ in terms of simple modules, i.e. we have in the Grothendieck group $[\mathcal{E}^{\mathfrak{g}}(\nu)] = \sum_{\nu} a(\lambda,\nu)[L^{\mathfrak{g}}(\lambda)]$.

Proof. We have $\dim \operatorname{Hom}_{\mathcal{F}'}(P^{\mathfrak{g}}(\lambda), P^{\mathfrak{g}}(\mu)) = [P^{\mathfrak{g}}(\mu) : L^{\mathfrak{g}}(\lambda)]$, where $[P^{\mathfrak{g}}(\mu) : L^{\mathfrak{g}}(\lambda)]$ denotes the Jordan-Hölder multiplicities of $L^{\mathfrak{g}}(\lambda)$ in $P^{\mathfrak{g}}(\mu)]$ or alternatively the coefficient of the class $[L^{\mathfrak{g}}(\lambda)]$ in the Grothendieck group when the class $[P^{\mathfrak{g}}(\lambda)]$ is expressed in the classes of the simple modules. On the other hand, the classes $[\mathcal{E}^{\mathfrak{g}}(\nu)]$ of the Euler characteristics (for tailless dominant ν) are also linearly independent in the Grothendieck group and

$$[P^{\mathfrak{g}}(\lambda)] = \sum_{\nu} a(\lambda,\nu)[\mathcal{E}^{\mathfrak{g}}(\nu)] \quad \text{and} \quad [\mathcal{E}^{\mathfrak{g}}(\lambda)] = \sum_{\mu} a(\mu,\nu)[L^{\mathfrak{g}}(\mu)]$$

by [GS13, Lemma 3, Theorem 1], so 8.59 claim holds. Formula 8.60 is just a concise reformulation of [GS13, Theorem 2, Theorem 3, Theorem 4]. □

8.2. **Dimensions of homomorphism spaces: positive formula.** We first show that the cancellations addressed in Remark 8.2 appear precisely if the corresponding space of homomorphisms vanishes completely. This allows us to get the following rather easy and explicit dimension formula.

Proposition 8.3. *Let* $\lambda,\mu \in X^+(\mathfrak{g})$ *with* $a_m \geq 0$ *and* $a'_m \geq 0$ *in the notation from* (4.21) *respectively* (4.26) *and let* $^{\ulcorner}\lambda, {}^{\ulcorner}\mu$ *be the corresponding hook partitions. Then the following are equivalent*

- *(I)* $\operatorname{Hom}_{\mathcal{F}'}(P^{\mathfrak{g}}(\lambda), P^{\mathfrak{g}}(\mu)) \neq 0$,
- *(II)* $\underline{({}^{\ulcorner}\lambda^{\oslash}, \overline{1})}\overline{({}^{\ulcorner}\mu^{\oslash}, \overline{1})}$ *has no non-propagating line and every component has an even number of dots.*

In these cases moreover, the following holds

- *(1)* $a(\lambda,\nu)a(\mu,\nu) \in \{0,1\}$ *for any tailless* $\nu \in X^+(\mathfrak{g})$, *and*
- *(2)* $\dim \operatorname{Hom}_{\mathcal{F}'}(P^{\mathfrak{g}}(\lambda), P^{\mathfrak{g}}(\mu)) = 2^c$, *where* c *is the number of closed components.*

Proof. For (I) $\Rightarrow$ (II) it is enough to show that $\operatorname{Hom}_{\mathcal{F}'}(P^{\mathfrak{g}}(\lambda), P^{\mathfrak{g}}(\mu)) \neq 0$ implies we have no non-propagating line and that each closed component has an even number of dots, since then only the leftmost line is allowed to carry dots and by definition of $({}^{\ulcorner}\lambda^{\oslash}, \overline{1})$ and $({}^{\ulcorner}\mu^{\oslash}, \overline{1})$ the total number of dots is even. By (8.59) we have $\dim \operatorname{Hom}_{\mathcal{F}'}(P^{\mathfrak{g}}(\lambda), P^{\mathfrak{g}}(\mu)) = \sum_{\nu} a(\lambda,\nu)a(\mu,\nu)$ and $a(\lambda,\nu)$ is non-zero if putting the GS-weight $\mathrm{GS}(\nu)$ on top of the cup diagram D associated with $\mathrm{GS}(\lambda)$ results in a picture where the labels $>$ and $<$ in λ and ν agree and each cup has the two symbols $\circ$, $\times$ in any order at its two endpoints. Clearly it is zero if there is a non-propagating line, since the line must have $\circ$'s at the end, but has an odd total number of cups and caps.

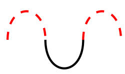
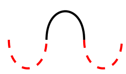
If non-zero, then it is $(-1)^{x+z+y}$, where x is the number of coloured cups in C and $y = y(\lambda, \nu)$ is the number of coloured cups with $\times$ $\circ$ in this order at the endpoints when putting ν on top, and $z = 1$ if λ has the indicator $(+)$ and ν has a $\times$ at position $\frac{1}{2}$. In particular, x doesn't depend on ν. Let K be a closed component of $\underline{\lambda}^{\oslash}\overline{\mu}^{\oslash}$. If $a(\lambda, \nu)a(\mu, \nu) \neq 0$ then we can find a weight ν' such that $\mathrm{GK}(\nu')$ agrees with $\mathrm{GK}(\nu)$ at all vertices not contained in K, but the symbols $\times$ and $\circ$ swapped for the vertices contained in K. Assume now that K has an odd total number of dots. If K does not contain the vertex $\frac{1}{2}$ then $(-1)^{y(\lambda,\nu)+y(\mu,\nu)} = -(-1)^{y(\lambda,\nu')+y(\mu,\nu')}$, hence $a(\lambda, \nu)a(\mu, \nu) = -a(\lambda, \nu')a(\mu, \nu')$ and so the two contributions cancel.

The same holds if K does contain the vertex $\frac{1}{2}$ but with the same indicator $(+)$ or $(-)$ in λ and μ. In case the symbols differ then we have an even number of coloured cups and caps in K, hence

$$(-1)^{y(\lambda,\nu)+y(\mu,\nu)} = (-1)^{y(\lambda,\nu')+y(\mu,\nu')} \quad \text{and}$$
$$(-1)^{z(\lambda,\nu)+z(\mu,\nu)} = -(-1)^{z(\lambda,\nu')+z(\mu,\nu')}$$

implying again $a(\lambda, \nu)a(\mu, \nu) = -a(\lambda, \nu')a(\mu, \nu')$. Hence each closed component requires an odd number of dots and so (I) implies (II).

For the converse note that (II) implies that the diagram is orientable, each line in a unique way and each closed component in exactly two ways. The same holds if we remove the dots. After applying T, any such orientation gives an allowed labelling ν in the sense of [GS13]. We claim that the corresponding value $A := a(\lambda, \nu)a(\mu, \nu)$ is equal to 1. By definition $A := (-1)^{x(\lambda)+x(\mu)+z(\lambda,\nu)+z(\mu,\nu)}(-1)^{y(\lambda,\nu)+y(\mu,\nu)}$. If $X := \underline{\lambda}^{\oslash}\overline{\mu}^{\oslash}$ is a small circle then it has either no dots, hence no coloured cups and there is nothing to check. Or two dots and two coloured cups (and the same indicator) and the statement is clear as well. Otherwise, if X contains a kink without coloured cups and caps then we can remove the kink to obtain a new λ and μ with the same value A attached. So we assume there is no such kink, but then it contains a configuration of the form (dashed lines indicate the colouring)

 or 

Removing the colouring and also the newly created uncoloured kink changes λ and μ, but not the corresponding value A. Hence it must be equal to 1.

Altogether every orientation of the circle diagram $\underline{\lambda}^{\oslash}\overline{\mu}^{\oslash}$ gives a contribution of 1 to $\dim \mathrm{Hom}_{\mathcal{F}'}(P^{\mathfrak{g}}(\lambda), P^{\mathfrak{g}}(\mu)) = \sum_{\nu} a(\lambda, \nu)a(\mu, \nu)$. But on the other hand the number of possible ν's is precisely the number of orientations. Therefore all the remaining statements follow. $\square$

8.3. **The Dimension Formula.** We finally use Proposition 8.3 to deduce:

Theorem 8.4 (Dimension formula)**.** *Consider $G = \mathrm{OSp}(r|2n)$ and $\lambda, \mu \in X^+(G)$. Then the dimension of $\mathrm{Hom}_{\mathcal{F}}(P(\lambda), P(\mu))$ equals the number of orientations $\underline{\lambda}\nu\overline{\mu}$ of $\underline{\lambda}\overline{\mu}$ if the circle diagram $\underline{\lambda}\overline{\mu}$ is defined and contains no non-propagating line, and the dimension is zero otherwise.*

Proof. For $\lambda \in X^+(G)$ let $\lambda^{\mathfrak{g}} \in X^+(\mathfrak{g})$ such that $a_m \geq 0$ in the notation from (4.21) (with λ replaced by $\lambda^{\mathfrak{g}}$) and the hook partitions underlying λ and $\lambda^{\mathfrak{g}}$ are the same. Now consider $H := \mathrm{Hom}_{\mathcal{F}}(P(\lambda), P(\mu))$ as in the theorem. We will freely use Proposition 4.18 to it to swap the roles of λ and μ.

If $\lambda = (\lambda^{\mathfrak{g}}, +)$ and $\mu = (\mu^{\mathfrak{g}}, -)$ (or the reversed signs), then $H = 0$ by Remark 4.7 and Lemma 4.16 and $\underline{\lambda}\overline{\mu}$ is not orientable by Proposition 6.15; thus the claim holds.

If $\lambda = (\lambda^{\mathfrak{g}}, \pm)$ and $\mu = (\mu^{\mathfrak{g}})^G$. Then $\dim H = \dim \mathrm{Hom}_{\mathcal{F}'}(P^{\mathfrak{g}}(\lambda^{\mathfrak{g}}), P^{\mathfrak{g}}(\mu^{\mathfrak{g}}))$ by Proposition 4.17, and the latter is given by Proposition 8.3. Then the claim follows by comparing Proposition 8.3 with Proposition 6.16.

If $\lambda = (\lambda^{\mathfrak{g}})^G$ and $\mu = (\mu^{\mathfrak{g}})^G$, then $\dim H = \dim \mathrm{Hom}_{\mathcal{F}'}(P^{\mathfrak{g}}(\lambda^{\mathfrak{g}}), P^{\mathfrak{g}}(\mu^{\mathfrak{g}}))$ again by Proposition 4.17. Moreover by Proposition 6.10 $\underline{\lambda}$ and $\underline{\mu}$ have a (dotted) ray at zero. In particular, the circle diagram $\underline{\lambda}\overline{\mu}$ has, apart from a straight line L passing through zero and built from two dotted rays, only closed components or rays containing no dots at all (since they are to the right of the propagating line). Hence by Remark 5.19 the diagram is orientable if and only if every component has an even number of dots. The number of orientations is then obviously equal to 2^c, where c is the number of closed components. Thus the claim follows with Proposition 8.3.

If finally $\lambda = (\lambda^{\mathfrak{g}}, \pm)$ and $\mu = (\mu^{\mathfrak{g}}, \pm)$. Then again it holds $\dim H = \dim \mathrm{Hom}_{\mathcal{F}'}(P^{\mathfrak{g}}(\lambda^{\mathfrak{g}}), P^{\mathfrak{g}}(\mu^{\mathfrak{g}}))$ by Proposition 4.17(3). Then the claim follows by comparing Proposition 8.3 with Corollary 6.11. □

Hence we also established the Theorem 7.1.

9. Examples

9.1. **The classical case:** $\mathrm{OSp}(r|0)$**.** We start with the case $\mathrm{OSp}(3|0)$. The irreducible modules in $\mathcal{F}$ are labelled by $(1,1)$-hook partitions, that means partitions which fit into one column, all with an attached sign, see Proposition 4.6 and Lemma 4.21. The tail length is always zero, see Definition 4.22. The table below shows in the first and third column the partitions together with their signs and next to it (on the right) the corresponding weight diagrams from Definition 6.6. Since all non-core symbols are frozen, i.e. the associated cup diagram consists consist of rays.

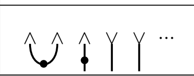
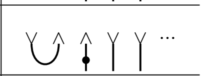
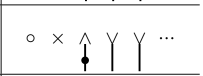
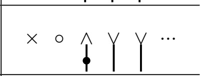
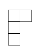
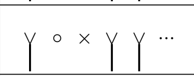
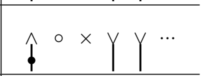
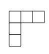
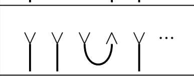
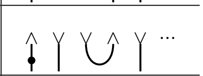
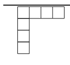
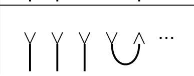
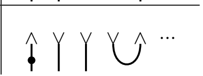
$(\varnothing, +)$	$\circ \wedge \vee \vee \vee \vee \vee \ldots$	$(\varnothing, -)$	$\circ \vee \vee \vee \vee \vee \vee \ldots$	$\square\square\square$
$(\square, +)$	$\wedge \circ \vee \vee \vee \vee \vee \ldots$	$(\square, -)$	$\vee \circ \vee \vee \vee \vee \vee \ldots$	$\square\square$
$\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}, +\right)$	$\wedge \vee \circ \vee \vee \vee \vee \ldots$	$\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}, -\right)$	$\vee \vee \circ \vee \vee \vee \vee \ldots$	$\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$
$\left(\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}, +\right)$	$\wedge \vee \vee \circ \vee \vee \vee \ldots$	$\left(\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}, -\right)$	$\vee \vee \vee \circ \vee \vee \vee \ldots$	$\begin{smallmatrix}\square\square\\\square\\\square\end{smallmatrix}$
$(1^a, +)$	$\wedge \vee \ldots \vee \overset{a+1}{\circ} \vee \ldots$	$(1^a, -)$	$\vee \vee \ldots \vee \overset{a+1}{\circ} \vee \ldots$	$(2, 1^{a-1})$

Additionally, the last column shows the unique partitions $^{\ulcorner}\gamma$ such that $^{\ulcorner}\gamma^{\infty}$, obtained via $\mathcal{S}(^{\ulcorner}\gamma)$, is the weight diagram in the fourth column. Note that the partitions in the first and last column together form a pair of *associated partitions* in the sense of Weyl, i.e. their first rows are of length less or equal to $r = 3$ and together sum up to $r = 3$, and the partitions coincide otherwise. Associated partitions correspond to irreducible OSp(3|0) representations that are isomorphic when restricted to SOSp(3|0). For more details on associated partitions see e.g. [FH91, § 19.5]. Hence our diagrammatics could be seen as extending Weyl's notion of associated partitions.

The same is true more generally for OSp($2m + 1|0$). The two weight diagrams attached to the two different signs for a given partition exactly differ at the first symbol. Changing this first symbol from an $\wedge$ to a $\vee$ exactly produces the associated partition. A pair of associated partitions corresponds to representations that differ by taking the tensor product with the sign representation, see [FH91, Exercise 19.23], which agrees with Proposition 4.6.

In the case OSp($2m|0$) the irreducible modules are labelled by partitions that have at most m columns and they have a sign iff the partition has strictly less than m columns, in which case the two partitions are associated in Weyl's sense. In case of a partition with m columns the partition is associated to itself, which corresponds to the fact that the weight diagram starts with the symbol $\diamond$ at position zero and therefore it does not obtain a sign in our convention.

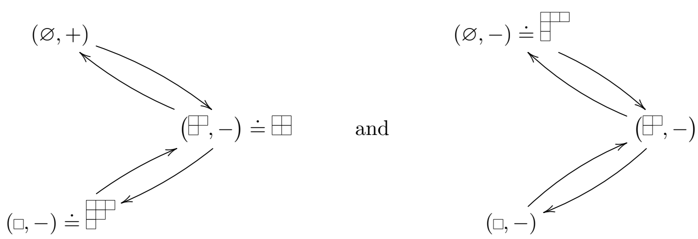
9.2. **The smallest non-semisimple case:** OSp(3|2)**.** Let us come back to the example in the introduction, the category $\mathcal{F}(\mathrm{OSp}(3|2))$. The various diagrammatic weights are listed in the table below. We first list the $(1,1)$-hook partitions, then the sequence $\mathcal{S}(^\ulcorner\lambda)$ and the corresponding weight diagrams.

$^\ulcorner\lambda$	$\mathcal{S}(^\ulcorner\lambda)$	$^\ulcorner\lambda^\infty$	$(^\ulcorner\lambda, +)$	$(^\ulcorner\lambda, -)$
$\varnothing$	$(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \ldots)$	$\wedge\wedge$⊘⊘⊘		
□	$(-\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \ldots)$	$\vee\wedge$⊘⊘⊘		
	$(-\frac{3}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \ldots)$	$\circ\times$⊘⊘⊘		
	$(-\frac{1}{2}, \frac{1}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \ldots)$	$\times\circ$⊘⊘⊘		
	$(-\frac{3}{2}, \frac{1}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \ldots)$	⊘$\vee\wedge$⊘⊘		
	$(-\frac{5}{2}, \frac{1}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \ldots)$	⊘$\circ\times$⊘⊘		
	$(-\frac{5}{2}, \frac{1}{2}, \frac{3}{2}, \frac{7}{2}, \frac{9}{2}, \ldots)$	⊘⊘$\vee\wedge$⊘		
	$(-\frac{7}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{9}{2}, \ldots)$	⊘⊘⊘$\vee\wedge$		

From the weight diagram one can read of (using Proposition 7.5) the blocks and obtain easily the cup diagrams (including those from the introduction) for the indecomposable projective modules. Using now Theorem 7.1, Theorem B and the multiplication rule for circle diagrams from [ES13a], one deduces the shape and relations for the quiver from Theorem A.

The block containing $L(\varnothing,-)$ is equivalent. All other blocks are obviously semisimple (and of atypicality 0).

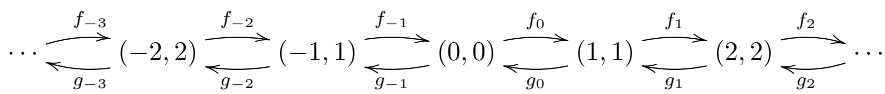
Remark 9.1. Although the category $\mathcal{F} = \mathcal{F}(\mathrm{OSp}(3|2))$ decomposes as $\mathcal{F}^+ \oplus \mathcal{F}^-$ with the summands equivalent to $\mathcal{F}(\mathrm{SOSp}(3|2))$, we still prefer to work with the whole $\mathcal{F}$ due to its connection to Deligne's category, see [Del96], [CH15] and to the Brauer algebras, in particular because (1.3) is not surjective for SOSp(3|2). To see this observe that



show pieces of the quiver corresponding to the two summands $\mathcal{F}^{\pm}$. On the vertices one can see the labellings of the indecomposable projective modules $P(\lambda)$ and the corresponding associated partition in case the sign is $-$. The number of boxes in the partitions corresponding to $+$ respectively to the associated partition equals the tensor power d such that $P(\lambda)$ appears as a summand in $V^{\otimes d}$. Observe that these numbers are always even for the quiver on the left and odd for the quiver on the right, in agreement with Remark 4.9. If one now restricts to G', then $\operatorname{res} P(\lambda,+) \cong \operatorname{res} P(\lambda,+)$ and of course all non-trivial homomorphisms stays non-trivial. Therefore there are non-trivial morphism from $V^{\otimes d}$ to $V^{\otimes d'}$ for some d, d' such that $d \not\equiv d' \bmod 2$. These morphisms cannot be controlled by the Brauer or Deligne category.

9.3. **The smallest even case:** OSp(2|2) **and** OSp **vs** SOSp. We chose now one of the most basic non-classical cases, to showcase the differences between the OSp and the SOSp situation.

In case of SOSp(2|2) the block containing the trivial representation $L^{\mathfrak{g}}(0)$ contains all irreducible representations of the form $L^{\mathfrak{g}}(\pm a\varepsilon_1 + a\delta_1)$. Abbreviating the $L^{\mathfrak{g}}(\pm a\varepsilon_1 + a\delta_1)$ by $(\pm a, a)$ we obtain for it the quiver

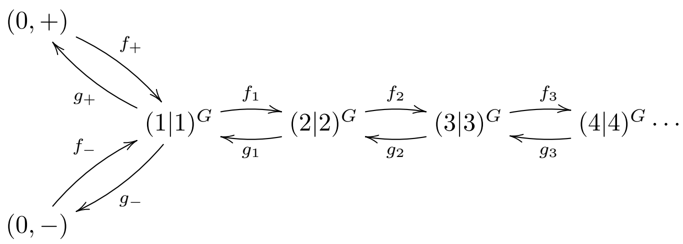


subject to the relations $f_{i+1} \circ f_i = 0 = g_i \circ g_{i-1}$ and $g_i \circ f_i = g_{i-1} \circ f_{i-1}$.

The shape of the quiver and the relations follow from Proposition 8.3. Alternatively one can also use translation functors studied in [GS13].

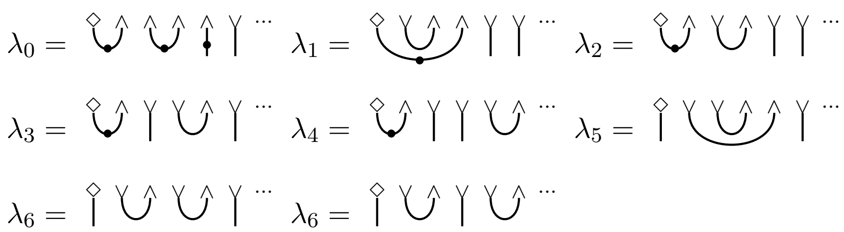
Switching to OSp(2|2) corresponds here to take in some sense the smash product of the original path algebra with the group $\mathbb{Z}/2\mathbb{Z}$ generated by the involution σ and consider the corresponding category of modules, see e.g. [RR85, Example 2.1] for an analogous situation. More precisely we obtain the following: the representation $L^{\mathfrak{g}}(0)$ is doubled up to $L(0,+)$ and $L(0,-)$ while $L^{\mathfrak{g}}(a\varepsilon_1 + a\delta_1)$ and $L^{\mathfrak{g}}(-a\varepsilon_1 + a\delta_1)$ give the same representation $L((a|a)^G)$, see Definition 4.10. This results is that the following quiver describes the principal block of $\mathcal{F}$ (where we used the elements from $X^+(G)$

as labels for the vertices),



subject to the same kind of zero relations as above. Moreover, the induced grading via Corollary C corresponds exactly to the grading given by the path lengths. Observe that the trivial block of atypicality 1 here is equivalent to the blocks of atypicality 1 for OSp(3|2) (in contrast to the SOSp-case). We expect that the passage to the smash product rings is a general procedure to pass between the representation theory of the orthosymplectic and special orthosymplectic group.

9.4. **Illustration of the Dimension Formula for** OSp(4|4)**:** In this section we apply Theorem 7.1 respectively the Dimension Formula to calculate the (graded) dimensions of the morphism spaces between certain projective indecomposable modules in the principal block for OSp(4|4). Below is a list of cup diagrams, whose weight sequences are all diagrammatically linked and in the same block as the trivial representation with sign +.



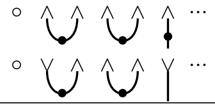
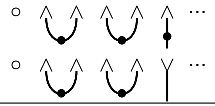
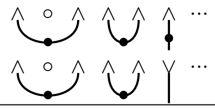
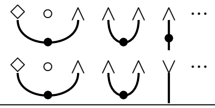
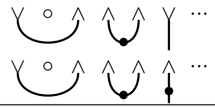
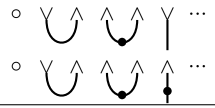
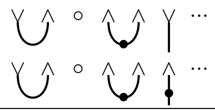
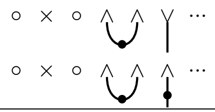
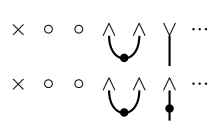
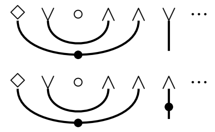
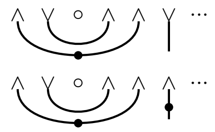
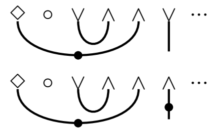
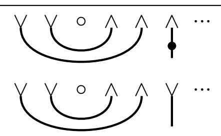
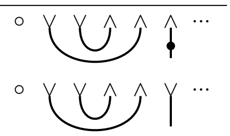
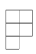
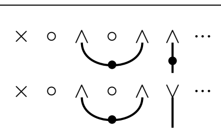
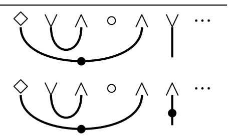
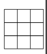
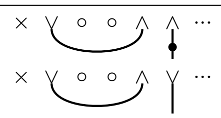
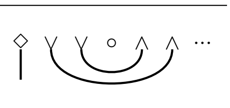
By building all possible circle diagrams by pairing these cup diagrams and checking the possible orientations and the degrees of those, one directly deduces the following table giving the Hilbert-Poincaré polynomials of the morphism spaces, where we abbreviate $E(q) = 1 + 2q + q^2$,

	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
λ_0	$E(q)$			$q+q^3$			q^2	
λ_1		$E(q)$	$q+q^3$	q^2		q^2	$q+q^3$	
λ_2		$q+q^3$	$E(q)$	$q+q^3$		$q+q^3$	q^2	
λ_3	$q+q^3$	q^2	$q+q^3$	$E(q)$	$q+q^3$	q^2	$q+q^3$	q^2
λ_4				$q+q^3$	$E(q)$		q^2	$q+q^3$
λ_5		q^2	$q+q^3$	q^2		$E(q)$	$q+q^3$	q^2
λ_6	q^2	$q+q^3$	q^2	$q+q^3$	q^2	$q+q^3$	$E(q)$	$q+q^3$
λ_7				q^2	$q+q^3$	q^2	$q+q^3$	$E(q)$

Note that the Hilbert-Poincaré polynomials of the endomorphism spaces are constant, namely always equal to $E(q)$. This is a general phenomenon. By Lemma 6.13, each block has a well-defined defect def which coincides with the number of cups in each cup diagram. Then by Theorem B and the definition of the diagram algebra [ES13a, Theorem 6.2 and Corollary 8.8] we always have an isomorphism of algebras $\mathrm{End}_{\mathcal{B}}(P(\lambda)) \cong \mathbb{C}[X]/(X^2)^{\otimes \mathrm{def}}$ with $\deg(X) = 2$.

As predicted by Proposition 4.18 the table is symmetric. Moreover one can convince oneself that the listed cup diagrams are the only ones such that the corresponding projective indecomposable can have a non-zero morphism to $P(\lambda_0)$. One observes that there is, up to scalars, only one degree 1 morphism from $P(\lambda_0)$ to $P(\lambda_3)$ and vice-versa, but the endomorphism ring of $P(\lambda_0)$ has dimension 2 in degree 2. Thus the algebra cannot be generated in degrees ≤ 1. In particular, the principal block in this case is *not Koszul*.

9.5. **Some higher rank examples:** $\mathrm{OSp}(7|4)$ **and** $\mathrm{OSp}(6|4)$**.** Finally we calculate the weight and cup diagrams for the two special cases of $\mathrm{OSp}(7|4)$, with $\frac{\delta}{2} = \frac{3}{2}$, and $\mathrm{OSp}(6|4)$, with $\frac{\delta}{2} = 1$. The first column lists the $(3,2)$-hook partition, follows by two columns showing first the sequence $\mathcal{S}(^\ulcorner\lambda)$ and then the associated weight and cup diagrams (both if it includes a sign).

	OSp(7|4)		OSp(6|4)	
$\ulcorner\lambda$	$\mathcal{S}(\ulcorner\lambda)$	$(\ulcorner\lambda,+)$ $(\ulcorner\lambda,-)$	$\mathcal{S}(\ulcorner\lambda)$	$(\ulcorner\lambda,+)$ $(\ulcorner\lambda,-)$ resp. $\ulcorner\lambda$
$\varnothing$	$(\frac{3}{2},\frac{5}{2},\frac{7}{2},\frac{9}{2},\frac{11}{2},\ldots)$		$(1,2,3,4,5,\ldots)$	
	$(\frac{1}{2},\frac{5}{2},\frac{7}{2},\frac{9}{2},\frac{11}{2},\ldots)$		$(0,2,3,4,5,\ldots)$	
	$(-\frac{1}{2},\frac{5}{2},\frac{7}{2},\frac{9}{2},\frac{11}{2},\ldots)$		$(-1,2,3,4,5,\ldots)$	
	$(-\frac{1}{2},\frac{3}{2},\frac{7}{2},\frac{9}{2},\frac{11}{2},\ldots)$		$(-1,1,3,4,5,\ldots)$	
	$(-\frac{1}{2},\frac{1}{2},\frac{7}{2},\frac{9}{2},\frac{11}{2},\ldots)$		$(-1,0,3,4,5,\ldots)$	
	$(-\frac{3}{2},\frac{1}{2},\frac{7}{2},\frac{9}{2},\frac{11}{2},\ldots)$		$(-2,0,3,4,5,\ldots)$	
	$(-\frac{3}{2},-\frac{1}{2},\frac{7}{2},\frac{9}{2},\frac{11}{2},\ldots)$		$(-2,-1,3,4,5,\ldots)$	
	$(-\frac{1}{2},\frac{1}{2},\frac{5}{2},\frac{9}{2},\frac{11}{2},\ldots)$		$(-1,0,2,4,5,\ldots)$	
	$(-\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{9}{2},\frac{11}{2},\ldots)$		$(-2,-1,0,4,5,\ldots)$	

In this case we have higher defect. We leave it to the reader to check that the blocks are all equivalent to blocks which we have seen already (namely to those with the same atypicality).

References

[Bal11] L. Balduzzi. *Supermanifolds, super Lie groups, and super Harish-Chandra pairs functorial methods and actions.* PhD thesis, Universita degli Studi di Genova, 2011.

[BD16] J. Brundan and N. Davidson. Categorical actions and crystals. to appear, 2016.

[BGS96] A. Beilinson, V. Ginzburg, and W. Soergel. Koszul duality patterns in representation theory. *J. Amer. Math. Soc.*, 9(2):473–527, 1996.

[BKN11] B. D. Boe, J. R. Kujawa, and D. K. Nakano. Complexity and module varieties for classical Lie superalgebras. *Int. Math. Res. Not. IMRN*, 3:696–724, 2011.

[BR87] A. Berele and A. Regev. Hook Young diagrams with applications to combinatorics and representations of Lie superalgebras. *Adv. in Math.*, 2:118–175, 1987.

[Bra37] R. Brauer. On algebras which are connected with the semisimple continuous groups. *Ann. of Math. (2)*, 38(4):857–872, 1937.

[Bru03] J. Brundan. Kazhdan-Lusztig polynomials and character formulae for the Lie superalgebra $\mathfrak{gl}(m|n)$,. *J. Amer. Math. Soc.*, 16(1):185–231, 2003.

[Bru14] J. Brundan. Schur-Weyl duality and categorification. In *Proceedings of the International Congress of Mathematicians (Seoul)*, 2014.

[BS11a] J. Brundan and C. Stroppel. Highest weight categories arising from Khovanov's diagram algebra I: cellularity. *Mosc. Math. J.*, 11(4):685–722, 2011.

[BS11b] J. Brundan and C. Stroppel. Highest weight categories arising from Khovanov's diagram algebra III: Category $\mathcal{O}$. *Repr. Theory*, 15:170–243, 2011.

[BS12a] J. Brundan and C. Stroppel. Gradings on walled Brauer algebras and Khovanov's arc algebra. *Adv. Math.*, 231(2):709–773, 2012.

[BS12b] J. Brundan and C. Stroppel. Highest weight categories arising from Khovanov's diagram algebra IV: the general linear supergroup. *J. Eur. Math. Soc. (JEMS)*, 14(2):373–419, 2012.

[BW13] H. Bao and W. Wang. A new approach to Kazhdan-Lusztig theory of type B via quantum symmetric pairs. arXiv:1310.0103, 2013.

[CDV11] A. Cox and M. De Visscher. Diagrammatic Kazhdan-Lusztig theory for the (walled) Brauer algebra. *J. Algebra*, 340:151–181, 2011.

[CDVM09] A. Cox, M. De Visscher, and P. Martin. The blocks of the Brauer algebra in characteristic zero. *Represent. Theory*, 13:272–308, 2009.

[CH15] J. Comes and T. Heidersdorf. Thick ideals in Deligne's category $\underline{\mathrm{Rep}(O_\delta)}$. arXiv:1507.06728, 2015.

[CL10] S. Cautis and A. Licata. Heisenberg categorification and Hilbert schemes. arXiv:1009.5147, 2010.

[CLW11] S. Cheng, N. Lam, and W. Wang. Super duality and irreducible characters of ortho-symplectic Lie superalgebras. *Invent. Math.*, 183(1):189–224, 2011.

[CW12] S.-J.-Jen Cheng and W. Wang. Dualities for Lie superalgebras. In *Lie theory and representation theory*, volume 2 of *Surv. Mod. Math.*, pages 1–46, 2012.

[Del96] P. Deligne. La série exceptionnelle de groupes de Lie. *C. R. Acad. Sci. Paris Sér. I Math.*, 322(4):321–326, 1996.

[Don98] S. Donkin. *The q-Schur algebra*, volume 253. LMS Lecture Notes, 1998.

[ES12] M. Ehrig and C. Stroppel. 2-row Springer fibres and Khovanov diagram algebras for type D. arXiv:1209.4998 , to appear in CJM, 2012.

[ES13a] M. Ehrig and C. Stroppel. Diagrammatic description for the categories of perverse sheaves on isotropic Grassmannians. arXiv:1511.04111, to appear in Selecta Math., 2013.

[ES13b] M. Ehrig and C. Stroppel. Nazarov-Wenzl algebras, coideal subalgebras and categorified skew Howe duality. *arXiv:1310.1972*, 2013.

[ES14] M. Ehrig and C. Stroppel. Schur-Weyl duality for the Brauer algebra and the ortho-symplectic Lie superalgebra. arXiv:1412.7853, to appear in Math. Zeitschrift, 2014.

[ES15] M. Ehrig and C. Stroppel. Koszul Gradings on Brauer algebras. arXiv:1504.03924, to appear in IMRN, 2015.

[FH91] William Fulton and Joe Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[Fio03] R. Fioresi. On algebraic supergroups and quantum deformations. *J. Algebra Appl.*, 2003.

[Ger00] J. Germoni. Indecomposable representations of osp(3, 2), $D(2,1;\alpha)$ and $G(3)$. *Bol. Acad. Nac. Cienc. (Córdoba)*, 65:147–163, 2000. Colloquium on Homology and Representation Theory (Spanish) (Vaquerías, 1998).

[GS10] C. Gruson and V. Serganova. Cohomology of generalized supergrassmannians and character formulae for basic classical Lie superalgebras. *Proc. Lond. Math. Soc. (3)*, 101(3):852–892, 2010.

[GS13] C. Gruson and V. Serganova. Bernstein-Gelfand-Gelfand reciprocity and indecomposable modules for classical algebraic supergroups. *Moscow Math journal*, 13(2):1–33, 2013.

[GW09] R. Goodman and N.R. Wallach. *Symmetry, representations, and invariants*, volume 255 of *Graduate Texts in Mathematics*. Springer, Dordrecht, 2009.

[Jan03] J.-C. Jantzen. *Representations of algebraic groups*. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2003.

[Kos77] B. Kostant. Graded manifolds, graded Lie theory, and prequantization. In *Differential geometrical methods in mathematical physics (Proc. Sympos., Univ. Bonn, Bonn, 1975)*, pages 177–306. Springer, 1977.

[KX98] S. König and C. Xi. On the structure of cellular algebras. In *Algebras and modules, II (Geiranger, 1996)*, volume 24 of *CMS Conf. Proc.*, pages 365–386. AMS, 1998.

[LS12] T. Lejczyk and C. Stroppel. A graphical description of (D_n, A_{n-1}) Kazhdan-Lusztig polynomials. *GMJ*, 55:313–340, 2012.

[LZ14a] G. Lehrer and R. Zhang. he first fundamental theorem of invariant theory for the orthosymplectic supergroup., 2014.

[LZ14b] G. Lehrer and R. Zhang. The second fundamental theorem of invariant theory for the orthosymplectic supergroup. arXiv:1407.1058, 2014.

[LZ15] G. Lehrer and R. Zhang. Invariants of the orthosymplectic Lie superalgebra and super Pfaffians. arXiv:1507.01329, 2015.

[Mar09] P. Martin. The decomposition matrices of the Brauer algebra over the complex field. arXiv:0908.1500, 2009.

[Mas13] A. Masuoka. Hopf algebraic techniques applied to super algebraic groups. arXiv:math/1311.1261, 2013.

[MOS09] V. Mazorchuk, S. Ovsienko, and C. Stroppel. Quadratic duals, Koszul dual functors, and applications. *Trans. Amer. Math. Soc.*, 361(3):1129–1172, 2009.

[Mus12] I. M. Musson. *Lie superalgebras and enveloping algebras*, volume 131 of *Graduate Studies in Mathematics*. American Mathematical Society, 2012.

[MW14] V. Mikhaylov and E. Witten. Branes And Supergroups. hep-th/arXiv:1410.1175, 2014.

[RR85] I. Reiten and C. Riedtmann. Skew group algebras in the representation theory of Artin algebras. *J. Algebra*, 92(3):224–282, 1985.

[Ser84] A. N. Sergeev. Tensor algebra of the identity representation as a module over the Lie superalgebras Gl(n, m) and $Q(n)$,. *Mat. Sb. (N.S.),*, 123(3):1984, 1984.

[Ser11] V. Serganova. Quasireductive supergroups. In *New developments in Lie theory and its applications*, volume 544 of *Contemp. Math.*, pages 141–159. Amer. Math. Soc., Providence, RI, 2011.

[Ser14] V. Serganova. Finite dimensional representations of algebraic supergroups. In *Proceedings of the International Congress of Mathematicians (Seoul)*, 2014.

[Vis11] E. G. Vishnyakova. On complex Lie supergroups and split homogeneous supermanifolds,. *Transform. Groups*, 16(1):265–285, 2011.

[Wil15] A. Wilbert. Topology of two-row Springer fibers for the even orthogonal and symplectic group. arXiv:1511.01961, 2015.

M.E.: Mathematisches Institut, Universität Bonn, Endenicher Allee 60, Room 1.003, 53115 Bonn, Germany

E-mail address: mehrig@math.uni-bonn.de

C.S.: Mathematisches Institut, Universität Bonn, Endenicher Allee 60, Room 4.007, 53115 Bonn, Germany

E-mail address: stroppel@math.uni-bonn.de