Proof of the Umbral Moonshine Conjecture

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2015 October 13

Abstract

The Umbral Moonshine Conjectures assert that there are infinite-dimensional graded modules, for prescribed finite groups, whose McKay-Thompson series are certain distinguished mock modular forms. Gannon has proved this for the special case involving the largest sporadic simple Mathieu group. Here we establish the existence of the umbral moonshine modules in the remaining 22 cases.

[*] *MSC2010:* 11F22, 11F37.

[*]The authors thank the NSF for its support. The first author also thanks the Simons Foundation (#316779), and the third author thanks the A. G. Candler Fund.

1 Introduction and Statement of Results

Monstrous moonshine relates distinguished modular functions to the representation theory of the Monster, $\mathbb{M}$, the largest sporadic simple group. This theory was inspired by the famous observations of McKay and Thompson in the late 1970s [18, 51] that

$$196884 = 1 + 196883,$$
$$21493760 = 1 + 196883 + 21296876.$$

The left hand sides here are familiar as coefficients of Klein's modular function (note $q := e^{2\pi i\tau}$),

$$J(\tau) = \sum_{n=-1}^{\infty} c(n)q^n := j(\tau) - 744 = q^{-1} + 196884q + 21493760q^2 + \dots.$$

The sums on the right hand sides involve the first three numbers arising as dimensions of irreducible representations of $\mathbb{M}$,

$$1,\ 196883,\ 21296876,\ 842609326,\ \dots,\ 258823477531055064045234375.$$

Thompson conjectured that there is a graded infinite-dimensional $\mathbb{M}$-module

$$V^\natural = \bigoplus_{n=-1}^{\infty} V_n^\natural,$$

satisfying $\dim(V_n^\natural) = c(n)$. For $g \in \mathbb{M}$, he also suggested [50] to consider the graded-trace functions

$$T_g(\tau) := \sum_{n=-1}^{\infty} \mathrm{tr}(g|V_n^\natural)q^n,$$

now known as the *McKay-Thompson series*, that arise from the conjectured $\mathbb{M}$-module $V^\natural$. Using the character table for $\mathbb{M}$, it was observed [18,50] that the first few coefficients of each $T_g(\tau)$ coincide with those of a generator for the function field of a discrete group $\Gamma_g < SL_2(\mathbb{R})$, leading Conway and Norton [18] to their famous *Monstrous Moonshine Conjecture*: This is the claim that for each $g \in \mathbb{M}$ there is a specific *genus zero* group Γ_g such that $T_g(\tau)$ is the unique normalized *hauptmodul* for Γ_g, i.e., the unique Γ_g-invariant holomorphic function on $\mathbb{H}$ which satisfies $T_g(\tau) = q^{-1} + O(q)$ as $\Im(\tau) \to \infty$.

In a series of ground-breaking works, Borcherds introduced vertex algebras [2], and generalized Kac–Moody Lie algebras [3, 4], and used these notions to prove [5] the Monstrous Moonshine Conjecture of Conway and Norton. He confirmed the conjecture for the module $V^\natural$ constructed by Frenkel, Lepowsky, and Meurman [30–32] in the early 1980s. These results provide much more than the predictions of monstrous moonshine. The $\mathbb{M}$-module $V^\natural$ is a vertex operator algebra, one whose automorphism group is precisely $\mathbb{M}$. The construction of Frenkel, Lepowsky and Meurman can be regarded as one of the first examples of an *orbifold conformal field theory.* (Cf. [23].) Here the orbifold in question is the quotient $\left(\mathbb{R}^{24}/\Lambda_{24}\right)/(\mathbb{Z}/2\mathbb{Z})$, of the 24-dimensional torus $\Lambda_{24} \otimes_{\mathbb{Z}} \mathbb{R}/\Lambda_{24} \simeq \mathbb{R}^{24}/\Lambda_{24}$ by the Kummer involution $x \mapsto -x$, where Λ_{24} denotes the Leech lattice.

We refer to [24, 32, 35, 36] for more on monstrous moonshine.

In 2010, Eguchi, Ooguri, and Tachikawa reignited moonshine with their observation [28] that dimensions of some representations of M_{24}, the largest sporadic simple Mathieu group (cf. e.g. [20, 21]), are multiplicities of superconformal algebra characters in the K3 elliptic genus. This observation suggested a manifestation of moonshine for M_{24}: Namely, there should be an infinite-dimensional graded M_{24}-module whose McKay-Thompson series are holomorphic parts of *harmonic Maass forms*, the so-called *mock modular forms*. (See [45, 54, 55] for introductory accounts of the theory of mock modular forms.)

Following the work of Cheng [10], Eguchi and Hikami [27], and Gaberdiel, Hohenegger, and Volpato [33, 34], Gannon established the existence of this infinite-dimensional graded M_{24}-module in [37].

It is natural to seek a general mathematical and physical setting for these results. Here we consider the mathematical setting, which develops from the close relationship between the monster group $\mathbb{M}$ and the Leech lattice Λ_{24}. Recall (cf. e.g. [20]) that the Leech lattice is even, unimodular, and positive-definite of rank 24. It turns out that M_{24} is closely related to another such lattice. Such observations led Cheng, Duncan and Harvey to further instances of moonshine within the setting of even unimodular positive-definite lattices of rank 24. In this way they arrived at the *Umbral Moonshine Conjectures* (cf. §5 of [15], §6 of [16], and §2 of [17]), predicting the existence of 22 further, graded infinite-dimensional modules, relating certain finite groups to distinguished mock modular forms.

To explain this prediction in more detail we recall Niemeier's result [43] that there are 24 (up to isomorphism) even unimodular positive-definite lattices of rank 24. The Leech lattice is the unique one with no root vectors (i.e. lattice vectors with norm-square 2), while the other 23 have root systems with full rank, 24. These *Niemeier root systems* are unions of simple simply-laced root systems with the same Coxeter numbers, and are given explicitly as

$$\begin{gathered} A_1^{24},\ A_2^{12},\ A_3^{8},\ A_4^{6},\ A_6^{4},\ A_{12}^{2}, \\ A_5^4D_4,\ A_7^2D_5^2,\ A_8^3,\ A_9^2D_6,\ A_{11}D_7E_6,\ A_{15}D_9,\ A_{17}E_7,\ A_{24}, \\ D_4^6,\ D_6^4,\ D_8^3,\ D_{10}E_7^2,\ D_{12}^2,\ D_{16}E_8,\ D_{24}, E_6^4,\ E_8^3, \end{gathered} \tag{1.1}$$

in terms of the standard ADE notation. (Cf. e.g. [20] or [39] for more on root systems.)

For each Niemeier root system X let N^X denote the corresponding unimodular lattice, let W^X denote the (normal) subgroup of $\mathrm{Aut}(N^X)$ generated by reflections in roots, and define the *umbral group* of X by setting

$$G^X := \mathrm{Aut}(N^X)/W^X. \tag{1.2}$$

(See §A.1 for explicit descriptions of the groups G^X.)

Let m^X denote the Coxeter number of any simple component of X. An association of distinguished $2m^X$-vector-valued mock modular forms $H_g^X(\tau) = (H_{g,r}^X(\tau))$ to elements $g \in G^X$ is described and analyzed in [15–17].

For $X = A_1^{24}$ we have $G^X \simeq M_{24}$ and $m^X = 2$, and the functions $H_{g,1}^X(\tau)$ are precisely the mock modular forms assigned to elements $g \in M_{24}$ in the works [10, 27, 33, 34] mentioned above. Generalizing the M_{24} moonshine initiated by Eguchi, Ooguri and Tachikawa, we have the following conjecture of Cheng, Duncan and Harvey (cf. §2 of [17] or §9.3 of [24]).

Conjecture (Umbral Moonshine Modules)**.** Let X be a Niemeier root system X and set $m := m^X$. There is a naturally defined bi-graded infinite-dimensional G^X-module

$$\check{K}^X = \bigoplus_{r \in I^X} \bigoplus_{\substack{D \in \mathbb{Z},\, D \leq 0, \\ D = r^2 \pmod{4m}}} \check{K}^X_{r,-D/4m} \tag{1.3}$$

such that the vector-valued mock modular form $H^X_g = (H^X_{g,r})$ is a McKay-Thompson series for $\check{K}^X$ related[1] to the graded trace of g on $\check{K}^X$ by

$$H^X_{g,r}(\tau) = -2q^{-1/4m}\delta_{r,1} + \sum_{\substack{D \in \mathbb{Z},\, D \leq 0, \\ D = r^2 \pmod{4m}}} \operatorname{tr}(g|\check{K}^X_{r,-D/4m})q^{-D/4m} \tag{1.4}$$

for $r \in I^X$.

In (1.3) and (1.4) the set $I^X \subset \mathbb{Z}/2m\mathbb{Z}$ is defined in the following way. If X has an A-type component then $I^X := \{1, 2, 3, \ldots, m-1\}$. If X has no A-type component but does have a D-type component then $m = 2 \mod 4$, and $I^X := \{1, 3, 5, \ldots, m/2\}$. The remaining cases are $X = E_6^4$ and $X = E_8^3$. In the former of these, $I^X := \{1, 4, 5\}$, and in the latter case $I^X := \{1, 7\}$.

Remark. The functions $H^X_g(\tau)$ are defined explicitly in §B.3. An alternative description in terms of Rademacher sums is given in §B.4.

Here we prove the following theorem.

Theorem 1.1. *The umbral moonshine modules exist.*

Two remarks.

1) Theorem 1.1 for $X = A_1^{24}$ is the main result of Gannon's work [37].

2) The vector-valued mock modular forms $H^X = (H^X_{g,r})$ have "minimal" *principal parts*. This minimality is analogous to the fact that the original McKay-Thompson series $T_g(\tau)$ for the Monster are hauptmoduln, and plays an important role in our proof.

Example. Many of Ramanujan's mock theta functions [46] are components of the vector-valued umbral McKay-Thompson series $H^X_g = (H^X_{g,r})$. For example, consider the root system $X = A_2^{12}$, whose umbral group is a double cover $2.M_{12}$ of the sporadic simple Mathieu group M_{12}. In terms

[1] In the statement of Conjecture 6.1 of [16] the function $H^X_{g,r}$ in (1.4) is replaced with $3H^X_{g,r}$ in the case that $X = A_8^3$. This is now known to be an error, arising from a misspecification of some of the functions H^X_g for $X = A_8^3$. Our treatment of the case $X = A_8^3$ in this work reflects the corrected specification of the corresponding H^X_g which is described and discussed in detail in [17].

of Ramanujan's 3rd order mock theta functions

$$f(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2},$$
$$\phi(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q^2)(1+q^4)\cdots(1+q^{2n})},$$
$$\chi(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q+q^2)(1-q^2+q^4)\cdots(1-q^n+q^{2n})}$$
$$\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(1-q)^2(1-q^3)^2\cdots(1-q^{2n+1})^2},$$
$$\rho(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(1+q+q^2)(1+q^3+q^6)\cdots(1+q^{2n+1}+q^{4n+2})},$$

we have that

$$\begin{aligned} H^X_{2B,1}(\tau) = H^X_{2C,1}(\tau) = H^X_{4C,1}(\tau) &= -2q^{-\frac{1}{12}} \cdot f(q^2), \\ H^X_{6C,1}(\tau) = H^X_{6D,1}(\tau) &= -2q^{-\frac{1}{12}} \cdot \chi(q^2), \\ H^X_{8C,1}(\tau) = H^X_{8D,1}(\tau) &= -2q^{-\frac{1}{12}} \cdot \phi(-q^2), \\ H^X_{2B,2}(\tau) = -H^X_{2C,2}(\tau) &= -4q^{\frac{2}{3}} \cdot \omega(-q), \\ H^X_{6C,2}(\tau) = -H^X_{6D,2}(\tau) &= 2q^{\frac{2}{3}} \cdot \rho(-q). \end{aligned}$$

See §5.4 of [16] for more coincidences between umbral McKay-Thompson series and mock theta functions identified by Ramanujan almost a hundred years ago.

Our proof of Theorem 1.1 involves the explicit determination of each G^X-module $\check{K}^X$ by computing the multiplicity of each irreducible component for each homogeneous subspace. It guarantees the existence and uniqueness of a $\check{K}^X$ which is compatible with the representation theory of G^X and the Fourier expansions of the vector-valued mock modular forms $H^X_g(\tau) = (H^X_{g,r}(\tau))$.

At first glance our methods do not appear to shed light on any deeper algebraic properties of the $\check{K}^X$, such as might correspond to the vertex operator algebra structure on $V^\natural$, or the monster Lie algebra introduced by Borcherds in [5]. However, we do determine, and utilize, specific recursion relations for the coefficients of the umbral McKay-Thompson series which are analogous to the replicability properties of monstrous moonshine formulated by Conway and Norton in §8 of [18] (cf. also [1]). More specifically, we use recent work [41] of Imamoğlu, Raum and Richter, as generalized [42] by Mertens, to obtain such recursions. These results are based on the process of *holomorphic projection.*

Theorem 1.2. *For each $g \in G^X$ and $0 < r < m$, the mock modular form $H^X_{g,r}(\tau)$ is replicable in the mock modular sense.*

A key step in Borcherds' proof [5] of the monstrous moonshine conjecture is the reformulation of replicability in Lie theoretic terms. We may speculate that the *mock modular replicability* utilized in this work will ultimately admit an analogous algebraic interpretation. Such a result remains an important goal for future work.

In the statement of Theorem 1.2, replicable means that there are explicit recursion relations for the coefficients of the vector-valued mock modular form in question. For example, we recall the recurrence formula for Ramanujan's third order mock theta function $f(q) = \sum_{n=0}^{\infty} c_f(n)q^n$ that was obtained recently by Imamoğlu, Raum and Richter [41]. If $n \in \mathbb{Q}$, then let

$$\sigma_1(n) := \begin{cases} \sum_{d|n} d & \text{if } n \in \mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathrm{sgn}^+(n) := \begin{cases} \mathrm{sgn}(n) & \text{if } n \neq 0, \\ 1 & \text{if } n = 0, \end{cases}$$

and then define

$$d(N, \widetilde{N}, t, \widetilde{t}) := \mathrm{sgn}^+(N) \cdot \mathrm{sgn}^+(\widetilde{N}) \cdot \left(|N + t| - |\widetilde{N} + \widetilde{t}| \right).$$

Then for positive integers n, we have that

$$\sum_{\substack{m \in \mathbb{Z} \\ 3m^2+m \leq 2n}} \left(m + \frac{1}{6} \right) c_f \left(n - \frac{3}{2}m^2 - \frac{1}{2}m \right)$$
$$= \frac{4}{3}\sigma(n) - \frac{16}{3}\sigma\left(\frac{n}{2}\right) - 2 \sum_{\substack{a,b \in \mathbb{Z} \\ 2n=ab}} d\left(N, \widetilde{N}, \frac{1}{6}, \frac{1}{6}\right),$$

where $N := \frac{1}{6}(-3a + b - 1)$ and $\widetilde{N} := \frac{1}{6}(3a + b - 1)$, and the sum is over integers a, b for which $N, \widetilde{N} \in \mathbb{Z}$. This is easily seen to be a recurrence relation for the coefficients $c_f(n)$. The replicability formulas for all of the $H^X_{g,r}(\tau)$ are similar (although some of these relations are slightly more complicated and involve the coefficients of weight 2 cusp forms).

It is important to emphasize that, despite the progress which is represented by our main results, Theorems 1.1 and 1.2, the following important question remains open in general.

Question. *Is there a "natural" construction of $\check{K}^X$? Is $\check{K}^X$ equipped with a deeper algebra structure as in the case of the monster module $V^\natural$ of Frenkel, Lepowsky and Meurman?*

We remark that this question has been answered positively, recently, in one special case: A vertex operator algebra structure underlying the umbral moonshine module $\check{K}^X$ for $X = E_8^3$ has been described explicitly in [25]. See also [14, 26], where the problem of constructing algebraic structures that illuminate the umbral moonshine observations is addressed from a different point of view.

The proof of Theorem 1.1 is not difficult. It is essentially a collection of tedious calculations. We use the theory of mock modular forms and the character table for each G^X (cf. §A.2) to solve for the multiplicities of the irreducible G^X-module constituents of each homogeneous subspace in the alleged G^X-module $\check{K}^X$. To prove Theorem 1.1 it suffices to prove that these multiplicities are non-negative integers. To prove Theorem 1.2 we apply recent work [42] of Mertens on the holomorphic projection of weight $\frac{1}{2}$ mock modular forms, which generalizes earlier work [41] of Imamoğlu, Raum and Richter.

In §2 we recall the facts about mock modular forms that we require, and we prove Theorem 1.2. We prove Theorem 1.1 in §3. The appendices furnish all the data that our method requires. In particular, the umbral groups G^X are described in detail in §A, and explicit definitions for the mock modular forms $H_g^X(\tau)$ are given in §B.

2 Harmonic Maass forms and Mock modular forms

Here we recall some very basic facts about harmonic Maass forms as developed by Bruinier and Funke [9] (see also [45]).

We begin by briefly recalling the definition of a *harmonic Maass form* of weight $k \in \frac{1}{2}\mathbb{Z}$ and multiplier ν (a generalization of the notion of a Nebentypus). If $\tau = x + iy$ with x and y real, we define the weight k hyperbolic Laplacian by

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \right). \tag{2.1}$$

Suppose Γ is a subgroup of finite index in $SL_2(\mathbb{Z})$ and $k \in \frac{1}{2}\mathbb{Z}$. Then a function $F(\tau)$ which is real-analytic on the upper half of the complex plane is a *harmonic Maass form* of weight k on Γ with multiplier ν if:

(a) The function $F(\tau)$ satisfies the weight k modular transformation,

$$F(\tau)|_k\gamma = \nu(\gamma)F(\tau)$$

for every matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, where $F(\tau)|_k\gamma := F(\gamma\tau)(c\tau+d)^{-k}$, and if $k \in \mathbb{Z} + \frac{1}{2}$, the square root is taken to be the principal branch.

(b) We have that $\Delta_k F(\tau) = 0$,

(c) There is a polynomial $P_F(q^{-1})$ and a constant $c > 0$ such that $F(\tau) - P_F(e^{-2\pi i\tau}) = O(e^{-cy})$ as $\tau \to i\infty$. Analogous conditions are required at each cusp of Γ.

We denote the $\mathbb{C}$-vector space of harmonic Maass forms of a given weight k, group Γ and multiplier ν by $H_k(\Gamma, \nu)$. If no multiplier is specified, we will take

$$\nu_0(\gamma) := \left(\left(\frac{c}{d}\right) \sqrt{\left(\frac{-1}{d}\right)}^{-1} \right)^{2k},$$

where $\left(\frac{*}{d}\right)$ is the Kronecker symbol.

2.1 Main properties

The Fourier expansion of harmonic Maass forms F (see Proposition 3.2 of [9]) splits into two components. As before, we let $q := e^{2\pi i\tau}$.

Lemma 2.1. *If $F(\tau)$ is a harmonic Maass form of weight $2-k$ for Γ where $\frac{3}{2} \leq k \in \frac{1}{2}\mathbb{Z}$, then*

$$F(\tau) = F^+(\tau) + F^-(\tau),$$

where F^+ is the holomorphic part of F, given by

$$F^+(\tau) = \sum_{n \gg -\infty} c_F^+(n) q^n$$

where the sum admits only finitely many non-zero terms with $n < 0$, and F^- is the nonholomorphic part, given by

$$F^-(\tau) = \sum_{n<0} c_F^-(n)\Gamma(k-1, 4\pi y|n|)q^n.$$

Here $\Gamma(s,z)$ is the upper incomplete gamma function.

The holomorphic part of a harmonic Maass form is called a *mock modular form.* We denote the space of harmonic Maass forms of weight $2-k$ for Γ and multiplier ν by $H_k(\Gamma,\nu)$. Similarly, we denote the corresponding subspace of holomorphic modular forms by $M_k(\Gamma,\nu)$, and the space of cusp forms by $S_k(\Gamma,\overline{\nu})$. The differential operator $\xi_w := 2iy^w\overline{\frac{\partial}{\partial\overline{\tau}}}$ (see [9]) defines a surjective map

$$\xi_{2-k} : H_{2-k}(\Gamma,\nu) \to S_k(\Gamma,\overline{\nu})$$

onto the space of weight k cusp forms for the same group but conjugate multiplier. The *shadow* of a Maass form $f(\tau) \in H_{2-k}(\Gamma,\nu)$ is the cusp form $g(\tau) \in S_k(\Gamma,\overline{\nu})$ (defined, for now, only up to scale) such that $\xi_{2-k}f(\tau) = \frac{g}{||g||}$, where $||\bullet||$ denotes the usual Petersson norm.

2.2 Holomorphic projection of weight $\frac{1}{2}$ mock modular forms

As noted above, the modular transformations of a weight $\frac{1}{2}$ harmonic Maass form may be simplified by multiplying by its shadow to obtain a weight 2 nonholomorphic modular form. One can use the theory of holomorphic projections to obtain explicit identities relating these nonholomorphic modular forms to classical quasimodular forms. In this way, we may essentially reduce many questions about the coefficients of weight $\frac{1}{2}$ mock modular forms to questions about weight 2 holomorphic modular forms. The following theorem is a special case of a more general theorem due to Mertens (cf. Theorem 6.3 of [42]). See also [41].

Theorem 2.2 (Mertens)**.** *Suppose $g(\tau)$ and $h(\tau)$ are both theta functions of weight $\frac{3}{2}$ contained in $S_{\frac{3}{2}}(\Gamma,\nu_g)$ and $S_{\frac{3}{2}}(\Gamma,\nu_h)$ respectively, with Fourier expansions*

$$g(\tau) := \sum_{i=1}^{s}\sum_{n\in\mathbb{Z}} n\chi_i(n)q^{n^2},$$

$$h(\tau) := \sum_{j=1}^{t}\sum_{n\in\mathbb{Z}} n\psi_j(n)q^{n^2},$$

where each χ_i and ψ_i is a Dirichlet character. Moreover, suppose $h(\tau)$ is the shadow of a weight $\frac{1}{2}$ harmonic Maass form $f(\tau) \in H_{\frac{1}{2}}(\Gamma, \overline{\nu_h})$. Define the function

$$D^{f,g}(\tau) := 2 \sum_{r=1}^{\infty} \sum_{\chi_i, \psi_j} \sum_{\substack{m,n \in \mathbb{Z}^+ \\ m^2 - n^2 = r}} \chi_i(m) \overline{\psi_j(n)} (m-n) q^r.$$

If $f(\tau)g(\tau)$ has no singularity at any cusp, then $f^+(\tau)g(\tau) + D^{f,g}(\tau)$ is a weight 2 quasimodular form. In other words, it lies in the space $\mathbb{C}E_2(\tau) \oplus M_2(\Gamma, \nu_g \overline{\nu_h})$, where $E_2(\tau)$ is the quasimodular Eisenstein series $E_2(\tau) := 1 - 24 \sum_{n \geq 1} \frac{nq^n}{1-q^n}$.

Two Remarks.
1) These identities give recurrence relations for the weight $\frac{1}{2}$ mock modular form f^+ in terms of the weight 2 quasimodular form which equals $f^+(\tau)g(\tau) + D^{f,g}(\tau)$. The example after Theorem 1.2 for Ramanujan's third order mock theta function f is an explicit example of such a relation.

2) Theorem 2.2 extends to vector-valued mock modular forms in a natural way.

Proof of Theorem 1.2. Fix a Niemeier lattice and its root system X, and let $M = m^X$ denote its Coxeter number. Each $H^X_{g,r}(\tau)$ is the holomorphic part of a weight $\frac{1}{2}$ harmonic Maass form $\widehat{H}^X_{g,r}(\tau)$. To simplify the exposition in the following section, we will emphasize the case that the root system X is of pure A-type. If the root system X is of pure A-type, the shadow function $S^X_{g,r}(\tau)$ is given by $\hat{\chi}^{X_A}_{g,r} S_{M,r}(\tau)$ (see §B.2), where

$$S_{M,r}(\tau) = \sum_{\substack{n \in \mathbb{Z} \\ m \equiv r \pmod{2M}}} n\, q^{\frac{n^2}{4M}},$$

and $\hat{\chi}^{X_A}_{g,r} = \chi^{X_A}_g$ or $\bar{\chi}^{X_A}_g$ depending on the parity of r is the twisted Euler character given in the appropriate table in §A.3, a character of G^X. (If X is not of pure A-type, then the shadow function $S^X_{g,r}(\tau)$ is a linear combination of similar functions as described in §B.2.)

Given X and g, the symbol $n_g|h_g$ given in the corresponding table in §A.3 defines the modularity for the vector-valued function $(\widehat{H}^X_{g,r}(\tau))$. In particular, if the shadow $(S^X_{g,r}(\tau))$ is nonzero, and if for $\gamma \in \Gamma_0(n_g)$ we have that

$$(S^X_{g,r}(\tau))|_{3/2}\gamma = \sigma_{g,\gamma}(S^X_{g,r}(\tau)),$$

then

$$(\widehat{H}^X_{g,r}(\tau))|_{1/2}\gamma = \overline{\sigma_{g,\gamma}}(\widehat{H}^X_{g,r}(\tau)).$$

Here, for $\gamma \in \Gamma_0(n_g)$, we have $\sigma_{g,\gamma} = \nu_g(\gamma)\sigma_{e,\gamma}$ where $\nu_g(\gamma)$ is a multiplier which is trivial on $\Gamma_0(n_g h_g)$. This identity holds even in the case that the shadow $S^X_{g,r}$ vanishes.

The vector-valued function $(H^X_{g,r}(\tau))$ has poles only at the infinite cusp of $\Gamma_0(n_g)$, and only at the component $H^X_{g,r}(\tau)$ where $r = 1$ if X has pure A-type, or at components where $r^2 \equiv 1 \pmod{2M}$ otherwise. These poles may only have order $\frac{1}{4M}$. This implies that the function $(\widehat{H}^X_{g,r}(\tau) S^X_{g,r}(\tau))$ has no pole at any cusp, and is therefore a candidate for an application of Theorem 2.2.

The modular transformation of $S_{M,r}(\tau)$ implies that

$$(\sigma_{e,S})^2 = (\sigma_{e,T})^{4M} = \mathbf{I}$$

where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and $\mathbf{I}$ is the identity matrix. Therefore $S^X_{M,r}(\tau)$, viewed as a scalar-valued modular function, is modular on $\Gamma(4M)$, and so $(\widehat{H}^X_{g,r}(\tau)S^X_{g,r}(\tau))$ is a weight 2 nonholomorphic scalar-valued modular form for the group $\Gamma(4M)\cap\Gamma_0(n_g)$ with trivial multiplier.

Applying Theorem 2.2, we obtain a function $F^X_{g,r}(\tau)$—call it the holomorphic projection of $\widehat{H^X_{g,r}}(\tau)S^X_{e,r}(\tau)$—which is a weight 2 quasimodular form on $\Gamma(4M)\cap\Gamma_0(n_g)$. In the case that $S^X_{g,r}(\tau)$ is zero, we substitute $S^X_{e,r}(\tau)$ in its place to obtain a function $\widetilde{F}^X_{g,r}(\tau) = H^X_{g,r}(\tau)S^X_{e,r}(\tau)$ which is a weight 2 holomorphic scalar-valued modular form for the group $\Gamma(4M)\cap\Gamma_0(n_g)$ with multiplier ν_g (alternatively, modular for the group $\Gamma(4M)\cap\Gamma_0(n_g h_g)$ with trivial multiplier).

The function $F^X_{g,r}(\tau)$ may be determined explicitly as the sum of Eisenstein series and cusp forms on $\Gamma(4M)\cap\Gamma_0(n_g h_g)$ using the standard arguments from the theory of holomorphic modular forms (i.e. the "first few" coefficients determine such a form). Therefore, we have the identity

$$F^X_{g,r}(\tau) = H^X_{g,r}(\tau)\cdot S^X_{g,r}(\tau) + D^X_{g,r}(\tau), \tag{2.2}$$

where the function $D^X_{g,r}(\tau)$ is the correction term arising in Theorem 2.2. If X has pure A-type, then

$$D^X_{g,r}(\tau) = (\hat{\chi}^{X_A}_{g,r})^2 \sum_{N=1}^{\infty} \sum_{\substack{m,n\in\mathbb{Z}_+ \\ m^2-n^2=N}} \phi_r(m)\phi_r(n)(m-n)q^{\frac{N}{4M}}, \tag{2.3}$$

where

$$\phi_r(\ell) = \begin{cases} \pm 1 & \text{if } \ell \equiv \pm r \pmod{2M} \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $H^X_{g,r}(\tau) = \sum_{n=0}^{\infty} A^X_{g,r}(n)q^{n-\frac{D}{4M}}$ where $0 < D < 4M$ and $D \equiv r^2 \pmod{4M}$, and $F^X_{g,r}(\tau) = \sum_{N=0}^{\infty} B^X_{g,r}(n)q^n$. Then by Theorem 2.2, we find that

$$B^X_{g,r}(N) = \hat{\chi}^{X_A}_{g,r} \sum_{\substack{m\in\mathbb{Z} \\ m\equiv r \pmod{2M}}} m\cdot A^X_{g,r}\left(N + \frac{D-m^2}{4M}\right) + (\hat{\chi}^{X_A}_{g,r})^2 \sum_{\substack{m,n\in\mathbb{Z}^+ \\ m^2-n^2=N}} \phi_r(m)\phi_r(n)(m-n). \tag{2.4}$$

The function $F^X_{g,r}(\tau)$ may be found in the following manner. Using the explicit prescriptions for $H^X_{g,r}(\tau)$ given in §B.3 and (2.2) above, we may calculate the first several coefficients of each component. The Eisenstein component is determined by the constant terms at cusps. Since $D^X_{g,r}(\tau)$ (and the corresponding correction terms at other cusps) has no constant term, these are the same as the constant terms of $\widehat{H^X_{g,r}}(\tau)S^X_{g,r}(\tau)$, which are determined by the poles of $\widehat{H^X_{g,r}}$. Call this Eisenstein component $E^X_{g,r}(\tau)$. The cuspidal component can be found by matching the initial coefficients of $F^X_{g,r}(\tau) - E^X_{g,r}(\tau)$.

Once the coefficients $B^X_{g,r}(n)$ are known, equation (2.4) provides a recursion relation which may be used to calculate the coefficients of $H^X_{g,r}(\tau)$. If the shadows $S^X_{g,r}(\tau)$ are zero, then we may apply a similar procedure in order to determine $\widetilde{F}^X_{g,r}(\tau)$. For example, suppose $\widetilde{F}^X_{g,r}(\tau) = \sum_{N=0}^{\infty} \widetilde{B}^X_{g,r}(n)q^n$, and X has pure A-type. Then we find that the coefficients $\widetilde{B}^X_{g,r}(N)$ satisfy

$$\widetilde{B}^X_{g,r}(N) = \hat{\chi}^{X_A}_{g,r} \sum_{\substack{m\in\mathbb{Z} \\ m\equiv s \pmod{2M}}} m \cdot A^X_{g,r}\left(N + \frac{D-m^2}{4M}\right) \tag{2.5}$$

Proceeding in this way we obtain the claimed results. □

3 Proof of Theorem 1.1

Here we prove Theorem 1.1. The idea is as follows. For each Niemeier root system X we begin with the vector-valued mock modular forms $(H^X_g(\tau))$ for $g \in G^X$. We use their q-expansions to solve for the q-series whose coefficients are the alleged multiplicities of the irreducible components of the alleged infinite-dimensional G^X-module

$$\check{K}^X = \bigoplus_{r \pmod{2m}} \bigoplus_{\substack{D\in\mathbb{Z},\, D\leq 0, \\ D=r^2 \pmod{4m}}} \check{K}^X_{r,-D/4m}.$$

These q-series turn out to be mock modular forms. The proof requires that we establish that these mock modular forms have non-negative integer coefficients.

Proof of Theorem 1.1. As in the previous section, we fix a root system X and set $M := m^X$, and we emphasize the case when X is of pure A-type.

The umbral moonshine conjecture asserts that

$$H^X_{g,r}(\tau) = \sum_{n=0}^{\infty}\sum_{\chi} m^X_{\chi,r}(n)\chi(g)q^{n-\frac{r^2}{4M}} \tag{3.1}$$

where the second sum is over the irreducible characters of G^X. Here we have rewritten the traces of the graded components $\check{K}^X_{r,n-\frac{r^2}{4M}}$ in 1.4 in terms of the values of the irreducible characters of G^X, where the $m^X_{\chi,r}(n)$ are the corresponding multiplicities. Naturally, if such a $\check{K}^X$ exists, these multiplicities must be non-negative integers for $n > 0$. Similarly, if the mock modular forms $H^X_{g,r}(\tau)$ can be expressed as in 3.1 with $m^X_{\chi,r}(n)$ non-negative integers, then we may construct the umbral moonshine module $\check{K}^X$ explicitly with $\check{K}^X_{r,n-r^2/4m}$ defined as the direct sum of irreducible components with the given multiplicities $m^X_{\chi,r}(n)$.

Let

$$H^X_{\chi,r}(\tau) := \frac{1}{|G^X|}\sum_g \overline{\chi(g)} H^X_{g,r}(\tau). \tag{3.2}$$

It turns out that the coefficients of $H^X_{\chi,r}(\tau)$ are precisely the multiplicities $m^X_{\chi,r}(n)$ required so that 3.1 holds: if

$$H^X_{\chi,r}(\tau) = \sum_{n=0}^{\infty} m^X_{\chi,r}(n) q^{n-\frac{r^2}{4M}}, \tag{3.3}$$

then

$$H^X_{g,r}(\tau) = \sum_{n=0}^{\infty} \sum_{\chi} m^X_{\chi,r}(n)\chi(g) q^{n-\frac{r^2}{4M}}.$$

Thus the umbral moonshine conjecture is true if and only if the Fourier coefficients of $H^X_{\chi,r}(\tau)$ are non-negative integers.

To see this fact, we recall the orthogonality of characters. For irreducible characters χ_i and χ_j,

$$\frac{1}{|G^X|} \sum_{g\in G^X} \overline{\chi_i(g)}\chi_j(g) = \begin{cases} 1 & \text{if } \chi_i = \chi_j, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

We also have the relation for g and $h \in G^X$,

$$\sum_{\chi} \overline{\chi_i(g)}\chi_i(h) = \begin{cases} |C_{G^X}(g)| & \text{if } g \text{ and } h \text{ are conjugate,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

Here $|C_{G^X}(g)|$ is the order of the centralizer of g in G^X. Since the order of the centralizer times the order of the conjugacy class of an element is the order of the group, (3.2) and (3.5) together imply the relation

$$H^X_{g,r}(\tau) = \sum_{\chi} \chi(g) H^X_{\chi,r}(\tau),$$

which in turn implies 3.3.

We have reduced the theorem to proving that the coefficients of certain weight $1/2$ mock modular forms are all non-negative integers. For holomorphic modular forms we may answer questions of this type by making use of Sturm's theorem [49] (see also Theorem 2.58 of [44]). This theorem provides a bound B associated to a space of modular forms such that if the first B coefficients of a modular form $f(\tau)$ are integral, then all of the coefficients of $f(\tau)$ are integral. This bound reduces many questions about the Fourier coefficients of modular forms to finite calculations.

Sturm's theorem relies on the finite dimensionality of certain spaces of modular forms, and so it can not be applied directly to spaces of mock modular forms. However, by making use of holomorphic projection we can adapt Sturm's theorem to this setting.

Let $\widehat{H^X_{\chi,r}}(\tau)$ be defined as above. Recall that the transformation matrix for the vector-valued function $\widehat{H^X_{g,r}}(\tau))$ is $\overline{\sigma_{g,\gamma}}$, the conjugate of the transformation matrix for $(S^X_{e,r}(\tau))$ when $\gamma \in \Gamma_0(n_g h_g)$, and $\sigma_{g,\gamma}$ is the identity for $\gamma \in \Gamma(4M)$. Therefore if

$$N^X_\chi := \operatorname{lcm}\{n_g h_g \mid g \in G, \chi(g) \neq 0\},$$

then the scalar-valued functions $\widehat{H^X}_{\chi,r}(\tau)$ are modular on $\Gamma(4M) \cap \Gamma_0(N^X_\chi)$.

Let

$$A_{\chi,r}(\tau) := H^X_{\chi,r}(\tau) S^X_{e,1}(\tau),$$

and let $\tilde{A}_{\chi,r}(\tau)$ be the holomorphic projection of $A_{\chi,r}(\tau)$. Suppose that $H^X_{\chi,r}(\tau)$ has integral coefficients up to some bound B. Formulas for the shadow functions (cf. §B.2) show that the leading coefficient of $S^X_{e,1}(\tau)$ is 1 and has integral coefficients. This implies that the function

$$A_{\chi,r}(\tau) := H^X_{\chi,r}(\tau) S^X_{e,1}(\tau)$$

also has integral coefficients up to the bound B. The shadow of $H^X_{\chi,r}(\tau)$ is given by

$$S^X_{\chi,r}(\tau) := \frac{1}{|G^X|} \sum_g \overline{\chi(g)} S^X_{g,r}(\tau).$$

If X is pure A-type, then $S^X_{g,r}(\tau) = \chi^{X_A}_{g,r} S_{M,r}(\tau) = (\chi'(g) + \chi''(g)) S_{M,r}(\tau)$ for some irreducible characters χ' and χ'', according to §A.3 and §B.2. Therefore,

$$S^X_{\chi,r}(\tau) = \begin{cases} S_{M,r}(\tau) & \text{if } \chi = \chi' \text{ or } \chi'', \\ 0 & \text{otherwise.} \end{cases}$$

When X is not of pure A-type the shadow is some sum of such functions, but in every case has integer coefficients, and so, applying Theorem 2.2 to $A_{\chi,r}(\tau)$, we find that $\tilde{A}_{\chi,r}(\tau)$ also has integer coefficients up to the bound B. In particular, since $\tilde{A}_{\chi,r}(\tau)$ is modular on $\Gamma(4M) \cap \Gamma_0(N^X_\chi)$, then if B is at least the Sturm bound for this group we have that every coefficient of $\tilde{A}_{\chi,r}(\tau)$ is integral. Since the leading coefficient of $S^X_{e,1}(\tau)$ is 1, we may reverse this argument and we have that every coefficient of $H^X{}_{\chi,r}(\tau)$. Therefore, in order to check that $H^X_{\chi,r}(\tau)$ has only integer coefficients, it suffices to check up to the Sturm bound for $\Gamma(4M) \cap \Gamma_0(N_\chi)$. These calculations were carried out using the `sage` mathematical software [47].

The calculations and argument given above shows that the multiplicities $m^X_{\chi,r}(n)$ are all integers. To complete the proof, it suffices to check that they are also are non-negative. The proof of this claim follows easily by modifying step-by-step the argument in Gannon's proof of non negativity in the M_{24} case [37] (i.e. $X = A_1^{24}$). Here we describe how this is done.

Expressions for the alleged McKay-Thompson series $H^X_{g,r}(\tau)$ in terms of Rademacher sums and unary theta functions are given in §B.4. Exact formulas are known for all the coefficients of Rademacher sums because they are defined by averaging the special function $\mathrm{r}^{[\alpha]}_{1/2}(\gamma, \tau)$ (see (B.114)) over cosets of a specific modular group modulo Γ_∞, the subgroup of translations. Therefore, Rademacher sums are standard Maass-Poincaré series, and as a result we have formulas for each of their coefficients as convergent infinite sums of Kloosterman-type sums weighted by values of the $I_{1/2}$ modified Bessel function. (For example, see [8] or [53] for the general theory, and [12] for the specific case that $X = A_1^{24}$.) More importantly, this means also that the generating function for the multiplicities $m^X_{\chi,r}(n)$ is a weight $\frac{1}{2}$ harmonic Maass form, which in turn means that exact formulas (modulo the unary theta functions) are also available in similar terms. For positive integers n, this then means that (cf. Theorem 1.1 of [8])

$$m^X_{\chi,r}(n) = \sum_\rho \sum_{m<0} \frac{a^X_\rho(m)}{n^{\frac{1}{4}}} \sum_{c=1}^{\infty} \frac{K^X_\rho(m,n,c)}{c} \cdot \mathbb{I}^X \left(\frac{4\pi\sqrt{|nm|}}{c} \right), \tag{3.6}$$

where the sums are over the cusps ρ of the group $\Gamma_0(N^X_g)$, and finitely many explicit negative rational numbers m. The constants $a^X_\rho(m)$ are essentially the coefficients which describe the generating function in terms of Maass-Poincaré series. Here $\mathbb{I}$ is a suitable normalization and change of variable for the standard $I_{1/2}$ modified Bessel-function.

The Kloosterman-type sums $K_\rho^X(m,n,c)$ are well known to be related to Salié-type sums (for example see Proposition 5 of [40]). These Salié-type sums are of the form

$$S_\rho^X(m,n,c) = \sum_{\substack{x \pmod c \\ x^2 \equiv -D(m,n) \pmod c}} \epsilon_\rho^X(m,n) \cdot e\left(\frac{\beta^X x}{c}\right),$$

where $\epsilon_\rho^X(m,n)$ is a root of unity, $-D(m,n)$ is a discriminant of a positive definite binary quadratic form, and β^X is a nonzero positive rational number.

These Salié sums may then be estimated using the equidistribution of CM points with discriminant $-D(m,n)$. This process was first introduced by Hooley [38], and it was first applied to the coefficients of weight $\frac{1}{2}$ mock modular forms by Bringmann and Ono [7]. Gannon explains how to make effective the estimates for sums of this shape in §4 of [37], thereby reducing the proof of the M_{24} case of umbral moonshine to a finite calculation. In particular, in equations (4.6-4.10) of [37] Gannon shows how to bound coefficients of the form (3.6) in terms of the Selberg–Kloosterman zeta function, which is bounded in turn in his proof of Theorem 3 of [37]. We follow Gannon's proof *mutatis mutandis.* We find, for each root system, that the coefficients of each multiplicity generating function are positive beyond the 390th coefficient. Moreover, the coefficients exhibit subexponential growth. A finite computer calculation in `sage` has verified the non-negativity of the finitely many remaining coefficients. □

Remark. It turns out that the estimates required for proving nonnegativity are the worst for the M_{24} case considered by Gannon.

A The Umbral Groups

In this section we present the facts about the umbral groups that we have used in establishing the main results of this paper. We recall (from [16]) their construction in terms of Niemeier root systems in §A.1, and we reproduce their character tables (appearing also in [16]) in §A.2. Note that we use the abbreviations $a_n := \sqrt{-n}$ and $b_n := (-1+\sqrt{-n})/2$ in the tables of §A.2.

The root system description of the umbral groups (cf. §A.1) gives rise to certain characters called *twisted Euler characters* which we recall (from [16]) in §A.3. The data appearing in §A.3 plays an important role in §B.2, where we use it to describe the shadows S^X_g of the umbral McKay-Thompson series H^X_g explicitly.

A.1 Construction

As mentioned in §1, there are exactly 24 self-dual even positive-definite lattices of rank 24 up to isomorphism, according to the classification of Niemeier [43] (cf. also [19, 52]). Such a lattice L is determined up to isomorphism by its *root system* $L_2 := \{\alpha \in L \mid \langle \alpha, \alpha \rangle = 2\}$. The unique example without roots is the Leech lattice. We refer to the remaining 23 as the *Niemeier lattices*, and we call a root system X a *Niemeier root system* if it occurs as the root system of a Niemeier lattice.

The simple components of Niemeier root systems are root systems of ADE type, and it turns out that the simple components of a Niemeier root system X all have the same Coxeter number. Define m^X to be the Coxeter number of any simple component of X, and call this the *Coxeter number* of X.

For X a Niemeier root system write N^X for the corresponding Niemeier lattice. The *umbral group* attached to X is defined by setting

$$G^X := \mathrm{Aut}(N^X)/W^X \tag{A.1}$$

where W^X is the normal subgroup of $\mathrm{Aut}(N^X)$ generated by reflections in root vectors.

Observe that G^X acts as permutations on the simple components of X. In general this action is not faithful, so define $\overline{G}^X$ to be the quotient of G^X by its kernel. It turns out that the level of the mock modular form H^X_g attached to $g \in G^X$ is given by the order, denoted n_g, of the image of g in $\bar{G}^X$. (Cf. §A.3 for the values n_g.)

The Niemeier root systems and their corresponding umbral groups are described in Table 1. The root systems are given in terms of their simple components of ADE type. Here $D_{10}E_7^2$, for example, means the direct sum of one copy of the D_{10} root system and two copies of the E_7 root system. The symbol ℓ is called the *lambency* of X, and the Coxeter number m^X appears as the first summand of ℓ.

In the descriptions of the umbral groups G^X, and their permutation group quotients $\bar{G}^X$, we write M_{24} and M_{12} for the sporadic simple groups of Mathieu which act quintuply transitively on 24 and 12 points, respectively. (Cf. e.g. [21].) We write $GL_n(q)$ for the general linear group of a vector space of dimension n over a field with q elements, and $SL_n(q)$ is the subgroup of linear transformations with determinant 1, &c. The symbols $AGL_3(2)$ denote the *affine general linear group*, obtained by adjoining translations to $GL_3(2)$. We write Dih_n for the dihedral group of order $2n$, and Sym_n denotes the symmetric group on n symbols. We use n as a shorthand for a cyclic group of order n.

Table 1: The Umbral Groups

X	A_1^{24}	A_2^{12}	A_3^8	A_4^6	$A_5^4D_4$	A_6^4	$A_7^2D_5^2$
ℓ	2	3	4	5	6	7	8
G^X	M_{24}	$2.M_{12}$	$2.AGL_3(2)$	$GL_2(5)/2$	$GL_2(3)$	$SL_2(3)$	Dih_4
$\bar{G}^X$	M_{24}	M_{12}	$AGL_3(2)$	$PGL_2(5)$	$PGL_2(3)$	$PSL_2(3)$	2^2

X	A_8^3	$A_9^2D_6$	$A_{11}D_7E_6$	A_{12}^2	$A_{15}D_9$	$A_{17}E_7$	A_{24}
ℓ	9	10	12	13	16	18	25
G^X	Dih_6	4	2	4	2	2	2
$\bar{G}^X$	Sym_3	2	1	2	1	1	1

X	D_4^6	D_6^4	D_8^3	$D_{10}E_7^2$	D_{12}^2	$D_{16}E_8$	D_{24}
ℓ	6+3	10+5	14+7	18+9	22+11	30+15	46+23
G^X	$3.Sym_6$	Sym_4	Sym_3	2	2	1	1
$\bar{G}^X$	Sym_6	Sym_4	Sym_3	2	2	1	1

X	E_6^4	E_8^3
ℓ	12+4	30+6,10,15
G^X	$GL_2(3)$	Sym_3
$\bar{G}^X$	$PGL_2(3)$	Sym_3

We also use the notational convention of writing $A.B$ to denote the middle term in a short exact sequence $1 \to A \to A.B \to B \to 1$. This introduces some ambiguity which is nonetheless easily navigated in practice. For example, $2.M_{12}$ is the unique (up to isomorphism) double cover of M_{12} which is not $2 \times M_{12}$. The group $AGL_3(2)$ naturally embeds in $GL_4(2)$, which in turn admits a unique (up to isomorphism) double cover $2.GL_4(2)$ which is not a direct product. The group we denote $2.AGL_3(2)$ is the preimage of $AGL_3(2) < GL_4(2)$ in $2.GL_4(2)$ under the natural projection.

A.2 Character Tables

Table 2: Character table of $G^X \simeq M_{24}$, $X = A_1^{24}$

$[g]$	FS	1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7A	7B	8A	10A	11A	12A	12B	14A	14B	15A	15B	21A	21B	23A	23B
$[g^2]$		1A	1A	1A	3A	3B	2A	2A	2B	5A	3A	3B	7A	7B	4B	5A	11A	6A	6B	7A	7B	15A	15B	21A	21B	23A	23B
$[g^3]$		1A	2A	2B	1A	1A	4A	4B	4C	5A	2A	2B	7B	7A	8A	10A	11A	4A	4C	14B	14A	5A	5A	7B	7A	23A	23B
$[g^5]$		1A	2A	2B	3A	3B	4A	4B	4C	1A	6A	6B	7B	7A	8A	2B	11A	12A	12B	14B	14A	3A	3A	21B	21A	23B	23A
$[g^7]$		1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	1A	1A	8A	10A	11A	12A	12B	2A	2A	15B	15A	3B	3B	23B	23A
$[g^{11}]$		1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7A	7B	8A	10A	1A	12A	12B	14A	14B	15B	15A	21A	21B	23B	23A
$[g^{23}]$		1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7A	7B	8A	10A	11A	12A	12B	14A	14B	15A	15B	21A	21B	1A	1A
χ_1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	+	23	7	−1	5	−1	−1	3	−1	3	1	−1	2	2	1	−1	1	−1	−1	0	0	0	0	−1	−1	0	0
χ_3	∘	45	−3	5	0	3	−3	1	1	0	0	−1	b_7	$\overline{b_7}$	−1	0	1	0	1	$-b_7$	$-\overline{b_7}$	0	0	b_7	$\overline{b_7}$	−1	−1
χ_4	∘	45	−3	5	0	3	−3	1	1	0	0	−1	$\overline{b_7}$	b_7	−1	0	1	0	1	$-\overline{b_7}$	$-b_7$	0	0	$\overline{b_7}$	b_7	−1	−1
χ_5	∘	231	7	−9	−3	0	−1	−1	3	1	1	0	0	0	−1	1	0	−1	0	0	0	b_{15}	$\overline{b_{15}}$	0	0	1	1
χ_6	∘	231	7	−9	−3	0	−1	−1	3	1	1	0	0	0	−1	1	0	−1	0	0	0	$\overline{b_{15}}$	b_{15}	0	0	1	1
χ_7	+	252	28	12	9	0	4	4	0	2	1	0	0	0	0	2	−1	1	0	0	0	−1	−1	0	0	−1	−1
χ_8	+	253	13	−11	10	1	−3	1	1	3	−2	1	1	1	−1	−1	0	0	1	−1	−1	0	0	1	1	0	0
χ_9	+	483	35	3	6	0	3	3	3	−2	2	0	0	0	−1	−2	−1	0	0	0	0	1	1	0	0	0	0
χ_{10}	∘	770	−14	10	5	−7	2	−2	−2	0	1	1	0	0	0	0	0	−1	1	0	0	0	0	0	0	b_{23}	$\overline{b_{23}}$
χ_{11}	∘	770	−14	10	5	−7	2	−2	−2	0	1	1	0	0	0	0	0	−1	1	0	0	0	0	0	0	$\overline{b_{23}}$	b_{23}
χ_{12}	∘	990	−18	−10	0	3	6	2	−2	0	0	−1	b_7	$\overline{b_7}$	0	0	0	0	1	b_7	$\overline{b_7}$	0	0	b_7	$\overline{b_7}$	1	1
χ_{13}	∘	990	−18	−10	0	3	6	2	−2	0	0	−1	$\overline{b_7}$	b_7	0	0	0	0	1	$\overline{b_7}$	b_7	0	0	$\overline{b_7}$	b_7	1	1
χ_{14}	+	1035	27	35	0	6	3	−1	3	0	0	2	−1	−1	1	0	1	0	0	−1	−1	0	0	−1	−1	0	0
χ_{15}	∘	1035	−21	−5	0	−3	3	3	−1	0	0	1	$2b_7$	$2\overline{b_7}$	−1	0	1	0	−1	0	0	0	0	$-b_7$	$-\overline{b_7}$	0	0
χ_{16}	∘	1035	−21	−5	0	−3	3	3	−1	0	0	1	$2\overline{b_7}$	$2b_7$	−1	0	1	0	−1	0	0	0	0	$-\overline{b_7}$	$-b_7$	0	0
χ_{17}	+	1265	49	−15	5	8	−7	1	−3	0	1	0	−2	−2	1	0	0	−1	0	0	0	0	0	1	1	0	0
χ_{18}	+	1771	−21	11	16	7	3	−5	−1	1	0	−1	0	0	−1	1	0	0	−1	0	0	1	1	0	0	0	0
χ_{19}	+	2024	8	24	−1	8	8	0	0	−1	−1	0	1	1	0	−1	0	−1	0	1	1	−1	−1	1	1	0	0
χ_{20}	+	2277	21	−19	0	6	−3	1	−3	−3	0	2	2	2	−1	1	0	0	0	0	0	0	0	−1	−1	0	0
χ_{21}	+	3312	48	16	0	−6	0	0	0	−3	0	−2	1	1	0	1	1	0	0	−1	−1	0	0	1	1	0	0
χ_{22}	+	3520	64	0	10	−8	0	0	0	0	−2	0	−1	−1	0	0	0	0	0	1	1	0	0	−1	−1	1	1
χ_{23}	+	5313	49	9	−15	0	1	−3	−3	3	1	0	0	0	−1	−1	0	1	0	0	0	0	0	0	0	0	0
χ_{24}	+	5544	−56	24	9	0	−8	0	0	−1	1	0	0	0	0	−1	0	1	0	0	0	−1	−1	0	0	1	1
χ_{25}	+	5796	−28	36	−9	0	−4	4	0	1	−1	0	0	0	0	1	−1	−1	0	0	0	1	1	0	0	0	0
χ_{26}	+	10395	−21	−45	0	0	3	−1	3	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	−1	−1

Table 3: Character table of $G^X \simeq 2.M_{12}$, $X = A_2^{12}$

$[g]$	FS	1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	5A	10A	12A	6C	6D	8A	8B	8C	8D	20A	20B	11A	22A	11B	22B
$[g^2]$		1A	1A	2A	1A	1A	3A	3A	3B	3B	2B	2B	5A	5A	6B	3A	3A	4B	4B	4C	4C	10A	10A	11B	11B	11A	11A
$[g^3]$		1A	2A	4A	2B	2C	1A	2A	1A	2A	4B	4C	5A	10A	4A	2B	2C	8A	8B	8C	8D	20A	20B	11A	22A	11B	22B
$[g^5]$		1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	1A	2A	12A	6C	6D	8B	8A	8D	8C	4A	4A	11A	22A	11B	22B
$[g^{11}]$		1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	5A	10A	12A	6C	6D	8A	8B	8C	8D	20B	20A	1A	2A	1A	2A
χ_1	$+$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	$+$	11	11	-1	3	3	2	2	-1	-1	-1	3	1	1	-1	0	0	-1	-1	1	1	-1	-1	0	0	0	0
χ_3	$+$	11	11	-1	3	3	2	2	-1	-1	3	-1	1	1	-1	0	0	1	1	-1	-1	-1	-1	0	0	0	0
χ_4	$\circ$	16	16	4	0	0	-2	-2	1	1	0	0	1	1	1	0	0	0	0	0	0	-1	-1	b_{11}	b_{11}	$\overline{b_{11}}$	$\overline{b_{11}}$
χ_5	$\circ$	16	16	4	0	0	-2	-2	1	1	0	0	1	1	1	0	0	0	0	0	0	-1	-1	$\overline{b_{11}}$	$\overline{b_{11}}$	b_{11}	b_{11}
χ_6	$+$	45	45	5	-3	-3	0	0	3	3	1	1	0	0	-1	0	0	-1	-1	-1	-1	0	0	1	1	1	1
χ_7	$+$	54	54	6	6	6	0	0	0	0	2	2	-1	-1	0	0	0	0	0	0	0	1	1	-1	-1	-1	-1
χ_8	$+$	55	55	-5	7	7	1	1	1	1	-1	-1	0	0	1	1	1	-1	-1	-1	-1	0	0	0	0	0	0
χ_9	$+$	55	55	-5	-1	-1	1	1	1	1	3	-1	0	0	1	-1	-1	-1	-1	1	1	0	0	0	0	0	0
χ_{10}	$+$	55	55	-5	-1	-1	1	1	1	1	-1	3	0	0	1	-1	-1	1	1	-1	-1	0	0	0	0	0	0
χ_{11}	$+$	66	66	6	2	2	3	3	0	0	-2	-2	1	1	0	-1	-1	0	0	0	0	1	1	0	0	0	0
χ_{12}	$+$	99	99	-1	3	3	0	0	3	3	-1	-1	-1	-1	-1	0	0	1	1	1	1	-1	-1	0	0	0	0
χ_{13}	$+$	120	120	0	-8	-8	3	3	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	-1	-1	-1	-1
χ_{14}	$+$	144	144	4	0	0	0	0	-3	-3	0	0	-1	-1	1	0	0	0	0	0	0	-1	-1	1	1	1	1
χ_{15}	$+$	176	176	-4	0	0	-4	-4	-1	-1	0	0	1	1	-1	0	0	0	0	0	0	1	1	0	0	0	0
χ_{16}	$\circ$	10	-10	0	-2	2	1	-1	-2	2	0	0	0	0	0	1	-1	a_2	$\overline{a_2}$	a_2	$\overline{a_2}$	0	0	-1	1	-1	1
χ_{17}	$\circ$	10	-10	0	-2	2	1	-1	-2	2	0	0	0	0	0	1	-1	$\overline{a_2}$	a_2	$\overline{a_2}$	a_2	0	0	-1	1	-1	1
χ_{18}	$+$	12	-12	0	4	-4	3	-3	0	0	0	0	2	-2	0	1	-1	0	0	0	0	0	0	1	-1	1	-1
χ_{19}	$-$	32	-32	0	0	0	-4	4	2	-2	0	0	2	-2	0	0	0	0	0	0	0	0	0	-1	1	-1	1
χ_{20}	$\circ$	44	-44	0	4	-4	-1	1	2	-2	0	0	-1	1	0	1	-1	0	0	0	0	a_5	$\overline{a_5}$	0	0	0	0
χ_{21}	$\circ$	44	-44	0	4	-4	-1	1	2	-2	0	0	-1	1	0	1	-1	0	0	0	0	$\overline{a_5}$	a_5	0	0	0	0
χ_{22}	$\circ$	110	-110	0	-6	6	2	-2	2	-2	0	0	0	0	0	0	0	a_2	$\overline{a_2}$	$\overline{a_2}$	a_2	0	0	0	0	0	0
χ_{23}	$\circ$	110	-110	0	-6	6	2	-2	2	-2	0	0	0	0	0	0	0	$\overline{a_2}$	a_2	a_2	$\overline{a_2}$	0	0	0	0	0	0
χ_{24}	$+$	120	-120	0	8	-8	3	-3	0	0	0	0	0	0	0	-1	1	0	0	0	0	0	0	-1	1	-1	1
χ_{25}	$\circ$	160	-160	0	0	0	-2	2	-2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	$-b_{11}$	b_{11}	$-\overline{b_{11}}$	$\overline{b_{11}}$
χ_{26}	$\circ$	160	-160	0	0	0	-2	2	-2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	$-\overline{b_{11}}$	$\overline{b_{11}}$	$-b_{11}$	b_{11}

Table 4: Character table of $G^X \simeq 2.AGL_3(2)$, $X = A_3^8$

$[g]$	FS	1A	2A	2B	4A	4B	2C	3A	6A	6B	6C	8A	4C	7A	14A	7B	14B
$[g^2]$		1A	1A	1A	2A	2B	1A	3A	3A	3A	3A	4A	2C	7A	7A	7B	7B
$[g^3]$		1A	2A	2B	4A	4B	2C	1A	2A	2B	2B	8A	4C	7B	14B	7A	14A
$[g^7]$		1A	2A	2B	4A	4B	2C	3A	6A	6B	6C	8A	4C	1A	2A	1A	2A
χ_1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	∘	3	3	3	−1	−1	−1	0	0	0	0	1	1	b_7	b_7	$\overline{b_7}$	$\overline{b_7}$
χ_3	∘	3	3	3	−1	−1	−1	0	0	0	0	1	1	$\overline{b_7}$	$\overline{b_7}$	b_7	b_7
χ_4	+	6	6	6	2	2	2	0	0	0	0	0	0	−1	−1	−1	−1
χ_5	+	7	7	7	−1	−1	−1	1	1	1	1	−1	−1	0	0	0	0
χ_6	+	8	8	8	0	0	0	−1	−1	−1	−1	0	0	1	1	1	1
χ_7	+	7	7	−1	3	−1	−1	1	1	−1	−1	1	−1	0	0	0	0
χ_8	+	7	7	−1	−1	−1	3	1	1	−1	−1	−1	1	0	0	0	0
χ_9	+	14	14	−2	2	−2	2	−1	−1	1	1	0	0	0	0	0	0
χ_{10}	+	21	21	−3	1	1	−3	0	0	0	0	−1	1	0	0	0	0
χ_{11}	+	21	21	−3	−3	1	1	0	0	0	0	1	−1	0	0	0	0
χ_{12}	+	8	−8	0	0	0	0	2	−2	0	0	0	0	1	−1	1	−1
χ_{13}	∘	8	−8	0	0	0	0	−1	1	a_3	$\overline{a_3}$	0	0	1	−1	1	−1
χ_{14}	∘	8	−8	0	0	0	0	−1	1	$\overline{a_3}$	a_3	0	0	1	−1	1	−1
χ_{15}	∘	24	−24	0	0	0	0	0	0	0	0	0	0	$\overline{b_7}$	$-\overline{b_7}$	b_7	$-b_7$
χ_{16}	∘	24	−24	0	0	0	0	0	0	0	0	0	0	b_7	$-b_7$	$\overline{b_7}$	$-\overline{b_7}$

Table 5: Character table of $G^X \simeq GL_2(5)/2$, $X = A_4^6$

$[g]$	FS	1A	2A	2B	2C	3A	6A	5A	10A	4A	4B	4C	4D	12A	12B
$[g^2]$		1A	1A	1A	1A	3A	3A	5A	5A	2A	2A	2C	2C	6A	6A
$[g^3]$		1A	2A	2B	2C	1A	2A	5A	10A	4B	4A	4D	4C	4B	4A
$[g^5]$		1A	2A	2B	2C	3A	6A	1A	2A	4A	4B	4C	4D	12A	12B
χ_1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	+	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1
χ_3	+	4	4	0	0	1	1	−1	−1	2	2	0	0	−1	−1
χ_4	+	4	4	0	0	1	1	−1	−1	−2	−2	0	0	1	1
χ_5	+	5	5	1	1	−1	−1	0	0	1	1	−1	−1	1	1
χ_6	+	5	5	1	1	−1	−1	0	0	−1	−1	1	1	−1	−1
χ_7	+	6	6	−2	−2	0	0	1	1	0	0	0	0	0	0
χ_8	∘	1	−1	1	−1	1	−1	1	−1	a_1	$-a_1$	a_1	$-a_1$	a_1	$-a_1$
χ_9	∘	1	−1	1	−1	1	−1	1	−1	$-a_1$	a_1	$-a_1$	a_1	$-a_1$	a_1
χ_{10}	∘	4	−4	0	0	1	−1	−1	1	$2a_1$	$-2a_1$	0	0	$-a_1$	a_1
χ_{11}	∘	4	−4	0	0	1	−1	−1	1	$-2a_1$	$2a_1$	0	0	a_1	$-a_1$
χ_{12}	∘	5	−5	1	−1	−1	1	0	0	a_1	$-a_1$	$-a_1$	a_1	a_1	$-a_1$
χ_{13}	∘	5	−5	1	−1	−1	1	0	0	$-a_1$	a_1	a_1	$-a_1$	$-a_1$	a_1
χ_{14}	+	6	−6	−2	2	0	0	1	−1	0	0	0	0	0	0

Table 6: Character table of $G^X \simeq GL_2(3)$, $X \in \{A_5^4 D_4, E_6^4\}$

$[g]$	FS	1A	2A	2B	4A	3A	6A	8A	8B
$[g^2]$		1A	1A	1A	2A	3A	3A	4A	4A
$[g^3]$		1A	2A	2B	4A	1A	2A	8A	8B
χ_1	$+$	1	1	1	1	1	1	1	1
χ_2	$+$	1	1	-1	1	1	1	-1	-1
χ_3	$+$	2	2	0	2	-1	-1	0	0
χ_4	$+$	3	3	-1	-1	0	0	1	1
χ_5	$+$	3	3	1	-1	0	0	-1	-1
χ_6	$\circ$	2	-2	0	0	-1	1	a_2	$\overline{a_2}$
χ_7	$\circ$	2	-2	0	0	-1	1	$\overline{a_2}$	a_2
χ_8	$+$	4	-4	0	0	1	-1	0	0

Table 7: Character table of $G^X \simeq 3.Sym_6$, $X = D_4^6$

$[g]$	FS	1A	3A	2A	6A	3B	3C	4A	12A	5A	15A	15B	2B	2C	4B	6B	6C
$[g^2]$		1A	3A	1A	3A	3B	3C	2A	6A	5A	15A	15B	1A	1A	2A	3B	3C
$[g^3]$		1A	1A	2A	2A	1A	1A	4A	4A	5A	5A	5A	2B	2C	4B	2B	2C
$[g^5]$		1A	3A	2A	6A	3B	3C	4A	12A	1A	3A	3A	2B	2C	4B	6B	6C
χ_1	$+$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	$+$	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1
χ_3	$+$	5	5	1	1	2	-1	-1	-1	0	0	0	3	-1	1	0	-1
χ_4	$+$	5	5	1	1	2	-1	-1	-1	0	0	0	-3	1	-1	0	1
χ_5	$+$	5	5	1	1	-1	2	-1	-1	0	0	0	-1	3	1	-1	0
χ_6	$+$	5	5	1	1	-1	2	-1	-1	0	0	0	1	-3	-1	1	0
χ_7	$+$	16	16	0	0	-2	-2	0	0	1	1	1	0	0	0	0	0
χ_8	$+$	9	9	1	1	0	0	1	1	-1	-1	-1	3	3	-1	0	0
χ_9	$+$	9	9	1	1	0	0	1	1	-1	-1	-1	-3	-3	1	0	0
χ_{10}	$+$	10	10	-2	-2	1	1	0	0	0	0	0	2	-2	0	-1	1
χ_{11}	$+$	10	10	-2	-2	1	1	0	0	0	0	0	-2	2	0	1	-1
χ_{12}	$\circ$	6	-3	-2	1	0	0	2	-1	1	b_{15}	$\overline{b_{15}}$	0	0	0	0	0
χ_{13}	$\circ$	6	-3	-2	1	0	0	2	-1	1	$\overline{b_{15}}$	b_{15}	0	0	0	0	0
χ_{14}	$+$	12	-6	4	-2	0	0	0	0	2	-1	-1	0	0	0	0	0
χ_{15}	$+$	18	-9	2	-1	0	0	2	-1	-2	1	1	0	0	0	0	0
χ_{16}	$+$	30	-15	-2	1	0	0	-2	1	0	0	0	0	0	0	0	0

Table 8: Character table of $G^X \simeq SL_2(3)$, $X = A_6^4$

$[g]$	FS	1A	2A	4A	3A	6A	3B	6B
$[g^2]$		1A	1A	2A	3B	3A	3A	3B
$[g^3]$		1A	2A	4A	1A	2A	1A	2A
χ_1	$+$	1	1	1	1	1	1	1
χ_2	$\circ$	1	1	1	b_3	$\overline{b_3}$	$\overline{b_3}$	b_3
χ_3	$\circ$	1	1	1	$\overline{b_3}$	b_3	b_3	$\overline{b_3}$
χ_4	$+$	3	3	-1	0	0	0	0
χ_5	$-$	2	-2	0	-1	1	-1	1
χ_6	$\circ$	2	-2	0	$-\overline{b_3}$	b_3	$-b_3$	$\overline{b_3}$
χ_7	$\circ$	2	-2	0	$-b_3$	$\overline{b_3}$	$-\overline{b_3}$	b_3

Table 9: Character table of $G^X \simeq Dih_4$, $X = A_7^2 D_5^2$

$[g]$	FS	1A	2A	2B	2C	4A
$[g^2]$		1A	1A	1A	1A	2A
χ_1	$+$	1	1	1	1	1
χ_2	$+$	1	1	-1	-1	1
χ_3	$+$	1	1	-1	1	-1
χ_4	$+$	1	1	1	-1	-1
χ_5	$+$	2	-2	0	0	0

Table 10: Character table of $G^X \simeq Dih_6$, $X = A_8^3$

$[g]$	FS	1A	2A	2B	2C	3A	6A
$[g^2]$		1A	1A	1A	1A	3A	3A
$[g^3]$		1A	2A	2B	2C	1A	2A
χ_1	$+$	1	1	1	1	1	1
χ_2	$+$	1	1	-1	-1	1	1
χ_3	$+$	2	2	0	0	-1	-1
χ_4	$+$	1	-1	-1	1	1	-1
χ_5	$+$	1	-1	1	-1	1	-1
χ_6	$+$	2	-2	0	0	-1	1

Table 11: Character table of $G^X \simeq 4$, for $X \in \{A_9^2D_6, A_{12}^2\}$

$[g]$	FS	1A	2A	4A	4B
$[g^2]$		1A	1A	2A	2A
χ_1	$+$	1	1	1	1
χ_2	$+$	1	1	-1	-1
χ_3	$\circ$	1	-1	a_1	$\overline{a_1}$
χ_4	$\circ$	1	-1	$\overline{a_1}$	a_1

Table 12: Character table of $G^X \simeq PGL_2(3) \simeq Sym_4$, $X = D_6^4$

$[g]$	FS	1A	2A	3A	2B	4A
$[g^2]$		1A	1A	3A	1A	2A
$[g^3]$		1A	2A	1A	2B	4A
χ_1	$+$	1	1	1	1	1
χ_2	$+$	1	1	1	-1	-1
χ_3	$+$	2	2	-1	0	0
χ_4	$+$	3	-1	0	1	-1
χ_5	$+$	3	-1	0	-1	1

Table 13: Character table of $G^X \simeq 2$, for $X \in \{A_{11}D_7E_6, A_{15}D_9, A_{17}E_7, A_{24}, D_{10}E_7^2, D_{12}^2\}$

$[g]$	FS	1A	2A
$[g^2]$		1A	1A
χ_1	$+$	1	1
χ_2	$+$	1	-1

Table 14: Character table of $G^X \simeq Sym_3$, $X \in \{D_8^3, E_8^3\}$

$[g]$	FS	1A	2A	3A
$[g^2]$		1A	1A	3A
$[g^3]$		1A	2A	1A
χ_1	$+$	1	1	1
χ_2	$+$	1	-1	1
χ_3	$+$	2	0	-1

A.3 Twisted Euler Characters

In this section we reproduce certain characters—the *twisted Euler characters*—which are attached to each group G^X, via its action on the root system X. (Their construction is described in detail in §2.4 of [16].)

To interpret the tables, write X_A for the (possibly empty) union of type A components of X, and interpret X_D and X_E similarly, so that if $m = m^X$ Then $X = A_{m-1}^d$ for some d, and $X = X_A \cup X_D \cup X_E$, for example. Then $g \mapsto \bar{\chi}_g^{X_A}$ denotes the character of the permutation representation attached to the action of $\bar{G}^X$ on the simple components of X_A. The characters $g \mapsto \bar{\chi}_g^{X_D}$ and $g \mapsto \bar{\chi}_g^{X_E}$ are defined similarly. The characters $\chi_g^{X_A}$, $\chi_g^{X_D}$, $\chi_g^{X_E}$ and $\check{\chi}_g^{X_D}$ incorporate outer automorphisms of simple root systems induced by the action G^X on X. We refer to §2.4 of [16] for full details of the construction. For the purposes of this work, it suffices to have the explicit descriptions in the tables in this section. The twisted Euler characters presented here will be used to specify the umbral shadow functions in §B.2.

The twisted Euler character tables also attach integers n_g and h_g to each $g \in G^X$. By definition, n_g is the order of the image of $g \in G^X$ in $\bar{G}^X$ (cf. §A.1). The integer h_g may be defined by setting $h_g := N_g/n_g$ where N_g is the product of the shortest and longest cycle lengths appearing in the cycle shape attached to g by the action of G^X on a (suitable) set of simple roots for X.

Table 15: Twisted Euler characters at $\ell = 2$, $X = A_1^{24}$

$[g]$	1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B
$n_g \vert h_g$	1|1	2|1	2|2	3|1	3|3	4|2	4|1	4|4	5|1	6|1	6|6
$\bar{\chi}_g^{X_A}$	24	8	0	6	0	0	4	0	4	2	0

$[g]$	7AB	8A	10A	11A	12A	12B	14AB	15AB	21AB	23AB
$n_g \vert h_g$	7|1	8|1	10|2	11|1	12|2	12|12	14|1	15|1	21|3	23|1
$\bar{\chi}_g^{X_A}$	3	2	0	2	0	0	1	1	0	1

Table 16: Twisted Euler characters at $\ell = 3$, $X = A_2^{12}$

$[g]$	1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	5A	10A	12A	6C	6D	8AB	8CD	20AB	11AB	22AB
$n_g \vert h_g$	1|1	1|4	2|8	2|1	2|2	3|1	3|4	3|3	3|12	4|2	4|1	5|1	5|4	6|24	6|1	6|2	8|4	8|1	10|8	11|1	11|4
$\bar{\chi}_g^{X_A}$	12	12	0	4	4	3	3	0	0	0	4	2	2	0	1	1	0	2	0	1	1
$\chi_g^{X_A}$	12	−12	0	4	−4	3	−3	0	0	0	0	2	−2	0	1	−1	0	0	0	1	−1

Table 17: Twisted Euler characters at $\ell = 4$, $X = A_3^8$

$[g]$	1A	2A	2B	4A	4B	2C	3A	6A	6BC	8A	4C	7AB	14AB
$n_g\|h_g$	1|1	1|2	2|2	2|4	4|4	2|1	3|1	3|2	6|2	4|8	4|1	7|1	7|2
$\bar{\chi}_g^{X_A}$	8	8	0	0	0	4	2	2	0	0	2	1	1
$\chi_g^{X_A}$	8	−8	0	0	0	0	2	−2	0	0	0	1	−1

Table 18: Twisted Euler characters at $\ell = 5$, $X = A_4^6$

$[g]$	1A	2A	2B	2C	3A	6A	5A	10A	4AB	4CD	12AB
$n_g\|h_g$	1|1	1|4	2|2	2|1	3|3	3|12	5|1	5|4	2|8	4|1	6|24
$\bar{\chi}_g^{X_A}$	6	6	2	2	0	0	1	1	0	2	0
$\chi_g^{X_A}$	6	−6	−2	2	0	0	1	−1	0	0	0

Table 19: Twisted Euler characters at $\ell = 6$, $X = A_5^4 D_4$

$[g]$	1A	2A	2B	4A	3A	6A	8AB
$n_g\|h_g$	1|1	1|2	2|1	2|2	3|1	3|2	4|2
$\bar{\chi}_g^{X_A}$	4	4	2	0	1	1	0
$\chi_g^{X_A}$	4	−4	0	0	1	−1	0
$\bar{\chi}_g^{X_D}$	1	1	1	1	1	1	1
$\chi_g^{X_D}$	1	1	−1	1	1	1	−1
$\check{\chi}_g^{X_D}$	2	2	0	2	−1	−1	0

Table 20: Twisted Euler characters at $\ell = 6 + 3$, $X = D_4^6$

$[g]$	1A	3A	2A	6A	3B	6C	4A	12A	5A	15AB	2B	2C	4B	6B	6C
$n_g\|h_g$	1|1	1|3	2|1	2|3	3|1	3|3	4|2	4|6	5|1	5|3	2|1	2|2	4|1	6|1	6|6
$\bar{\chi}_g^{X_D}$	6	6	2	2	3	0	0	0	1	1	4	0	2	1	0
$\chi_g^{X_D}$	6	6	2	2	3	0	0	0	1	1	−4	0	−2	−1	0
$\check{\chi}_g^{X_D}$	12	−6	4	−2	0	0	0	0	2	−1	0	0	0	0	0

Table 21: Twisted Euler characters at $\ell = 7$, $X = A_6^4$

$[g]$	1A	2A	4A	3AB	6AB
$n_g\|h_g$	1|1	1|4	2|8	3|1	3|4
$\bar{\chi}_g^{X_A}$	4	4	0	1	1
$\chi_g^{X_A}$	4	-4	0	1	-1

Table 22: Twisted Euler characters at $\ell = 8$, $X = A_7^2 D_5^2$

$[g]$	1A	2A	2B	2C	4A
$n_g\|h_g$	1|1	1|2	2|1	2|1	2|4
$\bar{\chi}_g^{X_A}$	2	2	0	2	0
$\chi_g^{X_A}$	2	-2	0	0	0
$\bar{\chi}_g^{X_D}$	2	2	2	0	0
$\chi_g^{X_D}$	2	-2	0	0	0

Table 23: Twisted Euler characters at $\ell = 9$, $X = A_8^3$

$[g]$	1A	2A	2B	2C	3A	6A
$n_g\|h_g$	1|1	1|4	2|1	2|2	3|3	3|12
$\bar{\chi}_g^{X_A}$	3	3	1	1	0	0
$\chi_g^{X_A}$	3	-3	1	-1	0	0

Table 24: Twisted Euler characters at $\ell = 10$, $X = A_9^2 D_6$

$[g]$	1A	2A	4AB
$n_g\|h_g$	1|1	1|2	2|2
$\bar{\chi}_g^{X_A}$	2	2	0
$\chi_g^{X_A}$	2	-2	0
$\bar{\chi}_g^{X_D}$	1	1	1
$\chi_g^{X_D}$	1	1	-1

Table 25: Twisted Euler characters at $\ell = 10+5$, $X = D_6^4$

$[g]$	1A	2A	3A	2B	4A
$n_g\|h_g$	1|1	2|2	3|1	2|1	4|4
$\bar{\chi}_g^{X_D}$	4	0	1	2	0
$\chi_g^{X_D}$	4	0	1	-2	0

Table 26: Twisted Euler characters at $\ell = 12$, $X = A_{11}D_7E_6$

$[g]$	1A	2A
$n_g\|h_g$	1|1	1|2
$\bar{\chi}_g^{X_A}$	1	1
$\chi_g^{X_A}$	1	-1
$\bar{\chi}_g^{X_D}$	1	1
$\chi_g^{X_D}$	1	-1
$\bar{\chi}_g^{X_E}$	1	1
$\chi_g^{X_E}$	1	-1

Table 27: Twisted Euler characters at $\ell = 12+4$, $X = E_6^4$

$[g]$	1A	2A	2B	4A	3A	6A	8AB
$n_g\|h_g$	1|1	1|2	2|1	2|4	3|1	3|2	4|8
$\bar{\chi}_g^{X_E}$	4	4	2	0	1	1	0
$\chi_g^{X_E}$	4	-4	0	0	1	-1	0

Table 28: Twisted Euler characters at $\ell = 13$, $X = A_{12}^2$

$[g]$	1A	2A	4AB
$n_g\|h_g$	1|1	1|4	2|8
$\bar{\chi}_g^{X_A}$	2	2	0
$\chi_g^{X_A}$	2	-2	0

Table 29: Twisted Euler characters at $\ell = 14 + 7$, $X = D_8^3$

$[g]$	1A	2A	3A
$n_g \vert h_g$	1|1	2|1	3|3
$\bar{\chi}_g^{X_D}$	3	1	0
$\chi_g^{X_D}$	3	1	0

Table 30: Twisted Euler characters at $\ell = 16$, $X = A_{15}D_9$

$[g]$	1A	2A
$n_g \vert h_g$	1|1	1|2
$\bar{\chi}_g^{X_A}$	1	1
$\chi_g^{X_A}$	1	−1
$\bar{\chi}_g^{X_D}$	1	1
$\chi_g^{X_D}$	1	−1

Table 31: Twisted Euler characters at $\ell = 18$, $X = A_{17}E_7$

$[g]$	1A	2A
$n_g \vert h_g$	1|1	1|2
$\bar{\chi}_g^{X_A}$	1	1
$\chi_g^{X_A}$	1	−1
$\bar{\chi}_g^{X_E}$	1	1

Table 32: Twisted Euler characters at $\ell = 18 + 9$, $X = D_{10}E_7^2$

$[g]$	1A	2A
$n_g \vert h_g$	1|1	2|1
$\bar{\chi}_g^{X_D}$	1	1
$\chi_g^{X_D}$	1	−1
$\bar{\chi}_g^{X_E}$	2	0

Table 33: Twisted Euler characters at $\ell = 22 + 11$, $X = D_{12}^2$

$[g]$	1A	2A
$n_g \vert h_g$	1|1	2|2
$\bar{\chi}_g^{X_D}$	2	0
$\chi_g^{X_D}$	2	0

Table 34: Twisted Euler characters at $\ell = 25$, $X = A_{24}$

$[g]$	1A	2A
$n_g \vert h_g$	1|1	1|4
$\bar{\chi}_g^{X_A}$	1	1
$\chi_g^{X_A}$	1	-1

Table 35: Twisted Euler characters at $\ell = 30 + 6, 10, 15$, $X = E_8^3$

$[g]$	1A	2A	3A
$n_g \vert h_g$	1|1	2|1	3|3
$\bar{\chi}_g^{X_E}$	3	1	0

B The Umbral McKay-Thompson Series

In this section we describe the umbral McKay-Thompson series in complete detail. In particular, we present explicit formulas for all the McKay-Thompson series attached to elements of the umbral groups by umbral moonshine in §B.3. Most of these expressions appeared first in [15, 16], but some appear for the first time in this work.

In order to facilitate explicit formulations we recall certain standard functions in §B.1. We then, using the twisted Euler characters of §A.3, explicitly describe the shadow functions of umbral moonshine in §B.2. The umbral McKay–Thompson series defined in §B.3 may also be described in terms of Rademacher sums, according to the results of [17]. We present this description in §B.4.

B.1 Special Functions

Throughout this section we assume $q := e^{2\pi i\tau}$, and $u := e^{2\pi iz}$, where $\tau, z \in \mathbb{C}$ with Im $\tau > 0$. The Dedekind eta function is $\eta(\tau) := q^{1/24}\prod_{n>0}(1-q^n)$, where . Write $\Lambda_M(\tau)$ for the function

$$\Lambda_M(\tau) := Mq\frac{\mathrm{d}}{\mathrm{d}q}\left(\log\frac{\eta(M\tau)}{\eta(\tau)}\right) = \frac{M(M-1)}{24} + M\sum_{k>0}\sum_{d|k} d\left(q^k - Mq^{Mk}\right),$$

which is a modular form of weight two for $\Gamma_0(N)$ if $M|N$.

Define the Jacobi theta function $\theta_1(\tau, z)$ by setting

$$\theta_1(\tau, z) := iq^{1/8}u^{-1/2}\sum_{n\in\mathbb{Z}}(-1)^n u^n q^{n(n-1)/2}. \tag{B.1}$$

According to the Jacobi triple product identity we have

$$\theta_1(\tau, z) = -iq^{1/8}u^{1/2}\prod_{n>0}(1-u^{-1}q^{n-1})(1-uq^n)(1-q^n). \tag{B.2}$$

The other Jacobi theta functions are

$$\begin{aligned}
\theta_2(\tau, z) &:= q^{1/8}u^{1/2}\prod_{n>0}(1+u^{-1}q^{n-1})(1+uq^n)(1-q^n),\\
\theta_3(\tau, z) &:= \prod_{n>0}(1+u^{-1}q^{n-1/2})(1+uq^{n-1/2})(1-q^n),\\
\theta_4(\tau, z) &:= \prod_{n>0}(1-u^{-1}q^{n-1/2})(1-uq^{n-1/2})(1-q^n).
\end{aligned} \tag{B.3}$$

Define $\Psi_{1,1}$ and $\Psi_{1,-1/2}$ by setting

$$\begin{aligned}
\Psi_{1,1}(\tau, z) &:= -i\frac{\theta_1(\tau, 2z)\eta(\tau)^3}{\theta_1(\tau, z)^2},\\
\Psi_{1,-1/2}(\tau, z) &:= -i\frac{\eta(\tau)^3}{\theta_1(\tau, z)}.
\end{aligned} \tag{B.4}$$

These are meromorphic Jacobi forms of weight one, with indexes 1 and $-1/2$, respectively. Here, the term meromorphic refers to the presence of simple poles in the functions $z \mapsto \Psi_{1,*}(\tau, z)$, for fixed $\tau \in \mathbb{H}$, at lattice points $z \in \mathbb{Z}\tau + \mathbb{Z}$. (Cf. §8 of [22].)

From §5 of [29] we recall the *index m theta functions*, for $m \in \mathbb{Z}$, defined by setting

$$\theta_{m,r}(\tau,z) := \sum_{k\in\mathbb{Z}} u^{2mk+r} q^{(2mk+r)^2/4m}, \tag{B.5}$$

where $r \in \mathbb{Z}$. Evidently, $\theta_{m,r}$ only depends on $r \mod 2m$. We set $S_{m,r}(\tau) := \frac{1}{2\pi i}\,\partial_z \theta_{m,r}(\tau,z)|_{z=0}$, so that

$$S_{m,r}(\tau) = \sum_{k\in\mathbb{Z}} (2mk+r) q^{(2mk+r)^2/4m}. \tag{B.6}$$

For a m a positive integer define

$$\mu_{m,0}(\tau,z) = \sum_{k\in\mathbb{Z}} u^{2km} q^{mk^2} \frac{uq^k+1}{uq^k-1} = \frac{u+1}{u-1} + O(q), \tag{B.7}$$

and observe that we recover $\Psi_{1,1}$ upon specializing (B.7) to $m = 1$. Observe also that

$$\mu_{m,0}(\tau,z+1/2) = \sum_{k\in\mathbb{Z}} u^{2km} q^{mk^2} \frac{uq^k-1}{uq^k+1} = \frac{u-1}{u+1} + O(q). \tag{B.8}$$

Define the even and odd parts of $\mu_{m,0}$ by setting

$$\mu^k_{m,0}(\tau,z) := \frac{1}{2}(\mu_{m,0}(\tau,z) + (-1)^k \mu_{m,0}(\tau,z+1/2)) \tag{B.9}$$

for $k \mod 2$.

For $m, r \in \mathbb{Z} + \frac{1}{2}$ with $m > 0$ define *half-integral index theta functions*

$$\theta_{m,r}(\tau,z) := \sum_{k\in\mathbb{Z}} e(mk+r/2) u^{2mk+r} q^{(2mk+r)^2/4m}, \tag{B.10}$$

and define also $S_{m,r}(\tau) := \frac{1}{2\pi i}\partial_z \theta_{m,r}(\tau,z)|_{z=0}$, so that

$$S_{m,r}(\tau) = \sum_{k\in\mathbb{Z}} e(mk+r/2)(2mk+r) q^{(2mk+r)^2/4m}. \tag{B.11}$$

As in the integral index case, $\theta_{m,r}$ depends only on $r \mod 2m$. We recover $-\theta_1$ upon specializing $\theta_{m,r}$ to $m = r = 1/2$.

For $m \in \mathbb{Z} + 1/2$, $m > 0$, define

$$\mu_{m,0}(\tau,z) := i \sum_{k\in\mathbb{Z}} (-1)^k u^{2mk+1/2} q^{mk^2+k/2} \frac{1}{1-uq^k} = \frac{-iu^{1/2}}{y-1} + O(q). \tag{B.12}$$

Given $\alpha \in \mathbb{Q}$ write $[\alpha]$ for the operator on q-series (in rational, possibility negative powers of q) that eliminates exponents not contained in $\mathbb{Z} + \alpha$, so that if $f = \sum_{\beta\in\mathbb{Q}} c(\beta) q^\beta$ then

$$[\alpha] f := \sum_{n\in\mathbb{Z}} c(n+\alpha) q^{n+\alpha} \tag{B.13}$$

B.2 Shadows

Let X be a Niemeier root system and let $m = m^X$ be the Coxeter number of X. For $g \in G^X$ we define the associated *shadow function* $S_g^X = (S_{g,r}^X)$ by setting

$$S_g^X := S_g^{X_A} + S_g^{X_D} + S_g^{X_E} \tag{B.14}$$

where the $S_g^{X_A}$, &c., are defined in the following way, in terms of the twisted Euler characters $\chi_g^{X_A}$, &c. given in §A.3, and the unary theta series $S_{m,r}$ (cf. (B.6)).

Note that if $m = m^X$ then $S_{g,r}^X = S_{g,r+2m}^X = -S_{g,-r}^X$ for all $g \in G^X$, so we need specify the $S_{g,r}^{X_A}$, &c., only for $0 < r < m$.

If $X_A = \emptyset$ then $S_g^{X_A} := 0$. Otherwise, we define $S_{g,r}^{X_A}$ for $0 < r < m$ by setting

$$S_{g,r}^{X_A} := \begin{cases} \chi_g^{X_A} S_{m,r} & \text{if } r = 0 \mod 2, \\ \bar{\chi}_g^{X_A} S_{m,r} & \text{if } r = 1 \mod 2. \end{cases} \tag{B.15}$$

If $X_D = \emptyset$ then $S_g^{X_D} := 0$. If $X_D \neq \emptyset$ then m is even and $m \geq 6$. If $m = 6$ then set

$$S_{g,r}^{X_D} := \begin{cases} 0 & \text{if } r = 0 \mod 2, \\ \bar{\chi}_g^{X_D} S_{6,r} + \chi_g^{X_D} S_{6,6-r} & \text{if } r = 1,5 \mod 6, \\ \check{\chi}_g^{X_D} S_{6,r} & \text{if } r = 3 \mod 6. \end{cases} \tag{B.16}$$

If $m > 6$ and $m = 2 \mod 4$ then set

$$S_{g,r}^{X_D} := \begin{cases} 0 & \text{if } r = 0 \mod 2, \\ \bar{\chi}_g^{X_D} S_{m,r} + \chi_g^{X_D} S_{m,m-r} & \text{if } r = 1 \mod 2. \end{cases} \tag{B.17}$$

If $m > 6$ and $m = 0 \mod 4$ then set

$$S_{g,r}^{X_D} := \begin{cases} \chi_g^{X_D} S_{m,m-r} & \text{if } r = 0 \mod 2, \\ \bar{\chi}_g^{X_D} S_{m,r} & \text{if } r = 1 \mod 2. \end{cases} \tag{B.18}$$

If $X_E = \emptyset$ then $S_g^{X_E} := 0$. Otherwise, m is 12 or 18 or 30. In case $m = 12$ define $S_{g,r}^{X_E}$ for $0 < r < 12$ by setting

$$S_{g,r}^{X_E} = \begin{cases} \bar{\chi}_g^{X_E}(S_{12,1} + S_{12,7}) & \text{if } r \in \{1,7\}, \\ \bar{\chi}_g^{X_E}(S_{12,5} + S_{12,11}) & \text{if } r \in \{5,11\}, \\ \chi_g^{X_E}(S_{12,4} + S_{12,8}) & \text{if } r \in \{4,8\}, \\ 0 & \text{else.} \end{cases} \tag{B.19}$$

In case $m = 18$ define $S_{g,r}^{X_E}$ for $0 < r < 18$ by setting

$$S_{g,r}^{X_E} = \begin{cases} \bar{\chi}_g^{X_E}(S_{18,r} + S_{18,18-r}) & \text{if } r \in \{1,5,7,11,13,17\}, \\ \bar{\chi}_g^{X_E} S_{18,9} & \text{if } r \in \{3,15\}, \\ \bar{\chi}_g^{X_E}(S_{18,3} + S_{18,9} + S_{18,15}) & \text{if } r = 9, \\ 0 & \text{else.} \end{cases} \tag{B.20}$$

In case $m = 30$ define $S^{X_E}_{g,r}$ for $0 < r < 30$ by setting

$$S^{X_E}_{g,r} = \begin{cases} \bar{\chi}^{X_E}_g (S_{30,1} + S_{30,11} + S_{30,19} + S_{30,29}) & \text{if } r \in \{1, 11, 19, 29\}, \\ \bar{\chi}^{X_E}_g (S_{30,7} + S_{30,13} + S_{30,17} + S_{30,23}) & \text{if } r \in \{7, 13, 17, 23\}, \\ 0 & \text{else}. \end{cases} \tag{B.21}$$

B.3 Explicit Prescriptions

Here we give explicit expressions for all the umbral McKay-Thompson series H^X_g. Most of these appeared first in [15, 16]. The expressions in §§B.3.3, B.3.4, B.3.7, B.3.14 are taken from [26]. The expressions in §§B.3.11, B.3.15, B.3.19, B.3.23 are taken from [14]. The expressions for H^X_g with $X = E_8^3$ appeared first in [25]. The expression for $H^{(6+3)}_{2B,1}$ in §B.3.6, and the expressions for $H^{(12+4)}_{4A,r}$ and $H^{(12+4)}_{8AB,r}$ in §B.3.13, appear here for the first time.

The labels for conjugacy classes in G^X are as in §A.2.

B.3.1 $\ell = 2$, $X = A_1^{24}$

We have $G^{(2)} = G^X \simeq M_{24}$ and $m^X = 2$. So for $g \in M_{24}$, the associated umbral McKay-Thompson series $H^{(2)}_g = (H^{(2)}_{g,r})$ is a 4-vector-valued function, with components indexed by $r \in \mathbb{Z}/4\mathbb{Z}$, satisfying $H^{(2)}_{g,r} = -H^{(2)}_{g,-r}$, and in particular, $H^{(2)}_{g,r} = 0$ for $r = 0 \mod 2$. So it suffices to specify the $H^{(2)}_{g,1}$ explicitly.

Define $H^{(2)}_g = (H^{(2)}_{g,r})$ for $g = e$ by requiring that

$$-2\Psi_{1,1}(\tau, z)\varphi^{(2)}_1(\tau, z) = -24\mu_{2,0}(\tau, z) + \sum_{r \mod 4} H^{(2)}_{e,r}(\tau)\theta_{2,r}(\tau, z), \tag{B.22}$$

where

$$\varphi^{(2)}_1(\tau, z) := 4\left(\frac{\theta_2(\tau, z)^2}{\theta_2(\tau, 0)^2} + \frac{\theta_3(\tau, z)^2}{\theta_3(\tau, 0)^2} + \frac{\theta_4(\tau, z)^2}{\theta_4(\tau, 0)^2}\right). \tag{B.23}$$

More generally, for $g \in G^{(2)}$ define

$$H^{(2)}_{g,1}(\tau) := \frac{\bar{\chi}^{(2)}_g}{24} H^{(2)}_{e,1}(\tau) - F^{(2)}_g(\tau)\frac{1}{S_{2,1}(\tau)}, \tag{B.24}$$

where $\bar{\chi}^{(2)}_g$ and $F^{(2)}_g$ are as specified in Table 36. Note that $\bar{\chi}^{(2)}_g = \bar{\chi}^{X_A}_g$, the latter appearing in Table 15. Also, $S_{2,1}(\tau) = \eta(\tau)^3$.

The functions $f_{23,a}$ and $f_{23,b}$ in Table 36 are cusp forms of weight two for $\Gamma_0(23)$, defined by

$$\begin{aligned} f_{23,a}(\tau) &:= \frac{\eta(\tau)^3\eta(23\tau)^3}{\eta(2\tau)\eta(46\tau)} + 3\eta(\tau)^2\eta(23\tau)^2 + 4\eta(\tau)\eta(2\tau)\eta(23\tau)\eta(46\tau) + 4\eta(2\tau)^2\eta(46\tau)^2, \\ f_{23,b}(\tau) &:= \eta(\tau)^2\eta(23\tau)^2. \end{aligned} \tag{B.25}$$

Note that the definition of $F^{(2)}_g$ appearing here for $g \in 23A \cup 23B$ corrects errors in [11, 12].

Table 36: Character Values and Weight Two Forms for $\ell = 2$, $X = A_1^{24}$

$[g]$	$\bar{\chi}_g^{(2)}$	$F_g^{(2)}(\tau)$
$1A$	24	0
$2A$	8	$16\Lambda_2(\tau)$
$2B$	0	$2\eta(\tau)^8\eta(2\tau)^{-4}$
$3A$	6	$6\Lambda_3(\tau)$
$3B$	0	$2\eta(\tau)^6\eta(3\tau)^{-2}$
$4A$	0	$2\eta(2\tau)^8\eta(4\tau)^{-4}$
$4B$	4	$4(-\Lambda_2(\tau)+\Lambda_4(\tau))$
$4C$	0	$2\eta(\tau)^4\eta(2\tau)^2\eta(4\tau)^{-2}$
$5A$	4	$2\Lambda_5(\tau)$
$6A$	2	$2(-\Lambda_2(\tau)-\Lambda_3(\tau)+\Lambda_6(\tau))$
$6B$	0	$2\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^{-2}$
$7AB$	3	$\Lambda_7(\tau)$
$8A$	2	$-\Lambda_4(\tau)+\Lambda_8(\tau)$
$10A$	0	$2\eta(\tau)^3\eta(2\tau)\eta(5\tau)\eta(10\tau)^{-1}$
$11A$	2	$2(\Lambda_{11}(\tau)-11\eta(\tau)^2\eta(11\tau)^2)/5$
$12A$	0	$2\eta(\tau)^3\eta(4\tau)^2\eta(6\tau)^3\eta(2\tau)^{-1}\eta(3\tau)^{-1}\eta(12\tau)^{-2}$
$12B$	0	$2\eta(\tau)^4\eta(4\tau)\eta(6\tau)\eta(2\tau)^{-1}\eta(12\tau)^{-1}$
$14AB$	1	$(-\Lambda_2(\tau)-\Lambda_7(\tau)+\Lambda_{14}(\tau)-14\eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau))/3$
$15AB$	1	$(-\Lambda_3(\tau)-\Lambda_5(\tau)+\Lambda_{15}(\tau)-15\eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau))/4$
$21AB$	0	$(7\eta(\tau)^3\eta(7\tau)^3\eta(3\tau)^{-1}\eta(21\tau)^{-1}-\eta(\tau)^6\eta(3\tau)^{-2})/3$
$23AB$	1	$(\Lambda_{23}(\tau)-23f_{23,a}(\tau)-69f_{23,b}(\tau))/11$

B.3.2 $\ell = 3$, $X = A_2^{12}$

We have $G^{(3)} = G^X \simeq 2.M_{12}$ and $m^X = 3$. So for $g \in 2.M_{12}$, the associated umbral McKay-Thompson series $H_g^{(3)} = (H_{g,r}^{(3)})$ is a 6-vector-valued function, with components indexed by $r \in \mathbb{Z}/6\mathbb{Z}$, satisfying $H_{g,r}^{(3)} = -H_{g,-r}^{(3)}$, and in particular, $H_{g,r}^{(3)} = 0$ for $r = 0 \mod 3$. So it suffices to specify the $H_{g,1}^{(3)}$ and $H_{g,2}^{(3)}$ explicitly.

Define $H_g^{(3)} = (H_{g,r}^{(3)})$ for $g = e$ by requiring that

$$-2\Psi_{1,1}(\tau,z)\varphi_1^{(3)}(\tau,z) = -12\mu_{3,0}(\tau,z) + \sum_{r \mod 6} H_{e,r}^{(3)}(\tau)\theta_{3,r}(\tau,z), \tag{B.26}$$

where

$$\varphi_1^{(3)}(\tau,z) := 2\left(\frac{\theta_3(\tau,z)^2}{\theta_3(\tau,0)^2}\frac{\theta_4(\tau,z)^2}{\theta_4(\tau,0)^2} + \frac{\theta_4(\tau,z)^2}{\theta_4(\tau,0)^2}\frac{\theta_2(\tau,z)^2}{\theta_2(\tau,0)^2} + \frac{\theta_2(\tau,z)^2}{\theta_2(\tau,0)^2}\frac{\theta_3(\tau,z)^2}{\theta_3(\tau,0)^2}\right). \tag{B.27}$$

More generally, for $g \in G^{(3)}$ define

$$H^{(3)}_{g,1}(\tau) := \frac{\bar{\chi}^{(3)}_g}{12} H^{(3)}_{e,1}(\tau) + \frac{1}{2}\left(F^{(3)}_g + F^{(3)}_{zg}\right)\frac{1}{S_{3,1}(\tau)}, \tag{B.28}$$

$$H^{(3)}_{g,2}(\tau) := \frac{\chi^{(3)}_g}{12} H^{(3)}_{e,1}(\tau) + \frac{1}{2}\left(F^{(3)}_g - F^{(3)}_{zg}\right)\frac{1}{S_{3,2}(\tau)}, \tag{B.29}$$

where $\chi^{(3)}_g$ and $F^{(3)}_g$ are as specified in Table 37, and z is the non-trivial central element of $G^{(3)}$. The action of $g \mapsto zg$ on conjugacy classes can be read off Table 37, for the horizontal lines indicate the sets $[g] \cup [zg]$.

Note the eta product identities, $S_{3,1}(\tau) = \eta(2\tau)^5/\eta(4\tau)^2$, and $S_{3,2}(\tau) = 2\eta(\tau)^2\eta(4\tau)^2/\eta(2\tau)$. Note also that $\bar{\chi}^{(3)}_g = \bar{\chi}^{X_A}_g$ and $\chi^{(3)}_g = \chi^{X_A}_g$, the latter appearing in Table 16.

Table 37: Character Values and Weight Two Forms for $\ell = 3$, $X = A_2^{12}$

$[g]$	$\bar{\chi}^{(3)}_g$	$\chi^{(3)}_g$	$F^{(3)}_g(\tau)$
1A	12	12	0
2A	12	−12	0
4A	0	0	$-2\eta(\tau)^4\eta(2\tau)^2/\eta(4\tau)^2$
2B	4	4	$-16\Lambda_2(\tau)$
2C	4	−4	$16\Lambda_2(\tau) - \frac{16}{3}\Lambda_4(\tau)$
3A	3	3	$-6\Lambda_3(\tau)$
6A	3	−3	$-9\Lambda_2(\tau) - 2\Lambda_3(\tau) + 3\Lambda_4(\tau) + 3\Lambda_6(\tau) - \Lambda_{12}(\tau)$
3B	0	0	$8\Lambda_3(\tau) - 2\Lambda_9(\tau) + 2\,\eta^6(\tau)/\eta^2(3\tau)$
6B	0	0	$-2\eta(\tau)^5\eta(3\tau)/\eta(2\tau)\eta(6\tau)$
4B	0	0	$-2\eta(2\tau)^8/\eta(4\tau)^4$
4C	4	0	$-8\Lambda_4(\tau)/3$
5A	2	2	$-2\Lambda_5(\tau)$
10A	2	−2	$\sum_{d\mid 20} c_{10A}(d)\Lambda_d(\tau) + \frac{20}{3}\eta(2\tau)^2\eta(10\tau)^2$
12A	0	0	$-2\eta(\tau)\eta(2\tau)^5\eta(3\tau)/\eta(4\tau)^2\eta(6\tau)$
6C	1	1	$2(\Lambda_2(\tau) + \Lambda_3(\tau) - \Lambda_6(\tau))$
6D	1	−1	$-5\Lambda_2(\tau) - 2\Lambda_3(\tau) + \frac{5}{3}\Lambda_4(\tau) + 3\Lambda_6(\tau) - \Lambda_{12}(\tau)$
8AB	0	0	$-2\eta(2\tau)^4\eta(4\tau)^2/\eta(8\tau)^2$
8CD	2	0	$-2\Lambda_2(\tau) + \frac{5}{3}\Lambda_4(\tau) - \Lambda_8(\tau)$
20AB	0	0	$-2\eta(2\tau)^7\eta(5\tau)/\eta(\tau)\eta(4\tau)^2\eta(10\tau)$
11AB	1	1	$-\frac{2}{5}\Lambda_{11}(\tau) - \frac{33}{5}\eta(\tau)^2\eta(11\tau)^2$
22AB	1	−1	$\sum_{d\mid 44} c_g(d)\Lambda_d(\tau) - \frac{11}{5}\sum_{d\mid 4} c'_g(d)\eta(d\tau)^2\eta(11d\tau)^2 + \frac{22}{3}f_{44}(\tau)$

The function f_{44} is the unique new cusp form of weight 2 for $\Gamma_0(44)$, normalized so that $f_{44}(\tau) =$

$q + O(q^3)$ as $\Im(\tau) \to \infty$. The coefficients $c_g(d)$ and $c'_g(d)$ for $g \in 10A \cup 22A \cup 22B$ are given by

$$c_{10A}(2) = -5,\ c_{10A}(4) = -\frac{5}{3},\ c_{10A}(5) = -\frac{2}{3},\ c_{10A}(10) = 1,\ c_{10A}(20) = -\frac{1}{3}, \tag{B.30}$$

$$c_{22AB}(2) = -\frac{11}{5},\ c_{22AB}(4) = \frac{11}{5},\ c_{22AB}(11) = -\frac{2}{15},\ c_{22AB}(22) = \frac{1}{5},\ c_{22AB}(44) = -\frac{1}{15}, \tag{B.31}$$

$$c'_{22AB}(1) = 1,\ c'_{22AB}(2) = 4,\ c'_{22AB}(4) = 8. \tag{B.32}$$

B.3.3 $\ell = 4$, $X = A_3^8$

We have $m^X = 4$, so the umbral McKay-Thompson series $H_g^{(4)} = (H_{g,r}^{(4)})$ associated to $g \in G^{(4)}$ is an 8-vector-valued function, with components indexed by $r \in \mathbb{Z}/8\mathbb{Z}$.

Define $H_g^{(4)} = (H_{g,r}^{(4)})$ for $g \in G^{(4)}$, $g \notin 4C$, by requiring that

$$\psi_g^{(4)}(\tau, z) = -\chi_g^{(4)}\mu_{4,0}^0(\tau, z) - \bar{\chi}_g^{(4)}\mu_{4,0}^1(\tau, z) + \sum_{r \mod 8} H_{g,r}^{(4)}(\tau)\theta_{4,r}(\tau, z), \tag{B.33}$$

where $\chi_g^{(4)} := \chi_g^{X_A}$ and $\bar{\chi}_g^{(4)} := \bar{\chi}_g^{X_A}$ (cf. Table 17), and the $\psi_g^{(4)}$ are meromorphic Jacobi forms of weight 1 and index 4 given explicitly in Table 38.

Table 38: Character Values and Meromorphic Jacobi Forms for $\ell = 4$, $X = A_3^8$

$[g]$	$\chi_g^{(4)}$	$\bar{\chi}_g^{(4)}$	$\psi_g^{(4)}(\tau, z)$
1A	8	8	$2i\theta_1(\tau, 2z)^3\theta_1(\tau, z)^{-4}\eta(\tau)^3$
2A	-8	8	$2i\theta_1(\tau, 2z)^3\theta_2(\tau, z)^{-4}\eta(\tau)^3$
2B	0	0	$-2i\theta_1(\tau, 2z)^3\theta_1(\tau, z)^{-2}\theta_2(\tau, z)^{-2}\eta(\tau)^3$
4A	0	0	$-2i\theta_1(\tau, 2z)\theta_2(\tau, 2z)^2\theta_2(2\tau, 2z)^{-2}\eta(2\tau)^2\eta(\tau)^{-1}$
4B	0	0	$-2i\theta_1(2\tau, 2z)\theta_3(2\tau, 2z)^2\theta_4(2\tau, 2z)\eta(2\tau)^2\eta(\tau)^{-2}\eta(4\tau)^{-2}$
2C	0	4	$2i\theta_1(\tau, 2z)\theta_2(\tau, 2z)^2\theta_1(\tau, z)^{-2}\theta_2(\tau, z)^{-2}\eta(\tau)^3$
3A	2	2	$2i\theta_1(3\tau, 6z)\theta_1(\tau, z)^{-1}\theta_1(3\tau, 3z)^{-1}\eta(\tau)^3$
6A	-2	2	$-2i\theta_1(3\tau, 6z)\theta_2(\tau, z)^{-1}\theta_2(3\tau, 3z)^{-1}\eta(\tau)^3$
6BC	0	0	cf. (B.34)
8A	0	0	$-2i\theta_1(\tau, 2z)\theta_2(2\tau, 4z)\theta_2(4\tau, 4z)^{-1}\eta(\tau)\eta(4\tau)\eta(2\tau)^{-1}$
4C	0	2	$2i\theta_1(\tau, 2z)\theta_2(2\tau, 4z)\theta_1(2\tau, 2z)^{-2}\eta(2\tau)^7\eta(\tau)^{-3}\eta(4\tau)^{-2}$
7AB	1	1	cf. (B.34)
14AB	-1	1	cf. (B.34)

$$\psi^{(4)}_{6BC} := \left(\theta_1(\tau,z+\tfrac{1}{3})\theta_1(\tau,z+\tfrac{1}{6})-\theta_1(\tau,z-\tfrac{1}{3})\theta_1(\tau,z-\tfrac{1}{6})\right)\frac{-i\theta_1(3\tau,6z)}{\theta_1(3\tau,3z)\theta_2(3\tau,3z)}\eta(3\tau)$$
$$\psi^{(4)}_{7AB} := \left(\prod_{j=1}^{3}\theta_1(\tau,2z+\tfrac{j^2}{7})\theta_1(\tau,z-\tfrac{j^2}{7})+\prod_{j=1}^{3}\theta_1(\tau,2z-\tfrac{j^2}{7})\theta_1(\tau,z+\tfrac{j^2}{7})\right)\frac{-i}{\theta_1(7\tau,7z)}\frac{\eta(7\tau)}{\eta(\tau)^4}$$
$$\psi^{(4)}_{14AB} := \left(\prod_{j=1}^{3}\theta_1(\tau,2z+\tfrac{j^2}{7})\theta_2(\tau,z-\tfrac{j^2}{7})+\prod_{j=1}^{3}\theta_1(\tau,2z-\tfrac{j^2}{7})\theta_2(\tau,z+\tfrac{j^2}{7})\right)\frac{i}{\theta_2(7\tau,7z)}\frac{\eta(7\tau)}{\eta(\tau)^4} \tag{B.34}$$

For use later on, note that $\psi^{(4)}_{1A} = -2\Psi_{1,1}\varphi^{(4)}_1$, where

$$\varphi^{(4)}_1(\tau,z) := \frac{\theta_1(\tau,2z)^2}{\theta_1(\tau,z)^2}. \tag{B.35}$$

B.3.4 $\ell = 5$, $X = A_4^6$

We have $m^X = 5$, so the umbral McKay-Thompson series $H^{(5)}_g = (H^{(5)}_{g,r})$ associated to $g \in G^{(5)}$ is a 10-vector-valued function, with components indexed by $r \in \mathbb{Z}/10\mathbb{Z}$.

Define $H^{(5)}_g = (H^{(5)}_{g,r})$ for $g \in G^{(5)}$, $g \notin 5A \cup 10A$, by requiring that

$$\psi^{(5)}_g(\tau,z) = -\chi^{(5)}_g\mu^0_{5,0}(\tau,z) - \bar{\chi}^{(5)}_g\mu^1_{5,0}(\tau,z) + \sum_{r \bmod 10} H^{(5)}_{g,r}(\tau)\theta_{5,r}(\tau,z), \tag{B.36}$$

where $\chi^{(5)}_g := \chi^{X_A}_g$ and $\bar{\chi}^{(5)}_g := \bar{\chi}^{X_A}_g$ (cf. Table 18), and the $\psi^{(5)}_g$ are meromorphic Jacobi forms of weight 1 and index 5 given explicitly in Table 39.

Table 39: Character Values and Meromorphic Jacobi Forms for $\ell = 5$, $X = A_4^6$

$[g]$	$\chi^{(5)}_g$	$\bar{\chi}^{(5)}_g$	$\psi^{(5)}_g(\tau,z)$
1A	6	6	$2i\theta_1(\tau,2z)\theta_1(\tau,3z)\theta_1(\tau,z)^{-3}\eta(\tau)^3$
2A	-6	6	$-2i\theta_1(\tau,2z)\theta_2(\tau,3z)\theta_2(\tau,z)^{-3}\eta(\tau)^3$
2B	-2	2	$-2i\theta_1(\tau,2z)\theta_1(\tau,3z)\theta_1(\tau,z)^{-1}\theta_2(\tau,z)^{-2}\eta(\tau)^3$
2C	2	2	$2i\theta_1(\tau,2z)\theta_2(\tau,3z)\theta_1(\tau,z)^{-2}\theta_2(\tau,z)^{-1}\eta(\tau)^3$
3A	0	0	$-2i\theta_1(\tau,2z)\theta_1(\tau,3z)\theta_1(3\tau,3z)^{-1}\eta(3\tau)$
6A	0	0	$-2i\theta_1(\tau,2z)\theta_2(\tau,3z)\theta_2(3\tau,3z)^{-1}\eta(3\tau)$
4AB	0	0	cf. (B.37)
4CD	0	2	cf. (B.37)
12AB	0	0	cf. (B.37)

$$\begin{aligned}\psi^{(5)}_{4AB}(\tau,z) &:= -i\theta_2(\tau,2z)\frac{\theta_1(\tau,z+\frac14)\theta_1(\tau,3z+\frac14)-\theta_1(\tau,z-\frac14)\theta_1(\tau,3z-\frac14)}{\theta_2(2\tau,2z)^2}\frac{\eta(2\tau)^2}{\eta(\tau)}\\ \psi^{(5)}_{4CD}(\tau,z) &:= -i\theta_2(\tau,2z)\frac{\theta_1(\tau,z+\frac14)\theta_1(\tau,3z-\frac14)+\theta_1(\tau,z-\frac14)\theta_1(\tau,3z+\frac14)}{\theta_1(2\tau,2z)\theta_2(2\tau,2z)}\frac{\eta(2\tau)^2}{\eta(\tau)}\\ \psi^{(5)}_{12AB}(\tau,z) &:= i\frac{\theta_2(\tau,2z)}{\theta_2(6\tau,6z)}\left(\theta_1(\tau,z+\tfrac{1}{12})\theta_1(\tau,z+\tfrac14)\theta_1(\tau,z+\tfrac{5}{12})\theta_1(\tau,3z-\tfrac14)\right.\\ &\qquad\left.-\,\theta_1(\tau,z-\tfrac{1}{12})\theta_1(\tau,z-\tfrac14)\theta_1(\tau,z-\tfrac{5}{12})\theta_1(\tau,3z+\tfrac14)\right)\frac{\eta(6\tau)}{\eta(\tau)^3}\end{aligned} \tag{B.37}$$

For $g\in 5A$ use the formulas of §B.3.20 to define

$$H^{(5)}_{5A,r}(\tau) := H^{(25)}_{1A,r}(\tau/5) - H^{(25)}_{1A,10-r}(\tau/5) + H^{(25)}_{1A,10+r}(\tau/5) - H^{(25)}_{1A,20-r}(\tau/5) + H^{(25)}_{1A,20+r}(\tau/5). \tag{B.38}$$

For $g\in 10A$ set $H^{(5)}_{10A,r}(\tau) := -(-1)^r H^{(5)}_{5A,r}(\tau)$.

For use later on we note that $\psi^{(5)}_{1A} = -2\Psi_{1,1}\varphi^{(5)}_1$, where

$$\varphi^{(5)}_1(\tau,z) := \frac{\theta_1(\tau,3z)}{\theta_1(\tau,z)}. \tag{B.39}$$

B.3.5 $\ell = 6$, $X = A_5^4D_4$

We have $m^X = 6$, so the umbral McKay-Thompson series $H^{(6)}_g = (H^{(6)}_{g,r})$ associated to $g\in G^{(6)}$ is a 12-vector-valued function with components indexed by $r\in\mathbb{Z}/12\mathbb{Z}$. We have $H^{(6)}_{g,r} = -H^{(6)}_{g,-r}$, so it suffices to specify the $H^{(6)}_{g,r}$ for $r\in\{1,2,3,4,5\}$.

To define $H^{(6)}_g = (H^{(6)}_{g,r})$ for $g=e$, first define $h(\tau) = (h_r(\tau))$ by requiring that

$$-2\Psi_{1,1}(\tau,z)\varphi^{(6)}_1(\tau,z) = -24\mu_{6,0}(\tau,z) + \sum_{r \bmod 12} h_r(\tau)\theta_{6,r}(\tau,z), \tag{B.40}$$

where

$$\varphi^{(6)}_1(\tau,z) := \varphi^{(2)}_1(\tau,z)\varphi^{(5)}_1(\tau,z) - \varphi^{(3)}_1(\tau,z)\varphi^{(4)}_1(\tau,z). \tag{B.41}$$

(Cf. (B.23), (B.27), (B.35), (B.39).) Now define the $H^{(6)}_{1A,r}$ by setting

$$\begin{aligned} H^{(6)}_{1A,1}(\tau) &:= \frac{1}{24}\left(5h_1(\tau)+h_5(\tau)\right),\\ H^{(6)}_{1A,2}(\tau) &:= \frac16 h_2(\tau),\\ H^{(6)}_{1A,3}(\tau) &:= \frac14 h_3(\tau),\\ H^{(6)}_{1A,4}(\tau) &:= \frac16 h_4(\tau),\\ H^{(6)}_{1A,5}(\tau) &:= \frac{1}{24}\left(h_1(\tau)+5h_5(\tau)\right). \end{aligned} \tag{B.42}$$

Define $H^{(6)}_{2A,r}$ by requiring

$$H^{(6)}_{2A,r}(\tau) := -(-1)^r H^{(6)}_{1A,r}(\tau). \tag{B.43}$$

For the remaining g, recall (B.13). The $H^{(6)}_{g,r}$ for $g \notin 1A \cup 2A$ are defined as follows for $r = 2$ and $r = 4$, noting that $H^{(3)}_{g,4} = H^{(3)}_{g,-2} = -H^{(3)}_{g,2}$.

$$\begin{aligned} H^{(6)}_{2B,r}(\tau) &:= [-\tfrac{r^2}{24}]H^{(3)}_{4C,r}(\tau/2) \\ H^{(6)}_{4A,r}(\tau) &:= [-\tfrac{r^2}{24}]H^{(3)}_{4B,r}(\tau/2) \\ H^{(6)}_{3A,r}(\tau) &:= [-\tfrac{r^2}{24}]H^{(3)}_{6C,r}(\tau/2) \\ H^{(6)}_{6A,r}(\tau) &:= [-\tfrac{r^2}{24}]H^{(3)}_{6D,r}(\tau/2) \\ H^{(6)}_{8AB,r}(\tau) &:= [-\tfrac{r^2}{24}]H^{(3)}_{8CD,r}(\tau/2) \end{aligned} \tag{B.44}$$

For the $H^{(6)}_{g,3}$ we define

$$\begin{aligned} H^{(6)}_{2B,3}(\tau), H^{(6)}_{4A,3}(\tau) &:= -[-\tfrac{9}{24}]H^{(2)}_{6A,1}(\tau/3), \\ H^{(6)}_{3A,3}(\tau), H^{(6)}_{6A,3}(\tau) &:= 0, \\ H^{(6)}_{8AB,3}(\tau) &:= -[-\tfrac{9}{24}]H^{(2)}_{12A,1}(\tau/3). \end{aligned} \tag{B.45}$$

Noting that $H^{(2)}_{g,5} = H^{(2)}_{g,1}$ and $H^{(3)}_{g,5} = -H^{(3)}_{g,1}$, the $H^{(6)}_{g,1}$ and $H^{(6)}_{g,5}$ are defined for $o(g) \neq 0 \mod 3$ by setting

$$\begin{aligned} H^{(6)}_{2B,r}(\tau) &:= [-\tfrac{1}{24}]\frac{1}{2}\left(H^{(2)}_{6A,r}(\tau/3) + H^{(3)}_{4C,r}(\tau/2)\right) \\ H^{(6)}_{4A,r}(\tau) &:= [-\tfrac{1}{24}]\frac{1}{2}\left(H^{(2)}_{6A,r}(\tau/3) + H^{(3)}_{4B,r}(\tau/2)\right) \\ H^{(6)}_{8AB,r}(\tau) &:= [-\tfrac{1}{24}]\frac{1}{2}\left(H^{(2)}_{12A,r}(\tau/3) + H^{(3)}_{8CD,r}(\tau/2)\right) \end{aligned} \tag{B.46}$$

It remains to specify the $H^{(6)}_{g,r}$ when $g \in 3A \cup 6A$ and r is 1 or 5. These cases are determined by using the formulas of §B.3.17 to set

$$\begin{aligned} H^{(6)}_{3A,1}(\tau), H^{(6)}_{6A,1}(\tau) &:= H^{(18)}_{1A,1}(3\tau) - H^{(18)}_{1A,11}(3\tau) + H^{(18)}_{1A,13}(3\tau), \\ H^{(6)}_{3A,5}(\tau), H^{(6)}_{6A,5}(\tau) &:= H^{(18)}_{1A,5}(3\tau) - H^{(18)}_{1A,7}(3\tau) + H^{(18)}_{1A,17}(3\tau). \end{aligned} \tag{B.47}$$

B.3.6 $\ell = 6 + 3$, $X = D_4^6$

We have $m^X = 6$, so the umbral McKay-Thompson series $H^{(6+3)}_g = (H^{(6+3)}_{g,r})$ associated to $g \in G^{(6+3)}$ is a 12-vector-valued function with components indexed by $r \in \mathbb{Z}/12\mathbb{Z}$. In addition to the identity $H^{(6+3)}_{g,r} = -H^{(6+3)}_{g,-r}$, we have $H^{(6+3)}_{g,r} = 0$ for $r = 0 \mod 2$. Thus it suffices to specify the $H^{(6+3)}_{g,r}$ for $r \in \{1, 3, 5\}$.

Recall (B.13). For $r = 1$, define

$$\begin{aligned}
H^{(6+3)}_{1A,1}(\tau), H^{(6+3)}_{3A,1}(\tau) &:= H^{(6)}_{1A,1}(\tau) + H^{(6)}_{1A,5}(\tau),\\
H^{(6+3)}_{2A,1}(\tau), H^{(6+3)}_{6A,1}(\tau) &:= H^{(6)}_{2B,1}(\tau) + H^{(6)}_{2B,5}(\tau),\\
H^{(6+3)}_{3B,1}(\tau) &:= H^{(6)}_{3A,1}(\tau) + H^{(6)}_{3A,5}(\tau),\\
H^{(6+3)}_{3C,1}(\tau) &:= -2\frac{\eta(\tau)^2}{\eta(3\tau)},\\
H^{(6+3)}_{4A,1}(\tau), H^{(6+3)}_{12A,1}(\tau) &:= H^{(6)}_{8AB,1}(\tau) + H^{(6)}_{8AB,5}(\tau),\\
H^{(6+3)}_{5A,1}(\tau), H^{(6+3)}_{15A,1}(\tau) &:= [-\tfrac{1}{24}]H^{(2)}_{15AB,1}(\tau/3),\\
H^{(6+3)}_{2C,1}(\tau) &:= H^{(6)}_{4A,1}(\tau) - H^{(6)}_{4A,5}(\tau),\\
H^{(6+3)}_{4B,1}(\tau) &:= H^{(6)}_{8AB,1}(\tau) - H^{(6)}_{8AB,5}(\tau),\\
H^{(6+3)}_{6B,1}(\tau) &:= H^{(6)}_{6A,1}(\tau) - H^{(6)}_{6A,5}(\tau),\\
H^{(6+3)}_{6C,1}(\tau) &:= -2\frac{\eta(2\tau)\,\eta(3\tau)}{\eta(6\tau)}.
\end{aligned} \tag{B.48}$$

Then define $H^{(6+3)}_{2B,1}$ by setting

$$H^{(6+3)}_{2B,1}(\tau) := 2H^{(6+3)}_{4B,1}(\tau) + 2\frac{\eta(\tau)^3}{\eta(2\tau)^2}. \tag{B.49}$$

For $r = 3$ set

$$\begin{aligned}
H^{(6+3)}_{1A,3}(\tau) &:= 2H^{(6)}_{1A,3}(\tau),\\
H^{(6+3)}_{3A,3}(\tau) &:= -H^{(6)}_{1A,3}(\tau),\\
H^{(6+3)}_{2A,3}(\tau) &:= 2H^{(6)}_{2B,3}(\tau),\\
H^{(6+3)}_{6A,3}(\tau) &:= -H^{(6)}_{2B,3}(\tau),\\
H^{(6+3)}_{4A,3}(\tau) &:= 2H^{(6)}_{8AB,3}(\tau),\\
H^{(6+3)}_{12A,3}(\tau) &:= -H^{(6)}_{8AB,3}(\tau),\\
H^{(6+3)}_{5A,3}(\tau) &:= -2[-\tfrac{9}{24}]H^{(2)}_{15AB,1}(\tau),\\
H^{(6+3)}_{15A,3}(\tau) &:= [-\tfrac{9}{24}]H^{(2)}_{15AB,1}(\tau),
\end{aligned} \tag{B.50}$$

and

$$H^{(6+3)}_{3B,3}(\tau), H^{(6+3)}_{3C,3}(\tau), H^{(6+3)}_{2B,3}(\tau), H^{(6+3)}_{2C,3}(\tau), H^{(6+3)}_{4B,3}(\tau), H^{(6+3)}_{6B,3}(\tau), H^{(6+3)}_{6C,3}(\tau) := 0. \tag{B.51}$$

For $r = 5$ define $H^{(6+3)}_{g,5}(\tau) := H^{(6+3)}_{g,1}(\tau)$ for $[g] \in \{1A, 3A, 2A, 6A, 3B, 3C, 4A, 12A, 5A, 15AB\}$, and set $H^{(6+3)}_{g,5}(\tau) := -H^{(6+3)}_{g,1}(\tau)$ for the remaining cases, $[g] \in \{2B, 2C, 4B, 6B, 6C\}$.

B.3.7 $\ell = 7$, $X = A_6^4$

We have $m^X = 7$, so the umbral McKay-Thompson series $H_g^{(7)} = (H_{g,r}^{(7)})$ associated to $g \in G^{(7)} = G^X \simeq SL_2(3)$ is a 14-vector-valued function, with components indexed by $r \in \mathbb{Z}/14\mathbb{Z}$.

Define $H_g^{(7)} = (H_{g,r}^{(7)})$ for $g \in G^{(7)}$ by requiring that

$$\psi_g^{(7)}(\tau, z) = -\chi_g^{(7)}\mu_{7,0}^0(\tau, z) - \bar{\chi}_g^{(7)}\mu_{7,0}^1(\tau, z) + \sum_{r \mod 14} H_{g,r}^{(7)}(\tau)\theta_{7,r}(\tau, z), \tag{B.52}$$

where $\chi_g^{(7)} := \chi_g^{X_A}$ and $\bar{\chi}_g^{(7)} := \bar{\chi}_g^{X_A}$ (cf. Table 21), and the $\psi_g^{(7)}$ are meromorphic Jacobi forms of weight 1 and index 7 given explicitly in Table 40.

Table 40: Character Values and Meromorphic Jacobi Forms for $\ell = 7$, $X = A_6^4$

$[g]$	$\chi_g^{(7)}$	$\bar{\chi}_g^{(7)}$	$\psi_g^{(7)}(\tau, z)$
1A	4	4	$2i\theta_1(\tau, 4z)\theta_1(\tau, z)^{-2}\eta(\tau)^3$
2A	−4	4	$-2i\theta_1(\tau, 4z)\theta_2(\tau, z)^{-2}\eta(\tau)^3$
4A	0	0	$-2i\theta_1(\tau, 4z)\theta_2(2\tau, 2z)^{-1}\eta(2\tau)\eta(\tau)$
3A	1	1	cf. (B.53)
6A	−1	1	cf. (B.53)

$$\begin{aligned}
\psi_{3A}^{(7)}(\tau, z) &:= -i\frac{\theta_1(\tau, 4z + \frac{1}{3})\theta_1(\tau, z - \frac{1}{3}) + \theta_1(\tau, 4z - \frac{1}{3})\theta_1(\tau, z + \frac{1}{3})}{\theta_1(3\tau, 3z)}\eta(3\tau) \\
\psi_{6A}^{(7)}(\tau, z) &:= -i\frac{\theta_1(\tau, 4z + \frac{1}{3})\theta_1(\tau, z - \frac{1}{6}) - \theta_1(\tau, 4z - \frac{1}{3})\theta_1(\tau, z + \frac{1}{6})}{\theta_2(3\tau, 3z)}\eta(3\tau)
\end{aligned} \tag{B.53}$$

For use later on we note that $\psi_{1A}^{(7)} = -2\Psi_{1,1}\varphi_1^{(7)}$, where

$$\varphi_1^{(7)}(\tau, z) := \frac{\theta_1(\tau, 4z)}{\theta_1(\tau, 2z)}. \tag{B.54}$$

B.3.8 $\ell = 8$, $X = A_7^2 D_5^2$

We have $m^X = 8$, so the umbral McKay-Thompson series $H_g^{(8)} = (H_{g,r}^{(8)})$ associated to $g \in G^{(8)}$ is a 16-vector-valued function with components indexed by $r \in \mathbb{Z}/16\mathbb{Z}$. We have $H_{g,r}^{(8)} = -H_{g,-r}^{(8)}$, so it suffices to specify the $H_{g,r}^{(8)}$ for $r \in \{1, 2, 3, 4, 5, 6, 7\}$.

To define $H_g^{(8)} = (H_{g,r}^{(8)})$ for $g = e$, first define $h(\tau) = (h_r(\tau))$ by requiring that

$$-2\Psi_{1,1}(\tau, z)\left(\varphi_1^{(8)}(\tau, z) + \frac{1}{2}\varphi_2^{(8)}(\tau, z)\right) = -24\mu_{8,0}(\tau, z) + \sum_{r \mod 16} h_r(\tau)\theta_{8,r}(\tau, z), \tag{B.55}$$

where

$$\begin{aligned}
\varphi_1^{(8)}(\tau, z) &:= \varphi_1^{(3)}(\tau, z)\varphi_1^{(6)}(\tau, z) - 5\varphi_1^{(4)}(\tau, z)\varphi_1^{(5)}(\tau, z), \\
\varphi_2^{(8)}(\tau, z) &:= \varphi_1^{(4)}(\tau, z)\varphi_1^{(5)}(\tau, z) - \varphi_1^{(8)}(\tau, z).
\end{aligned} \tag{B.56}$$

(Cf. (B.27), (B.35), (B.39), (B.41).) Now define the $H^{(8)}_{1A,r}$ by setting

$$H^{(8)}_{1A,r}(\tau) := \frac{1}{6}h_r(\tau), \tag{B.57}$$

for $r \in \{1,3,4,5,7\}$, and

$$H^{(8)}_{1A,2}(\tau), H^{(8)}_{1A,6}(\tau) := \frac{1}{12}\left(h_2(\tau)+h_6(\tau)\right). \tag{B.58}$$

Define $H^{(8)}_{2A,r}$ for $1 \leq r \leq 7$ by requiring

$$H^{(8)}_{2A,r}(\tau) := -(-1)^r H^{(8)}_{1A,r}(\tau). \tag{B.59}$$

For the remaining g, recall (B.13). The $H^{(8)}_{g,r}$ for $g \in 2B \cup 2C \cup 4A$ are defined as follows for $r \in \{1,3,5,7\}$, noting that $H^{(4)}_{g,7} = H^{(4)}_{g,-1} = -H^{(4)}_{g,1}$, &c.

$$\begin{aligned} H^{(8)}_{2BC,r}(\tau) &:= [-\tfrac{r^2}{32}]H^{(4)}_{4C,r}(\tau/2) \\ H^{(8)}_{4A,r}(\tau) &:= [-\tfrac{r^2}{32}]H^{(4)}_{4B,r}(\tau/2) \end{aligned} \tag{B.60}$$

The $H^{(8)}_{2BC,r}$ and $H^{(8)}_{4A,r}$ vanish for $r = 0 \mod 2$.

B.3.9 $\ell = 9$, $X = A_8^3$

We have $m^X = 9$, so for $g \in G^{(9)}$ the associated umbral McKay-Thompson series $H^{(9)}_g = (H^{(9)}_{g,r})$ is a 18-vector-valued function, with components indexed by $r \in \mathbb{Z}/18\mathbb{Z}$, satisfying $H^{(9)}_{g,r} = -H^{(9)}_{g,-r}$, and in particular, $H^{(9)}_{g,r} = 0$ for $r = 0 \mod 9$. So it suffices to specify the $H^{(9)}_{g,r}$ for $r \in \{1,2,3,4,5,6,7,8\}$.

Define $H^{(9)}_g = (H^{(9)}_{g,r})$ for $g = e$ by requiring that

$$-\Psi_{1,1}(\tau,z)\varphi^{(9)}_1(\tau,z) = -3\mu_{9,0}(\tau,z) + \sum_{r \mod 18} H^{(9)}_{e,r}(\tau)\theta_{9,r}(\tau,z), \tag{B.61}$$

where

$$\varphi^{(9)}_1(\tau,z) := \varphi^{(3)}_1(\tau,z)\varphi^{(7)}_1(\tau,z) - \varphi^{(5)}_1(\tau,z)^2. \tag{B.62}$$

(Cf. (B.27), (B.39), (B.54).)

Recall (B.13). The $H^{(9)}_{2B,r}$ are defined for $r \in \{1,2,4,5,7,8\}$ by setting

$$H^{(9)}_{2B,r}(\tau) := [-\tfrac{r^2}{36}]H^{(3)}_{6C,r}(\tau/3), \tag{B.63}$$

where we note that $H^{(3)}_{g,4} = H^{(3)}_{g,-2} = -H^{(3)}_{g,2}$, &c. We determine $H^{(9)}_{2B,3}$ and $H^{(9)}_{2B,6}$ by using §B.3.17 to set

$$H^{(9)}_{2B,r}(\tau) := H^{(18)}_{1A,r}(2\tau) - H^{(18)}_{1A,18-r}(2\tau) \tag{B.64}$$

for $r \in \{3,6\}$.

The $H^{(9)}_{3A,r}$ are defined by the explicit formulas

$$\begin{aligned}
H^{(9)}_{3A,1}(\tau) &:= [-\tfrac{1}{36}]f^{(9)}_1(\tau/3),\\
H^{(9)}_{3A,2}(\tau) &:= [-\tfrac{4}{36}]f^{(9)}_2(\tau/3),\\
H^{(9)}_{3A,3}(\tau) &:= -\theta_{3,3}(\tau,0),\\
H^{(9)}_{3A,4}(\tau) &:= -[-\tfrac{16}{36}]f^{(9)}_2(\tau/3),\\
H^{(9)}_{3A,5}(\tau) &:= -[-\tfrac{25}{36}]f^{(9)}_1(\tau/3),\\
H^{(9)}_{3A,6}(\tau) &:= \theta_{3,0}(\tau,0),\\
H^{(9)}_{3A,7}(\tau) &:= [-\tfrac{13}{36}]f^{(9)}_1(\tau/3),\\
H^{(9)}_{3A,8}(\tau) &:= [-\tfrac{28}{36}]f^{(9)}_2(\tau/3),
\end{aligned} \tag{B.65}$$

where

$$\begin{aligned}
f^{(9)}_1(\tau) &:= -2\frac{\eta(\tau)\eta(12\tau)\eta(18\tau)^2}{\eta(6\tau)\eta(9\tau)\eta(36\tau)},\\
f^{(9)}_2(\tau) &:= \frac{\eta(2\tau)^6\eta(12\tau)\eta(18\tau)^2}{\eta(\tau)\eta(4\tau)^4\eta(6\tau)\eta(9\tau)\eta(36\tau)} - \frac{\eta(\tau)\eta(2\tau)\eta(3\tau)^2}{\eta(4\tau)^2\eta(9\tau)}.
\end{aligned} \tag{B.66}$$

Finally, the $H^{(9)}_{g,r}$ are determined for $g \in 2A \cup 2C \cup 6A$ by setting

$$\begin{aligned}
H^{(9)}_{2A,r}(\tau) &:= (-1)^{r+1}H^{(9)}_{1A,r}(\tau),\\
H^{(9)}_{2C,r}(\tau) &:= (-1)^{r+1}H^{(9)}_{2B,r}(\tau),\\
H^{(9)}_{6A,r}(\tau) &:= (-1)^{r+1}H^{(9)}_{3A,r}(\tau).
\end{aligned} \tag{B.67}$$

B.3.10 $\ell = 10$, $X = A_9^2D_6$

We have $m^X = 10$, so the umbral McKay-Thompson series $H^{(10)}_g = (H^{(10)}_{g,r})$ associated to $g \in G^{(10)}$ is a 20-vector-valued function with components indexed by $r \in \mathbb{Z}/20\mathbb{Z}$. We have $H^{(10)}_{g,r} = -H^{(10)}_{g,-r}$, so it suffices to specify the $H^{(10)}_{g,r}$ for $1 \le r \le 9$.

To define $H^{(10)}_g = (H^{(10)}_{g,r})$ for $g = e$, first define $h(\tau) = (h_r(\tau))$ by requiring that

$$-6\Psi_{1,1}(\tau,z)\varphi^{(10)}_1(\tau,z) = -24\mu_{10,0}(\tau,z) + \sum_{r \bmod 20} h_r(\tau)\theta_{10,r}(\tau,z), \tag{B.68}$$

where

$$\varphi^{(10)}_1(\tau,z) := 5\varphi^{(4)}_1(\tau,z)\varphi^{(7)}_1(\tau,z) - \varphi^{(5)}_1(\tau,z)\varphi^{(6)}_1(\tau,z). \tag{B.69}$$

(Cf. (B.35), (B.39), (B.41), (B.54).) Now define the $H^{(10)}_{1A,r}$ for r odd by setting

$$\begin{aligned} H^{(10)}_{1A,1}(\tau) &:= \frac{1}{24}\left(3h_1(\tau)+h_9(\tau)\right), \\ H^{(10)}_{1A,3}(\tau) &:= \frac{1}{24}\left(3h_3(\tau)+h_7(\tau)\right), \\ H^{(10)}_{1A,5}(\tau) &:= \frac{1}{6}h_5(\tau), \\ H^{(10)}_{1A,3}(\tau) &:= \frac{1}{24}\left(h_3(\tau)+3h_7(\tau)\right), \\ H^{(10)}_{1A,9}(\tau) &:= \frac{1}{24}\left(h_1(\tau)+3h_9(\tau)\right). \end{aligned} \tag{B.70}$$

For $r=0 \mod 2$ set

$$H^{(10)}_{1A,r}(\tau) := \frac{1}{12}h_r(\tau), \tag{B.71}$$

and define $H^{(10)}_{2A,r}$ for $1\leq r\leq 9$ by requiring

$$H^{(10)}_{2A,r}(\tau) := -(-1)^r H^{(10)}_{1A,r}(\tau). \tag{B.72}$$

It remains to specify $H^{(10)}_{g,r}$ for $g\in 4A\cup 4B$. For $r=0 \mod 2$ set

$$H^{(10)}_{4AB,r}(\tau) := 0. \tag{B.73}$$

For r odd, recall (B.13), and define

$$H^{(10)}_{4A,r}(\tau) := [-\tfrac{r^2}{40}]\frac{1}{2}\left(H^{(2)}_{10A,r}(\tau/5)+H^{(5)}_{4CD,r}(\tau/2)\right). \tag{B.74}$$

B.3.11 $\ell=10+5,\ X=D_6^4$

We have $m^X=10$, so the umbral McKay-Thompson series $H^{(10+5)}_g=(H^{(10+5)}_{g,r})$ associated to $g\in G^{(10+5)}$ is a 20-vector-valued function with components indexed by $r\in\mathbb{Z}/20\mathbb{Z}$. We have $H^{(10+5)}_{g,r}=0$ for $r=0 \mod 2$, so it suffices to specify the $H^{(10+5)}_{g,r}$ for r odd. Observing that $H^{(10+5)}_{g,r}=-H^{(10+5)}_{g,-r}$ we may determine $H^{(10+5)}_g$ by requiring that

$$\psi^{(5/2)}_g(\tau,z) = -2\chi^{(5/2)}_g i\mu_{5/2,0}(\tau,z) + \sum_{\substack{r\in\mathbb{Z}+1/2\\ r \mod 5}} e(-r/2)H^{(10+5)}_{g,2r}(\tau)\theta_{5/2,r}(\tau,z), \tag{B.75}$$

where $\chi^{(5/2)}_g := \bar{\chi}^{X_D}_g$ as in Table 25, and the $\psi^{(5/2)}_g$ are the meromorphic Jacobi forms of weight 1 and index 5/2 defined as follows.

Table 41: Character Values and Meromorphic Jacobi Forms for $\ell = 10+5$, $X = D_6^4$

$[g]$	$\bar{\chi}_g^{(5/2)}$	$\psi_g^{(5/2)}(\tau,z)$
1A	4	$2i\theta_1(\tau,2z)^2\theta_1(\tau,z)^{-3}\eta(\tau)^3$
2A	0	$-2i\theta_1(\tau,2z)^2\theta_1(\tau,z)^{-1}\theta_2(\tau,z)^{-2}\eta(\tau)^3$
3A	1	$2i\theta_1(3\tau,6z)\theta_1(\tau,2z)^{-1}\theta_1(3\tau,3z)^{-1}\eta(\tau)^3$
2B	2	$2i\theta_1(\tau,2z)\theta_2(\tau,2z)\theta_1(\tau,z)^{-2}\theta_2(\tau,z)^{-1}\eta(\tau)^3$
4A	0	$-2i\theta_1(\tau,2z)\theta_2(\tau,2z)\theta_2(2\tau,2z)^{-1}\eta(\tau)\eta(2\tau)$

B.3.12 $\ell = 12$, $X = A_{11}D_7E_6$

We have $m^X = 12$, so the umbral McKay-Thompson series $H_g^{(12)} = (H_{g,r}^{(12)})$ associated to $g \in G^{(12)} \simeq \mathbb{Z}/2\mathbb{Z}$ is a 24-vector-valued function with components indexed by $r \in \mathbb{Z}/24\mathbb{Z}$. We have $H_{g,r}^{(12)} = -H_{g,-r}^{(12)}$, so it suffices to specify the $H_{g,r}^{(12)}$ for $1 \le r \le 11$.

To define $H_e^{(12)} = (H_{e,r}^{(12)})$, first define $h(\tau) = (h_r(\tau))$ by requiring that

$$-2\Psi_{1,1}(\tau,z)\left(\varphi_1^{(12)}(\tau,z) + \varphi_2^{(12)}(\tau,z)\right) = -24\mu_{12,0}(\tau,z) + \sum_{r \bmod 24} h_r(\tau)\theta_{12,r}(\tau,z), \tag{B.76}$$

where

$$\begin{aligned}\varphi_1^{(12)}(\tau,z) &:= 3\varphi_1^{(3)}(\tau,z)\varphi_1^{(10)}(\tau,z) - 8\varphi_1^{(4)}(\tau,z)\varphi_1^{(9)}(\tau,z) + \varphi_1^{(5)}(\tau,z)\varphi_1^{(8)}(\tau,z),\\ \varphi_2^{(12)}(\tau,z) &:= 4\varphi_1^{(4)}(\tau,z)\varphi_1^{(9)}(\tau,z) - \varphi_1^{(5)}(\tau,z)\varphi_1^{(8)}(\tau,z) - \varphi_1^{(12)}(\tau,z).\end{aligned} \tag{B.77}$$

(Cf. (B.27), (B.35), (B.39), (B.54), (B.56), (B.62), (B.69).) Now define the $H_{1A,r}^{(12)}$ for $r \neq 0 \mod 3$ by setting

$$\begin{aligned} H_{1A,1}^{(12)}(\tau) &:= \frac{1}{24}\left(3h_1(\tau) + h_7(\tau)\right),\\ H_{1A,2}^{(12)}(\tau), H_{1A,10}^{(12)}(\tau) &:= \frac{1}{24}\left(h_2(\tau) + h_{10}(\tau)\right),\\ H_{1A,4}^{(12)}(\tau), H_{1A,8}^{(12)}(\tau) &:= \frac{1}{12}\left(h_4(\tau) + h_8(\tau)\right),\\ H_{1A,5}^{(12)}(\tau) &:= \frac{1}{24}\left(3h_5(\tau) + h_{11}(\tau)\right),\\ H_{1A,7}^{(12)}(\tau) &:= \frac{1}{24}\left(h_1(\tau) + 3h_7(\tau)\right),\\ H_{1A,11}^{(12)}(\tau) &:= \frac{1}{24}\left(h_5(\tau) + 3h_{11}(\tau)\right). \end{aligned} \tag{B.78}$$

For $r = 0 \mod 3$ set

$$H_{1A,r}^{(12)}(\tau) := \frac{1}{12}h_r(\tau), \tag{B.79}$$

and define $H_{2A,r}^{(12)}$ by requiring

$$H_{2A,r}^{(12)}(\tau) := -(-1)^r H_{1A,r}^{(12)}(\tau). \tag{B.80}$$

B.3.13 $\ell = 12+4$, $X = E_6^4$

We have $m^X = 12$, so the umbral McKay-Thompson series $H_g^{(12+4)} = (H_{g,r}^{(12+4)})$ associated to $g \in G^{(12+4)}$ is a 24-vector-valued function with components indexed by $r \in \mathbb{Z}/24\mathbb{Z}$. In addition to the identity $H_{g,r}^{(12+4)} = -H_{g,-r}^{(12+4)}$, we have $H_{g,r}^{(12+4)} = 0$ for $r \in \{2,3,6,9,10\}$, $H_{g,1}^{(12+4)} = H_{g,7}^{(12+4)}$, $H_{g,4}^{(12+4)} = H_{g,8}^{(12+4)}$, and $H_{g,5}^{(12+4)} = H_{g,11}^{(12+4)}$. Thus it suffices to specify the $H_{g,1}^{(12+4)}$, $H_{g,4}^{(12+4)}$ and $H_{g,5}^{(12+4)}$.

Recall (B.13). Also, set $S_1^{E_6}(\tau) := S_{12,1}(\tau) + S_{12,7}(\tau)$, and $S_5^{E_6}(\tau) := S_{12,5}(\tau) + S_{12,11}(\tau)$. For $r = 1$ define

$$\begin{aligned}
H_{1A,1}^{(12+4)}(\tau) &:= H_{1A,1}^{(12)}(\tau) + H_{1A,7}^{(12)}(\tau),\\
H_{2B,1}^{(12+4)}(\tau) &:= [-\tfrac{1}{48}]\left(H_{8AB,1}^{(6)}(\tau/2) - H_{8AB,5}^{(6)}(\tau/2)\right),\\
H_{4A,1}^{(12+4)}(\tau) &:= \frac{1}{S_1^{E_6}(\tau)^2 - S_5^{E_6}(\tau)^2}\left(-2\frac{\eta(2\tau)^8}{\eta(\tau)^4}S_1^{E_6}(\tau) + 8\frac{\eta(\tau)^4\eta(4\tau)^4}{\eta(2\tau)^4}S_5^{E_6}(\tau)\right),\\
H_{3A,1}^{(12+4)}(\tau) &:= [-\tfrac{1}{48}]\left(H_{3A,1}^{(6)}(\tau/2) - H_{3A,5}^{(6)}(\tau/2)\right),\\
H_{8AB,1}^{(12+4)}(\tau) &:= \frac{1}{S_1^{E_6}(\tau)^2 - S_5^{E_6}(\tau)^2}\left(-2F_{8AB,1}^{(12+4)}(\tau)S_1^{E_6}(\tau) + 12F_{8AB,5}^{(12+4)}(\tau/2)S_5^{E_6}(\tau)\right).
\end{aligned} \tag{B.81}$$

In the expression for $g \in 8AB$, we write $F_{8AB,1}^{(12+4)}$ for the unique modular form of weight 2 for $\Gamma_0(32)$ such that

$$F_{8AB,1}^{(12+4)}(\tau) = 1 + 12q + 4q^2 - 24q^5 - 16q^6 - 8q^8 + O(q^9), \tag{B.82}$$

and we write $F_{8AB,5}^{(12+4)}$ for the unique modular form of weight 2 for $\Gamma_0(64)$ such that

$$F_{8AB,5}^{(12+4)}(\tau) = 3q + 4q^3 + 6q^5 - 8q^7 - 9q^9 + 12q^{11} - 18q^{13} - 24q^{15} + O(q^{17}). \tag{B.83}$$

For $r = 4$ define

$$\begin{aligned}
H_{1A,4}^{(12+4)}(\tau) &:= H_{1A,4}^{(12)}(\tau) + H_{1A,8}^{(12)}(\tau),\\
H_{3A,4}^{(12+4)}(\tau) &:= H_{3A,2}^{(6)}(\tau/2) + H_{3A,4}^{(6)}(\tau/2),
\end{aligned} \tag{B.84}$$

and set $H_{g,4}^{(12+4)}(\tau) := 0$ for $g \in 2B \cup 4A \cup 8AB$.

For $r = 5$ define

$$\begin{aligned}
H_{1A,5}^{(12+4)}(\tau) &:= H_{1A,5}^{(12)}(\tau) + H_{1A,11}^{(12)}(\tau),\\
H_{2B,5}^{(12+4)}(\tau) &:= [-\tfrac{25}{48}]\left(H_{8AB,5}^{(6)}(\tau/2) - H_{8AB,1}^{(6)}(\tau/2)\right),\\
H_{4A,5}^{(12+4)}(\tau) &:= \frac{1}{S_1^{E_6}(\tau)^2 - S_5^{E_6}(\tau)^2}\left(2\frac{\eta(2\tau)^8}{\eta(\tau)^4}S_5^{E_6}(\tau) - 8\frac{\eta(\tau)^4\eta(4\tau)^4}{\eta(2\tau)^4}S_1^{E_6}(\tau)\right),\\
H_{3A,5}^{(12+4)}(\tau) &:= [-\tfrac{25}{48}]\left(H_{3A,5}^{(6)}(\tau/2) - H_{3A,1}^{(6)}(\tau/2)\right),\\
H_{8AB,5}^{(12+4)}(\tau) &:= \frac{1}{S_1^{E_6}(\tau)^2 - S_5^{E_6}(\tau)^2}\left(2F_{8AB,1}^{(12+4)}(\tau)S_5^{E_6}(\tau) - 12F_{8AB,5}^{(12+4)}(\tau/2)S_1^{E_6}(\tau)\right).
\end{aligned} \tag{B.85}$$

Finally, define $H^{(12+4)}_{g,r}$ for $g \in 2A \cup 6A$ by setting

$$\begin{aligned} H^{(12+4)}_{2A,r}(\tau) &:= -(-1)^r H^{(12+4)}_{1A,r}(\tau), \\ H^{(12+4)}_{6A,r}(\tau) &:= -(-1)^r H^{(12+4)}_{3A,r}(\tau). \end{aligned} \tag{B.86}$$

B.3.14 $\ell = 13$, $X = A_{12}^2$

We have $m^X = 13$, so the umbral McKay-Thompson series $H^{(13)}_g = (H^{(13)}_{g,r})$ associated to $g \in G^{(13)} = G^X \simeq \mathbb{Z}/4\mathbb{Z}$ is a 26-vector-valued function, with components indexed by $r \in \mathbb{Z}/26\mathbb{Z}$.

Define $H^{(13)}_g = (H^{(13)}_{g,r})$ for $g \in G^{(13)}$ by requiring that

$$\psi^{(13)}_g(\tau,z) = -\chi^{(13)}_g \mu^0_{13,0}(\tau,z) - \bar{\chi}^{(13)}_g \mu^1_{13,0}(\tau,z) + \sum_{r \bmod 26} H^{(13)}_{g,r}(\tau)\theta_{13,r}(\tau,z), \tag{B.87}$$

where $\chi^{(13)}_g := \chi^{X_A}_g$ and $\bar{\chi}^{(13)}_g := \bar{\chi}^{X_A}_g$ (cf. Table 28), and the $\psi^{(13)}_g$ are meromorphic Jacobi forms of weight 1 and index 13 given explicitly in Table 42.

Table 42: Character Values and Meromorphic Jacobi Forms for $\ell = 13$, $X = A_{12}^2$

$[g]$	$\chi^{(13)}_g$	$\bar{\chi}^{(13)}_g$	$\psi^{(13)}_g(\tau,z)$
1A	2	2	$2i\theta_1(\tau,6z)\theta_1(\tau,z)^{-1}\theta_1(\tau,3z)^{-1}\eta(\tau)^3$
2A	-2	2	$-2i\theta_1(\tau,6z)\theta_2(\tau,z)^{-1}\theta_2(\tau,3z)^{-1}\eta(\tau)^3$
4A	0	0	cf. (B.88)

$$\psi^{(13)}_{4AB}(\tau,z) := -i\theta_2(\tau,6z)\frac{\theta_1(\tau,z+\frac14)\theta_1(\tau,3z+\frac14) - \theta_1(\tau,z-\frac14)\theta_1(\tau,3z-\frac14)}{\theta_2(2\tau,2z)\theta_2(2\tau,6z)}\frac{\eta(2\tau)^2}{\eta(\tau)} \tag{B.88}$$

For use later on we note that $\psi^{(13)}_{1A} = -2\Psi_{1,1}\varphi^{(13)}_1$, where

$$\varphi^{(13)}_1(\tau,z) := \frac{\theta_1(\tau,z)\theta_1(\tau,6z)}{\theta_1(\tau,2z)\theta_1(\tau,3z)}. \tag{B.89}$$

B.3.15 $\ell = 14+7$, $X = D_8^3$

We have $m^X = 14$, so the umbral McKay-Thompson series $H^{(14+7)}_g = (H^{(14+7)}_{g,r})$ associated to $g \in G^{(14+7)}$ is a 28-vector-valued function with components indexed by $r \in \mathbb{Z}/28\mathbb{Z}$. We have $H^{(14+7)}_{g,r} = 0$ for $r = 0 \mod 2$, so it suffices to specify the $H^{(14+7)}_{g,r}$ for r odd. Observing that $H^{(14+7)}_{g,r} = -H^{(14+7)}_{g,-r}$ we may determine $H^{(14+7)}_g$ by requiring that

$$\psi^{(7/2)}_g(\tau,z) = -2\bar{\chi}^{(7/2)}_g i\mu_{7/2,0}(\tau,z) + \sum_{\substack{r\in\mathbb{Z}+1/2 \\ r \bmod 7}} e(-r/2)H^{(14+7)}_{g,2r}(\tau)\theta_{7/2,r}(\tau,z), \tag{B.90}$$

where $\bar{\chi}^{(7/2)}_g := \bar{\chi}^{X_D}_g$ is the number of fixed points of $g \in G^{(14+7)} \simeq S_3$ in the defining permutation representation on 3 points. The $\psi^{(7/2)}_g$ are the meromorphic Jacobi forms of weight 1 and index 7/2 defined in Table 43.

Table 43: Character Values and Meromorphic Jacobi Forms for $\ell = 14+7$, $X = D_8^3$

$[g]$	$\bar{\chi}_g^{(7/2)}$	$\psi_g^{(7/2)}(\tau,z)$
1A	3	$2i\theta_1(\tau,3z)\theta_1(\tau,z)^{-2}\eta(\tau)^3$
2A	1	$2i\theta_2(\tau,3z)\theta_1(\tau,z)^{-1}\theta_2(\tau,z)^{-1}\eta(\tau)^3$
3A	0	$-2i\theta_1(\tau,z)\theta_1(\tau,3z)\theta_1(3\tau,3z)^{-1}\eta(3\tau)$

B.3.16 $\ell = 16$, $X = A_{15}D_9$

We have $m^X = 16$, so the umbral McKay-Thompson series $H_g^{(16)} = (H_{g,r}^{(16)})$ associated to $g \in G^{(16)} \simeq \mathbb{Z}/2\mathbb{Z}$ is a 32-vector-valued function with components indexed by $r \in \mathbb{Z}/32\mathbb{Z}$. We have $H_{g,r}^{(16)} = -H_{g,-r}^{(16)}$, so it suffices to specify the $H_{g,r}^{(16)}$ for $1 \leq r \leq 15$.

To define $H_g^{(16)} = (H_{g,r}^{(16)})$ for $g = e$, first define $h(\tau) = (h_r(\tau))$ by requiring that

$$-6\Psi_{1,1}(\tau,z)\left(\varphi_1^{(16)}(\tau,z) + \frac{1}{2}\varphi_2^{(16)}(\tau,z)\right) = -24\mu_{16,0}(\tau,z) + \sum_{r \bmod 32} h_r(\tau)\theta_{16,r}(\tau,z), \tag{B.91}$$

where

$$\begin{aligned}\varphi_1^{(16)}(\tau,z) &:= 8\varphi_1^{(4)}(\tau,z)\varphi_1^{(13)}(\tau,z) - \varphi_1^{(5)}(\tau,z)\varphi_1^{(12)}(\tau,z) + \varphi_1^{(7)}(\tau,z)\varphi_1^{(10)}(\tau,z),\\ \varphi_2^{(16)}(\tau,z) &:= 12\varphi_1^{(4)}(\tau,z)\varphi_1^{(13)}(\tau,z) - \varphi_1^{(5)}(\tau,z)\varphi_1^{(12)}(\tau,z) - 3\varphi_1^{(16)}(\tau,z).\end{aligned} \tag{B.92}$$

(Cf. (B.35), (B.39), (B.54), (B.69), (B.77), (B.89).) Now define the $H_{1A,r}^{(16)}$ by setting

$$H_{1A,r}^{(16)}(\tau) := \frac{1}{12}h_r(\tau) \tag{B.93}$$

for r odd. For r even, $2 \leq r \leq 14$, use

$$H_{1A,r}^{(16)}(\tau) := \frac{1}{24}\left(h_r(\tau) + h_{16-r}(\tau)\right). \tag{B.94}$$

Define $H_{2A,r}^{(16)}$ by requiring

$$H_{2A,r}^{(16)}(\tau) := -(-1)^r H_{1A,r}^{(16)}(\tau). \tag{B.95}$$

B.3.17 $\ell = 18$, $X = A_{17}E_7$

We have $m^X = 18$, so the umbral McKay-Thompson series $H_g^{(18)} = (H_{g,r}^{(18)})$ associated to $g \in G^{(18)} \simeq \mathbb{Z}/2\mathbb{Z}$ is a 36-vector-valued function with components indexed by $r \in \mathbb{Z}/36\mathbb{Z}$. We have $H_{g,r}^{(18)} = -H_{g,-r}^{(18)}$, so it suffices to specify the $H_{g,r}^{(18)}$ for $1 \leq r \leq 17$.

To define $H_g^{(18)} = (H_{g,r}^{(18)})$ for $g = e$, first define $h(\tau) = (h_r(\tau))$ by requiring that

$$-24\Psi_{1,1}(\tau,z)\phi^{(18)}(\tau,z) = -24\mu_{18,0}(\tau,z) + \sum_{r \bmod 36} h_r(\tau)\theta_{18,r}(\tau,z), \tag{B.96}$$

where

$$\phi^{(18)} := \frac{1}{12}\left(\varphi_1^{(18)} + \frac{1}{3}\varphi_3^{(18)} + 4\frac{\theta_1^{12}}{\eta^{12}}\left(\varphi_1^{(12)} + 2\varphi_2^{(12)} + \frac{1}{3}\varphi_3^{(12)}\right)\right). \tag{B.97}$$

For the definition of $\phi^{(18)}$ we require

$$\begin{aligned}
\varphi_2^{(9)}(\tau,z) &:= \varphi_1^{(4)}(\tau,z)\varphi_1^{(6)}(\tau,z) - 4\varphi_1^{(5)}(\tau,z)^2 - 4\varphi_1^{(9)}(\tau,z),\\
\varphi_1^{(11)}(\tau,z) &:= 3\varphi_1^{(5)}(\tau,z)\varphi_1^{(7)}(\tau,z) + 2\varphi_1^{(3)}(\tau,z)\varphi_1^{(9)}(\tau,z) - \varphi_1^{(4)}(\tau,z)\varphi_1^{(8)}(\tau,z),\\
\varphi_3^{(12)}(\tau,z) &:= \varphi_1^{(4)}(\tau,z)\varphi_2^{(9)}(\tau,z),\\
\varphi_1^{(14)}(\tau,z) &:= 3\varphi_1^{(5)}(\tau,z)\varphi_1^{(10)}(\tau,z) + \varphi_1^{(3)}(\tau,z)\varphi_1^{(12)}(\tau,z) - 4\varphi_1^{(4)}(\tau,z)\varphi_1^{(11)}(\tau,z),\\
\varphi_1^{(15)}(\tau,z) &:= \varphi_1^{(5)}(\tau,z)\varphi_1^{(11)}(\tau,z) + 6\varphi_1^{(3)}(\tau,z)\varphi_1^{(13)}(\tau,z) - \varphi_1^{(4)}(\tau,z)\varphi_1^{(12)}(\tau,z),\\
\varphi_2^{(15)}(\tau,z) &:= \varphi_1^{(4)}(\tau,z)\varphi_1^{(12)}(\tau,z) - 2\varphi_1^{(5)}(\tau,z)\varphi_1^{(11)}(\tau,z) - 2\varphi_1^{(15)}(\tau,z),\\
\varphi_1^{(18)}(\tau,z) &:= \varphi_1^{(5)}(\tau,z)\varphi_1^{(14)}(\tau,z) + 3\varphi_1^{(3)}(\tau,z)\varphi_1^{(16)}(\tau,z) - 4\varphi_1^{(4)}(\tau,z)\varphi_1^{(15)}(\tau,z),\\
\varphi_3^{(18)}(\tau,z) &:= \varphi_1^{(4)}(\tau,z)\varphi_2^{(15)}(\tau,z),
\end{aligned} \tag{B.98}$$

in addition to the other $\varphi_k^{(m)}$ that have appeared already. Now define the $H_{1A,r}^{(18)}$ by setting

$$H_{1A,r}^{(18)}(\tau) := \frac{1}{24}h_r(\tau) \tag{B.99}$$

for r even. For r odd, use

$$\begin{aligned}
H_{1A,1}^{(18)}(\tau) &:= \frac{1}{24}\left(2h_1(\tau) + h_{17}(\tau)\right),\\
H_{1A,3}^{(18)}(\tau) &:= \frac{1}{24}\left(h_3(\tau) + h_9(\tau)\right),\\
H_{1A,5}^{(18)}(\tau) &:= \frac{1}{24}\left(2h_5(\tau) + h_{13}(\tau)\right),\\
H_{1A,7}^{(18)}(\tau) &:= \frac{1}{24}\left(2h_7(\tau) + h_{11}(\tau)\right),\\
H_{1A,9}^{(18)}(\tau) &:= \frac{1}{24}\left(h_3(\tau) + 2h_9(\tau) + h_{15}(\tau)\right),\\
H_{1A,11}^{(18)}(\tau) &:= \frac{1}{24}\left(h_7(\tau) + 2h_{11}(\tau)\right),\\
H_{1A,13}^{(18)}(\tau) &:= \frac{1}{24}\left(h_5(\tau) + 2h_{13}(\tau)\right),\\
H_{1A,15}^{(18)}(\tau) &:= \frac{1}{24}\left(h_{15}(\tau) + h_9(\tau)\right),\\
H_{1A,17}^{(18)}(\tau) &:= \frac{1}{24}\left(h_1(\tau) + 2h_{17}(\tau)\right).
\end{aligned} \tag{B.100}$$

Define $H_{2A,r}^{(18)}$ in the usual way for root systems with a type A component, by requiring

$$H_{2A,r}^{(18)}(\tau) := -(-1)^r H_{1A,r}^{(18)}(\tau). \tag{B.101}$$

B.3.18 $\ell = 18+9$, $X = D_{10}E_7^2$

We have $m^X = 18$, so the umbral McKay-Thompson series $H_g^{(18+9)} = (H_{g,r}^{(18+9)})$ associated to $g \in G^{(18+9)} \simeq \mathbb{Z}/2\mathbb{Z}$ is a 36-vector-valued function with components indexed by $r \in \mathbb{Z}/36\mathbb{Z}$. We have $H_{g,r}^{(18+9)} = -H_{g,-r}^{(18+9)}$, $H_{g,r}^{(18+9)} = H_{g,18-r}^{(18+9)}$ for $1 \leq r \leq 17$, and $H_{g,r}^{(18+9)} = 0$ for $r = 0 \mod 2$, so it suffices to specify the $H_{g,r}^{(18+9)}$ for $r \in \{1,3,5,7,9\}$.

Define

$$\begin{aligned} H_{1A,r}^{(18+9)}(\tau) &:= H_{1A,r}^{(18)}(\tau) + H_{1A,18-r}^{(18)}(\tau), \\ H_{2A,r}^{(18+9)}(\tau) &:= H_{1A,r}^{(18)}(\tau) - H_{1A,18-r}^{(18)}(\tau), \end{aligned} \tag{B.102}$$

for $r \in \{1,3,5,7,9\}$.

B.3.19 $\ell = 22+11$, $X = D_{12}^2$

We have $m^X = 22$, so the umbral McKay-Thompson series $H_g^{(22+11)} = (H_{g,r}^{(22+11)})$ associated to $g \in G^{(22+11)} \simeq \mathbb{Z}/2\mathbb{Z}$ is a 44-vector-valued function with components indexed by $r \in \mathbb{Z}/44\mathbb{Z}$. We have $H_{g,r}^{(22+11)} = -H_{g,-r}^{(22+11)}$ and $H_{g,r}^{(22+11)} = 0$ for $r = 0 \mod 2$, so it suffices to specify the $H_{g,r}^{(22+11)}$ for r odd. Observing that $H_{g,r}^{(22+11)} = -H_{g,-r}^{(22+11)}$ we may determine $H_g^{(22+11)}$ by requiring that

$$\psi_g^{(11/2)}(\tau,z) = -2\bar{\chi}_g^{(11/2)} i\mu_{11/2,0}(\tau,z) + \sum_{\substack{r\in\mathbb{Z}+1/2 \\ r \mod 11}} e(-r/2) H_{g,2r}^{(22+11)}(\tau)\theta_{11/2,r}(\tau,z), \tag{B.103}$$

where $\bar{\chi}_{1A}^{(11/2)} := 2$, $\bar{\chi}_{2A}^{(11/2)} := 0$, and the $\psi_g^{(11/2)}$ are the meromorphic Jacobi forms of weight 1 and index 11/2 defined as follows.

$$\begin{aligned} \psi_{1A}^{(11/2)}(\tau,z) &:= 2i\frac{\theta_1(\tau,4z)}{\theta_1(\tau,z)\theta_1(\tau,2z)}\eta(\tau)^3 \\ \psi_{2A}^{(11/2)}(\tau,z) &:= -2i\frac{\theta_1(\tau,4z)}{\theta_2(\tau,z)\theta_2(\tau,2z)}\eta(\tau)^3 \end{aligned} \tag{B.104}$$

B.3.20 $\ell = 25$, $X = A_{24}$

We have $m^X = 25$, so for $g \in G^{(25)} \simeq \mathbb{Z}/2\mathbb{Z}$, the associated umbral McKay-Thompson series $H_g^{(25)} = (H_{g,r}^{(25)})$ is a 50-vector-valued function, with components indexed by $r \in \mathbb{Z}/50\mathbb{Z}$, satisfying $H_{g,r}^{(25)} = -H_{g,-r}^{(25)}$, and in particular, $H_{g,r}^{(25)} = 0$ for $r = 0 \mod 25$. So it suffices to specify the $H_{g,r}^{(25)}$ for $1 \leq r \leq 24$.

Define $H_g^{(25)} = (H_{g,r}^{(25)})$ for $g = e$ by requiring that

$$-\Psi_{1,1}(\tau,z)\varphi_1^{(25)}(\tau,z) = -\mu_{25,0}(\tau,z) + \sum_{r \mod 50} H_{e,r}^{(25)}(\tau)\theta_{25,r}(\tau,z), \tag{B.105}$$

where

$$\varphi_1^{(25)}(\tau,z) := \frac{1}{2}\varphi_1^{(5)}(\tau,z)\varphi_1^{(21)}(\tau,z) - \varphi_1^{(7)}(\tau,z)\varphi_1^{(19)}(\tau,z) + \frac{1}{2}\varphi_1^{(13)}(\tau,z)^2. \tag{B.106}$$

For the definition of $\varphi_1^{(25)}$ we require

$$\begin{aligned}
\varphi_1^{(17)}(\tau,z) &:= 4\varphi_1^{(5)}(\tau,z)\varphi_1^{(13)}(\tau,z) - \varphi_1^{(9)}(\tau,z)^2,\\
\varphi_1^{(19)}(\tau,z) &:= \varphi_1^{(4)}(\tau,z)\varphi_1^{(16)}(\tau,z) + 2\varphi_1^{(7)}(\tau,z)\varphi_1^{(13)}(\tau,z) - \varphi_1^{(5)}(\tau,z)\varphi_1^{(15)}(\tau,z),\\
\varphi_1^{(21)}(\tau,z) &:= \varphi_1^{(5)}(\tau,z)\varphi_1^{(17)}(\tau,z) - 2\varphi_1^{(9)}(\tau,z)\varphi_1^{(13)}(\tau,z),
\end{aligned} \tag{B.107}$$

in addition to the other $\varphi_k^{(m)}$ that have appeared already. Define $H_{2A,r}^{(25)}$ in the usual way for root systems with a type A component, by requiring

$$H_{2A,r}^{(18)}(\tau) := -(-1)^r H_{1A,r}^{(18)}(\tau). \tag{B.108}$$

B.3.21 $\ell = 30+15$, $X = D_{16}E_8$

We have $m^X = 30$, so the umbral McKay-Thompson series $H_g^{(30+15)} = (H_{g,r}^{(30+15)})$ associated to $g \in G^{(30+15)} = \{e\}$ is a 60-vector-valued function with components indexed by $r \in \mathbb{Z}/60\mathbb{Z}$. We have $H_{e,r}^{(30+15)} = -H_{e,-r}^{(30+15)}$, $H_{e,r}^{(30+15)} = H_{e,30-r}^{(30+15)}$ for $1 \le r \le 29$, and $H_{e,r}^{(30+15)} = 0$ for $r = 0 \mod 2$, so it suffices to specify the $H_{e,r}^{(30+15)}$ for $r \in \{1,3,5,7,9,11,13,15\}$.

Define

$$\begin{aligned}
H_{1A,1}^{(30+15)}(\tau) &:= \frac{1}{2}\left(H_{1A,1}^{(30+6,10,15)} + [-\tfrac{1}{120}]H_{3A,1}^{(10+5)}(\tau/3)\right),\\
H_{1A,3}^{(30+15)}(\tau) &:= [-\tfrac{9}{120}]H_{3A,3}^{(10+5)}(\tau/3),\\
H_{1A,5}^{(30+15)}(\tau) &:= [-\tfrac{25}{120}]H_{3A,5}^{(10+5)}(\tau/3),\\
H_{1A,7}^{(30+15)}(\tau) &:= \frac{1}{2}\left(H_{1A,7}^{(30+6,10,15)} + [-\tfrac{49}{120}]H_{3A,3}^{(10+5)}(\tau/3)\right),\\
H_{1A,11}^{(30+15)}(\tau) &:= \frac{1}{2}\left(H_{1A,1}^{(30+6,10,15)} - [-\tfrac{1}{120}]H_{3A,1}^{(10+5)}(\tau/3)\right),\\
H_{1A,13}^{(30+15)}(\tau) &:= \frac{1}{2}\left(H_{1A,7}^{(30+6,10,15)} - [-\tfrac{49}{120}]H_{3A,3}^{(10+5)}(\tau/3)\right),\\
H_{1A,15}^{(30+15)}(\tau) &:= -[-\tfrac{105}{120}]H_{3A,5}^{(10+5)}(\tau/3).
\end{aligned} \tag{B.109}$$

B.3.22 $\ell = 30+6,10,15$, $X = E_8^3$

We have $m^X = 30$, and $G^{(30+6,10,15)} = G^X \simeq S_3$. The umbral McKay-Thompson series $H^{(30+6,10,15)}$ is a 60-vector-valued function with components indexed by $r \in \mathbb{Z}/60\mathbb{Z}$. We have

$$H_{g,r}^{(30+6,10,15)}(\tau) = \begin{cases} \pm H_{g,1}^{(30+6,10,15)} & \text{if } r = \pm1,\pm11,\pm19,\pm29 \mod 60,\\ \pm H_{g,7}^{(30+6,10,15)} & \text{if } r = \pm7,\pm13,\pm17,\pm27 \mod 60,\\ 0 & \text{else,}\end{cases} \tag{B.110}$$

so it suffices to specify the $H^{(30+6,10,15)}_{g,r}$ for $r=1$ and $r=7$. These functions may be defined as follows.

$$
\begin{aligned}
H^{(30+6,10,15)}_{1A,1} &:= -2\frac{1}{\eta(\tau)^2}\left(\sum_{k,l,m\geq 0}+\sum_{k,l,m<0}\right)(-1)^{k+l+m}q^{(k^2+l^2+m^2)/2+2(kl+lm+mk)+(k+l+m)/2+3/40}\\
H^{(30+6,10,15)}_{2A,1} &:= -2\frac{1}{\eta(2\tau)}\left(\sum_{k,m\geq 0}-\sum_{k,m<0}\right)(-1)^{k+m}q^{3k^2+m^2/2+4km+(2k+m)/2+3/40}\\
H^{(30+6,10,15)}_{3A,1} &:= -2\frac{\eta(\tau)}{\eta(3\tau)}\sum_{k\in\mathbb{Z}}(-1)^k q^{15k^2/2+3k/2+3/40}\\
H^{(30+6,10,15)}_{1A,7} &= -2\frac{1}{\eta(\tau)^2}\left(\sum_{k,l,m\geq 0}+\sum_{k,l,m<0}\right)(-1)^{k+l+m}q^{(k^2+l^2+m^2)/2+2(kl+lm+mk)+3(k+l+m)/2+27/40}\\
H^{(30+6,10,15)}_{2A,7} &= 2\frac{1}{\eta(2\tau)}\left(\sum_{k,m\geq 0}-\sum_{k,m<0}\right)(-1)^{k+m}q^{3k^2+m^2/2+4km+3(2k+m)/2+27/40}\\
H^{(30+6,10,15)}_{3A,7} &= -2\frac{\eta(\tau)}{\eta(3\tau)}\sum_{k\in\mathbb{Z}}(-1)^k q^{15k^2/2+9k/2+27/40}
\end{aligned}
\tag{B.111}
$$

B.3.23 $\ell = 46+23$, $X = D_{24}$

We have $m^X = 22$, and $G^{(46+23)} = \{e\}$. The umbral McKay-Thompson series $H^{(46+23)}_e = (H^{(46+23)}_{e,r})$ is a 92-vector-valued function with components indexed by $r \in \mathbb{Z}/92\mathbb{Z}$. We have $H^{(46+23)}_{e,r} = -H^{(46+23)}_{e,-r}$ and $H^{(46+23)}_{e,r} = 0$ for $r = 0 \mod 2$, so it suffices to specify the $H^{(46+23)}_{e,r}$ for r odd. Observing that $H^{(46+23)}_{e,r} = -H^{(46+23)}_{e,-r}$ we may determine $H^{(46+23)}_e$ by requiring that

$$\psi^{(23/2)}_e(\tau,z) = -2i\mu_{23/2,0}(\tau,z) + \sum_{\substack{r\in\mathbb{Z}+1/2\\ r \mod 23}} e(-r/2)H^{(46+23)}_{g,2r}(\tau)\theta_{23/2,r}(\tau,z), \tag{B.112}$$

where $\psi^{(23/2)}_e$ is the meromorphic Jacobi forms of weight 1 and index 23/2 defined by setting

$$\psi^{(23/2)}_e(\tau,z) := 2i\frac{\theta_1(\tau,6z)}{\theta_1(\tau,2z)\theta_1(\tau,3z)}\eta(\tau)^3. \tag{B.113}$$

B.4 Rademacher Sums

Let Γ_∞ denote the subgroup of upper-triangular matrices in $SL_2(\mathbb{Z})$. Given $\alpha\in\mathbb{R}$ and $\gamma\in SL_2(\mathbb{Z})$, define $\mathrm{r}^{[\alpha]}_{1/2}(\gamma,\tau) := 1$ if $\gamma\in\Gamma_\infty$. Otherwise, set

$$\mathrm{r}^{[\alpha]}_{1/2}(\gamma,\tau) := e(-\alpha(\gamma\tau-\gamma\infty))\sum_{k\geq 0}\frac{(2\pi i\alpha(\gamma\tau-\gamma\infty))^{n+1/2}}{\Gamma(n+3/2)}, \tag{B.114}$$

where $e(x) := e^{2\pi ix}$. Let n be a positive integer, and suppose that ν is a multiplier system for vector-valued modular forms of weight 1/2 on $\Gamma = \Gamma_0(n)$. Assume that $\nu = (\nu_{ij})$ satisfies $\nu_{11}(T) = e^{\pi i/2m}$, for some basis $\{\mathfrak{e}_i\}$, for some positive integer m, where $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. To this data, attach the Rademacher sum

$$R_{\Gamma,\nu}(\tau) := \lim_{K\to\infty} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_{K,K^2}} \nu(\gamma)e\left(-\frac{\gamma\tau}{4m}\right)\mathfrak{e}_1\,\mathrm{j}(\gamma,\tau)^{1/2}\,\mathrm{r}^{[-1/4m]}_{1/2}(\gamma,\tau), \tag{B.115}$$

where $\Gamma_{K,K^2} := \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \mid 0 \le c < K,\ |d| < K^2\right\}$, and $\mathrm{j}(\gamma,\tau) := (c\tau + d)^{-1}$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. If the expression (B.115) converges then it defines a mock modular form of weight 1/2 for Γ whose shadow is given by an explicitly identifiable Poincaré series. We refer to [13] for a review of this, and to [53] for a more general and detailed discussion.

Convergence of (B.115) can be shown by rewriting the Fourier expansion as in [53, Theorem 2] in terms of a sum of Kloostermann sums weighted by Bessel functions. This expression converges at $w = 1/2$ by the analysis discussed at the end of §3, following the method of Hooley as adapted by Gannon. That analysis requires not only establishing that the expressions converge, but also explicitly bounding the rates of convergence.

For the special case that $X = A_8^3$ we require 8-vector-valued functions $\check{t}^{(9)}_g = (\check{t}^{(9)}_{g,r})$ for $g \in G^X$ with order 3 or 6. For such g, define $\check{t}^{(9)}_{g,r}$, for $0 < r < 9$, by setting

$$\check{t}^{(9)}_{3A,r}(\tau) := \begin{cases} 0, & \text{if } r \neq 0 \mod 3, \\ -\theta_{3,3}(\tau,0), & \text{if } r = 3, \\ \theta_{3,0}(\tau,0), & \text{if } r = 6, \end{cases} \tag{B.116}$$

in the case that g has order 3, and

$$\check{t}^{(9)}_{6A,r}(\tau) := \begin{cases} 0, & \text{if } r \neq 0 \mod 3, \\ -\theta_{3,3}(\tau,0), & \text{if } r = 3, \\ -\theta_{3,0}(\tau,0), & \text{if } r = 6, \end{cases} \tag{B.117}$$

when $o(g) = 6$. Here $\theta_{m,r}(\tau,z)$ is as defined in (B.5).

The following result is proved in [17], using an analysis of representations of the metaplectic double cover of $SL_2(\mathbb{Z})$.

Theorem B.1 ([17]). Let X be a Niemeier root system and let $g \in G^X$. Assume that the Rademacher sum $R^X_{\Gamma_0(n_g),\check{\nu}^X_g}$ converges. If $X \neq A_8^3$, or if $X = A_8^3$ and $g \in G^X$ does not satisfy $o(g) = 0 \mod 3$, then we have

$$\check{H}^X_g(\tau) = -2R^X_{\Gamma_0(n_g),\check{\nu}^X_g}. \tag{B.118}$$

If $X = A_8^3$ and $g \in G^X$ satisfies $o(g) = 0 \mod 3$ then

$$\check{H}^X_{g,r}(\tau) = -2R^X_{\Gamma_0(n_g),\check{\nu}^X_g}(\tau) + \check{t}^{(9)}_g(\tau). \tag{B.119}$$

The $X = A_1^{24}$ case of Theorem B.1 was proven first in [12], via different methods.

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