B.V. Solutions to Evolution Problems with Time-Dependent Domains

M. Kunze[1] & M.D.P. Monteiro Marques[2]
[1] Mathematisches Institut der Universität Köln
Weyertal 86, D - 50931 Köln, Germany
[2] C.M.A.F. and Faculdade de Ciências da Universidade de Lisboa
Av. Prof. Gama Pinto 2, P - 1699 Lisboa, Portugal

Abstract

In a Hilbert space H we consider evolution problems $-du(t)\in A(t)u(t)$ on some interval $[0,T]$, where every $A(t):D(A(t))\to 2^H$ is a maximal monotone operator, and the correspondance $t\mapsto A(t)$ is – in a suitable sense – of bounded variation or absolutely continuous.

1 Introduction and Main Results

In this paper, we first study evolutionary problems of the type

$$-\frac{du}{dr}(t)\in A(t)u(t)\quad dr-\text{a.e. in}\quad [0,T], \tag{1}$$

where $A(t):D(A(t))\subset H\to 2^H$ is a maximal monotone operator (mmop) in a Hilbert space H for every $t\in[0,T]$, and the dependence $t\mapsto A(t)$ is - in some sense - of bounded variation (bv). To be more precise, we assume:

(H1) There exists a function $r:[0,T]\to[0,\infty)$ which is right-continuous on $[0,T[$ and non-decreasing with $r(T)<\infty$ such that

$$\operatorname{dis}(A(t),A(s))\le dr(]s,t])=r(t)-r(s)\quad\text{for}\quad 0\le s\le t\le T, \tag{2}$$

where $\operatorname{dis}(\cdot,\cdot)$ is the pseudo-distance between mmops introduced in [9], and given through

$$\operatorname{dis}(A,B) = \sup\left\{\frac{\langle y-\bar y,\bar x-x\rangle}{1+|y|+|\bar y|}:x\in D(A),y\in Ax,\bar x\in D(B),\bar y\in B\bar x\right\} \tag{3}$$

for mmops A and B in H with domains $D(A)$ resp. $D(B)$.

We are mainly interested in this notion of distance, since if $C(t)\subset H$ is closed and convex for $t\in[0,T]$, then with $A(t)=\partial\delta_{C(t)}=N_{C(t)}$ (the cone of normals) we have

$$\operatorname{dis}(A(t),A(s))=d_H(C(t),C(s)) \quad \text{for} \quad t,s\in[0,T], \tag{4}$$

cf. [9, Lemma 3.5], where d_H denotes the Hausdorff distance. Thus existence results for (1) will in particular generalize the known results for the sweeping process

$$-\frac{du}{dr}(t)\in N_{C(t)}(u(t)) \quad dr-\text{a.e. in} \quad [0,T]. \tag{5}$$

We also let $A^\circ(t,x)$ denote the element of minimal norm of $A(t)x$ and suppose that

(H2) There exists a nonnegative $c\in L^1([0,T];dr)$ such that

$$|A^\circ(t,x)|\le c(t)\,(1+|x|) \quad \text{for} \quad t\in[0,T],\, x\in D(A(t)). \tag{6}$$

Note that this condition is trivially satisfied for the sweeping process (5), since in this case every $N_{C(t)}x$ is a cone, and hence $0\in N_{C(t)}x$ for $x\in C(t)$.

Our main result concerning (1), which will be proved in Section 3 below, is the following

THEOREM 1 *Let for every $t\in[0,T]$ a mmop $A(t):D(A(t))\to 2^H$ be given such that (2) and (6) are satisfied. Then for all $u_0\in D(A(0))$ the evolution problem (1) with initial value $u(0)=u_0$ has a unique bv solution.*

If one is satisfied with local existence only, then (6) can be relaxed to a local boundedness condition. In this spirit, we will obtain also in Section 3

THEOREM 2 *Let the assumptions of Theorem 1 be satisfied, but with $(H2)$ replaced by the local condition*

(H2-loc) For every $R>0$ there exists a $M_R\ge 0$ such that

$$|A^\circ(t,x)|\le M_R \quad \textit{for} \quad t\in[0,T] \quad \textit{and} \quad x\in D(A(t)) \quad \textit{with} \quad |x|\le R. \tag{7}$$

Then for every $u_0\in D(A(0))$ there exists $T_1\in(0,T]$ such that (1) with initial value $u(0)=u_0$ has a unique bv solution on $[0,T_1]$.

Afterwards, in Section 4 we will consider the case of operators $A(t)$ such that $t \mapsto A(t)$ is absolutely continuous (ac) on $[0,T]$. More exactly, we will assume that $(H2)$ holds and that

(H3) There exists an $a \in W^{1,1}([0,T])$ with

$$\operatorname{dis}(A(t),A(s)) \leq |a(t)-a(s)| \quad \text{for} \quad t,s \in [0,T]. \tag{8}$$

Then the corresponding existence result reads as

THEOREM 3 *Let for every $t \in [0,T]$ a mmop $A(t): D(A(t)) \to 2^H$ be given such that (6) and (8) are satisfied. Then for all $u_0 \in D(A(0))$ the problem*

$$-\frac{du}{dt}(t) \in A(t)u(t) \quad dt - a.e. \ in \quad [0,T] \tag{9}$$

with initial value $u(0) = u_0$ has a unique ac solution.

Several remarks are in order:

(i) To compare our results with the main theorem [9, Thm. 7.1], we note that in this paper $t \mapsto A(t)$ was assumed to be "regular", cf. [9, Def. 5.3]. The most reasonable condition to ensure "regularity" is that $\operatorname{int}(D(A(t)) \neq \emptyset$ for every $t \in [0,T]$ and that $t \mapsto A(t)$ is continuous w.r. to dis, cf. [9, Thm. 5.1]. Thus one difference is that we consider the case of $t \mapsto A(t)$ being of bounded variation or absolutely continuous. In addition we don't require that $\operatorname{int}(D(A(t)) \neq \emptyset$, a condition which restricts the possible applications of the theorems to PDEs or even to sweeping processes. As an example, one can consider the sweeping process which arises from a quasi-variational inequality in the investigation of sandpiles or river networks, cf. [8]. There one has to study the sweeping process in $H = L^2([0,1])$, where for a suitable uniformly bounded function $g: [0,T] \times [0,1] \to \mathbb{R}$

$$C(t) = \left\{ u \in H : \int_0^1 u(x)\,dx = 0, \quad |u(x)| \leq g(t,x) \text{ a.e. in } [0,1] \right\}$$

for $t \in [0,T]$. Thus with $A(t) = \partial \delta_{C(t)}$ we have $D(A(t)) = C(t)$, and hence $\operatorname{int}(D(A(t))) = \emptyset$. Therefore, taking the regularity of $t \mapsto A(t)$ (cf. (4) above) as granted, an application of the results from [9] is not possible.

(ii) Our Theorems 1, 2, and 3 also have inhomogeneous versions, i.e. on the right-hand side of the inclusions (1) resp. (9) there can be added a function $f \in L^1([0,T],H;dr)$ resp. $f \in L^1([0,T],H)$.

We start by introducing the notation and giving some preliminary lemmas in Section 2.

2 Preliminaries

Throughout this paper, H will be a Hilbert space of arbitrary dimension with norm $|x|$ and inner product $x \cdot y$ or $\langle x,y\rangle$. For sets $D_1, D_2 \subset H$

$$d_H(D_1,D_2) = \max\Big\{ \sup_{x\in D_2} \mathrm{dist}\,(x,D_1),\ \sup_{x\in D_1} \mathrm{dist}\,(x,D_2)\Big\}$$

with $\mathrm{dist}\,(x,D) = \inf\{|x-y| : y \in D\}$ is the Hausdorff-distance between D_1 and D_2. For $D \subset H$ closed and convex, $\mathrm{proj}\,(x,D)$ denotes the projection of $x \in H$ onto D, i.e. $y = \mathrm{proj}\,(x,D)$ iff $y \in D$ and $\langle x-y, y-z\rangle \geq 0$ for all $z \in D$.

If $A : D(A) \to 2^H \setminus \emptyset$ is a mmop, then $A^\circ x \in Ax$ will be the element of Ax of smallest norm, i.e. $|A^\circ x| = \mathrm{dist}\,(0, Ax)$. Moreover, $J_\lambda = (I+\lambda A)^{-1}$ resp. $A_\lambda = \lambda^{-1}(I - J_\lambda)$ will denote the resolvent resp. the Yosida-approximation of A for $\lambda > 0$, and we write J_λ^A if we are dealing with more than one mmop. For properties and terminology related to mmops (and also to convex functionals) we refer to [1], cf. also [3] and [10].

First we will list some elementary properties of $\mathrm{dis}\,(\cdot,\cdot)$, defined in (3).

LEMMA 1 *Let A, B be mmops. Then*

(a) $\mathrm{dis}\,(A,B) \in [0,\infty]$ *,* $\mathrm{dis}\,(A,B) = \mathrm{dis}\,(B,A)$ *and* $\mathrm{dis}\,(A,B) = 0$ *iff* $A = B$.

(b) $|x - \mathrm{proj}\,\big(x, \overline{D(B)}\big)| \leq \mathrm{dis}\,(A,B)$ *for* $x \in \overline{D(A)}$.

(c) $d_H(D(A), D(B)) \leq \mathrm{dis}\,(A,B)$.

Proof: Cf. [9, Lemma 3.1, (3.2), Lemma 3.2]. □

We shall also need the following result [1, Prop. 2.7], which says that A° is a principal section of A.

LEMMA 2 *Let A be a mmop. If $x\in\overline{D(A)}$ and $y\in H$ are such that*

$$\langle A^\circ\eta-y,\eta-x\rangle\geq 0 \quad for \quad \eta\in D(A),$$

then $x\in D(A)$ and $y\in Ax$.

The next lemma will give a possibility to obtain information about A in case that $\mathrm{dis}(A_n,A)\to 0$ for mmops A_n and A.

LEMMA 3 *Let A_n be a mmop for $n\in\mathbb{N}$ such that $\mathrm{dis}(A_n,A)\to 0$ for some mmop A. Suppose also that $x_n\in D(A_n)$ with $x_n\to x$ and that $y_n\in A_nx_n$ with $y_n\to y$ weakly for some $x,y\in H$. Then $x\in D(A)$ and $y\in Ax$.*

Proof: Let $w\in D(A)$ and $z\in Aw$ be given. By definition of dis we have

$$\begin{aligned}\langle x-w,z-y\rangle &= \lim_{n\to\infty}\langle x_n-w,z-y_n\rangle\\ &\leq \lim_{n\to\infty}[(1+|z|+|y_n|)\,\mathrm{dis}(A_n,A)]=0,\end{aligned}$$

and this gives the claim, since A is a mmop. □

The following result states some elementary estimates relating the pseudo-distance $\mathrm{dis}(A,B)$ to the norm-distance between the corresponding regularizations of A and B.

LEMMA 4 *Let A,B be mmops. Then*

(a) for $\lambda>0$ and $x\in D(A)$

$$|x-J^B_\lambda x|\leq\lambda|A^\circ x|+\mathrm{dis}(A,B)+\sqrt{\lambda(1+|A^\circ x|)\,\mathrm{dis}(A,B)}.$$

(b) for $\lambda>0$ and $x,\bar{x}\in H$

$$|J^A_\lambda x-J^B_\lambda\bar{x}|^2\leq|x-\bar{x}|^2+2\lambda(1+|A_\lambda x|+|B_\lambda\bar{x}|)\,\mathrm{dis}(A,B).$$

(c) for $\lambda>0$ and $x,\bar{x}\in H$

$$|A_\lambda x-B_\lambda\bar{x}|^2\leq\frac{1}{\lambda^2}|x-\bar{x}|^2+\frac{2}{\lambda}(1+|A_\lambda x|+|B_\lambda\bar{x}|)\,\mathrm{dis}(A,B).$$

Proof: (a) We have $J_\lambda^B x \in D(B)$ and $B_\lambda x \in B(J_\lambda^B x)$. Therefore by definition of dis for every $y \in Ax$

$$\operatorname{dis}(A,B) \geq \frac{\langle y - B_\lambda x, J_\lambda^B x - x\rangle}{1 + |y| + |B_\lambda x|} = \frac{|x - J_\lambda^B x|^2 - \lambda\langle y, x - J_\lambda^B x\rangle}{\lambda(1+|y|) + |x - J_\lambda^B x|},$$

and thus, taking $y = A^\circ x$, in particular

$$|x - J_\lambda^B x|^2 \leq \Big(\lambda(1+|A^\circ x|) + |x - J_\lambda^B x|\Big)\operatorname{dis}(A,B) + \lambda |A^\circ x|\,|x - J_\lambda^B x|.$$

Hence we obtain

$$\Big(|x - J_\lambda^B x| - \tfrac{1}{2}[\operatorname{dis}(A,B) + \lambda|A^\circ x|]\Big)^2 \leq$$
$$\tfrac{1}{4}[\operatorname{dis}(A,B) + \lambda|A^\circ x|]^2 + \lambda(1+|A^\circ x|)\operatorname{dis}(A,B),$$

and therefore the claim by using $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for $a, b \geq 0$. Concerning (b) note that $A_\lambda x \in A(J_\lambda^A x)$ and $B_\lambda \bar{x} \in B(J_\lambda^B \bar{x})$ imply by definition of dis that

$$\langle J_\lambda^B \bar{x} - J_\lambda^A x, A_\lambda x - B_\lambda \bar{x}\rangle \leq (1 + |A_\lambda x| + |B_\lambda \bar{x}|)\operatorname{dis}(A,B). \tag{10}$$

Therefore

$$\begin{aligned}|J_\lambda^A x - J_\lambda^B \bar{x}|^2 = {} & |x - \bar{x}|^2 - |[J_\lambda^A x - x] - [J_\lambda^B \bar{x} - \bar{x}]|^2 \\ & + 2\langle J_\lambda^B \bar{x} - J_\lambda^A x, [x - J_\lambda^A x] - [\bar{x} - J_\lambda^B \bar{x}]\rangle\end{aligned}$$

yields the claim of (b). Finally, (c) follows from

$$\begin{aligned}&\lambda^2 |A_\lambda x - B_\lambda \bar{x}|^2 \\ &= |x - \bar{x}|^2 - |J_\lambda^A x - J_\lambda^B \bar{x}|^2 + 2\lambda\langle A_\lambda x - B_\lambda \bar{x}, J_\lambda^B \bar{x} - J_\lambda^A x\rangle\end{aligned}$$

and (10). □

From the next lemma only the mentioned particular case will be used in the sequel, and this is closely related to [2]. Nevertheless, since the result is interesting in its own, we include it here in full generality.

LEMMA 5 *Let A_n be a mmop for $n \in \mathbb{N}$ such that $\operatorname{dis}(A_n, A) \to 0$ for some mmop A. Suppose further that $\eta_n \in D(A_n)$ with $\eta_n \to \eta$ for some $\eta \in D(A)$*

and that $M = \sup_{n\in\mathbb{N}} |A_n^\circ \eta_n| < \infty$. *Then there exists a sequence* (ξ_n) *such that*

$$\xi_n \in D(A_n), \quad \xi_n \to \eta \quad \text{and} \quad A_n^\circ \xi_n \to A^\circ \eta. \tag{11}$$

To be more precise, one can take $\xi_n = J_{\lambda_n}^{A_n} \eta_n$ *with*

$$\lambda_n = (|\eta_n - \eta| + \operatorname{dis}(A_n, A))^{1/2} \to 0.$$

In particular, if $\operatorname{dis}(A_n, A) \to 0$ *and* $|A_n^\circ x| \le c(1+|x|)$ *for some* $c>0$*, all* $n \in \mathbb{N}$ *and* $x \in D(A_n)$*, then for every* $\eta \in D(A)$ *there exists a sequence* (ξ_n) *such that (11) is satisfied.*

Proof: We have

$$\begin{aligned} |A_{\lambda_n} \eta_n| &\le |A_{\lambda_n} \eta_n - A_{\lambda_n} \eta| + |A_{\lambda_n} \eta| \\ &\le \frac{1}{\lambda_n} |\eta_n - \eta| + |A^\circ \eta| \le |\eta_n - \eta|^{1/2} + |A^\circ \eta|, \end{aligned}$$

and since $|A_{n,\lambda_n} \eta_n| \le |A_n^\circ \eta_n| \le M$ therefore by Lemma 4(a) and (b)

$$\begin{aligned} |\eta_n - \xi_n| &\le |\eta_n - J_{\lambda_n}^A \eta_n| + |J_{\lambda_n}^A \eta_n - J_{\lambda_n}^{A_n} \eta_n| \\ &\le \lambda_n |A_n^\circ \eta_n| + \operatorname{dis}(A_n, A) + \sqrt{\lambda_n (1 + |A_n^\circ \eta_n|) \operatorname{dis}(A_n, A)} \\ &\quad + \Big[2\lambda_n (1 + |A_{n,\lambda_n} \eta_n| + |A_{\lambda_n} \eta_n|) \operatorname{dis}(A_n, A) \Big]^{1/2} \to 0, \end{aligned}$$

and thus $|\xi_n - \eta| \to 0$. Moreover, since $|A_{\lambda_n} \eta| \le |A^\circ \eta|$, we obtain from Lemma 4(c)

$$\begin{aligned} &|A_{n,\lambda_n} \eta_n - A_{\lambda_n} \eta|^2 \\ &\le \frac{1}{\lambda_n^2} |\eta_n - \eta|^2 + \frac{2}{\lambda_n} (1 + |A_{n,\lambda_n} \eta_n| + |A_{\lambda_n} \eta|) \operatorname{dis}(A_n, A) \to 0, \end{aligned}$$

and hence

$$A_{n,\lambda_n} \eta_n \to A^\circ \eta \quad \text{as} \quad n \to \infty.$$

Because of $A_{n,\lambda_n} \eta_n \in A_n (J_{\lambda_n}^{A_n} \eta_n) = A_n \xi_n$ we get $|A_n^\circ \xi_n| \le |A_{n,\lambda_n} \eta_n|$ and thus

$$\limsup_{n\to\infty} |A_n^\circ \xi_n| \le \limsup_{n\to\infty} |A_{n,\lambda_n} \eta_n| = |A^\circ \eta|. \tag{12}$$

To prove that $A_n^\circ \xi_n \to A^\circ \eta$ we consequently may assume that $y_n = A_n^\circ \xi_n \to y$ weakly for some $y \in H$ and apply Lemma 3 to $x_n = \xi_n \to \eta$ to get $y \in A\eta$.

Hence $|A^\circ\eta| \le |y| \le \liminf_{n\to\infty} |A_n^\circ \xi_n|$ together with (12) implies that in fact we must have $y = A^\circ\eta$ and $A_n^\circ\xi_n \to A^\circ\eta$.

The particular case follows by taking $\eta_n = J_{\mu_n}^{A_n}\eta \in D(A_n)$ for an arbitrary sequence $\mu_n \to 0$, since then $\eta_n \to \eta$ by Lemma 4(a) and also $|A_n^\circ\eta_n| \le c(1 + |\eta_n - \eta| + |\eta|) \le M$ for some constant $M > 0$. □

Finally, we state a discrete version of Gronwall's lemma.

LEMMA 6 *Let $(\alpha_i), (\beta_i), (\gamma_i)$ and (a_i) be sequences of non-negative real numbers such that $a_{i+1} \le \alpha_i + \beta_i\,(a_0 + \ldots + a_{i-1}) + (1+\gamma_i)\,a_i$ for $i \in \mathbb{N}_0$. Then*

$$a_j \le \Big(a_0 + \sum_{k=0}^{j-1} \alpha_k\Big) \cdot \exp\Big(\sum_{k=0}^{j-1} [k\beta_k + \gamma_k]\Big) \quad \textit{for} \quad j \in \mathbb{N}_0\,.$$

Proof: Easy induction. □

3 The bv case

In this section we intend to prove Theorems 1 and 2. For that, we first remark that we will assume $c(t) \equiv c$ in (6) to avoid further additional technicalities. For the general case one may use an argument analogous to [4, Lemma 3.3.1] to approximate $c \in L^1([0,T]; dr)$ through suitable step-functions with integrals close to the corresponding integrals of c. We also remark that the inhomogeneous problem

$$-\frac{du}{dr}(t) \in A(t)u(t) + f(t) \quad dr-\text{a.e. in}\,[0,T]$$

with $f \in L^1([0,T], H; dr)$ may be treated in the same way by appropriately modifying the discretization described in more detail below.

Let us also fix that $u : [0,T] \to H$ will be called a solution of (1) on $[0,T]$, if u is rcbv with $u(0) = u_0$ and $u(t) \in D(A(t))$ for $t \in [0,T]$. Moreover, there has to exist a density u' of du w.r.t. dr such that

$$-u'(t) \in A(t)u(t) \quad \text{for} \quad dr-\text{a.e. } t \in [0,T]\,. \tag{13}$$

To discretize (1) completely, we choose a sequence $(\varepsilon_n)_{n\in\mathbb{N}}\subset(0,1]$ with $\varepsilon_n\searrow 0$ and partitions $0=t_0^n<t_1^n<\cdots<t_{k_n}^n=T$ of $[0,T]$ such that

$$|t_{i+1}^n-t_i^n|+dr(]t_i^n,t_{i+1}^n[)\le\varepsilon_n \quad \text{for} \quad i=0,\ldots,k_n-1. \tag{14}$$

We remark that for given $(\varepsilon_n)_{n\in\mathbb{N}}$ a corresponding sequence of partitions can be obtained analogously to [6, Prop. 3b] by applying the arguments given there to the measure $dt+dr$. We also note that because (2) is not affected by subtracting constants from r, we may assume that $r(0)=0$. Let

$$\delta_{i+1}^n=dr(]t_i^n,t_{i+1}^n])=r(t_{i+1}^n)-r(t_i^n) \quad \text{for} \quad i=0,\ldots,k_n-1, \tag{15}$$

and let $u_0\in D(A(0))$ be given. We define a sequence of rc step-functions $u_n:[0,T]\to H$ through $u_n(t)=u_0$ for $t\in[0,t_1^n[$, and

$$u_n(t)=u_i^n \quad \text{for} \quad i\ge 1, \quad t\in[t_i^n,t_{i+1}^n[, \tag{16}$$

where for $i=0,\ldots,k_n-1$

$$u_{i+1}^n=J_{i+1}^n(u_i^n) \quad \text{with} \quad J_{i+1}^n x=\Big(I+\delta_{i+1}^n A(t_{i+1}^n)\Big)^{-1}x \tag{17}$$

for $x\in H$ (and thus $J_{i+1}^n=I$ if $\delta_{i+1}^n=0$). Moreover, we let $u_n(T)=u_{k_n}^n$.

We are going to show first that this sequence of step approximations is uniformly bounded in norm and variation.

LEMMA 7 *There exists a constant $K>0$ such that for all $n\in\mathbb{N}$ and $i=0,\ldots,k_n$ resp. $i=0,\ldots,k_n-1$*

$$|u_i^n|\le K \quad \textit{resp.} \quad |u_{i+1}^n-u_i^n|\le K\,dr(]t_i^n,t_{i+1}^n]). \tag{18}$$

In particular,

$$\sup_{n\in\mathbb{N}}|u_n|_\infty\le K \quad \textit{and} \quad \sup_{n\in\mathbb{N}}\operatorname{var}(u_n)=\sup_{n\in\mathbb{N}}\left(\sum_{i=0}^{k_n-1}|u_{i+1}^n-u_i^n|\right)\le K\,dr([0,T]) \tag{19}$$

as well as, for $n\in\mathbb{N}$ and $0\le s\le t\le T$,

$$|u_n(t)-u_n(s)|\le K\Big(dr(]s,t])+\varepsilon_n\Big). \tag{20}$$

Proof; We have $\operatorname{dis}(A(t_i^n), A(t_{i+1}^n)) \le \delta_{i+1}^n$ for $i=0,\ldots,k_n-1$ by (2). Hence by Lemma 4(a) and (6)

$$\begin{aligned} |u_{i+1}^n - u_i^n| &\le \delta_{i+1}^n\,|A^\circ(t_i^n)u_i^n| + \operatorname{dis}(A(t_i^n), A(t_{i+1}^n)) \\ &\quad + \Big(\delta_{i+1}^n\Big(1+|A^\circ(t_i^n)u_i^n|\Big)\operatorname{dis}(A(t_i^n), A(t_{i+1}^n))\Big)^{1/2} \\ &\le \Big[1+|A^\circ(t_i^n)u_i^n| + \sqrt{1+|A^\circ(t_i^n)u_i^n|}\Big]\,\delta_{i+1}^n \\ &\le 2\,[1+|A^\circ(t_i^n)u_i^n|]\,\delta_{i+1}^n \\ &\le 2\,[1+c\,(1+|u_i^n|)]\,\delta_{i+1}^n \end{aligned} \tag{21}$$

for $i=0,\ldots,k_n-1$ in both cases $\delta_{i+1}^n>0$ or $\delta_{i+1}^n=0$. Thus we obtain

$$|u_{i+1}^n| \le 2\,[1+c\,(1+|u_i^n|)]\,\delta_{i+1}^n + |u_i^n| = [1+2\,c\,\delta_{i+1}^n]\,|u_i^n| + 2\,(1+c)\,\delta_{i+1}^n,$$

and consequently by means of Lemma 6 for $n\in\mathbb{N}$ and $i=0,\ldots,k_n$

$$|u_i^n| \le \Big(|u_0| + 2\,(1+c)\sum_{k=0}^{i-1}\delta_{k+1}^n\Big)\cdot\exp\Big(2\,c\sum_{k=0}^{i-1}\delta_{k+1}^n\Big).$$

Since $\sum_{k=0}^{i-1}\delta_{k+1}^n = dr(]0,t_i^n])$ by (15), it follows that

$$|u_i^n| \le \Big(|u_0| + 2\,(1+c)\,dr([0,T])\Big)\cdot\exp\Big(2\,c\,dr([0,T])\Big).$$

Hence the second claimed estimate in (18) follows from (21), and (19) is implied by (18). To show (20), fix $s\in[t_i^n, t_{i+1}^n[$ and $t\in[t_j^n, t_{j+1}^n[$ with $j>i$. Then by (18)

$$\begin{aligned} |u_n(t)-u_n(s)| = |u_j^n - u_i^n| &\le \sum_{k=0}^{j-i-1}|u_{i+k+1}^n - u_{i+k}^n| \\ &\le K\sum_{k=0}^{j-i-1} dr(]t_{i+k}^n, t_{i+k+1}^n]) = K\,dr(]t_i^n, t_j^n]) \\ &\le K\Big(dr(]t_i^n, s]) + dr(]s,t])\Big) \\ &\le K\Big(dr(]s,t] + dr(]t_i^n, t_{i+1}^n[)\Big). \end{aligned}$$

Since by (14) in particular $dr(]t_i^n, t_{i+1}^n[) \le \varepsilon_n$, we obtain (20). $\square$

Because the sequence (u_n) of rcbv functions is uniformly bounded in variation and norm, we may assume that there is a bv function $u:[0,T]\to H$ such that

$$u_n(t)\to u(t) \quad \text{weakly} \quad \text{for} \quad t\in[0,T], \tag{22}$$

cf. [5, Thm. 0.2.1]. Hence in particular

$$u(0)=u_0, \tag{23}$$

and by taking $\liminf_{n\to\infty}$ of (20)

$$|u(t)-u(s)|\le K\,dr(]s,t]) \quad \text{for} \quad 0\le s\le t\le T. \tag{24}$$

Thus u is rc on $[0,T[$. Moreover, $|du|\le K\,dr$ in the sense of ordering of real measures. We let u' denote a density of du w.r.t. dr. Because u' is bounded by K, in particular $u'\in L^2([0,T],H;dr)$.

Following [6] we define functions $v_n:[0,T]\to H$ through

$$v_n(t)=\frac{r(t)-r(t_i^n)}{r(t_{i+1}^n)-r(t_i^n)}(u_{i+1}^n-u_i^n)+u_i^n \quad \text{for} \quad t\in[t_i^n,t_{i+1}^n[$$

and $v_n(T)=u_{k_n}^n$. Here, and also in the following, we use the convention $\dfrac{0}{0}=0$, i.e. $v_n(t)=u_i^n=u_{i+1}^n$ in $[t_i^n,t_{i+1}^n[$ in case that $\delta_{i+1}^n=0$. Then $v_n(0)=u_0$, and by (18) and (14)

$$|u_n(t)-v_n(t)|\le K\varepsilon_n \quad \text{for} \quad n\in\mathbb{N} \quad \text{and} \quad t\in[0,T] \tag{25}$$

as well as

$$|v_n(t)-v_n(s)|\le K\Big(r(t)-r(s)\Big) \quad \text{for} \quad 0\le s\le t\le T, \tag{26}$$

as is shown by direct calculation. Moreover, it follows analogously to [6, Lemma 1] that

$$dv_n=v_n'\,dr \quad \text{with} \quad v_n'(t)=\frac{u_{i+1}^n-u_i^n}{r(t_{i+1}^n)-r(t_i^n)} \quad \text{for} \quad t\in]t_i^n,t_{i+1}^n] \tag{27}$$

and $v_n'(0)=0$. Consequently,

$$|v_n'|_\infty\le K \quad \text{for} \quad n\in\mathbb{N}. \tag{28}$$

Since the sequence (v_n) of rcbv functions is also uniformly bounded in variation and norm, by means of (25) and (22) we may assume that

$$v_n(t) \to u(t) \quad \text{weakly} \quad \text{for} \quad t \in [0,T]. \tag{29}$$

Because $(v_n') \subset L^2([0,T],H;dr)$ is also bounded, we may assume that $v_n' \to w$ weakly in $L^2([0,T],H;dr)$ for some $w \in L^2([0,T],H;dr)$. Thus for $t \in]0,T]$ and $x \in H$

$$\begin{aligned}\langle x, u(t)-u(0)\rangle &= \lim_{n\to\infty}[\langle x, v_n(t)\rangle - \langle x, v_n(0)\rangle] = \lim_{n\to\infty} x \cdot dv_n(]0,t]) \\ &= \lim_{n\to\infty}\int \langle \mathbf{1}_{]0,t]}x, v_n'\rangle\, dr = \int \langle \mathbf{1}_{]0,t]}x, w\rangle\, dr \\ &= \left\langle x, \int_{]0,t]} w\, dr \right\rangle\end{aligned}$$

and therefore

$$(u'\,dr)(]0,t]) = du(]0,t]) = u(t)-u(0) = \int_{]0,t]} w\, dr = (w\,dr)(]0,t])$$

for $t \in [0,T]$. Thus $u' = w$ dr-a.e. in $[0,T]$, and hence $v_n' \to u'$ weakly in $L^2([0,T],H;dr)$.

Define $\varphi_n : [0,T] \to [0,T]$ through $\varphi_n(t) = t_{i+1}^n$ for $t \in]t_i^n, t_{i+1}^n]$ and $\varphi_n(0) = 0$. Then $\varphi_n(t) \geq t$ for $t \in [0,T]$, and (16), (17), and (27) just say that

$$-v_n'(t) \in A\Big(\varphi_n(t), v_n(\varphi_n(t))\Big) \quad \text{for} \quad dr-\text{a.e. } t \in [0,T], \tag{30}$$

since $dr(\{0\}) = 0$ and $dr(]t_i^n, t_{i+1}^n]) = 0$ in case that $\delta_{i+1}^n = 0$. Moreover,

$$|r(\varphi_n(t)) - r(t)| \to 0 \quad \text{as} \quad n \to \infty \quad \text{for all} \quad t \in [0,T], \tag{31}$$

because $|\varphi_n(t) - t| \leq \max_{i=0,\ldots,k_n-1} |t_{i+1}^n - t_i^n| \leq \varepsilon_n \to 0$ by (14), and since r is right-continuous. By (26) and (2), it follows for all $t \in [0,T]$ that as $n \to \infty$

$$|v_n(\varphi_n(t)) - v_n(t)| + \operatorname{dis}(A(\varphi_n(t)), A(t)) \to 0. \tag{32}$$

By definition of dis we obtain from (30) for $t \in]0,T]$, if we also use (28) and (2),

$$\begin{aligned}&\Big\langle v_n(\varphi_n(t)) - v_m(\varphi_m(t)), v_n'(t) - v_m'(t)\Big\rangle \\ &\leq (1 + |v_n'(t)| + |v_m'(t)|)\operatorname{dis}\Big(A(\varphi_n(t)), A(\varphi_m(t))\Big) \\ &\leq (1+2K)\,|r(\varphi_n(t)) - r(\varphi_m(t))| \\ &\leq (1+2K)\Big(|r(\varphi_n(t)) - r(t)| + |r(\varphi_m(t)) - r(t)|\Big).\end{aligned}$$

Thus by (28) for $t \in]0,T]$

$$\begin{aligned}
&\Big\langle v_n(t)-v_m(t), v_n'(t)-v_m'(t)\Big\rangle \\
&\le \Big|\Big\langle v_n(t)-v_n(\varphi_n(t)), v_n'(t)-v_m'(t)\Big\rangle\Big| \\
&\quad + \Big\langle v_n(\varphi_n(t))-v_m(\varphi_m(t)), v_n'(t)-v_m'(t)\Big\rangle \\
&\quad + \Big|\Big\langle v_m(\varphi_m(t))-v_m(t), v_n'(t)-v_m'(t)\Big\rangle\Big| \\
&\le \Big\langle v_n(\varphi_n(t))-v_m(\varphi_m(t)), v_n'(t)-v_m'(t)\Big\rangle \\
&\quad + 2K\Big(|v_n(\varphi_n(t))-v_n(t)|+|v_m(\varphi_m(t))-v_m(t)|\Big) \\
&\le (1+2K)\Big(|r(\varphi_n(t))-r(t)|+|r(\varphi_m(t))-r(t)| \\
&\qquad\qquad + |v_n(\varphi_n(t))-v_n(t)|+|v_m(\varphi_m(t))-v_m(t)|\Big) \\
&=: \Delta_{n,m}(t).
\end{aligned} \tag{33}$$

This estimate will be used in the following Lemma to show that in fact the convergence of $(v_n)_{n\in\mathbb{N}}$ to u is strong and uniform, and not only pointwise and weak, as was obtained in (29).

LEMMA 8 *We have $|v_n-u|_\infty \to 0$ as $n\to\infty$.*

Proof: We apply Moreau's formula $dw^2 \le 2w^+\cdot dw^+$, which is valid for a bv function w (cf. [7] or [5, p. 9]), to the rcbv function $w=v_n-v_m$. Then $dw= dv_n-dv_m$ and $\dfrac{dw}{dr}=v_n'-v_m'$ by (27). Consequently, since $v_n(0)=u_0=v_m(0)$, by (33)

$$\begin{aligned}
\frac{1}{2}|v_n(t)-v_m(t)|^2 &= \frac{1}{2}[w^2(t)-w^2(0)] = \frac{1}{2}\Big[\big(w^2\big)^+(t)-\big(w^2\big)^+(0)\Big] \\
&= \frac{1}{2}\int_{]0,t]} dw^2 \le \int_{]0,t]} w^+\cdot dw^+ = \int_{]0,t]} w\cdot dw \\
&= \int_{]0,t]} \langle v_n-v_m, v_n'-v_m'\rangle\, dr \\
&\le \int_{]0,t]} \Delta_{n,m}(s)\, dr(s) \le \int_{[0,T]} \Delta_{n,m}(s)\, dr(s),
\end{aligned} \tag{34}$$

and thus also

$$|v_n-v_m|_\infty^2 \le 2\int_{[0,T]} \Delta_{n,m}(s)\, dr(s).$$

By (25), (31), (32), and Lebesgue's dominated convergence theorem this yields the claim, since the $\Delta_{n,m}$ are also uniformly bounded. $\square$

Therefore, by Lemma 8 and (32), we have for all $t \in [0,T]$

$$|u(t) - v_n(\varphi_n(t))| \leq |u - v_n|_\infty + |v_n(t) - v_n(\varphi_n(t))| \to 0 \quad \text{as} \quad n \to \infty. \tag{35}$$

After these preparations, we can proceed to the

Proof of Theorem 1: We already know that the function u which was obtained above is rcbv and ac w.r.t. dr such that (23) is satisfied. We will prove first that $u(t) \in D(A(t))$ on $[0,T]$. For that, define $\psi_n : [0,T] \to [0,T]$ through $\psi_n(t) = t_i^n$ on $[t_i^n, t_{i+1}^n[$. Moreover, we let $\psi_n(T) = T$. Then $u_n(t) \in D(A(\psi_n(t)))$ for all $t \in [0,T]$, or equivalently $u_i^n \in D(A(t_i^n))$ for $i = 0, \ldots, k_n$. This is obvious by (17) in case that $\delta_{i+1}^n > 0$, but for $0 = \delta_{i+1}^n = dr(]t_i^n, t_{i+1}^n]) \geq \operatorname{dis}(A(t_{i+1}^n), A(t_i^n))$ we must have $A(t_{i+1}^n) = A(t_i^n)$ by Lemma 1(a), and hence also $u_{i+1}^n = u_i^n \in D(A(t_i^n)) = D(A(t_{i+1}^n))$. In addition, by (2), $\operatorname{dis}(A(\phi_n(t)), A(t)) \leq \varepsilon_n$ for $n \in \mathbb{N}$ and $t \in [0,T]$, since by (14) for $t \in [t_i^n, t_{i+1}^n[$ we have $r(t) - r(\phi_n(t)) = r(t) - r(t_i^n) = dr(]t_i^n, t]) \leq dr(]t_i^n, t_{i+1}^n[) \leq \varepsilon_n$. Consequently, for fixed $t \in [0,T]$ we get $A_n \to A$ w.r.t. dis, where $A_n = A(\psi_n(t))$ and $A = A(t)$. Because the sequence $y_n = A^\circ(\psi_n(t), u_n(t))$ is bounded by (6) and (18), hence relatively weakly compact, we can apply Lemma 3 to $x_n = u_n(t) \to u(t)$ at least on a subsequence to obtain in particular $u(t) \in D(A(t))$.

Next we will verify (13) on a set of full dr-measure. For that, we will use Mazur's theorem, as in [6]. By this theorem, we may choose convex combinations $w_j = \sum_{k=j}^{k_j} \mu_k^j v_k' \in L^2([0,T], H; dr)$ such that $w_j \to u'$ strongly. Hence there exists a $N_0 \subset [0,T]$ with $dr(N_0) = 0$ and a subsequence (j_p) of $\mathbb{N}$ satisfying

$$t \in [0,T] \setminus N_0 \Longrightarrow w_{j_p}(t) \to u'(t) \quad \text{as} \quad p \to \infty. \tag{36}$$

For $n \in \mathbb{N}$ let $[0,T] \setminus N_n$ denote the set on which (30) holds. Since every $A(t)$ is monotone, in particular for $t \in [0,T] \setminus N_n$ and $\xi \in D(A(\varphi_n(t)))$

$$\Big\langle v_n'(t), v_n(\varphi_n(t)) - \xi \Big\rangle \leq \Big\langle A(\varphi_n(t), \xi), \xi - v_n(\varphi_n(t)) \Big\rangle. \tag{37}$$

Since $u(t) \in D(A(t))$ on $[0,T]$, by Lemma 2 it is enough to show that

$$\langle u'(t), u(t) - \eta \rangle \leq \langle A^\circ(t,\eta), \eta - u(t) \rangle \qquad \text{for} \quad t \in [0,T] \setminus \Big(\bigcup_{n=0}^{\infty} N_n \Big) \tag{38}$$

and $\eta \in D(A(t))$. For that, by (32) and (6) we may apply the particular case of Lemma 5 to $A_n = A(\varphi_n(t))$ and $A = A(t)$ to find a sequence (ξ_n) such that $\xi_n \in D(A(\varphi_n(t)))$, $\xi_n \to \eta$ and $A^\circ(\varphi_n(t), \xi_n) \to A^\circ(t, \eta)$. Consequently, by (37) and (28)

$$\begin{aligned}
&\langle v_n'(t), u(t) - \eta\rangle \\
&= \left\langle v_n'(t), v_n(\varphi_n(t)) - \xi_n \right\rangle + \left\langle v_n'(t), [u(t) - v_n(\varphi_n(t))] - [\eta - \xi_n] \right\rangle \\
&\leq \left\langle A^\circ(\varphi_n(t), \xi_n), \xi_n - v_n(\varphi_n(t)) \right\rangle + K\,[|\xi_n - \eta| + |u(t) - v_n(\varphi_n(t))|].
\end{aligned}$$

Hence (35) implies

$$\limsup_{n\to\infty} \langle v_n'(t), u(t) - \eta\rangle \leq \langle A^\circ(t,\eta), \eta - u(t)\rangle,$$

and this clearly yields

$$\begin{aligned}
\langle u'(t), u(t) - \eta\rangle &= \lim_{p\to\infty} \langle w_{j_p}(t), u(t) - \eta\rangle \\
&= \lim_{p\to\infty} \left(\sum_{k=j_p}^{k_{j_p}} \mu_k^{j_p} \langle v_k'(t), u(t) - \eta\rangle \right) \\
&\leq \langle A^\circ(t,\eta), \eta - u(t)\rangle,
\end{aligned}$$

i.e. (38).

Finally, to prove uniqueness, suppose that there are two solutions u and $\bar{u}$ of (13). Then the monotonicity of $A(t)$ and (13) imply that $\langle u'(t) - v'(t), u(t) - v(t)\rangle \leq 0$ dr-a.e. in $[0,T]$. Hence we may proceed analogously to (34) to obtain

$$|u(t) - \bar{u}(t)|^2 - |u(0) - \bar{u}(0)|^2 \leq 2 \int_{]0,t]} \langle u' - v', u - v\rangle \, dr \leq 0,$$

and therefore $u = \bar{u}$ in $[0,T]$. This completes the proof of Theorem 1. □

Now we turn to the local variant of Theorem 1, i.e. to the

Proof of Theorem 2 : By the previous proof, it is clear that we only have to obtain uniform bounds for step-approximations defined as in (16) and (17). Let $R = 2|u_0| + 1$, choose the corresponding $M = M_R$ from (7) and $T_1 \in (0,T]$ such that $r(T_1) \leq (1 + |u_0|)/2(1 + M)$. Moreover, fix again partitions

$0=t_0^n<t_1^n<\dots<t_{k_n}^n=T_1$ of $[0,T_1]$ with the properties needed before. We claim that for $i=0,\dots,k_n$

$$|u_i^n|\le 2(1+M)\sum_{j=1}^{i}\delta_j^n+|u_0| \quad \text{and} \quad |A^\circ(t_i^n,u_i^n)|\le M. \tag{39}$$

Since $|u_0^n|=|u_0|\le R$, this follows from (7) for $i=0$. If (39) already holds for some i, then analogously to (21) we obtain

$$|u_{i+1}^n-u_i^n|\le 2\,[1+|A^\circ(t_i^n)u_i^n|]\,\delta_{i+1}^n\le 2\,[1+M]\,\delta_{i+1}^n, \tag{40}$$

and hence with $|u_{i+1}^n|\le|u_i^n|+|u_{i+1}^n-u_i^n|$ the first part of (39) for $i+1$. In particular, due to (14),

$$\begin{aligned}|u_{i+1}^n| &\le 2(1+M)\sum_{j=1}^{k_n}\delta_j^n+|u_0|\le 2(1+M)\,dr(]0,T_1])+|u_0|\\ &=2(1+M)\,r(T_1)+|u_0|\le R \end{aligned}\tag{41}$$

by choice of T_1. Therefore (7) may be applied to give (39) also for $i+1$.

Since (40) and (41) show that the corresponding sequence of step-approximations $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in norm and variation, we may proceed now analogously to the proof of Theorem 1. □

4 The ac case

In this section, we will consider the case that $t\mapsto A(t)$ is absolutely continuous (ac), and prove Theorem 3.

Let us fix first that $u:[0,T]\to H$ will be called a solution of (9), if u is ac on $[0,T]$, $u(0)=u_0$, $u(t)\in D(A(t))$ for $t\in[0,T]$, and if the differential inclusion is satisfied for Lebesgue-a.e. $t\in[0,T]$.

Note that existence does not follow directly from the bv case, since there we would have solved $-du/da\in A(t)u(t)$ instead of $-du/dt\in A(t)u(t)$. On the other hand, a proof of existence for (9) with a continuous $t\mapsto A(t)$ (w.r.t. dis$(\cdot,\cdot)$) was given in [9], and this proof can be modified also to work in our case. Therefore we do not want to go into details too much, but only show how uniform a-priori bounds in norm and variation can be derived from (6) (again with $c(t) \equiv c$ for simplicity) and (8) for a suitable sequence of approximations.

So we fix partitions $0 = t_0^n < t_1^n < \cdots < t_{k_n}^n = T$ of $[0,T]$ and let

$$\delta_{i+1}^n = \int_{t_i^n}^{t_{i+1}^n} |a'(t)|\,dt \quad \text{for} \quad i = 0,\dots,k_n - 1. \tag{42}$$

To construct approximants u_n for $n \in \mathbb{N}$, we may proceed as follows, cf. [9, Lemma 6.2]: Since $A(0)$ is a mmop, by [1, Thm. 3.1] we can solve $-v'(t) = A(0,v(t))$, $v(0) = u_0 \in D(A(0))$, on $[0,\infty[$ to define $u_n(t) = v(t) \in D(A(0))$ on $[0,t_1^n[$. Then we let $u_n(t_1^n) = u_1^n$, where $u_1^n \in D(A(t_1^n))$ is such that

$$|u_1^n - \operatorname{proj}(v(t_1^n), \overline{D(A(t_1^n))})| = |u_1^n - \operatorname{proj}(u_n^-(t_1^n), \overline{D(A(t_1^n))})| \le \delta_1^n.$$

Then we solve again $-v'(t) = A(t_1^n, v(t))$, $v(t_1^n) = u_1^n$, on $[t_1^n,\infty[$, and so on. In this way we obtain $u_n : [0,T] \to H$, such that for $i = 0,\dots,k_n - 1$

$$-\frac{d^+u_n}{dt}(t) = A^\circ(t_i^n, u_n(t)) \quad \text{and} \quad u_n(t) \in D(A(t_i^n)), \quad t \in [t_i^n, t_{i+1}^n[, \tag{43}$$

cf. [1, Thm. 3.1 (5)], and with $u_n(t_i^n) = u_i^n$ for $i = 1,\dots,k_n$

$$|u_i^n - \operatorname{proj}(u_n^-(t_i^n), \overline{D(A(t_i^n))})| \le \delta_i^n. \tag{44}$$

By (44), Lemma 1(b), (8), and (42) we have

$$\begin{aligned}
|u_n^+(t_i^n) - u_n^-(t_i^n)| &= |u_i^n - u_n^-(t_i^n)| \\
&\le \delta_i^n + |u_n^-(t_i^n) - \operatorname{proj}(u_n^-(t_i^n), \overline{D(A(t_i^n))})| \\
&\le \delta_i^n + \operatorname{dis}(A(t_{i-1}^n), A(t_i^n)) \\
&\le \delta_i^n + |a(t_{i-1}^n) - a(t_i^n)| \le 2\,\delta_i^n.
\end{aligned} \tag{45}$$

Hence it follows from (43), [1, Thm. 3.1 (6)], and (6) that for $t \in [t_i^n, t_{i+1}^n[$

$$\begin{aligned}
|u_n(t)| &\le |u_n^+(t_i^n)| + \int_{t_i^n}^{t} |A^\circ(t_i^n, u_n(s))|\,ds \\
&\le |u_n^+(t_i^n)| + (t - t_i^n)\,|A^\circ(t_i^n, u_n^+(t_i^n))| \\
&\le |u_n^+(t_i^n)| + c\,(t_{i+1}^n - t_i^n)\,(1 + |u_n^+(t_i^n)|) \\
&= c\,(t_{i+1}^n - t_i^n) + [1 + c\,(t_{i+1}^n - t_i^n)]\,|u_n^+(t_i^n)| \\
&\le c\,(t_{i+1}^n - t_i^n) + 2\,[1 + c\,(t_{i+1}^n - t_i^n)]\,\delta_i^n \\
&\quad + [1 + c\,(t_{i+1}^n - t_i^n)]\,|u_n^-(t_i^n)|,
\end{aligned} \tag{46}$$

and therefore in particular

$$
\begin{aligned}
&|u_n^-(t_{i+1}^n)| \\
&\le c(t_{i+1}^n - t_i^n) + 2\,[1 + c(t_{i+1}^n - t_i^n)]\,\delta_i^n + [1 + c(t_{i+1}^n - t_i^n)]\,|u_n^-(t_i^n)|\,.
\end{aligned}
$$

Consequently, Lemma 6 implies, if we let $u_n^-(t_0^n) = u_n^-(0) = u_0$,

$$
\begin{aligned}
|u_n^-(t_i^n)| &\le \Big(|u_0| + \sum_{k=0}^{i-1}[\,c(t_{k+1}^n - t_k^n) + 2\,[1 + c(t_{k+1}^n - t_k^n)]\,\delta_i^n]\Big) \\
&\qquad \times \exp\Big(c\sum_{k=0}^{i-1}(t_{k+1}^n - t_k^n)\Big) \\
&\le [\,|u_0| + cT + (1+2cT)\,|a'|_{L^1([0,T];H)}]\,\exp(cT) \qquad (47)
\end{aligned}
$$

for all $n \in \mathbb{N}$ and $i = 0, \ldots, k_n$. Thus $\sup_{n\in\mathbb{N}} |u_n|_\infty \le K$ for some $K > 0$ by (46). Since by construction of the u_n, by (43), (6), and (45) we have

$$
\begin{aligned}
\operatorname{var}(u_n) &\le T\sup\left\{\left|\frac{d^+u_n}{dt}(t)\right| : t \ne t_i^n\right\} \\
&\qquad + \sum_{i=0}^{k_n} |u_n^+(t_i^n) - u_n^-(t_i^n)| \\
&\le cT\,(1 + |u_n|_\infty) + 2\,|a'|_{L^1([0,T];H)},
\end{aligned}
$$

we also obtain a uniform bound $\sup_{n\in\mathbb{N}} \operatorname{var}(v_n) \le K$, as it was claimed.

Acknowledgements: The authors gratefully acknowledge support by Deutscher Akademischer Austauschdienst (DAAD) and Conselho de Reitores das Universidades Portuguesas (CRUP), project "Acção Integrada Luso-Alemã, No. A–8/95, Inclusões Diferenciais em Sistemas Dinâmicos Não-Regulares".

References

[1] Brézis, H.: *Opérateurs Maximaux Monotones*, North Holland Publ. Company, Amsterdam-London, 1973

[2] Brézis, H.: New results concerning monotone operators and nonlinear semigroups, in *Proceedings of the RIMS*, Kyoto University, 1975

[3] Deimling, K.: *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985

[4] Lakshmikantham, V. and Leela, S.: *Nonlinear Differential Equations in Abstract Spaces*, Pergamon Press, Oxford, 1991

[5] Monteiro Marques, M.D.P.: *Differential Inclusions in Nonsmooth Mechanical Problems-Shocks and Dry Friction*, Birkhäuser-Verlag, Basel, 1993

[6] Moreau, J.J.: Evolution problem associated with a moving convex set in a Hilbert space, *J. Differential Equations* **26** (1977), 347-374

[7] Moreau, J.J.: Sur les mesures différentielles de fonctions vectorielles et certains problèmes d'évolution, *C. R. Acad. Sci. Paris, Série A* **282** (1976), 837-840

[8] Prigozhin, L.: Sandpiles and river networks: Extended systems with nonlocal interactions, *Phys. Rev. E* **49** (1994), 1161-1167

[9] Vladimirov, A.A.: Nonstationary dissipative evolution equations in a Hilbert space, *Nonlinear Anal.* **17** (1991), 499-518

[10] Zeidler, E.: *Nonlinear Functional Analysis and its Applications II/B: Nonlinear Monotone Operators*, Springer-Verlag, Berlin, 1990