Linear Complementary Problems and the Simulation of the Motion of Rigid Body Systems Subject to Coulomb Friction

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Abstract

We describe how to systematically develop an algorithm to simulate the nonsmooth dynamics of multibody systems with impacts and friction. Our appoach works by using the theory of linear complementary problems.

1 Introduction

In this paper we are concerned with mechanical systems consisting of $l \geq 2$ rigid bodies which are subject to impacts and friction. Our aim is to describe how to develop an efficient numerical algorithm (related to [16, 17, 3, 1]), which is not specially taylored to treat only one model, but which is general enough to apply to most rigid body systems with Coulomb friction. The main difficulty results from the fact that in multibody systems the change of one contact usually will also lead to a change in some (or all) other contact points.

The equation of motion in its most general form for such systems may be written as

$$M(q,t)\,\ddot{q} = f(q,\dot{q},t) \quad \text{for} \quad t \geq 0, \tag{1}$$

where $q = q(t) \in \mathbb{R}^m$ denotes the coordinates of one of the $l \geq 2$ bodies which is thought of to be marked. Here the nonlinearity $f : \mathbb{R}^{2m+1} \to \mathbb{R}^m$ contains all kind of forces acting on the system, like driving forces, inertia forces, contact forces, friction forces, etc., and $M : \mathbb{R}^{m+1} \to L(\mathbb{R}^m, \mathbb{R}^m)$ is the symmetric and positive definite mass matrix. Despite (1), a solution also has to satisfy some constraint conditions, given through functions $\phi_i : \mathbb{R}^m \to \mathbb{R}$ describing the distances from the marked body to all other bodies, so that we require

$$\phi_i(q(t)) \geq 0 \quad \text{for} \quad t \geq 0 \quad \text{and} \quad 1 \leq i \leq p. \tag{2}$$

Since we allow measurement of multiple distances from the marked body to the others, we have $p \geq l - 1$. (One may imagine a slab falling down to the floor, where we can measure simultaneously e.g. the distance of the beginning and of the end of the slab to the floor.) Following [7], we let

$$J_N(q) := \{i \in \{1,\dots,p\} : \phi_i(q) = 0\} \quad \text{for} \quad q \in \mathbb{R}^m, \tag{3}$$

so that $i_0 \in J_N(q(t_0))$ means that at time $t = t_0$ there was an impact between the marked body and the body labelled with i_0. In this case of an impact, there may be also additional friction forces, due to sticking or sliding. So if we let

$$v_i(q,\dot{q}) = g_{F_i}(q)^T \dot{q} + w_i(q), \quad q,\dot{q} \in \mathbb{R}^m, \tag{4}$$

denote the corresponding relative velocities in tangential directions, with $g_{F_i} : \mathbb{R}^m \to \mathbb{R}^m$ and $w_i : \mathbb{R}^m \to \mathbb{R}^m$, then the index sets

$$J_{F_0}(q,\dot{q}) := \{i \in J_N(q) : v_i(q,\dot{q}) = 0\} = \{i \in \{1,\dots,p\} : \phi_i(q) = 0 \text{ and } v_i(q,\dot{q}) = 0\} \tag{5}$$

and

$$J_{FW}(q,\dot{q}) := \{i \in J_N(q) : v_i(q,\dot{q}) \neq 0\} = \{i \in \{1,\dots,p\} : \phi_i(q) = 0 \text{ and } v_i(q,\dot{q}) \neq 0\} \tag{6}$$

are useful in an analogous way as J_N is: if q is a solution of (1), then $i_0 \in J_F(q(t_0),\dot{q}(t_0))$ [resp. $i_0 \in J_{FW}(q(t_0),\dot{q}(t_0))$] is equivalent to stating that at time $t = t_0$ there is a sticking [resp. slipping] contact between the body labelled with i_0 and the marked body.

By specifying at every time $t = t_0$ the forces arising due to contacts, sticking or slipping, (1) now may be written as

$$\begin{aligned} \ddot{q}(t) = f(q(t),\dot{q}(t),t) \quad &+ \sum_{i \in J_N(q(t))} \lambda_{N_i}(t)\, g_{N_i}(q(t)) \\ + \sum_{i \in J_{F_0}(q(t),\dot{q}(t))} \lambda_{F_i}(t)\, g_{F_i}(q(t)) \quad &+ \sum_{i \in J_{FW}(q(t),\dot{q}(t))} \lambda_{F_i}(t)\, g_{F_i}(q(t)), \end{aligned} \tag{7}$$

cf. e.g. [14, Section 2.2] or [19, Section 2]. Here $g_{N_i} := \phi_i'$, and $\lambda_{N_i} \geq 0$ resp. λ_{F_i} are the Lagrange multipliers corresponding to the constraints. (They are proportional to the normal force resp. the friction force, both with the same nonnegative factor.) In contrast to (1), in (7) we chose $M(\dot{q},t)$ to be the identity matrix (this may be achieved by a suitable change of coordinate system), and f in (7) is assumed to contain only the external forces and the inertia forces.

For $i \in J_{FW}(q(t),\dot{q}(t))$ we have $v_i((q(t),\dot{q}(t)) \neq 0$. Now Coulomb's law in its general form says (cf. [6]), if N, F resp. v denote the normal force, the friction force resp. the relative velocity at an impact point,

$$N \geq 0, \quad v \neq 0 \Longrightarrow |F| = \mu N \text{ and } vF \leq 0, \quad v = 0 \Longrightarrow |F| \leq \mu N. \tag{8}$$

Thus, if we drop the argument t for the moment,

$$v_i(q,\dot{q})\,\lambda_{F_i} \leq 0 \quad \text{and} \quad \lambda_{F_i} = s_i\,\mu_i\,\lambda_{N_i} \quad \text{with} \quad s_i = \begin{cases} 1 & \text{for} \quad v_i(q,\dot{q}) < 0 \\ -1 & \text{for} \quad v_i(q,\dot{q}) > 0 \end{cases} \tag{9}$$

and suitable friction coefficients $\mu_i > 0$. Inserting this into (7) we finally obtain the basic equation

$$\begin{aligned}\ddot{q}(t) = f(q(t),\dot{q}(t),t) \quad &+ \sum_{i\in J_N(q(t))} \lambda_{N_i}(t)\, g_{N_i}(q(t)) \\ + \sum_{i\in J_{F_0}(q(t),\dot{q}(t))} \lambda_{F_i}(t)\, g_{F_i}(q(t)) \quad &+ \sum_{i\in J_{FW}(q(t),\dot{q}(t))} s_i(t)\,\mu_i\,\lambda_{N_i}(t)\, g_{F_i}(q(t)), \end{aligned} \tag{10}$$

where we of course let $\sum_\emptyset := 0$.

From (10), the inherent problem in numerical simulations of the dynamics is obvious, cf. [17]: in multibody systems with multiple contacts the change of one contact may lead to a change in all other contact points, i.e. at every time $t=t_0$ the "active" index sets $J_N(q(t))$, $J_{F_0}(q(t),\dot{q}(t))$ and $J_{FW}(q(t),\dot{q}(t))$ have to be determined anew. Therefore, to model the dynamics, the following questions arise:

a) Suppose that $(q_0,\dot{q}_0) := (q(t_0),\dot{q}(t_0))$ are known at some time $t=t_0$, and hence also $J_N(q_0)$, $J_{F_0}(q_0,\dot{q}_0)$ and $J_{FW}(q_0,\dot{q}_0)$ are known. How is it possible to modify these active index sets in such a way that they (at best) don't change in infinitesimal time, i.e. that they remain constant on some $[t_0,t_0+h]$?

b) How to incorporate the effects of possible changes of the active set ? The following possibilties have to be taken into account:

   i) two bodies impact, i.e. J_N has to be enlarged;
   ii) two bodies loose contact, i.e. J_N has to be made smaller;
   iii) there is a transition from slipping to sticking, i.e. an index from J_{FW} is moving to J_{F_0};
   iv) there is a transition from sticking to slipping, i.e. an index from J_{F_0} is moving to J_{FW};

c) How to solve (10) in time-intervals $[t_0,t_0+h]$ where the active sets are constant?

While c) is standard, the problems a) and b) may be transformed to solving (extended) linear complementary problems (LCPs), cf. e.g. [7, 8, 9, 11, 2, 3, 17, 19, 10]. In Section 2 we describe this exemplarly for a) above, and b)ii), iii) and iv) are similiar. Nevertheless, b)i) is more complicated because in this case the analysis has to be carried out on the impuls level (cf. [4] or [13]) instead of the force level. Despite the fact that the transformations to LCPs may be expensive from a numerical point of view, they are desirable to "stabilize" the active sets for (10), and the solution of LCPs is well-understood, cf. [1, 12]. In Figure 1 we schematically describe the algorithm for implementing the numerical solution of (10), taking into accont the possiblity (and necessity) to improve active sets. Finally, in Section 3 we consider an example.

2 Determination of an improved active index set at some time $t=t_0$

The situation we examine in more detail is the following: the solution of (10) has been calculated up to some time $t=t_0$, and we let $(q_0,\dot{q}_0) = (q(t_0),\dot{q}(t_0))$. (Here $t_0=0$ means that we start from the given initial conditions.) Hence the active sets $J_N := J_N(q_0)$, $J_{F_0} := J_{F_0}(q_0,\dot q_0)$ and $J_{FW} := J_{FW}(q_0,\dot q_0)$ from (3), (5) and (6) can be determined, and one could already start with the simulation of the dynamics. But this would not be very convenient, since these active sets may change in infinitesimal time. Hence they have to be improved as follows. Firstly, because of $\phi_i \geq 0$ we have for $i \in J_N$

$$0 \leq \frac{d}{dt}\phi_i(q(t))\,|_{t=t_0} = \phi_i'(q(t))\,\dot q(t)|_{t=t_0} = g_{N_i}(q_0)\,\dot q_0\,, \tag{11}$$

and thus this $i \in J_N$ would leave J_N at once in case that $g_{N_i}(q_0)\,\dot q_0 > 0$. So it certainly will be reasonable to improve the active sets at time $t = t_0$ to

$$J_N^* := J_N^*(q_0,\dot q_0) := J_N \cap \{i \in \{1,\dots,p\} : g_{N_i}(q_0)\,\dot q_0 = 0\}\,,\ J_{F_0}^* := J_N^* \cap J_{F_0} \text{ and } J_{FW}^* := J_N^* \cap J_{FW}\,. \tag{12}$$

To obtain a more precise picture, we need to analyze

$$\delta_{N_i}(q(t),\dot q(t),\ddot q(t)) := \frac{d^2}{dt^2}\phi_i(q(t)) = g_{N_i}(q(t))\,\ddot q(t) + g_{N_i}'(q(t))\,(\dot q(t),\dot q(t)) \tag{13}$$

and also

$$\delta_{F_i}(q(t),\dot q(t),\ddot q(t)) := \frac{d}{dt}v_i(q(t),\dot q(t)) = g_{F_i}(q(t))\,\ddot q(t) + g_{F_i}'(q(t))\,(\dot q(t),\dot q(t)) + w_i'(q(t))\,\dot q(t)\,. \tag{14}$$

at time $t = t_0$. Since $\ddot q_0 := \ddot q(t_0)$ cannot be obtained directly from the differential equation (10) (note that also the Lagrange mutipliers $\lambda_{N_i}(t_0)$ and $\lambda_{F_i}(t_0)$ are unknown in (10)), it is not possible to calculate $\delta_{N_i} := \delta_{N_i}(q_0,\dot q_0,\ddot q_0)$ resp. $\delta_{F_i} := \delta_{F_i}(q_0,\dot q_0,\ddot q_0)$ directly. We will sketch how one can overcome this difficulty by deriving a suitable LCP.

Replacing the active sets in (10) already by the better J_N^*, $J_{F_0}^*$ and J_{FW}^* from (12), (10) may be written as a system. For this, we introduce $n = \# J_N^*$ and $k = \# J_{F_0}^*$, hence $\# J_{FW}^* = n-k$. Moreover, we let

$$\begin{aligned} \lambda_N &:= ((\lambda_{N_i})_{i\in J_{F_0}^*},(\lambda_{N_i})_{i\in J_{FW}^*}) = (\lambda_{N_i})_{i\in J_N^*} \in \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n\,,\\ \lambda_{F_0} &:= (\lambda_{F_i})_{i\in J_{F_0}^*} \in \mathbb{R}^k, \quad \lambda = (\lambda_N,\lambda_{F_0}) \in \mathbb{R}^{n+k}, \quad \text{and} \quad \lambda_{FW} := (\lambda_{N_i})_{i\in J_{FW}^*} \in \mathbb{R}^{n-k}. \end{aligned} \tag{15}$$

In the same way, we define matrices

$$\begin{aligned} G(q_0) &:= (g_N(q_0),g_{F_0}(q_0)) = ((g_{N_i}(q_0))_{i\in J_N^*},(g_{F_i}(q_0))_{i\in J_{F_0}^*}) \in L(\mathbb{R}^{n+k},\mathbb{R}^m)\,,\\ H(q_0) &:= (g_{F_i}(q_0))_{i\in J_{FW}^*} \in L(\mathbb{R}^{n-k},\mathbb{R}^m)\,, \end{aligned}$$

and $C \in L(\mathbb{R}^{n+k},\mathbb{R}^{n-k})$ through

$$C := \left(\underbrace{\begin{matrix} 0 & \dots & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & 0\end{matrix}}_{k}\ \left|\ \underbrace{\mathrm{Diag}(s_i\,\mu_i)_{i\in J_{FW}^*}}_{n-k}\ \right|\ \underbrace{\begin{matrix} 0 & \dots & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & 0\end{matrix}}_{k}\right).$$

Then (10) reads as

$$\ddot q_0 = f(q_0,\dot q_0,t_0) + (G(q_0) + H(q_0)\,C)\,\lambda\,,$$

and by multiplying this with $G(q_0)^T \in L(\mathrm{I\!R}^m, \mathrm{I\!R}^{n+k})$ we obtain

$$G(q_0)^T \ddot{q}_0 = G(q_0)^T f(q_0, \dot{q}_0, t_0) + G(q_0)^T (G(q_0) + H(q_0) C)\lambda. \tag{16}$$

Now the point is that $G(q_0)^T \ddot{q}_0$ may be eliminated by means of (13) and (14). For this, we let

$$\delta := (\delta_N, \delta_{F_0}) := ((\delta_{N_i})_{i \in J_N^*}, (\delta_{F_i})_{i \in J_{F_0}^*}) \in \mathrm{I\!R}^n \times \mathrm{I\!R}^k = \mathrm{I\!R}^{n+k} \quad \text{and} \tag{17}$$

$$w(q_0, \dot{q}_0) := \Big((g_{N_i}'(q_0)(\dot{q}_0, \dot{q}_0))_{i \in J_N^*}, (g_{F_i}'(q_0)(\dot{q}_0, \dot{q}_0) + w_i'(q_0)\dot{q}_0)_{i \in J_{F_0}^*} \Big) \in \mathrm{I\!R}^n \times \mathrm{I\!R}^k = \mathrm{I\!R}^{n+k}\ .$$

(Note that we are interested only in the δ_{F_i} with $i \in J_{F_0}^*$, since by definition that means $v_i(q_0, \dot{q}_0) = 0$, and we thus want to determine $\delta_{F_i} = \dfrac{d}{dt} v_i(q(t), \dot{q}(t))|_{t=t_0}$ to find out whether this i is going to leave $J_{F_0}^*$ at once.)

Then (13) and (14) imply

$$\delta = G(q_0)^T \ddot{q}_0 + w(q_0, \dot{q}_0),$$

and consequently by (16) we obtain the inhomogeneous $(n+k) \times (n+k)$ – system

$$\delta = A\lambda + b \tag{18}$$

with

$$A = G(q_0)^T (G(q_0) + H(q_0) C) \in L(\mathrm{I\!R}^{n+k}, \mathrm{I\!R}^{n+k}), \quad b = G(q_0)^T f(q_0, \dot{q}_0, t_0) + w(q_0, \dot{q}_0) \in \mathrm{I\!R}^{n+k}. \tag{19}$$

We remark that A resp. b can be calculated only with knowledge of q_0 and $\dot{q}_0$. Hence we are looking for a solution (δ, λ) of (18) which satisfies the additional conditions:

$$\lambda_{N_i} \geq 0, \quad \delta_{N_i} \geq 0, \quad \lambda_{N_i}\delta_{N_i} = 0 \quad (i \in J_N^*), \quad \text{and} \tag{20}$$

$$|\lambda_{F_i}| \leq \mu_i \lambda_{N_i}, \quad |\lambda_{F_i}| < \mu_i \lambda_{N_i} \Longrightarrow \delta_{F_i} = 0, \quad |\lambda_{F_i}| = \mu_i \lambda_{N_i} \Longrightarrow \lambda_{F_i}\delta_{F_i} \leq 0 \quad (i \in J_{F_0}^*). \tag{21}$$

[The (slightly heuristic) arguments leading to (20) and (21) are as follows: since λ_{N_i} is proportial with a nonnegative factor to a normal force at a contact point, $\lambda_{N_i} \geq 0$ by Coulomb's law, cf. (8). Next, let $\Phi_i(t) = \phi_i(q(t))$. Then $\Phi_i(t_0) = 0 = \Phi_i'(t_0)$ for $i \in J_N^*$ by definition of J_N^*, cf. (11). In case that $\delta_{N_i} = \Phi_i''(t_0) < 0$ we would obtain $\Phi_i'(t) < 0$ for $t \in (t_0, t_0 + \varepsilon)$ with a suitable $\varepsilon > 0$, and thus also $\Phi_i(t) < 0$ for $t \in (t_0, t_0 + \varepsilon)$, a contradiction to (2). To prove $\lambda_{N_i}\delta_{N_i} = 0$, we consider two possible cases: if $\Phi_i(t) = 0$ for $t \in (t_0, t_0 + \varepsilon)$ with some $\varepsilon > 0$, then $\Phi_i''(t) = 0$ in this interval, and consequently also $\delta_{N_i} = \Phi_i''(t_0) = 0$. In the opposite case, $\Phi_i(t_k) > 0$ for some $t_k \searrow t_0$. This means that at times $t = t_k$ there is no contact, and hence no normal force, i.e. $\lambda_{N_i}(t_k) = 0$, so that we obtain $\lambda_{N_i} = \lambda_{N_i}(t_0) = 0$ in the limit. To prove (21), we remark that the first inequality follows from (8). To prove the second claim, assume that $\delta_{F_i} \neq 0$. Since $i \in J_{F_0}$, this yields either $V_i(t) := v_i(q(t), \dot{q}(t)) > 0$ or $V_i(t) < 0$ for t in some $(t_0, t_0 + \varepsilon)$, in particular $V_i(t) \neq 0$. Thus by (8) $|\lambda_{F_i}(t)| = \mu_i |\lambda_{N_i}(t)|$ for $t \in (t_0, t_0 + \varepsilon)$ gives a contradiction as $t \searrow t_0$. Finally, we suppose that $|\lambda_{F_i}| = \mu_i \lambda_{N_i}$, but $\lambda_{F_i}\delta_{F_i} > 0$. Then again $V_i(t) \neq 0$ for t in some $(t_0, t_0 + \varepsilon)$, and hence by (8) $[V_i(t) - V_i(t_0)]\lambda_{F_i}(t) = V_i(t)\lambda_{F_i}(t) \leq 0$, contradicting $V_i'(t_0)\lambda_{F_i}(t_0) = \delta_{F_i}\lambda_{F_i} > 0$ as $t \searrow t_0$.]

To sum up, we have shown that (λ, δ) with $\lambda \in \mathrm{I\!R}^{n+k}$ from (15) and $\delta \in \mathrm{I\!R}^{n+k}$ from (17) have to satisfy (18) with additional constraints (20) and (21), i.e. an (extended) LCP. Now a numerical algorithm has to be interrupted in case that this LCP does not have a solution. (This may happen, and also solutions may be not unique; cf. [7] or [13] for a discussion and for criteria assuring existence and/or uniqueness.)

Assuming that the LCP (18), (20), (21) can be solved for (λ,δ), we can now continue our improvement of the active sets J_N^*, $J_{F_0}^*$ and J_{FW}^* from (12). For $i \in J_N^*$, we have, by definition of this set and by (20),

$$\phi_i(q_0)=0, \quad \frac{d}{dt}\phi_i(q(t))\,|_{t=t_0}=0 \quad \text{and also} \quad \frac{d^2}{dt^2}\phi_i(q(t))\,|_{t=t_0}=\delta_{N_i}\geq 0.$$

Consequently, an $i\in J_N^*$ will leave J_N^* (and hence also J_N) in infinitesimal time, if $\delta_{N_i}>0$, so that this case has to be excluded. Therefore we let

$$J_N^{**} := J_N \cap \{i \in \{1,\ldots,p\} : g_{N_i}(q_0)\,\dot q_0 = 0,\, \delta_{N_i}=0\}\,,\ J_{F_0}^{**} := J_N^{**}\cap J_{F_0} \text{ and } J_{FW}^{*} := J_N^{**}\cap J_{FW}\,.$$

The last possibility to improve these sets by using our knowledge of (λ,δ) consists in preventing indices from $J_{F_0}^*$ to move to J_{FW}^* immediately. Since for $i\in J_{FW}^*$ we have

$$v_i(q_0,\dot q_0)=0 \quad \text{as well as} \quad \frac{d}{dt}v_i(q(t),\dot q(t))\,|_{t=t_0}=\delta_{F_i}\,,$$

we finally stabilize $J_{F_0}^{**}$ by defining

$$\begin{aligned} J_N^{***} &:= J_N^{**} = \{i\in\{1,\ldots,p\} : \phi_i(q_0)=0,\, g_{N_i}(q_0)\,\dot q_0=0,\, \delta_{N_i}=0\}\,,\\ J_{F_0}^{***} &:= \{i\in\{1,\ldots,p\} : \phi_i(q_0)=0,\, v_i(q_0,\dot q_0)=0,\, \delta_{F_i}=0\}\,,\\ \text{and} \quad J_{FW}^{***} &:= \{i\in\{1,\ldots,p\} : \phi_i(q_0)=0,\, v_i(q_0,\dot q_0)\neq 0\}\,. \end{aligned}$$

We remark that these sets can be calculated explicitely from q_0, $\dot q_0$, and the solution (λ,δ) of (18), (20) and (21). The same method can also be applied to determine the higher derivatives of $t\mapsto\phi_i(q(t))$ and $t\mapsto v_i(q(t),\dot q(t))$ at $t=t_0$, and that would lead to a further improvement of the active sets at $t=t_0$. From a practical point of view, it is suitable to use (the time-dependent) active sets J_N^{***}, $J_{F_0}^{***}$ and J_{FW}^{***} and to assume that these sets does not change for $t\in[t_0,t_0+h]$, so that the numerical determination of the trajectory can be continued up to time $t_1=t_0+h$.

The corresponding algorithm for the numerical simulation of rigid body systems may be summarized as is shown in Figure 1 below.

3 Examples

Based on the flow diagram of Figure 1, we developed a program for the numerical simulation of the motion of rigid body systems with Coulomb friction. In this section, we consider as an easy example the friction oscillator, as shown in Figure 2 below. In fact, the dynamical behaviour of the system is well-known, cf. [21, 20, 5], but we chose this example as a test to compare our solution trajectories to those already obtained in the aforementioned references. It is a main advantage of our program that it can be used to treat most other rigid body systems with Coulomb firction as well, with only minor changes. (Cf. [13] for additional examples, namely a slab falling down to the floor and a turbine blade damper, cf. also [18, 14, 15].) Since we worked via the LCP approach, additionally at every time step we are

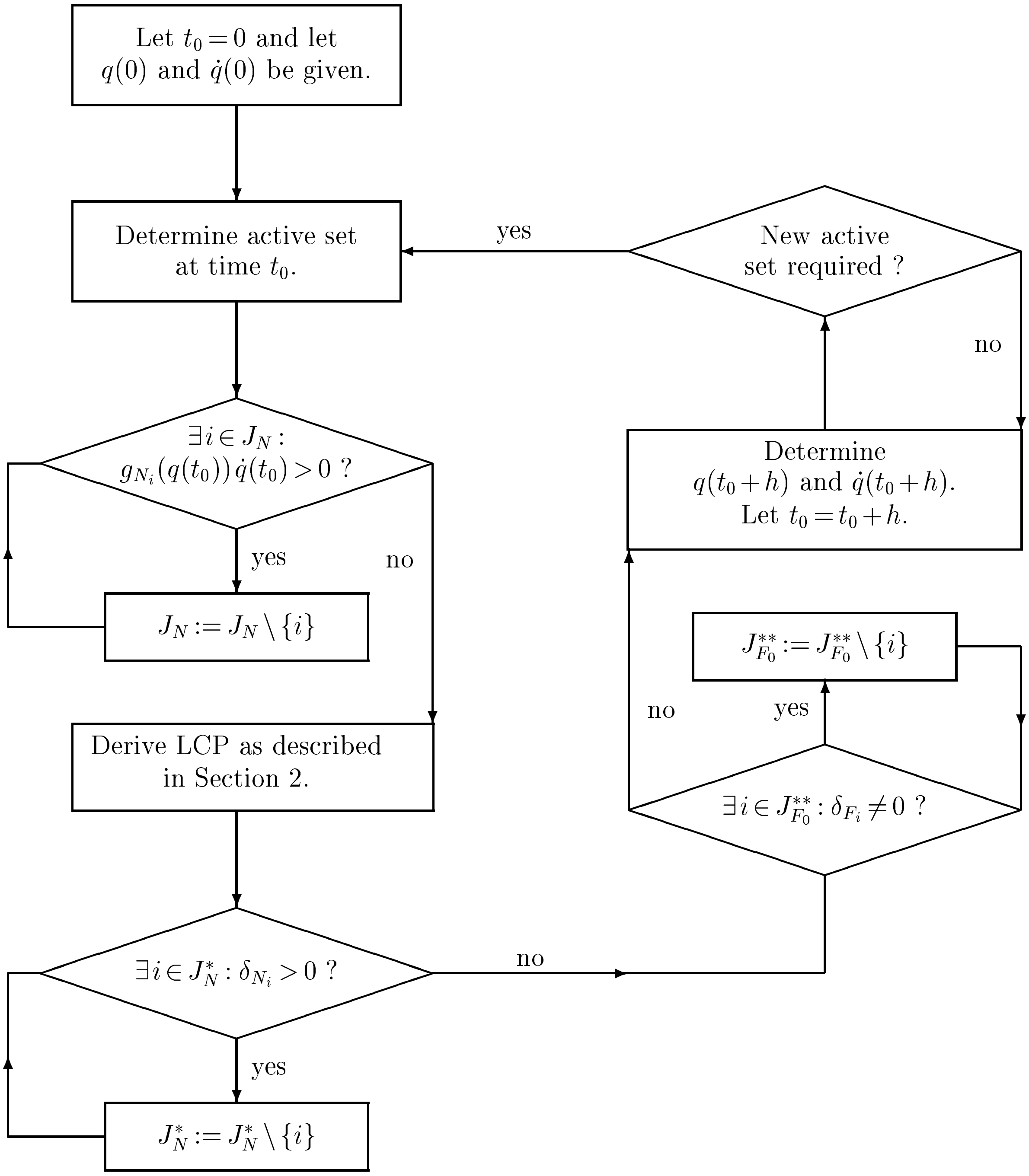

Figure 1: Schematic overview of the algorithm

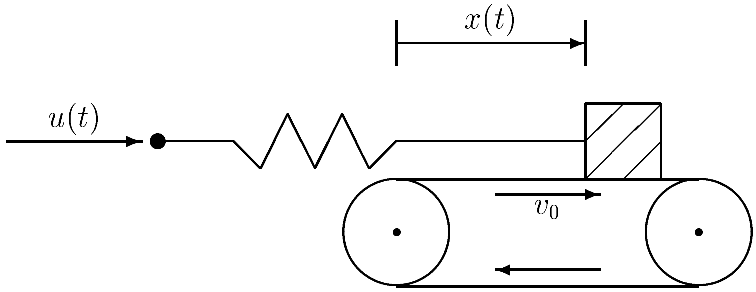

Figure 2: The friction oscillator

provided with the Lagrange multipliers, so that we also obtain a clear picture of the change of normal forces and friction forces.

In Figure 2, the friction oscillator is shown. It consists of a block, sitting on a rugged belt which turns around with constant velocity $v_0 = 1$. The block is attached to a spring, being periodically forced with $u(t) = \alpha \cos(t)$. In this example, as a simple governing equation one may choose (cf. [20])

$$\ddot{x}(t) + x(t) + \beta \operatorname{sgn}(\dot{x}(t) - 1) = \alpha \cos(t),$$

with parameters α, $\beta \geq 0$.

Some phase portraits (in x-$\dot{x}$-plane) obtained by means of our program are given in Figures 3 and 4, and we also plotted the corresponding normal forces resp. the friction forces vs. time.

Figure 3: Phase portrait and Lagrange multipliers for $\alpha = 1$, $q(0) = (2,2)$ and $\dot{q}(0) = (0.85, 0)$

For amplitude $\alpha = 1$, in Figure 3 the block repeatedly sticks on the belt. During such period of time, the friction force attains its maximal value. Afterwards, the block passes over to slipping, with velocity being now less than v_0, cf. also the corresponding friction forces. (Note that the normal force remains constant, since the block is sitting on the belt all the time.)

The picture changes if the amplitude is choosen to be large, cf. Figure 4 below for $\alpha = 10$. In that case, it is almost not possible for the block to stick on the belt (i.e. to have relative velocity $v = 0$), and this behaviour is reflected by the shape of the friction force, since we nearly obtained $\lambda_F = \mu \lambda_N$ for $v < 0$ and $\lambda_F = -\mu \lambda_N$ for $v > 0$, cf. also (9), with our choice of the friction coefficient $\mu = 0.55$.

Figure 4: Phase portrait and Lagrange multipliers for $\alpha = 10$, $q(0) = (2,2)$ and $\dot{q}(0) = (2,0)$

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