# On the asymptotic behaviour of a one-dimensional, monocharged plasma and a rescaling method

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**Abstract**

We consider a one-dimensional, monocharged plasma as described by the Vlasov-Poisson system and investigate the behaviour of the solutions for large times. Using a rescaling method we are able to determine an explicit solution of the system which corresponds to a globally attractive steady state for the rescaled system. We investigate in which sense and at which rate the solutions of the rescaled system converge to this global attractor and interpret the results for the original system.

## 1 Introduction

Consider a large ensemble of particles which are all of the same charge and mass and interact by the electrostatic force field which they create collectively. If we neglect collisional as well as relativistic effects and restrict ourselves to a one-dimensional setting such a monocharged plasma is modelled by the following system of partial differential equations, known as the Vlasov-Poisson system:

$$\partial_t f + v\,\partial_x f + E\,\partial_v f = 0,$$

$$\partial_x E = \rho,\ \rho(t,x) = \int f(t,x,v)\,dv,\ t \geq 0,\ x,v \in \mathbb{R}.$$

Here $t$, $x$, and $v$ denote time, position, and velocity, $f = f(t,x,v)$ denotes the particle distribution function on phase space, $\rho = \rho(t,x)$ is the spatial charge density, and $E = E(t,x)$ the electric field.

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We need to supplement the Vlasov-Poisson system by an initial condition for $f$ and a boundary condition for $E$:

$$f(0)=\mathring{f}$$

for

$$\mathring{f}\in\mathcal{D}:=\left\{g\in C^1(\mathbb{R}^2)\,|\,0\le g\le \mathring{d},\ g(x,v)=0 \text{ for } |x|>\mathring{r} \text{ or } |v|>\mathring{u}\right\};$$

$\mathring{d}>0$, $\mathring{r}>0$, $\mathring{u}>0$ are arbitrary constants which we keep fixed throughout the rest of the paper. As boundary condition for $E$ we request that $E(t,-\infty)=-E(t,\infty)$, and if we denote by

$$\mathring{m}:=\iint \mathring{f}(x,v)\,dv\,dx,\ \mathring{f}\in\mathcal{D},$$

the total charge of the system, this boundary condition for $E$ implies that

$$E(t,x)=-\frac{\mathring{m}}{2}+\int_{-\infty}^{x}\rho(t,y)\,dy,\ t\ge 0,\ x\in\mathbb{R}, \tag{1.1}$$

which yields the a-priori-estimate

$$-\frac{\mathring{m}}{2}\le E(t,x)\le\frac{\mathring{m}}{2},\ t\ge 0,\ x\in\mathbb{R}. \tag{1.2}$$

In the investigation of the Vlasov-Poisson system much progress has been made concerning the question of global existence of classical solutions, and global existence is now known even for the much more difficult three-dimensional case, cf. [1, 6, 10, 12, 14, 16]. These results also give bounds on the possible growth of $\rho$ and $E$ with respect to the supremum norm, both for the plasma physics and the stellar dynamics cases. For the plasma physics case it can actually be shown that these quantities decay as $t\to\infty$. Partial results in this direction were contained in [4, 5], and more recently decay of $\|\rho(t)\|_{5/3}$ and $\|E(t)\|_2$ was established in [8, 11]. The latter results have been used to improve the bounds in the supremum norm for the plasma physics case, cf. [13].

The present paper is devoted to a more detailed investigation of the asymptotic behaviour of the solutions to the relatively simple form of the Vlasov-Poisson system presented above: We wish to determine explicitly the asymptotic profile of $f$, $\rho$, and $E$ for large times.

In Section 2 we prove that $\rho(t)$ is asymptotically $O(t^{-1})$ in the supremum norm. The remaining part of the paper is concerned with the application of the rescaling method outlined by J. R. Burgan, M. R. Feix, E. Fijalkow, and A. Munier in [2]. While rescaling methods are widely used in different areas of mathematical analysis, we do not know of any other application to the area of Vlasov-Poisson or related systems than the recent investigation [3], in which the asymptotic convergence of the solutions of the Vlasov-Fokker-Planck-Poisson system to an equilibrium state was shown.

We prove the conjecture stated in [2], that all solutions of the rescaled system tend to a steady state as a global attractor—this is the aim of Sections 3–5. We thoroughly investigate the kind of convergence and the convergence rates. Section 6 interprets the obtained results for the original system. The rescaling method yields a new explicit solution of the original system which is obtained by applying the inverse transformation to the global attractor of the rescaled system. We disprove the natural conjecture that this is a globally asymptotically stable solution of the original system, which shows that the rescaling method may change the stability of a solution.

Throughout the paper we use the following

**Convention on constants:** Constants denoted by $C$ depend only on the initial data set $\mathcal{D}$, i. e., on the parameters $\mathring{d}$, $\mathring{r}$, $\mathring{u}$, and may change from line to line.

## 2 $L^\infty$-decay of the spatial density

The purpose of this section is to investigate the asymptotic behaviour of the solutions of the Vlasov-Poisson system directly, i. e., without referring to any rescaled system. We prove:

**Theorem 2.1** *For every solution of the Vlasov-Poisson system with $\mathring{f} \in \mathcal{D}$ the following estimate holds:*

$$\|\rho(t)\|_\infty \leq \frac{C}{1+t},\ t \geq 0.$$

*Proof :* Let $(X,V)(s,t,x,v)$ denote the solution of the characteristic system

$$\dot{x} = v,\ \dot{v} = E(s,x)$$

with initial condition $(X,V)(t,t,x,v) = (x,v)$, $t \geq 0$, $x,v \in \mathbb{R}$. It is well known that

$$f(t,x,v) = \mathring{f}((X,V)(0,t,x,v)),\ t \geq 0,\ x,v \in \mathbb{R},$$

and thus

$$\rho(t,x) = \int \mathring{f}((X,V)(0,t,x,v))\, dv,\ t \geq 0,\ x \in \mathbb{R}. \tag{2.1}$$

The idea of the proof is to estimate the derivative $|\partial_v X(0,t,x,v)|$. To this end we define for fixed $t,x,v$,

$$\xi(s) := \partial_v X(s,t,x,v),\ 0 \leq s \leq t.$$

From the characteristic system we conclude that

$$\ddot{\xi} = g(s)\xi,\ \xi(t) = 0,\ \dot{\xi}(t) = 1$$

with $g(s) := \partial_x E(s, X(s,t,x,v)) = \rho(s, X(s,t,x,v)) \geq 0$; at this point the fact that the plasma is monocharged enters. Now choose $a \in [0,t[$ minimal with the property that $\xi < 0$ on the interval $]a,t[$. Then $\ddot{\xi} \leq 0$ on $]a,t[$, and therefore $\dot{\xi} \geq 1$ on $]a,t]$, in particular, $a = 0$ and $\xi < 0$ on $[0,t[$. This in turn implies that $\dot{\xi} \geq 1$ on $[0,t]$ and

$$\xi(s) = -\int_s^t \dot{\xi}(\tau)\, d\tau \leq -(t-s),\ 0 \leq s \leq t,$$

in particular, $\xi(0) \leq -t$ so that

$$|\partial_v X(0,t,x,v)| \geq t,\ t \geq 0.$$

The transformation of variables $v \mapsto X = X(0,t,x,v)$ in (2.1) completes the proof. □

Since $\|\rho(t)\|_1 = \mathring{m}$ for all $t \geq 0$ we obtain the following corollary by interpolation:

**Corollary 2.2** *For every solution of the Vlasov-Poisson system with $\mathring{f} \in \mathcal{D}$ the estimate*

$$\|\rho(t)\|_p \leq C(1+t)^{\frac{1}{p}-1}$$

*holds for all $t \geq 0$ and $p \in [1,\infty]$.*

# 3 The rescaled Vlasov-Poisson system; bounds on the support of the solutions

We introduce the following scaled variables:

$$\tau = \ln(1+t),\ \xi = \frac{x}{(1+t)^2},\ \eta = \frac{v}{1+t} - \frac{2x}{(1+t)^2}.$$

The inverse transformation is given by

$$t = e^\tau - 1,\ x = e^{2\tau}\xi,\ v = e^\tau\eta + 2e^\tau\xi,$$

which implies

$$\left|\frac{\partial(x,v)}{\partial(\xi,\eta)}\right| = e^{3\tau}.$$

If we define

$$\tilde f(\tau,\xi,\eta) := (1+t)^3 f(t,x,v),\ \tilde\rho(\tau,\xi) := (1+t)^2\rho(t,x),\ \tilde E(\tau,\xi) := E(t,x),$$

then the Vlasov-Poisson system takes the form

$$\partial_\tau \tilde f + \eta\partial_\xi \tilde f + \partial_\eta((\tilde E - 2\xi - 3\eta)\tilde f) = 0, \tag{3.1}$$

$$\partial_\xi \tilde E = \tilde\rho,\ \tilde\rho(\tau,\xi) = \int \tilde f(\tau,\xi,\eta)\,d\eta. \tag{3.2}$$

Note that the rescaled Vlasov equation contains a frictional term which will slow the particles down and an external force which will push them into a spatial interval centered at $\xi = 0$; this will be shown in this and the next section. We still have

$$\tilde E(\tau,-\infty) = -\tilde E(\tau,\infty),$$

and since $\|\tilde f(\tau)\|_1 = \|f(t)\|_1 = \|\mathring f\|_1 = \mathring m$ we retain (1.1) and (1.2) for the new system.

We denote by $(\Xi,\mathrm{H})(\sigma,\tau,\xi,\eta)$ the solution of the characteristic system

$$\dot\xi = \eta,\ \dot\eta = \tilde E(\sigma,\xi) - 2\xi - 3\eta$$

with initial condition $(\Xi,\mathrm{H})(\tau,\tau,\xi,\eta) = (\xi,\eta)$, $\tau \geq 0$, $\xi,\eta \in \mathbb{R}$. Then

$$\tilde f(\tau,\xi,\eta) = e^{3\tau}\tilde{\mathring f}\,((\Xi,\mathrm{H})(0,\tau,\xi,\eta)),$$

in particular,

$$\operatorname{supp}\tilde f(\tau) = (\Xi,\mathrm{H})\left(\tau,0,\operatorname{supp}\tilde{\mathring f}\,\right), \tag{3.3}$$

where we define

$$\tilde{\mathring f}\,(\xi,\eta) := \mathring f(\xi,2\xi+\eta)$$

for $\mathring f \in \mathcal{D}$. Using this information we can prove the following estimate on the support of $\tilde f(\tau)$:

**Lemma 3.1** *For every solution $\tilde{f}$ of the rescaled Vlasov-Poisson system with $\mathring{f} \in \mathcal{D}$ we have*

$$\operatorname{supp} \tilde{f}(\tau) \subset [-b(\tau), b(\tau)] \times \mathbb{R}, \ \tau \geq 0,$$

*where*

$$b(\tau) := \frac{\mathring{m}}{4} + \mathring{C} e^{-\tau}$$

*and*

$$\mathring{C} := \sup \left\{ \max(|2\xi + \eta|, |\xi + \eta|) \mid (\xi, \eta) \in \operatorname{supp} \tilde{\mathring{f}} \right\}.$$

*Proof :* Let $(\xi(\tau), \eta(\tau))$ denote a solution of the characteristic system and let

$$h(\tau) := \tilde{E}(\tau, \xi(\tau)) \tag{3.4}$$

so that

$$-\frac{\mathring{m}}{2} \leq h \leq \frac{\mathring{m}}{2}.$$

The characteristic system can then be rewritten in the form

$$\frac{d}{d\tau} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} + \begin{pmatrix} 0 \\ h(\tau) \end{pmatrix}.$$

We solve this linear, inhomogeneous system and obtain the following formula for $\xi(\tau)$ in terms of the function $h$:

$$\xi(\tau) = c_1 e^{-\tau} + c_2 e^{-2\tau} + \int_0^\tau \left( e^{\sigma - \tau} - e^{2(\sigma - \tau)} \right) h(\sigma) \, d\sigma$$

where

$$c_1 = 2\xi(0) + \eta(0), \ c_2 = -\xi(0) - \eta(0).$$

If $(\xi(0), \eta(0)) \in \operatorname{supp} \tilde{\mathring{f}}$ then

$$\begin{aligned} |\xi(\tau)| &\leq \mathring{C} e^{-\tau} + \frac{\mathring{m}}{2} \int_0^\tau \left( e^{\sigma - \tau} - e^{2(\sigma - \tau)} \right) d\sigma \\ &= \mathring{C} e^{-\tau} + \frac{\mathring{m}}{2} \left( \frac{1}{2} - e^{-\tau} + \frac{1}{2} e^{-2\tau} \right) \\ &\leq \frac{\mathring{m}}{4} + \mathring{C} e^{-\tau}, \end{aligned}$$

and the assertion follows. □

# 4 Exponential decay of energy-type quantities

For a solution $\tilde{f}$ of the rescaled Vlasov-Poisson system we define

$$E_{\text{kin}}(\tau) := \iint \eta^2 \tilde{f}(\tau, \xi, \eta) \, d\eta \, d\xi \tag{4.1}$$

and

$$E_{\rm pot}(\tau) := \int_{-b(\tau)}^{b(\tau)} (\tilde{E}(\tau,\xi)-2\xi)^2 d\xi, \tag{4.2}$$

where $b(\tau)$ is defined in Lemma 3.1. The purpose of the present section is to show that these quantities decay exponentially fast in rescaled time. From a physics point of view this is a consequence of the frictional term and the external force in the rescaled Vlasov equation. As a first step we derive a system of differential equations for these quantities which is set up in such a way that the inhomogeneities either have a definite sign or decay exponentially fast.

**Lemma 4.1** *For every solution $\tilde{f}$ of the rescaled Vlasov-Poisson system the following identities hold:*

$$\begin{aligned}
\dot{E}_{\rm kin} &= -6E_{\rm kin} - \dot{E}_{\rm pot} + R_1(\tau),\\
\ddot{E}_{\rm pot} &= -3\dot{E}_{\rm pot} - 4E_{\rm pot} + 4E_{\rm kin}\\
&\quad +3R_1(\tau) + \dot{R}_1(\tau) + R_2(\tau) + R_3(\tau)
\end{aligned}$$

*where*

$$\begin{aligned}
R_1(\tau) &:= \frac{1}{2}(\mathring{m} - 4b(\tau))^2 \dot{b}(\tau),\\
R_2(\tau) &:= 2\int_{-b(\tau)}^{b(\tau)} (\tilde{E}(\tau,\xi)-2\xi)^2 (2-\tilde{\rho}(\tau,\xi))\, d\xi,\\
R_3(\tau) &:= 2\int \left(\tilde{\jmath}^2(\tau,\xi) - \tilde{\rho}(\tau,\xi)\int \eta^2 \tilde{f}(\tau,\xi,\eta)\, d\eta\right) d\xi,
\end{aligned}$$

*and*

$$\tilde{\jmath}(\tau,\xi) := \int \eta \tilde{f}(\tau,\xi,\eta)\, d\eta.$$

*Proof :* Using the rescaled Vlasov equation we have

$$\begin{aligned}
\dot{E}_{\rm kin} &= \iint \eta^2 \left(-\eta\partial_\xi \tilde{f} - \partial_\eta((\tilde{E}-2\xi-3\eta)\tilde{f})\right) d\eta\, d\xi\\
&= 2\int (\tilde{E}-2\xi)\int \eta \tilde{f}\, d\eta\, d\xi - 6\iint \eta^2 \tilde{f}\, d\eta\, d\xi\\
&= 2\int (\tilde{E}-2\xi)\tilde{\jmath}\, d\xi - 6E_{\rm kin}\,.
\end{aligned} \tag{4.3}$$

Integrating the rescaled Vlasov equation with respect to $\eta$ yields the conservation law

$$\partial_\tau \tilde{\rho} + \partial_\xi \tilde{\jmath} = 0,$$

so that by (1.1)

$$\partial_\tau \tilde{E}(\tau,\xi) = \int_{-\infty}^{\xi} \partial_\tau \tilde{\rho}(\tau,\zeta)\, d\zeta = -\int_{-\infty}^{\xi} \partial_\zeta \tilde{\jmath}(\tau,\zeta)\, d\zeta = -\tilde{\jmath}(\tau,\xi),$$

and

$$\begin{aligned}
\dot{E}_{\rm pot} &= (\tilde{E}-2\xi)^2\Big|_{\xi=b(\tau)} \dot{b}(\tau) - (\tilde{E}-2\xi)^2\Big|_{\xi=-b(\tau)} (-\dot{b}(\tau))\\
&\quad +2\int_{-b(\tau)}^{b(\tau)} (\tilde{E}-2\xi)\,\partial_\tau \tilde{E}\, d\xi\\
&= R_1(\tau) - 2\int (\tilde{E}-2\xi)\tilde{\jmath}\, d\xi;
\end{aligned} \tag{4.4}$$

note that $\tilde{E}(\tau,\pm b(\tau)) = \pm\mathring{m}/2$. By adding (4.3) and (4.4) we obtain the first assertion of the lemma. Differentiating (4.4) again we have

$$\ddot{E}_{\rm pot} = \dot{R}_1 + 2\int \tilde{\jmath}^2 d\xi - 2\int (\tilde{E} - 2\xi)\partial_\tau \tilde{\jmath}\, d\xi.$$

The rescaled Vlasov equation yields

$$\begin{aligned}\partial_\tau \tilde{\jmath} &= -\int \eta\Big(\eta\partial_\xi \tilde{f} + \partial_\eta((\tilde{E} - 2\xi - 3\eta)\tilde{f})\Big)\, d\eta \\ &= -\partial_\xi \int \eta^2 \tilde{f}\, d\eta + \int (\tilde{E} - 2\xi - 3\eta)\tilde{f}\, d\eta \\ &= -\partial_\xi \int \eta^2 \tilde{f}\, d\eta + (\tilde{E} - 2\xi)\,\tilde{\rho} - 3\tilde{\jmath},\end{aligned}$$

so that

$$\begin{aligned}\ddot{E}_{\rm pot} &= \dot{R}_1 + 2\int \tilde{\jmath}^2 d\xi + 2\int (\tilde{E} - 2\xi)\partial_\xi \int \eta^2 \tilde{f}\, d\eta\, d\xi \\ &\quad -2\int (\tilde{E} - 2\xi)^2 \tilde{\rho}\, d\xi + 6\int (\tilde{E} - 2\xi)\,\tilde{\jmath}\, d\xi \\ &= \dot{R}_1 + 2\int \tilde{\jmath}^2 d\xi - 2\int \tilde{\rho}\int \eta^2 \tilde{f}\, d\eta\, d\xi + 4E_{\rm kin} \\ &\quad -2\int (\tilde{E} - 2\xi)^2 \tilde{\rho}\, d\xi + 3R_1(\tau) - 3\dot{E}_{\rm pot},\end{aligned}$$

and the proof is complete. $\square$

Next we establish some estimates on the quantities $R_1,\ R_2,\ R_3$ introduced above.

**Lemma 4.2** *Let $\tilde{f}$ be a solution of the rescaled Vlasov-Poisson system with $\mathring{f} \in \mathcal{D}$ and let $R_1,\ R_2$, and $R_3$ be defined as in Lemma 4.1. Then the estimates*

$$|R_1(\tau)| + |\dot{R}_1(\tau)| + |R_2(\tau)| \le Ce^{-3\tau},$$

*and*

$$R_3(\tau) \le 0$$

*hold for $\tau \ge 0$.*

*Proof :* The estimates for $R_1$ and $\dot{R}_1$ follow immediately from the definitions of $R_1$ and $b$, cf. Lemma 3.1. Next observe that

$$\begin{aligned}|R_2(\tau)| &= \left|2\int_{-b(\tau)}^{b(\tau)} (\tilde{E} - 2\xi)^2(\tilde{\rho} - 2)\, d\xi\right| = \left|2\int_{-b(\tau)}^{b(\tau)} (\tilde{E} - 2\xi)^2 \partial_\xi(\tilde{E} - 2\xi)\, d\xi\right| \\ &= \frac{2}{3}\left|\int_{-b(\tau)}^{b(\tau)} \partial_\xi(\tilde{E} - 2\xi)^3 d\xi\right| \\ &\le \frac{2}{3}\left|(\tilde{E} - 2\xi)^3\Big|_{\xi = b(\tau)}\right| + \frac{2}{3}\left|(\tilde{E} - 2\xi)^3\Big|_{\xi = -b(\tau)}\right| \\ &= \frac{4}{3}\left(2b(\tau) - \frac{\mathring{m}}{2}\right)^3 \le Ce^{-3\tau}.\end{aligned}$$

Finally the Cauchy-Schwarz inequality implies that

$$|\tilde{\jmath}(\tau,\xi)| \leq \int |\eta| \tilde{f}(\tau,\xi,\eta)\, d\eta \leq \left(\int \eta^2 \tilde{f}(\tau,\xi,\eta)\, d\eta\right)^{1/2} \left(\int \tilde{f}(\tau,\xi,\eta)\, d\eta\right)^{1/2},$$

whence $R_3 \leq 0$, and the proof is complete. $\square$

Combining the estimates in the previous lemma with the differential equations in Lemma 4.1 we are now able to establish the desired exponential decay of $E_{\rm kin}$ and $E_{\rm pot}$.

**Lemma 4.3** *For every solution $\tilde{f}$ of the rescaled Vlasov-Poisson system with $\mathring{f} \in \mathcal{D}$ the estimate*

$$E_{\rm kin}(\tau) + E_{\rm pot}(\tau) \leq C e^{-2\tau}$$

*holds for all $\tau \geq 0$.*

*Proof :* By Lemma 4.1 the triple $Z := (E_{\rm pot}, \dot{E}_{\rm pot}, E_{\rm kin})$ satisfies an inhomogeneous linear system of differential equations of the form

$$\dot{Z} = AZ + W(\tau)$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -4 & -3 & 4 \\ 0 & -1 & -6 \end{pmatrix}$$

and

$$W(\tau) = \Big(0,\ 3R_1(\tau) + \dot{R}_1(\tau) + R_2(\tau) + R_3(\tau),\ R_1(\tau)\Big).$$

Thus

$$Z(\tau) = X(\tau) Z_0 + \int_0^\tau X(\tau - \sigma) W(\sigma)\, d\sigma,$$

where $X(\tau)$ is the solution of $\dot{X} = AX$, $X(0) = \mathrm{id}$ and $Z_0$ is determined by $\tilde{\mathring{f}}$. We are only interested in the first component of $Z$, and observing that the eigenvalues of the matrix $A$ are $-2, -3, -4$ we obtain

$$\begin{aligned} E_{\rm pot}(\tau) &= c_1 e^{-2\tau} + c_2 e^{-3\tau} + c_3 e^{-4\tau} \\ &\quad + \int_0^\tau \left(2e^{-2(\tau-\sigma)} - 3e^{-3(\tau-\sigma)} + e^{-4(\tau-\sigma)}\right) \\ &\qquad\qquad \Big(3R_1(\sigma) + \dot{R}_1(\sigma) + R_2(\sigma) + R_3(\sigma)\Big)\, d\sigma \\ &\quad + \int_0^\tau \left(2e^{-2(\tau-\sigma)} - 4e^{-3(\tau-\sigma)} + 2e^{-4(\tau-\sigma)}\right) R_1(\sigma)\, d\sigma \\ &\leq C e^{-2\tau} + C \int_0^\tau e^{-2(\tau-\sigma)} e^{-3\sigma} d\sigma \\ &\leq C e^{-2\tau}. \end{aligned}$$

Here we have used the estimates from Lemma 4.2; note that with $e^\tau = 1 + \vartheta$ where $\vartheta \geq 0$ for $\tau \geq 0$ we have

$$2e^{-2\tau} - 3e^{-3\tau} + e^{-4\tau} = \frac{\vartheta(1 + 2\vartheta)}{(1+\vartheta)^4} \geq 0,$$

so that the product of this term with $R_3$ in the above estimate can be dropped. Therefore, the estimate for $E_{\rm pot}$ is established. Integrating the differential equation for $E_{\rm kin}$ in Lemma 4.1 we obtain

$$\begin{aligned}
E_{\rm kin}(\tau) &= E_{\rm kin}(0)e^{-6\tau} + \int_0^\tau e^{-6(\tau-\sigma)}\left(R_1(\sigma) - \dot{E}_{\rm pot}(\sigma)\right) d\sigma \\
&= (E_{\rm kin}(0) + E_{\rm pot}(0))e^{-6\tau} - E_{\rm pot}(\tau) \\
&\quad + \int_0^\tau e^{-6(\tau-\sigma)}(R_1(\sigma) + 6E_{\rm pot}(\sigma))\, d\sigma \\
&\leq Ce^{-6\tau} + C\int_0^\tau e^{-6(\tau-\sigma)}\left(e^{-3\sigma} + e^{-2\sigma}\right) d\sigma \\
&\leq Ce^{-2\tau},
\end{aligned}$$

and the proof is complete. □

As an obvious corollary of the decay of the kinetic energy and the conservation of $\|\tilde{f}(\tau)\|_1$ we obtain by the Cauchy-Schwarz inequality:

**Corollary 4.4** *For every solution $\tilde{f}$ of the rescaled Vlasov-Poisson system with $\mathring{f} \in \mathcal{D}$,*

$$\iint |\eta|\tilde{f}(\tau,\xi,\eta)\, d\eta\, d\xi \leq Ce^{-\tau},\ \tau \geq 0.$$

# 5 The asymptotic behaviour of the rescaled system

Intuitively, we may interpret Lemma 3.1 and the estimates for $E_{\rm kin}(\tau)$ and $E_{\rm pot}(\tau)$ in Lemma 4.3 as saying that the final state of the evolution governed by the rescaled system is given by

$$\tilde{f}_\infty(\xi,\eta) := 2\chi_{[-\mathring{m}/4,\mathring{m}/4]}(\xi)\delta(\eta),$$

$$\tilde{\rho}_\infty(\xi) := 2\chi_{[-\mathring{m}/4,\mathring{m}/4]}(\xi),$$

$$\tilde{E}_\infty(\xi) := \begin{cases} -\mathring{m}/2 \ , x \leq -\mathring{m}/4, \\ \quad 2\xi \quad , -\mathring{m}/4 \leq \xi \leq \mathring{m}/4, \\ \ \mathring{m}/2 \quad , \xi \geq \mathring{m}/4. \end{cases}$$

Here $\chi_M$ denotes the characteristic function of the set $M$, and $\delta$ is the Dirac measure centered at $\eta = 0$. This triple is a measure solution of the rescaled system: the equations (3.2) are satisfied classically—the first one with the exception of $\xi = \pm\mathring{m}/4$, where $\tilde{E}_\infty$ is not differentiable—whereas (3.1) is satisfied in the sense that for all $\varphi \in C^1(\mathbb{R}^2)$,

$$\begin{aligned}
&\iint \left(\eta\partial_\xi\varphi(\xi,\eta) + (\tilde{E}_\infty(\xi) - 2\xi - 3\eta)\partial_\eta\varphi(\xi,\eta)\right)\tilde{f}_\infty(\xi,\eta)\, d\eta\, d\xi \\
&\qquad = 2\int_{-\mathring{m}/4}^{\mathring{m}/4} \left(\eta\partial_\xi\varphi(\xi,\eta) + (\tilde{E}_\infty(\xi) - 2\xi - 3\eta)\partial_\eta\varphi(\xi,\eta)\right)\Big|_{\eta=0} d\xi = 0.
\end{aligned}$$

In particular, $\tilde{f}_\infty$ is a distributional solution of (3.1), and the points in $\operatorname{supp}\tilde{f}_\infty = [-\mathring{m}/4, \mathring{m}/4] \times 0$ are all stationary with respect to the flow induced by the characteristic system of (3.1).

The following theorem and its corollaries show that $(\tilde{f}_\infty, \tilde{\rho}_\infty, \tilde{E}_\infty)$ is indeed a global attractor for the rescaled system, and we investigate in which sense and at which rate $(\tilde{f}(\tau), \tilde{\rho}(\tau), \tilde{E}(\tau))$ tends to $(\tilde{f}_\infty, \tilde{\rho}_\infty, \tilde{E}_\infty)$ as $\tau \to \infty$; $\|\cdot\|_p$ denotes the norm in $L^p(\mathbb{R})$ or $L^p(\mathbb{R}^2)$ respectively for $p \in [1,\infty]$.

**Theorem 5.1** *Let $\tilde{f}$ be a solution of the rescaled Vlasov-Poisson system with $\mathring{f} \in \mathcal{D}$.*

(a) *For $\tau \geq 0$,*

$$\|\tilde{E}(\tau) - \tilde{E}_\infty\|_2 \leq C e^{-\tau}.$$

(b) *For all $\varphi \in C^1(\mathbb{R})$ such that the norms appearing below are finite and for all $\tau \geq 0$,*

$$\left| \int (\tilde{\rho}(\tau,\xi) - \tilde{\rho}_\infty(\xi))\, \varphi(\xi)\, d\xi \right| \leq C \|\varphi'\|_2 e^{-\tau},$$

*and*

$$\left| \int (\tilde{\rho}(\tau,\xi) - \tilde{\rho}_\infty(\xi))\, \varphi(\xi)\, d\xi \right| \leq C \|\varphi'\|_\infty e^{-\tau}.$$

(c) *For all $\varphi \in C^1(\mathbb{R}^2)$ such that the norms appearing below are finite and for all $\tau \geq 0$,*

$$\left| \iint \tilde{f}(\tau,\xi,\eta)\, \varphi(\xi,\eta)\, d\eta\, d\xi - 2\int_{-\mathring{m}/4}^{\mathring{m}/4} \varphi(\xi,0)\, d\xi \right| \leq C \|\partial_{(\xi,\eta)}\varphi\|_\infty e^{-\tau}.$$

*Proof :* To derive (a) from Lemma 4.3 we use the estimate

$$\int_{-b(\tau)}^{b(\tau)} (\tilde{E}_\infty(\xi) - 2\xi)^2\, d\xi \leq \frac{8}{3} \mathring{C}^3 e^{-3\tau}.$$

Using Lemma 3.1 and integration by parts we obtain for all $\varphi \in C^1(\mathbb{R})$,

$$\begin{aligned} \left| \int \big(\tilde{\rho}(\tau,\xi) - \tilde{\rho}_\infty(\xi)\big)\, \varphi(\xi)\, d\xi \right| &= \left| \int \partial_\xi (\tilde{E}(\tau,\xi) - \tilde{E}_\infty(\xi))\, \varphi(\xi)\, d\xi \right| \\ &= \left| \int (\tilde{E}(\tau,\xi) - \tilde{E}_\infty(\xi))\, \varphi'(\xi)\, d\xi \right| \\ &\leq \|\tilde{E}(\tau) - \tilde{E}_\infty\|_2 \|\varphi'\|_2 \leq C e^{-\tau} \|\varphi'\|_2 \end{aligned}$$

cf. Lemma 4.3. This proves the first estimate in (b). The second estimate in (b) is obtained if instead of using the Cauchy-Schwarz inequality one takes $\|\varphi'\|_\infty$ out of the integral above and uses the fact that $\|\tilde{E}(\tau) - \tilde{E}_\infty\|_1$ can be estimated by the $L^2$-norm of $\tilde{E}(\tau) - \tilde{E}_\infty$, since by Lemma 3.1 this difference has compact support, uniformly in $\tau$. As to (c), we have

$$\begin{aligned} &\left| \iint \tilde{f}(\tau,\xi,\eta)\, \varphi(\xi,\eta)\, d\eta\, d\xi - 2\int_{-\mathring{m}/4}^{\mathring{m}/4} \varphi(\xi,0)\, d\xi \right| \\ &\qquad \leq \iint \tilde{f}(\tau,\xi,\eta) |\varphi(\xi,\eta) - \varphi(\xi,0)|\, d\eta\, d\xi \\ &\qquad\quad + \left| \int \tilde{\rho}(\tau,\xi)\, \varphi(\xi,0)\, d\xi - 2\int_{-\mathring{m}/4}^{\mathring{m}/4} \varphi(\xi,0)\, d\xi \right| \\ &\qquad \leq \|\partial_\eta \varphi\|_\infty \iint |\eta|\, \tilde{f}(\tau,\xi,\eta)\, d\eta\, d\xi + C \|\partial_\xi \varphi(\cdot,0)\|_\infty e^{-\tau} \\ &\qquad \leq C \Big( \|\partial_\xi \varphi(\cdot,0)\|_\infty + \|\partial_\eta \varphi\|_\infty \Big) e^{-\tau} \end{aligned}$$

for every $\varphi \in C^1(\mathbb{R}^2)$ by Corollary 4.4. $\square$

There are now various possible topologies in which to express the convergence result contained in parts (b) and (c) of the theorem above, and we give three such possibilities. Using the fact that the continuously differentiable functions are dense in $W_2^1(\mathbb{R}^n)$ and $W_\infty^1(\mathbb{R}^n)$ $(n=1,2)$ we get as a first possibility

**Corollary 5.2** *Under the above assumptions on $\tilde{f}$ the estimates*

$$\|\tilde{\rho}(\tau)-\tilde{\rho}_\infty\|_{W_2^1(\mathbb{R})'} \le C e^{-\tau},$$

$$\|\tilde{\rho}(\tau)-\tilde{\rho}_\infty\|_{W_\infty^1(\mathbb{R})'} \le C e^{-\tau},$$

*and*

$$\|\tilde{f}(\tau)-\tilde{f}_\infty\|_{W_\infty^1(\mathbb{R}^2)'} \le C e^{-\tau},$$

*hold for all $\tau \ge 0$.*

Next, if we consider the metric $d$ on the space $\mathcal{M}_0(\mathbb{R}^n)$ of positive measures with total mass $\mathring{m}$ $(n=1,2)$, which is given by the so-called *bounded Lipschitz distance*

$$d(\mu,\nu) := \sup\Big\{\Big|\int \varphi\, d\mu - \int \varphi\, d\nu\Big| \;\Big|\; \varphi \in B(\mathbb{R}^n)\Big\},$$

where

$$B(\mathbb{R}^n) := \Big\{\varphi : \mathbb{R}^n \to [0,1] \;\big|\; |\varphi(x)-\varphi(y)| \le |x-y|,\ x,y \in \mathbb{R}^n\Big\},$$

we obtain

**Corollary 5.3** *Under the above assumptions on $\tilde{f}$ the estimate*

$$d(\tilde{\rho}(\tau),\tilde{\rho}_\infty) + d(\tilde{f}(\tau),\tilde{f}_\infty) \le C e^{-\tau}$$

*holds for all $\tau \ge 0$.*

Let $C_b(\mathbb{R}^n)$ denote the space of bounded continuous functions on $\mathbb{R}^n$ with the supremum norm and $C_b^1(\mathbb{R}^n)$ the space of continuously differentiable functions which are bounded together with their first order derivatives. In our third corollary to Theorem 5.1 we consider $\mathcal{M}_0(\mathbb{R}^n)$ as a subset of the dual space of $C_b(\mathbb{R}^n)$ $(n=1,2)$:

**Corollary 5.4** *Under the above assumptions on $\tilde{f}$,*

$$\tilde{\rho}(\tau) \to \tilde{\rho}_\infty,\ \tilde{f}(\tau) \to \tilde{f}_\infty$$

*for $\tau \to \infty$ in the weak*-topology on $\mathcal{M}_0(\mathbb{R})$ or $\mathcal{M}_0(\mathbb{R}^2)$, respectively.*

*Proof :* For $\varphi \in C_b(\mathbb{R})$ there exists $\psi \in C_b^1(\mathbb{R})$ such that $\|\varphi-\psi\|_\infty$ is arbitrarily small. From Theorem 5.1 (b) we obtain the estimate

$$\begin{aligned}\Big|\int (\tilde{\rho}(\tau)-\tilde{\rho}_\infty)\varphi\Big| &\le \Big|\int (\tilde{\rho}(\tau)-\tilde{\rho}_\infty)\psi\Big| + \Big|\int (\tilde{\rho}(\tau)-\tilde{\rho}_\infty)(\psi-\varphi)\Big| \\ &\le C\|\psi'\|_\infty e^{-\tau} + 2\mathring{m}\|\varphi-\psi\|_\infty,\end{aligned}$$

and this implies the assertion for $\tilde{\rho}$. The proof for $\tilde{f}$ is completely analogous. $\square$

Note that this result does not give an estimate for the rate of convergence, but it has the advantage that the test functions which are used need not be differentiable or Lipschitz.

## 6 The asymptotic behaviour of the original system

In this final section we exploit the asymptotic estimates obtained for the rescaled system to describe the asymptotic behaviour of the original Vlasov-Poisson system. As a direct consequence of Theorem 5.1 and its corollaries we obtain the following two results:

**Theorem 6.1** *Let $f$ be a solution of the Vlasov-Poisson system with $\mathring{f}\in\mathcal{D}$.*

(a) *For every $t\geq 0$:*

$$\|E(t,(1+t)^2 x)-\tilde{E}_\infty\|_2\leq\frac{C}{1+t}.$$

(b) *For all $\varphi\in C^1(\mathrm{IR})$ such that the norms appearing below are finite and for all $t\geq 0$,*

$$\left|\int(1+t)^2\rho(t,(1+t)^2x)\varphi(x)\,dx-2\int_{-\mathring{m}/4}^{\mathring{m}/4}\varphi(x)\,dx\right|\leq C\,\|\varphi'\|_2\frac{1}{1+t},$$

*and*

$$\left|\int(1+t)^2\rho(t,(1+t)^2x)\varphi(x)\,dx-2\int_{-\mathring{m}/4}^{\mathring{m}/4}\varphi(x)\,dx\right|\leq C\,\|\varphi'\|_\infty\frac{1}{1+t}.$$

(c) *For all $\varphi\in C^1(\mathrm{IR}^2)$ such that the norms appearing below are finite and for all $t\geq 0$,*

$$\left|\iint(1+t)^3 f(t,(1+t)^2x,(1+t)v)\varphi(x,v)\,dv\,dx-2\int_{-\mathring{m}/4}^{\mathring{m}/4}\varphi(x,2x)\,dx\right|$$
$$\leq C\,\|\partial_{(x,v)}\varphi\|_\infty\frac{1}{1+t}.$$

*Proof :* The assertions in (a) and (b) follow directly from the corresponding assertions in Theorem 5.1. As to (c), we have by definition of the rescaling transformation

$$\left|\iint(1+t)^3 f(t,(1+t)^2\xi,(1+t)\eta+2(1+t)\xi)\varphi(\xi,\eta)\,d\eta\,d\xi-2\int_{-\mathring{m}/4}^{\mathring{m}/4}\varphi(\xi,0)\,d\xi\right|$$
$$\leq C\,\|\partial_{(\xi,\eta)}\varphi\|_\infty\frac{1}{1+t}.$$

With $x=\xi$, $v=2\xi+\eta$, and $\psi(x,v)=\varphi(x,v-2x)$ this estimate can be rewritten as

$$\left|\iint(1+t)^3 f(t,(1+t)^2x,(1+t)v)\psi(x,v)\,dv\,dx-2\int_{-\mathring{m}/4}^{\mathring{m}/4}\psi(x,2x)\,dx\right|$$
$$\leq C\,\|\partial_{(x,v)}\varphi\|_\infty\frac{1}{1+t},$$

and since the mapping $\varphi\mapsto\psi$ is bijective on $C^1(\mathrm{IR}^2)$ with $\|\partial_{(x,v)}\varphi\|_\infty\leq 2\|\partial_{(x,v)}\psi\|_\infty$ the assertion in (c) follows. $\square$

**Corollary 6.2** *Let $f$ be a solution of the Vlasov-Poisson system $\mathring{f}\in\mathcal{D}$. Then the estimates*

$$\left\|(1+t)^2\rho(t,(1+t)^2\cdot)-2\chi_{[-\mathring{m}/4,\mathring{m}/4]}\right\|_{W_2^1(\mathrm{IR})'}\leq\frac{C}{1+t},$$

$$\left\|(1+t)^2\rho(t,(1+t)^2\cdot)-2\chi_{[-\mathring{m}/4,\mathring{m}/4]}\right\|_{W^1_\infty(\mathrm{I\!R})'}\leq\frac{C}{1+t},$$

$$\left\|(1+t)^3 f(t,(1+t)^2\cdot,(1+t)\cdot)-2\chi_{[-\mathring{m}/4,\mathring{m}/4]}\delta(v-2x)\right\|_{W^1_\infty(\mathrm{I\!R})'}\leq\frac{C}{1+t},$$

$$d\Big((1+t)^2\rho(t,(1+t)^2\cdot),2\chi_{[-\mathring{m}/4,\mathring{m}/4]}\Big)\leq\frac{C}{1+t},$$

*and*

$$d\Big((1+t)^3 f(t,(1+t)^2\cdot,(1+t)\cdot),2\chi_{[-\mathring{m}/4,\mathring{m}/4]}\delta(v-2x)\Big)\leq\frac{C}{1+t}$$

*hold for all* $t\geq 0$. *Moreover,*

$$(1+t)^2\rho(t,(1+t)^2\cdot)\to 2\chi_{[-\mathring{m}/4,\mathring{m}/4]}$$

*and*

$$(1+t)^3 f(t,(1+t)^2 x,(1+t)v)\to 2\chi_{[-\mathring{m}/4,\mathring{m}/4]}\delta(v-2x)$$

*for* $t\to\infty$ *in the weak*$^*$*-topology on* $\mathcal{M}_0(\mathrm{I\!R})$ *or* $\mathcal{M}_0(\mathrm{I\!R}^2)$, *respectively.*

We note that the direct translation of the results of the previous section gives us the asymptotic behaviour for $f$, $\rho$, and $E$ in substituted variables. The attempt to deduce asymptotic results for $f$, $\rho$, and $E$ in the original variables fails. For example, if we substitute $y:=(1+t)^2x$ and $\psi(y):=\varphi(y/(1+t)^2)$ in the first estimate of Theorem 6.1 (b) we get

$$\left|\int\rho(t,y)\psi(y)\,dy-\frac{2}{(1+t)^2}\int_{-\mathring{m}(1+t)^2/4}^{\mathring{m}(1+t)^2/4}\psi(y)\,dy\right|\leq C\,\|\psi'\|_2.$$

This observation leads to the conjecture that the triple

$$t\mapsto(f_\infty(t),\rho_\infty(t),E_\infty(t)),$$

which corresponds to the attractor $(\tilde f_\infty,\tilde\rho_\infty,\tilde E_\infty)$ in the rescaled system, might not be globally asymptotically stable. In the subsequent part of this section we calculate $(f_\infty(t),\rho_\infty(t),E_\infty(t))$ explicitly, show that this triple is a solution of the original system in a well defined sense, and prove that for $T>0$,

$$f_\infty(t+T)-f_\infty(t)$$

and

$$E_\infty(t+T)-E_\infty(t)$$

do not tend to zero for $t\to\infty$; note that since the Vlasov-Poisson system is autonomous, we obtain another solution by shifting the solution $(f_\infty,\rho_\infty,E_\infty)$ in time.

Since

$$\eta=0\Leftrightarrow v=2x/(1+t)$$

we get

$$f_\infty(t,x,v)=\frac{2}{(1+t)^2}\,\chi_{[-\mathring{m}(1+t)^2/4,\mathring{m}(1+t)^2/4]}(x)\delta\Big(v-\frac{2x}{1+t}\Big),$$

$$\rho_\infty(t,x)=\frac{2}{(1+t)^2}\,\chi_{[-\mathring{m}(1+t)^2/4,\mathring{m}(1+t)^2/4]}(x),$$

$$E_\infty(t,x) = \begin{cases} -\mathring{m}/2 & , x \le -\mathring{m}(1+t)^2/4, \\ 2x/(1+t)^2 & , -\mathring{m}(1+t)^2/4 \le x \le \mathring{m}(1+t)^2/4, \\ \mathring{m}/2 & , x \ge \mathring{m}(1+t)^2/4. \end{cases}$$

Denote by $\mu_t$ the positive measure on $\mathbb{R}^2$ associated to $f_\infty(t)$. Now $E_\infty$ is uniformly Lipschitz, in particular $E_\infty(t)$ is absolutely continuous, and $E_\infty$ and $\rho_\infty$ satisfy Poisson's equation a. e. The characteristic system

$$\dot{x} = v, \;\; \dot{v} = E_\infty(t,x)$$

is classically uniquely solvable for given initial data, and the measure $\mu_t$ satisfies Vlasov's equation in the sense that it is invariant under the induced flow: Let $M \subset \mathbb{R}^2$ be measurable. Then

$$\mu_t(M) = \frac{2}{(1+t)^2} \mathrm{vol} \left\{ x \in [-\mathring{m}(1+t)^2/4, \mathring{m}(1+t)^2/4] \,|\, (x, 2x/(1+t)) \in M \right\}$$

whereas

$$\begin{aligned} \mu_0((X,V)(0,t,M)) &= 2\mathrm{vol}\, \{ x \in [-\mathring{m}/4, \mathring{m}/4] \,|\, (x,2x) \in (X,V)(0,t,M) \} \\ &= 2\mathrm{vol}\, \{ x \in [-\mathring{m}/4, \mathring{m}/4] \,|\, (X,V)(t,0,x,2x) \in M \} \\ &= 2\mathrm{vol} \left\{ x \in [-\mathring{m}/4, \mathring{m}/4] \,|\, ((1+t)^2 x, 2(1+t)x) \in M \right\} \\ &= 2\mathrm{vol} \left\{ y/(1+t)^2 \in [-\mathring{m}/4, \mathring{m}/4] \,|\, (y, 2y/(1+t)) \in M \right\} \\ &= \mu_t(M); \end{aligned}$$

one can immediately verify that for $x \in [-\mathring{m}/4, \mathring{m}/4]$,

$$(X,V)(t,0,x,2x) = ((1+t)^2 x, 2(1+t)x).$$

Finally, the measure $\mu_t$ induces, through integration with respect to $v$, a measure on $\mathbb{R}$ with density $\rho_\infty$.

Therefore, $(f_\infty, \rho_\infty, E_\infty)$ is indeed a solution of the Vlasov-Poisson system, which is of a very special structure, namely it is a fluid solution with pressure zero. By this we mean that if we define the velocity field $u_\infty$ by

$$u_\infty(t,x) := \frac{2x}{1+t}$$

so that

$$\rho_\infty(t,x)\, u_\infty(t,x) = \int v\, f_\infty(t,x,v)\, dv$$

then the triple $(\rho_\infty, u_\infty, E_\infty)$ satisfies the Euler-Poisson system with pressure zero

$$\partial_t \rho + \partial_x (u\rho) = 0,$$

$$\partial_t u + u\, \partial_x u = E,$$

$$\partial_x E = \rho$$

in a corresponding sense. For further information on the Euler-Poisson system with pressure zero and on its relation to the Vlasov-Poisson system we refer to [15].

Now we show that $t \mapsto (f_\infty(t), \rho_\infty(t), E_\infty(t))$ is not globally asymptotically stable:

**Proposition 6.3** *Let $T>0$. Then*

(a) $\|E_\infty(t+T)-E_\infty(t)\|_2 \geq C\dfrac{T^2}{(1+T)^4},\ t\geq 0,$

(b) $\exists \varphi\in C_b^1(\mathrm{I\!R}^2):\ \langle f_\infty(t+T)-f_\infty(t),\varphi\rangle \not\to 0,\ t\to\infty.$

*Proof :* As to (a), we have the estimate

$$\begin{aligned}
\|E_\infty(t+T)-E_\infty(t)\|_2^2 &\geq 2\int_0^{\mathring{m}(1+t)^2/4}\left(\frac{2x}{(1+t)^2}-\frac{2x}{(1+t+T)^2}\right)^2 dx\\
&= 8\left(\frac{1}{(1+t)^2}-\frac{1}{(1+t+T)^2}\right)^2 \frac{x^3}{3}\Big|_0^{\mathring{m}(1+t)^2/4}\\
&= C(1+t)^6\left(\frac{2T(1+t)+T^2}{(1+t)^2(1+t+T)^2}\right)^2\\
&\geq C\frac{T^2}{(1+T)^4},\ t\geq 0.
\end{aligned}$$

Next we prove (b). Beginning with $t_1:=1$ we inductively define a sequence $t_n\nearrow\infty$ with the property

$$1+\frac{\mathring{m}}{4}(1+t_{n-1}+T)^2\leq \frac{\mathring{m}}{12}(1+t_n)^2,\ n=2,3,\ldots.$$

For $t\geq 0$ define the interval

$$I(t):=[0,\mathring{m}(1+t)^2/4].$$

Then

$$I(t_{n-1})\subset I(t_{n-1}+T)\subset \frac{1}{3}I(t_n),\ n=2,3,\ldots,$$

and with respect to the Euclidean distance dist in $\mathrm{I\!R}^2$ the vertical strips

$$I(t_{n-1}+T)\times \mathrm{I\!R}\ \text{and}\ \left(I(t_n)\setminus\frac{1}{3}I(t_n)\right)\times\mathrm{I\!R}$$

have distance greater or equal to 1. The support of $f_\infty(t)$ is the straight line between the points

$$\pm\left(\frac{\mathring{m}}{4}(1+t)^2,\frac{\mathring{m}}{2}(1+t)\right)$$

in the $(x,v)$-plane. Hence, for $x\in I(t_n)\setminus\frac{1}{3}I(t_n)$,

$$\begin{aligned}
u_\infty(t_n,x)-u_\infty(t_n+T,x) &= 2x\left(\frac{1}{1+t_n}-\frac{1}{1+t_n+T}\right)\\
&\geq 2\frac{\mathring{m}}{12}(1+t_n)^2\left(\frac{1}{1+t_n}-\frac{1}{1+t_n+T}\right)\\
&= \frac{\mathring{m}}{6}T\frac{1+t_n}{1+t_n+T}\\
&\to \frac{\mathring{m}}{6}T,\ n\to\infty.
\end{aligned}$$

Since the slopes

$$\frac{\partial u_\infty}{\partial x}(t_n,x) = \frac{2}{1+t_n} > \frac{2}{1+t_n+T} = \frac{\partial u_\infty}{\partial x}(t_n+T,x)$$

tend to zero we may assume that

$$\begin{aligned}
\mathrm{dist}\Big(&\mathrm{supp}\, f_\infty(t_n) \cap \Big[\Big(I(t_n) \setminus \frac{1}{3} I(t_n)\Big) \times \mathbb{R}\Big],\\
&\qquad\qquad \mathrm{supp}\, f_\infty(t_n+T) \cap \Big[\Big(I(t_n+T) \setminus \frac{1}{3} I(t_n)\Big) \times \mathbb{R}\Big]\Big)\\
&\qquad\qquad \geq \frac{\mathring{m}}{12} T > 0, \; n \in \mathbb{N}.
\end{aligned}$$

Therefore, there exists a nonnegative function $\varphi \in C_b^1(\mathbb{R}^2)$ with $\|\varphi\|_\infty = 1$ and

$$\begin{aligned}
&\varphi = 0 \;\text{ on }\; \mathrm{supp}\, f_\infty(t_n) \cap \Big[\Big(I(t_n) \setminus \tfrac{1}{3} I(t_n)\Big) \times \mathbb{R}\Big]\\
&\varphi = 0 \;\text{ on }\; ]-\infty, 0] \times \mathbb{R}\\
&\varphi = 1 \;\text{ on }\; \mathrm{supp}\, f_\infty(t_n+T) \cap \Big[\Big(I(t_n+T) \setminus \tfrac{1}{3} I(t_n)\Big) \times \mathbb{R}\Big]
\end{aligned}$$

for every $n \in \mathbb{N}$. This implies that for $n \in \mathbb{N}$,

$$\begin{aligned}
&\iint \Big(f_\infty(t_n+T,x,v) - f_\infty(t_n,x,v)\Big) \varphi(x,v)\, dv\, dx\\
&\quad = \int_{I(t_n+T)} \frac{2}{(1+t_n+T)^2} \varphi(x, u_\infty(t_n+T,x))\, dx\\
&\qquad - \int_{I(t_n)} \frac{2}{(1+t_n)^2} \varphi(x, u_\infty(t_n,x))\, dx\\
&\quad = \frac{2}{(1+t_n+T)^2} \left( \int_{I(t_n+T) \setminus \frac{1}{3} I(t_n)} + \int_{\frac{1}{3} I(t_n)} \right) \varphi(x, u_\infty(t_n+T,x))\, dx\\
&\qquad - \frac{2}{(1+t_n)^2} \left( \int_{I(t_n) \setminus \frac{1}{3} I(t_n)} + \int_{\frac{1}{3} I(t_n)} \right) \varphi(x, u_\infty(t_n,x))\, dx\\
&\quad \geq \frac{2}{(1+t_n+T)^2} \int_{I(t_n+T) \setminus \frac{1}{3} I(t_n)} dx - \frac{2}{(1+t_n)^2} \int_{\frac{1}{3} I(t_n)} dx\\
&\quad \geq \frac{2}{(1+t_n+T)^2} \frac{2}{3} \frac{\mathring{m}}{4} (1+t_n+T)^2 - \frac{2}{(1+t_n)^2} \frac{1}{3} \frac{\mathring{m}}{4} (1+t_n)^2\\
&\quad = \frac{\mathring{m}}{6} > 0,
\end{aligned}$$

and the proof is complete. □

The omission of $\rho_\infty$ in the proposition above is not accidental: it is easily seen that

$$\|\rho_\infty(t+T) - \rho_\infty(t)\|_p \to 0, \; t \to \infty$$

for any $p \in [1, \infty]$.

To conclude our investigation we remark that besides $(f_\infty, \rho_\infty, E_\infty)$ there exists another family of explicit solutions, which are constructed in [7] by a method going back to Kurth [9], namely

$$f_K(t,x,v) := \frac{\mathring{m}}{2\pi} \Big(1 - (x/r(t))^2 - (r(t)v - \dot{r}(t)x)^2\Big)_+^{-1/2}$$

$$\rho_K(t,x) := \frac{\mathring{m}}{2r(t)}\,\chi_{[-r(t),r(t)]}(x),$$

$$E_K(t,x) := \frac{\mathring{m}}{2}\begin{cases} -1 & , x \leq -r(t), \\ x/r(t) & , -r(t) \leq x \leq r(t), \\ 1 & , x \geq r(t), \end{cases}$$

where $(\cdot)_+$ denotes the positive part of the argument and $r : [0,\infty[ \to ]0,\infty[$ is any solution of the differential equation

$$\ddot{r} = \frac{1}{r^3} + \frac{\mathring{m}}{2} \tag{6.1}$$

with initial data $r(0) > 0$ and $\dot{r}(0) \in \mathbb{R}$. With respect to Theorem 2.1 we note that all these solutions satisfy the estimates

$$\|\rho_\infty(t)\|_\infty,\ \|\rho_K(t)\|_\infty \leq \frac{C}{(1+t)^2},\ t \geq 0.$$

The latter estimate follows from the fact that for every solution of (6.1) with initial data as stated the energy

$$\frac{1}{2}\dot{r}^2 + \frac{1}{2r^2} - \frac{\mathring{m}}{2}r$$

is conserved. This implies that $r(t) \geq C > 0$ for $t \geq 0$, therefore by (6.1) $r(t) \geq C(1+t)^2$ for $t \geq 0$ and

$$\left|\ddot{r}(t) - \frac{\mathring{m}}{2}\right| \leq \frac{C}{(1+t)^6}$$

which shows that there exists a constant $C > 0$ such that

$$\left|r(t) - \frac{\mathring{m}}{4}t^2\right| \leq C(1+t),\ t \geq 0.$$

This estimate implies that

$$\|\rho_K(t) - \rho_\infty(t)\|_p \to 0,\ t \to \infty$$

for any $p \in [1,\infty]$. It remains an open question whether for every solution $(f,\rho,E)$ one has $\rho(t) - \rho_\infty(t) \to 0$ as $t \to \infty$. The fact that this convergence holds in the supremum norm by Theorem 2.1 can obviously not be considered as an adequate answer to this question. Given the asymptotic estimates for the explicit examples above a further open problem is whether the estimate for $\|\rho(t)\|_\infty$ in Theorem 2.1 can be improved in general.

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