Instability of the Periodic Motion of a Particle Interacting with a Scalar Wave Field

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Abstract

We consider the system of a point particle, accelerated through a confining potential, and interacting with a scalar wave field. We prove that the periodic solutions of the linearized system (which has an eigenvalue embedded in the continuous spectrum) do not survive perturbation through the nonlinearity, if some "mode" of the wave field does not couple to the particle.

1 Introduction and Main Results

We consider the physical system of a mechanical point particle coupled in a translation invariant manner to a scalar wave field, described through the equations

$$\begin{aligned} &\dot{\phi}(t,x)=\pi(t,x)\,, &&\dot{\pi}(t,x)=\Delta\phi(t,x)-\rho(x-q(t))\,,\\ &\dot{q}(t)=(1+p(t)^2)^{-1/2}p(t)\,, &&\dot{p}(t)=-\nabla V(q(t))-\int\nabla\phi(t,x)\rho(x-q(t))\,dx\,, \end{aligned}\tag{1}$$

for $t\in\mathbb{R}$ and $x\in\mathbb{R}^3$. Here $\phi=\phi(t,x)$ is the scalar wave field, and $q=q(t)\in\mathbb{R}^3$ is the position of the particle, whereas $\pi=\pi(t,x)$ and $p=p(t)$ represent the velocity field and the momentum of the particle, respectively. The motion of the particle is subjected to a potential $V=V(q)$, and the coupling is realized through a radially symmetric and compactly supported form factor ρ, thus allowing only finite-distance interactions. We assume that the potential V satisfies

$$V\in C^3(\mathbb{R}^3)\,,\quad \lim_{|q|\to\infty}V(q)=\infty\,,\tag{V}$$

and for the charge distribution ρ we require

$$\rho\in C_c^\infty(\mathbb{R}^3)\,,\quad \rho(x)=\rho_r(|x|)\,.\tag{C}$$

System (1) describes radiative damping, where the accelerated charge generates fields which partially are radiated to infinity, cf. [2], [3], [5], [11]. There has also been a lot of work on the question whether there is a consistent, relativistically covariant, theory of point charges and field, cf. [4], [18], [14], [6], a question which on the classical level seems to be undecided, see [19]. It should also be remarked that instead of the scalar wave field in (1), an analogous system with the full Maxwell field was looked at in [9], whereas the asymptotic of soliton solutions to (1) was considered in [8].

In this paper we are interested in the qualitative behaviour of solutions to the Hamiltonian system (1) for large times. For this, we first note that the stationary solutions of (1) are

$$\mathcal{S}=\{(\phi,q,\pi,p)=(\phi_{q^*},q^*,0,0)=:Y_{q^*}\,|\,q^*\in S\},$$

where $S=\{q\in\mathbb{R}^3:\nabla V(q)=0\}$ is the set of critical points for V, and

$$\phi_{q^*}(x)=\Delta^{-1}\rho(x-q^*)=-\frac{1}{4\pi}\int\frac{d^3y}{|x-y|}\rho(y-q^*).$$

In [10] we supposed that the Wiener condition

$$\hat{\rho}(k)=(2\pi)^{-3/2}\int e^{-ik\cdot x}\rho(x)\,d^3x\neq 0,\quad k\in\mathbb{R}^3, \tag{W}$$

holds, and under this condition we proved that solutions of finite energy converge to $\mathcal{S}$ in suitable local energy norms. In addition, we detected the rate of convergence of solutions $Y(t)=(\phi(t),q(t),\pi(t),p(t))\to Y_{q^*}$, provided that $Y(0)$ decays sufficiently fast in x-space, and provided that $Y(0)$ is close enough to Y_{q^*}, with q^* being "stable" for V, i.e., $D^2V(q^*)$ is positiv definite.

So the natural question arises whether there are generated periodic solutions in the neighborhood of some equilibrium, if (W) is violated on some sphere $|k|=|k_0|$, a condition which can be interpreted to say that some "mode" of the wave field does not couple to the particle. Since (W) is used mainly to exclude purely imaginary eigenvalues of the linearization of (1) around an equilibrium, we ask more precisely if the periodic solutions of the linearized system generated by an eigenvalue (due to a zero of $\hat{\rho}$) create periodic solutions of the full non-linear system.

To explain this in greater detail, we restrict ourselves to the case $q^*=0\in S$ and $D^2V(0)=\omega_0^2E_3$ (E_3 being the unit matrix), i.e., $q^*=0$ is "stable" for the linearization. We consider the system around the corresponding equilibrium $Y_{q^*}=Y_0\in\mathcal{S}$. With $u=\phi-\phi_{q^*}=\phi-\phi_0$ the second order system for $\psi=(u,q)$ reads as

$$\ddot{\psi}=-A\psi+F(\psi) \tag{2}$$

with linearization

$$A\psi=A(u,q)=\begin{pmatrix}-\Delta u(x)-\nabla\rho(x)\cdot q\\ (\omega_0^2+\omega_1^2)q-\int\nabla\rho(x)u(x)\,dx\end{pmatrix} \tag{3}$$

in $D(A)=H^2\oplus\mathbb{C}^3$, $H^m=H^m(\mathbb{R}^3)=W^{m,2}(\mathbb{R}^3)$ being the usual Sobolev space of functions having m'th order distributional derivatives in $L^2=L^2(\mathbb{R}^3)$, and nonlinearity

$$F(\psi)=\begin{pmatrix}\rho(x)-\rho(x-q)-\nabla\rho(x)\cdot q,\\ [\nabla V(0)-\nabla V(q)+\omega_0^2q]\\ +\int[\rho(x)-\rho(x-q)-\nabla\rho(x)\cdot q]\nabla\phi_0(x)\,dx\\ +\int[\nabla\rho(x-q)-\nabla\rho(x)]u(x)\,dx\end{pmatrix}\in C_c^\infty(\mathbb{R}^3)\oplus\mathbb{C}^3. \tag{4}$$

Here we let $\omega_1^2 = \frac{1}{3}|\hat\rho|_{L^2}^2$, and for simplicity of notation we considered (1) with $\dot q = p$ instead of $\dot q = (1+p^2)^{-1/2}p$; note also that $\nabla V(0) = 0 = \int \rho(x)\nabla\phi_0(x)\,dx$.

Suppose now that the Wiener condition (W) is violated, in that

$$\hat\rho(k) = 0 \quad \text{for} \quad |k| = |k_0| > 0, \quad \text{but} \quad \hat\rho(k) \neq 0 \quad \text{for} \quad |k| \neq |k_0|. \tag{5}$$

Then $[0,\infty[$ is the spectrum of A, with continuous spectrum $[0,\infty[\setminus\{|k_0|^2\}$ and an isolated imbedded eigenvalue $\lambda_0 = |k_0|^2$, cf. Lemma 3 below. The corresponding eigenspace is

$$\mathcal{E} = \{\psi : A\psi = |k_0|^2\psi\} = \left\{\psi = \psi_q := \begin{pmatrix} u_q \\ q \end{pmatrix} : q \in \mathbb{C}^3, \hat u_q(k) = \frac{i\,(k\cdot q)\,\hat\rho(k)}{k^2 - |k_0|^2}\right\}. \tag{6}$$

Since $A\psi = |k_0|^2\psi$ consists of two equations, in order to obtain an eigenvalue we have to assume additionally that ρ is adjusted in the right way, i.e., $\rho = \gamma\rho_0$ with some parameter γ which can be choosen such that

$$\alpha(\pm i|k_0|) = 0, \quad \text{with} \quad \alpha(\lambda) = \omega_0^2 + \lambda^2\left(1 + \frac{1}{3}\gamma^2\int dk\,\frac{|\hat\rho_0(k)|^2}{k^2+\lambda^2}\right), \tag{7}$$

see again Section 2, since otherwise A will have empty point spectrum. Note that it was exactly the fact that α had no zeros on the imaginary axis which allowed us to prove in [10] that $(u,q) = (0,0)$ was exponentially attracting (in a suitable sense), if (W) is assumed to hold.

Returning to the problem at hand, we hence find that the linear equation $\ddot\psi + A\psi = 0$ has gained the $2\pi/|k_0|$-periodic solutions $\sin(|k_0|t)\psi_q$ with $\psi_q \in \mathcal{E}$, or stated differently, the operator

$$K_0\psi = |k_0|^2\frac{\partial^2}{\partial t^2}\psi + A\psi$$

has kernel

$$\ker K_0 = \{e^{-it}\psi_- + e^{it}\psi_+ : \psi_\pm \in \mathcal{E}\}. \tag{8}$$

In general, when investigating non-linear equations, often it can be shown that the fully non-linear problem is somehow dominated by the linearized system, at least near to the equilibria. Thus we fix $q \in \mathbb{R}^3$, small $\varepsilon > 0$, and the initial value $\varepsilon(0,|k_0|)\psi_q$ of the periodic solution $(\psi(t),\dot\psi(t)) = \varepsilon(\sin(|k_0|t), |k_0|\cos(|k_0|t))\psi_q$ of the linearized equation, and we ask whether there are initial values $(\psi_\varepsilon^0, \dot\psi_\varepsilon^0)$ for the full system (2) close to $\varepsilon(0,|k_0|)\psi_q$ which generate periodic solutions of (2) with period $\tau_\varepsilon = 2\pi/\omega_\varepsilon$ close to $2\pi/|k_0|$. We fix this in more mathematical terms in the following

Definition 1 *Let $\mathcal{E}$ be defined through (6). An $\psi_q \in \mathcal{E}$ is called stable for (2), if there are initial values $(\psi_\varepsilon^0, \dot\psi_\varepsilon^0)$ such that the corresponding solutions $(\psi_\varepsilon(t), \dot\psi_\varepsilon(t))$ of (2) are periodic with period $\tau_\varepsilon = 2\pi/\omega_\varepsilon$, and*

(a) $\|\psi_\varepsilon^0\|_{L^2\oplus\mathbb{C}^3} + \|\dot\psi_\varepsilon^0 - \varepsilon|k_0|\psi_q\|_{L^2\oplus\mathbb{C}^3} = o(\varepsilon)$ *as* $\varepsilon \to 0^+$,

(b) $|\omega_\varepsilon - |k_0|| = o(\varepsilon)$ *as* $\varepsilon \to 0^+$, *and*

(c) $\|\psi_\varepsilon^0\|_{H^2_{1,\Delta}\oplus\mathbb{C}^3} + \|\dot\psi_\varepsilon^0\|_{H^1_1\oplus\mathbb{C}^3} = O(\varepsilon)$ *as* $\varepsilon \to 0^+$.

Here (c) is a technical assumption, which appears due to the fact that we have to derive certain bounds in weighted L^2-spaces in order to apply a suitable limit absorption principle, cf. the remarks below; the spaces $H^2_{1,\Delta}$ and H^1_1 are defined in (20) resp. (19) below.

Our main result is that *no* ψ_q with $q \neq 0$ is stable for (2).

Theorem 1 *Let assumptions (C) on ρ and (V) on V be satisfied. Suppose furthermore that (W) is violated, i.e., (5), and that $\rho = \gamma\rho_0$ can be adjusted to fulfil (7). Then every $\psi_q \in \mathcal{E}$ with $q \neq 0$ is unstable for (2), in the sense of Definition 1.*

Remarks. (i) With regard to the original system (1), Theorem 1 may be summarized that the periodic solutions of the linear system near $Y_{q^*} = Y_0$ do not survive the perturbation through the nonlinearity, even if $q^* = 0$ is assumed to be stable, in the sense that $D^2V(0) > 0$ as a quadratic form (we had $D^2V(0) = \omega_1^2 E_3$ only for simplicity). This transfers to the other equilibria $Y_{q^*} \in \mathcal{S}$ with $0 \neq q^* \in S$ under the respective assumptions.

(ii) To simplify some technicalities, during the proof we will assume $V \in C^4(\mathbb{R}^3)$, but in fact $V \in C^3(\mathbb{R}^3)$ is enough.

(iii) It is also not important that we considered $\dot{q} = p$ instead of $\dot{q} = (1+p^2)^{-1/2}p$ in (1), since in case of a relativistic kinetic energy the linearization remains unchanged, while in the second component of F there appears an additional term of order four, which will not affect the function $f_1(\psi) = (\partial/\partial\varepsilon)(\varepsilon^{-1}F(\varepsilon\psi))\Big|_{\varepsilon=0}$, and it is mainly the properties of this functions that enter the proof of the theorem.

(iv) We prove that the periodic solutions of the linear system do not generate periodic solutions of the non-linear system, but this a priori does not exclude the existence of periodic solutions to (1) in case that (W) is violated. However, actually we expect that there are no such periodic solutions, at least for some potentials.

(v) For the proof of this theorem we rely heavily on the ideas of Sigal [15]. There it was shown that for non-linear wave equations $-\frac{\partial^2 u}{\partial t^2} = Hu + f_\varepsilon(u)$ with a Schrödinger-like operator H having an isolated eigenvalue $\lambda_0 > 0$, periodic solutions of the linearized equation are unstable to non-linear perturbations in a sense similar to Definition 1. The main assumptions were that H satisfies the Mourre estimate, and that $n^2\lambda_0 \notin \operatorname{disc\,spec} H \cup \operatorname{thresholds} H$ for $n \geq 2$. Additionally, it was important to impose a further condition in order to exclude the possibility of $\delta(H - n^2\lambda_0)f_n = 0$ for all $n \geq 2$, cf. [15, Equ. (2.15)]. Here f_n are the Fourier coefficients of $\partial f_\varepsilon/\partial\varepsilon$ at $\varepsilon = 0$, evaluated at the periodic solution of the linearized system, and δ is related to a certain limiting absorption principle explained in more detail below. We note that the analogon to this latter condition will already be guaranteed by

$$\hat{\rho}(k) \neq 0 \quad \text{on the sphere} \quad |k| = 2|k_0| \tag{9}$$

in our present case, since this will imply $\delta(A - 4|k_0|^2)f_2 \neq 0$. In (5) it is assumed more restrictively that $\hat{\rho}(k) \neq 0$ for $|k| \neq |k_0|$ only to avoid additional eigenvalues besides $\lambda_0 = |k_0|^2$ and to ensure the validity of the limiting absorption principle for A for all $\lambda \in]0, \infty[\setminus \{|k_0|^2\}$.

(vi) In (b) of Definition 1 we impose the condition $|\omega_\varepsilon - |k_0|| = o(\varepsilon)$ as $\varepsilon \to 0^+$, which is stronger than $|\omega_\varepsilon - |k_0|| \to 0$, as was assumed in (i) of [15, Def. 2.1]. This is compensated through the fact that we only have to require $L^2 \oplus \mathbb{C}^3$-convergence of the initial data in (a). Theorem 1 remains true, if (b) is relaxed to $|\omega_\varepsilon - |k_0|| \to 0$, and the convergence in (a) is strengthened to convergence in $H^2_{1,\Delta}\oplus\mathbb{C}^3$-norm for ψ^0_ε resp. in $H^1_1\oplus\mathbb{C}^3$-norm for $\dot\psi^0_\varepsilon$. We preferred the conditions chosen, since they fit more into the framework considered earlier in [10].

Next we describe the strategy of the proof of Theorem 1, following [15]. Consider $\tilde\psi_\varepsilon(t,x)=\varepsilon^{-1}\psi_\varepsilon(t/\omega_\varepsilon,x)$ with a sequence ψ_ε according to Definition 1. Then $\tilde\psi_\varepsilon$ is 2π-periodic in t and satisfies, dropping the tilde again for notational simplicity,

$$\omega_\varepsilon^2\ddot\psi_\varepsilon+A\psi_\varepsilon=\varepsilon^{-1}F(\varepsilon\psi_\varepsilon)\tag{10}$$

as well as $(\psi^0_\varepsilon,\dot\psi^0_\varepsilon)\in(H^2_{1,\Delta}\oplus\mathbb{C}^3)\oplus(H^1_1\oplus\mathbb{C}^3)$,

$$\left.\begin{array}{ll}\text{(a)} & \|\psi^0_\varepsilon\|_{L^2\oplus\mathbb{C}^3}+\|\dot\psi^0_\varepsilon-\psi_q\|_{L^2\oplus\mathbb{C}^3}\to 0 \quad\text{as}\quad \varepsilon\to 0^+,\\ \text{(b)} & |\omega_\varepsilon-|k_0||=o(\varepsilon)\quad\text{as}\quad\varepsilon\to 0^+,\quad\text{and}\\ \text{(c)} & \sup_{\varepsilon\in]0,\varepsilon_0]}\|\psi^0_\varepsilon\|_{H^2_{1,\Delta}\oplus\mathbb{C}^3}+\sup_{\varepsilon\in]0,\varepsilon_0]}\|\dot\psi^0_\varepsilon\|_{H^1_1\oplus\mathbb{C}^3}\le C.\end{array}\right\}\tag{11}$$

We write (10) as

$$0=\Big[|k_0|^2\frac{\partial^2}{\partial t^2}\psi_\varepsilon+A\psi_\varepsilon\Big]+\Big[(\omega_\varepsilon^2-|k_0|^2)\frac{\partial^2}{\partial t^2}\psi_\varepsilon+f_\varepsilon(\psi_\varepsilon)\Big]=:K_0\psi_\varepsilon+I_\varepsilon(\psi_\varepsilon)\tag{12}$$

with $f_\varepsilon(\psi)=\varepsilon^{-1}F(\varepsilon\psi)$ and try to "invert" K_0. For this we have to avoid $\ker K_0$ from (8) and thus introduce the orthogonal projections

$$P_0:\mathcal{H}^2\to\ker K_0\quad\text{and}\quad\bar P_0=\mathbf{1}-P_0:\mathcal{H}^2\to(\ker K_0)^\perp$$

with $\mathcal{H}^2=L^2(S^1\times\mathbb{R}^3)\oplus L^2(S^1;\mathbb{C}^3)$. By e.g. Fourier series expansion it may be seen that some "invertability" of $\bar P_0H_0\bar P_0$, i.e., of K_0 outside $\ker K_0$, will be an immediate consequence of some kind of "invertability" of $(A-n^2|k_0|^2)$ for $n\ge 2$; note that in our case $n^2|k_0|^2\notin$ disc spec$A$ for $n\ge 2$, as is a basic assumption of [15, Thm. 2.2]. Hence an adaption of the limiting absorptions principle, cf. [1, Section 4], is needed for $\lambda=n^2|k_0|^2$, $n\ge 2$, and this works as follows. By formally solving $(u_1,q_1)=(A-\lambda)(u,q)$ for (u,q) with given (u_1,q_1), we obtain from the first equation by Fourier transform

$$\hat u(k)=\frac{\hat u_1(k)+i(k\cdot q)\hat\rho(k)}{k^2-\lambda}.$$

Because of the singularity we hence cannot expect a solution $u\in L^2$, but if – argueing very heuristically – $\hat u_1$ had a derivative, then in integrals the $1/(k^2-\lambda)$ singularity could be improved to a logarithmic singularity through integration by parts, and this logarithmic singularity is harmless. The property of $\hat u_1$ to have a derivative corresponds to u_1 being an element of the weighted L^2-space $L^2_1(\mathbb{R}^3)$, with

$$L^2_s=L^2_s(\mathbb{R}^3)=\Big\{u=u(x):\|u\|^2_{L^2_s(\mathbb{R}^3)}=\int(1+x^2)^s|u(x)|^2\,dx<\infty\Big\},\quad s\in\mathbb{R}.$$

By using the classical limiting absorption principle for $-\Delta$ from [1, Thm. 4.1], this can be made precise to yield (cf. Lemma 6 below) the following

Lemma 1 *For $\lambda \in]0,\infty[$ such that $\hat{\rho}(k) \neq 0$ on the sphere $|k| = \sqrt{\lambda}$, the limits*

$$R_A^{\pm}(\lambda) = \lim_{\eta \to 0^+} (A - \lambda \pm i\eta)^{-1}$$

exist in the strong operator topology $B(L_1^2 \oplus \mathbb{C}^3, L_{-1}^2 \oplus \mathbb{C}^3)$.

As noted above this in turn implies

Lemma 2 *The limits*

$$\mathcal{R}_{K_0}^{\pm}(0) = (\bar{P}_0 K_0 \bar{P}_0 \pm i0)^{-1} := \lim_{\eta \to 0^+} (\bar{P}_0 K_0 \bar{P}_0 \pm i\eta)^{-1}$$

exist in the strong operator topology $B(\bar{P}_0 \mathcal{H}_1^2, \bar{P}_0 \mathcal{H}_{-1}^2)$.

Here we defined $\mathcal{H}_s^2 = L_s^2(S^1 \times \mathbb{R}^3) \oplus L^2(S^1; \mathbb{C}^3)$ with

$$L_s^2(S^1 \times \mathbb{R}^3) = \left\{ u = u(t,x) : \|u\|^2_{L_s^2(S^1 \times \mathbb{R}^3)} = \int_0^{2\pi} \int (1+x^2)^s |u(t,x)|^2 \, dx \, dt < \infty \right\}, \quad s \in \mathbb{R}.$$

This can be used as follows. Projection of (12) yields

$$\bar{P}_0 K_0 \bar{P}_0 \psi_\varepsilon = -\bar{P}_0 I_\varepsilon(\psi_\varepsilon), \quad \text{and thus} \quad (\bar{P}_0 K_0 \bar{P}_0 \pm i\eta) \bar{P}_0 \psi_\varepsilon = \bar{P}_0(\pm i\eta \psi_\varepsilon - I_\varepsilon(\psi_\varepsilon)).$$

Hence if our assumptions (11) on the initial values $(\psi_\varepsilon^0, \dot{\psi}_\varepsilon^0)$ can be used to ensure

$$\psi_\varepsilon, I_\varepsilon(\psi_\varepsilon) \in \mathcal{H}_1^2, \tag{13}$$

then Lemma 2 gives

$$(\bar{P}_0 K_0 \bar{P}_0 \pm i0)^{-1} (\bar{P}_0 I_\varepsilon(\psi_\varepsilon)) = -\bar{P}_0 \psi_\varepsilon,$$

and hence

$$\delta(\bar{P}_0 K_0 \bar{P}_0)(\bar{P}_0 I_\varepsilon(\psi_\varepsilon)) = 0, \tag{14}$$

where

$$\delta(\bar{P}_0 K_0 \bar{P}_0) = \frac{1}{2\pi i} [(\bar{P}_0 K_0 \bar{P}_0 - i0)^{-1} - (\bar{P}_0 K_0 \bar{P}_0 + i0)^{-1}].$$

Suppose moreover that we can show from (11) that

$$\frac{1}{\varepsilon} I_\varepsilon(\psi_\varepsilon) = \frac{(\omega_\varepsilon^2 - |k_0|^2)}{\varepsilon} \left(\frac{\partial^2}{\partial t^2} \psi_\varepsilon \right) + \varepsilon^{-2} F(\varepsilon \psi_\varepsilon) \to f_1(\psi_q) \quad \text{in} \quad \mathcal{H}_1^2, \tag{15}$$

with

$$f_1(\psi) = \frac{\partial}{\partial \varepsilon} f_\varepsilon(\psi) \Big|_{\varepsilon=0}, \quad \text{and} \quad f_\varepsilon(\psi) = \varepsilon^{-1} F(\varepsilon \psi).$$

Then by continuity of $\delta(\bar{P}_0 K_0 \bar{P}_0) : \bar{P}_0 \mathcal{H}_1^2 \to \bar{P}_0 \mathcal{H}_{-1}^2$ we obtain from (14) as $\varepsilon \to 0^+$

$$\delta(\bar{P}_0 K_0 \bar{P}_0)(\bar{P}_0 f_1(\psi_q)) = 0.$$

This corresponds exactly to equation (5.8) of [15]. Thus exactly as in this paper we derive

$$\delta(A - n^2 |k_0|^2) f_n = 0, \quad n \geq 2, \tag{16}$$

where

$$f_n(x)=\int_0^{2\pi} f_1(\sin t\,\psi_q)(x)e^{-int}\,dt=\int_0^{2\pi} f_1\Big(\sin t\begin{pmatrix}u_q\\ q\end{pmatrix}\Big)(x)e^{-int}\,dt.$$

It will turn out, cf. Section 4 below, that $f_1(\sin t\,\psi_q)(x)=f_1(\psi_q)(x)\sin^2 t$, and because of $\int_0^{2\pi}\sin^2 te^{-int}\,dt\neq 0$ iff $n=2$, (16) is equivalent to

$$\delta(A-4|k_0|^2)\,f_1(\psi_q)=0. \tag{17}$$

Contrary to this we will incorporate (9) to complete the proof of the theorem, in that we can show

$$[R_A^-(\lambda)-R_A^+(\lambda)]\,f_1(\psi_q)=2\pi i\,\delta(A-\lambda)\,f_1(\psi_q)\neq 0 \tag{18}$$

with $\lambda=4|k_0|^2$, thus contradicting (17).

Note that (18) is the key point of the whole argument, being a kind of non-linear version of the Fermi Golden Rule from quantum mechanics, cf., e.g., [16, Section 4] or [13]. Since $\lambda=4|k_0|^2$ lies in the continuous spectrum, (18) may be interpreted as saying that there is a non-vanishing coupling of the periodic solutions of the linear system to this continuous spectrum. This allows energy to be transported to infinity by radiation, thus excluding the possibility of periodic solutions to persist under the influence of the nonlinearity.

The paper is organized as follows. In Section 2 we first analyze the spectrum of A. Afterwards in Section 3, the limiting absorption principle for A is shown, while in Section 4 the outline of proof from above is completed. Since in [10] we didn't treat (1) in weighted L^2-spaces, we added an appendix Section 5 where we derive existence and useful estimates for the dynamics of (1) in such setting.

We end this section by introducing some further notation. Let

$$H_1^1=H_1^1(\mathbb{R}^3)=\{u=u(x)\in H^1: u,\nabla u\in L_1^2\},\quad \|u\|_{H_1^1}^2=\|u\|_{L_1^2}^2+\|\nabla u\|_{L_1^2}^2, \tag{19}$$

and

$$H_{1,\Delta}^2=H_{1,\Delta}^2(\mathbb{R}^3)=\{u=u(x)\in H^2: u,\nabla u,\Delta u\in L_1^2\},\quad \|u\|_{H_{1,\Delta}^2}^2=\|u\|_{H^2}^2+\|u\|_{H_1^1}^2+\|\Delta u\|_{L_1^2}^2. \tag{20}$$

Recall also that the Fourier transform in L^2 is an isometric isomorphism from H^s to L_s^2, $s\geq 0$, cf. [17, Section 10.2]. In general, only standard notations will be used, as may be found in many basic references like [7], [17]. E.g., we define $\langle u,v\rangle_{L^2}=\int u(x)\overline{v(x)}\,dx$ to be the usual inner product in L^2, with conjugation in the second argument.

2 On the spectrum of the linearization

From (3) above we recall that

$$A\psi=A(u,q)=\begin{pmatrix}-\Delta u(x)-\nabla\rho(x)\cdot q\\ (\omega_0^2+\omega_1^2)q-\int dx\,\nabla\rho(x)u(x)\end{pmatrix}$$

in $D(A)=H^2\oplus\mathbb{C}^3\subset L^2\oplus\mathbb{C}^3$. Thus, trying to solve $\begin{pmatrix}u_1\\ q_1\end{pmatrix}=(A-\lambda)\begin{pmatrix}u\\ q\end{pmatrix}$ with given $(u_1,q_1)\in L^2\oplus\mathbb{C}^3$ for $(u,q)\in D(A)$, we obtain by Fourier transform

$$\hat u(k)=\frac{\hat u_1(k)+i(k\cdot q)\hat\rho(k)}{k^2-\lambda}, \tag{21}$$

and also, recalling $\omega_1^2=\frac{1}{3}|\hat\rho|^2_{L^2}$, due to the spherical symmetry of $\hat\rho$,

$$
\begin{aligned}
q_1 &= (\omega_0^2-\lambda)q+i\int k\,\overline{\hat\rho(k)}\,\hat u(k)-\omega_1^2 q\\
&= (\omega_0^2-\lambda)q+\Big(-\frac{1}{3}\int\frac{k^2}{k^2-\lambda}|\hat\rho(k)|^2\,dk+\frac{1}{3}\int|\hat\rho(k)|^2\,dk\Big)\,q+i\int k\,\overline{\hat\rho(k)}\,\frac{\hat u_1(k)}{k^2-\lambda}\,dk\\
&= \beta(-\lambda)\,q+i\int k\,\overline{\hat\rho(k)}\,\frac{\hat u_1(k)}{k^2-\lambda}\,dk, \qquad (22)
\end{aligned}
$$

where

$$
\beta(\lambda)=\omega_0^2+\lambda\Big(1+\frac{1}{3}\int\frac{|\hat\rho(k)|^2}{k^2+\lambda}\,dk\Big); \qquad (23)
$$

note that $\alpha(\lambda)=\beta(\lambda^2)$ with α as in (7), hence $\beta(-|k_0|^2)=0$. From (21) and (22) it is seen that $\mathbb{C}\setminus[0,\infty[$ is contained in the resolvent set of A. Concerning the point spectrum of A, (22) and (21) imply that $\lambda\in[0,\infty[$ is an eigenvalue iff $\beta(-\lambda)=0$ and $\hat\rho(k)=0$ for $|k|=\sqrt{\lambda}$. Thus (5) and $\beta(-|k_0|^2)=0$ yield that $\lambda_0=|k_0|^2>0$ is the only eigenvalue of A, and consequently it follows from (21) and (22) that under our assumptions we have the following

Lemma 3 *If (5) and (7) hold, then A has resolvent set $\mathbb{C}\setminus[0,\infty[$, continuous spectrum $[0,\infty[\setminus\{|k_0|^2\}$, and an embedded eigenvalue $\lambda_0=|k_0|^2$ with eigenspace $\mathcal{E}$ from (6).*

Note that corresponding to this, the linearization of the original problem (1) around the equilibrium $Y_{q^*}=Y_0\in\mathcal{S}$ has continuous spectrum $i\mathbb{R}\setminus\{\pm i|k_0|\}$ and two conjugate imbedded eigenvalues $\pm i|k_0|$.

A further property of the eigenelements $\psi_q=(u_q,q)\in\mathcal{E}$, where $\hat u_q(k)=\dfrac{i\,(k\cdot q)\,\hat\rho(k)}{k^2-|k_0|^2}$, is given in

Lemma 4 $\mathcal{E}\subset H^2_{1,\Delta}\oplus\mathbb{C}^3$.

Proof: We have to show $u_q\in H^2\cap L^2_1$, $\nabla u_q\in L^2_1$, and $\Delta u_q\in L^2_1$, or equivalently $\hat u_q\in L^2_2\cap H^1$, $k_i\hat u_q\in H^1$, and $k_i^2\hat u_q(k)\in H^1$ for $i=1,2,3$. Since $\hat\rho(k)=0$ for $|k|=|k_0|$, obviously $\hat u_q\in L^2_2$. To prove the rest, we apply [1, Thm. 3.2] with $f(k)=\widehat{\nabla\rho\cdot q}(k)=i\,(k\cdot q)\,\hat\rho(k)$. Then $f\in\mathcal{S}$, the Schwartz space of rapidly decreasing functions, and hence the quoted theorem can be used with $\alpha=0$, $\alpha=\delta_i$, and $\alpha=2\delta_i$ (in the notation used there) to imply $\hat u_q=v_0\in H^1$, $k_i\hat u_q=v_{\delta_i}\in H^1$, and $k_i^2\hat u_q=v_{2\delta_i}\in H^1$. Note that by this theorem in fact even $u_q,\nabla u_q,\Delta u_q\in L^2_s$ for every $s\ge 1$. □

3 Limiting absorption principle for A

In this section we prove a limiting absorption principle for A. This is readily obtained by means of the following lemma, which is a special case of a theorem of Agmon [1, Thm. 4.1].

Lemma 5 (Limiting Absorption Principle for $-\Delta$) *Let $R_0(\lambda)=(-\Delta-\lambda)^{-1}$ for $\lambda\in\mathbb{C}\setminus[0,\infty[$. Then for $\lambda\in]0,\infty[$*

(a) *the limits*
$$R_0^\pm(\lambda)=\lim_{\eta\to 0^+} R_0(\lambda\pm i\eta)$$
exist in the strong operator topology $B(L_1^2,L_{-1}^2)$.

(b) *the identities*
$$\mathrm{Im}\langle R_0^\pm(\lambda)u,u\rangle_{L^2}=\pm\frac{\pi}{2\sqrt{\lambda}}\int_{|k|=\sqrt{\lambda}}|\hat{u}(k)|^2\,d\sigma$$
hold.

This can be used to derive the corresponding limiting absorption principle for A, which was partially stated already in Lemma 1 from the introduction.

Lemma 6 *Let A be defined through (3). Then for $\lambda\in]0,\infty[$ such that $\hat{\rho}(k)\neq 0$ on the sphere $|k|=\sqrt{\lambda}$ the following holds.*

(a) *The limits*
$$R_A^\pm(\lambda)=\lim_{\eta\to 0^+}(A-\lambda\pm i\eta)^{-1}$$
exist in the strong operator topology $B(L_1^2\oplus\mathbb{C}^3,L_{-1}^2\oplus\mathbb{C}^3)$.

(b) *If additionally $\hat{u}_1(-k)=\hat{u}_1(k)$ $(k\in\mathbb{R}^3)$, we have the identities*
$$\left\langle R_A^\pm(\lambda)\begin{pmatrix}u_1\\ q_1\end{pmatrix},\begin{pmatrix}u_1\\ q_1\end{pmatrix}\right\rangle_{L^2\oplus\mathbb{C}^3}=\langle R_0^\mp(\lambda)u_1,u_1\rangle_{L^2}+\beta(-\lambda\pm i0)^{-1}|q_1|^2,$$
with $R_0^\mp(\lambda)$ from Lemma 5 and
$$\beta(-\lambda\pm i0)=\omega_0^2-\lambda\Big(1+\frac{1}{3}\langle R_0^\mp(\lambda)\rho,\rho\rangle_{L^2}\Big). \tag{24}$$

Proof: Let $(u_1,q_1)\in L_1^2\oplus\mathbb{C}^3$. Solving $\begin{pmatrix}u_1\\ q_1\end{pmatrix}=(A-\lambda\pm i\eta)\begin{pmatrix}u_\eta^\pm\\ q_\eta^\pm\end{pmatrix}$ for $(u_\eta^\pm,q_\eta^\pm)$, we obtain from the first equation
$$u_\eta^\pm=R_0(\lambda\mp i\eta)(u_1+\nabla\rho\cdot q_\eta^\pm). \tag{25}$$
Moreover, plugging the Fourier transform
$$\hat{u}_\eta^\pm(k)=\frac{\hat{u}_1(k)+i(k\cdot q_\eta^\pm)\hat{\rho}(k)}{k^2-\lambda\pm i\eta}$$
into the second equation $q_1=(\omega_0^2+\omega_1^2)q_\eta^\pm+i\int dkk\overline{\hat{\rho}(k)}\hat{u}_\eta^\pm(k)-(\lambda\mp i\eta)q_\eta^\pm$ and taking into account $\omega_1^2=\frac{1}{3}|\hat{\rho}|_{L^2}^2$, we compute
$$\begin{aligned}q_\eta^\pm&=\beta(-\lambda\pm i\eta)^{-1}\Big(q_1-i\int dkk\overline{\hat{\rho}(k)}\frac{\hat{u}_1(k)}{k^2-\lambda\pm i\eta}\Big)\\ &=\beta(-\lambda\pm i\eta)^{-1}\Big(q_1+\int dk\overline{\widehat{\nabla}\rho(k)}\frac{\hat{u}_1(k)}{k^2-\lambda\pm i\eta}\Big)\\ &=\beta(-\lambda\pm i\eta)^{-1}(q_1+\langle R_0(\lambda\mp i\eta)u_1,\nabla\rho\rangle_{L^2(\mathbb{R}^3,\mathbb{C}^3)}),\end{aligned} \tag{26}$$

cf. formula (4.6) of [1]. Here we had

$$\beta(\lambda)=\omega_0^2+\lambda\Big(1+\frac{1}{3}\int\frac{|\hat\rho(k)|^2}{k^2+\lambda}\,dk\Big)=\omega_0^2+\lambda(1+\frac{1}{3}\langle R_0(-\lambda)\rho,\rho\rangle_{L^2}),\quad \lambda\in\mathbb{C}\setminus]-\infty,0],$$

recall (23) above. Hence Lemma 5(a) implies that

$$\lim_{\eta\to 0^+}\beta(-\lambda\pm i\eta)=\omega_0^2-\lambda\Big(1+\frac{1}{3}\langle R_0^{\mp}(\lambda)\rho,\rho\rangle_{L^2}\Big)=:\beta(-\lambda\pm i0)\in\mathbb{C}$$

exist, and Lemma 5(b) gives

$$\operatorname{Im}\beta(-\lambda\pm i0)=\mp\frac{\pi}{6}\sqrt{\lambda}\int_{|k|=\sqrt{\lambda}}|\hat\rho(k)|^2\,d\sigma\neq 0,$$

by assumption on λ. Using again Lemma 5 we thus conclude from (26) and (25) that (a) holds, with

$$R_A^{\pm}(\lambda)\begin{pmatrix}u_1\\ q_1\end{pmatrix}=:\begin{pmatrix}u_0^{\pm}\\ q_0^{\pm}\end{pmatrix}=\begin{pmatrix}R_0^{\mp}(\lambda)(u_1+\nabla\rho\cdot q_0^{\pm})\\ \beta(-\lambda\pm i0)^{-1}(q_1+\langle R_0^{\mp}(\lambda)u_1,\nabla\rho\rangle_{L^2(\mathbb{R}^3,\mathbb{C}^3)})\end{pmatrix}\in L^2_{-1}\oplus\mathbb{C}^3.$$

Concerning (b), it hence suffices to show that

$$\lim_{\eta\to 0^+}\langle R_0(\lambda\mp i\eta)(\nabla\rho\cdot q_0^{\pm}),u_1\rangle_{L^2}=0=\lim_{\eta\to 0^+}\langle R_0(\lambda\mp i\eta)u_1,\nabla\rho\rangle_{L^2(\mathbb{R}^3,\mathbb{C}^3)}\cdot q_1,$$

but this follows from the additional assumption $\hat u_1(-k)=\hat u_1(k)$, since also $\hat\rho(-k)=\hat\rho(k)$, and the Fourier transform of $\nabla\rho$ produces an extra k, so that the integrand is an odd function of k; here we use again formula (4.6) of [1]. □

4 Proof of Theorem 1

In this section we fill in some of the details missing in the scetch of proof in Section 1. Let $f_1(\psi)=f_1(u,q)=\frac{\partial}{\partial\varepsilon}(\varepsilon^{-1}F(\varepsilon\psi))\Big|_{\varepsilon=0}\in C_c^\infty(\mathbb{R}^3)\oplus\mathbb{C}^3$ be given through

$$f_1(\psi)=\begin{pmatrix}-\frac{1}{2}D^2\rho(x)(q,q)\\ -\frac{1}{2}D^3V(0)(q,q)-\frac{1}{2}\int\nabla\phi_0(x)\,D^2\rho(x)(q,q)\,dx-\int D^2\rho(x)(q)\,u(x)\,dx\end{pmatrix}\tag{27}$$

for $\psi\in L^2\oplus\mathbb{C}^3$, with D^2 denoting the Hessian. In the appendix we will show that (13) and (15) hold, cf. Corollaries A.1, A.2, and A.3. By (27) we have $f_1(\sin t\,\psi_q)(x)=f_1(\psi_q)(x)\sin^2 t$, and therefore we only need to prove

$$[R_A^-(\lambda)-R_A^+(\lambda)]\,f_1(\psi_q)=2\pi i\,\delta(A-\lambda)\,f_1(\psi_q)\neq 0$$

with $\lambda=4|k_0|^2$, cf. (18). To see this, let $(u_1,q_1)=f_1(\psi_q)$. Then $\hat u_1(k)=\frac{1}{2}(k\cdot q)^2\hat\rho(k)$, and hence $\hat u_1(-k)=\hat u_1(k)$ implies that Lemma 6(b) can be used to yield

$$\begin{aligned}\operatorname{Im}\langle R_A^{\pm}(\lambda)f_1(\psi_q),f_1(\psi_q)\rangle_{L^2\oplus\mathbb{C}^3}&=\operatorname{Im}\langle R_0^{\mp}(\lambda)u_1,u_1\rangle_{L^2}+\operatorname{Im}(\beta(-\lambda\pm i0)^{-1}|q_1|^2)\\&=\operatorname{Im}\langle R_0^{\mp}(\lambda)u_1,u_1\rangle_{L^2}-\frac{|q_1|^2}{|\beta(-\lambda\pm i0)|^2}\operatorname{Im}\beta(-\lambda\pm i0).\end{aligned}$$

We have

$$\operatorname{Im}\langle R_0^{\mp}(\lambda)u_1,u_1\rangle_{L^2}=\mp\frac{\pi}{8\sqrt{\lambda}}\int_{|k|=\sqrt{\lambda}}\left|(k\cdot q)^2\,\hat\rho(k)\right|^2d\sigma$$

and

$$\operatorname{Im}\beta(-\lambda\pm i0)=-\frac{\lambda}{3}\operatorname{Im}\langle R_0^{\mp}(\lambda)\rho,\rho\rangle_{L^2}=\pm\frac{\pi}{6}\sqrt{\lambda}\int_{|k|=\sqrt{\lambda}}|\hat\rho(k)|^2\,d\sigma$$

by (24) and Lemma 5(b), whence

$$\begin{aligned}&\operatorname{Im}\langle R_A^{\pm}(\lambda)f_1(\psi_q),f_1(\psi_q)\rangle_{L^2\oplus\mathbb{C}^3}\\&=\mp\frac{\pi}{8\sqrt{\lambda}}\int_{|k|=\sqrt{\lambda}}\left|(k\cdot q)^2\,\hat\rho(k)\right|^2d\sigma\mp\frac{\pi}{6}\,\frac{|q_1|^2\sqrt{\lambda}}{|\beta(-\lambda\pm i0)|^2}\int_{|k|=\sqrt{\lambda}}|\hat\rho(k)|^2\,d\sigma\,.\end{aligned}$$

Therefore

$$\begin{aligned}&\operatorname{Im}\Big\langle[R_A^-(\lambda)-R_A^+(\lambda)]f_1(\psi_q),f_1(\psi_q)\Big\rangle_{L^2\oplus\mathbb{C}^3}\\&=\frac{\pi}{8\sqrt{\lambda}}\int_{|k|=\sqrt{\lambda}}\left|(k\cdot q)^2\,\hat\rho(k)\right|^2d\sigma+\frac{\pi}{6}\,\frac{|q_1|^2\sqrt{\lambda}}{|\beta(-\lambda-i0)|^2}\int_{|k|=\sqrt{\lambda}}|\hat\rho(k)|^2\,d\sigma\\&\quad-\Big(-\frac{\pi}{8\sqrt{\lambda}}\int_{|k|=\sqrt{\lambda}}\left|(k\cdot q)^2\,\hat\rho(k)\right|^2d\sigma-\frac{\pi}{6}\,\frac{|q_1|^2\sqrt{\lambda}}{|\beta(-\lambda+i0)|^2}\int_{|k|=\sqrt{\lambda}}|\hat\rho(k)|^2\,d\sigma\Big)\\&=\frac{\pi}{4\sqrt{\lambda}}\int\limits_{|k|=\sqrt{\lambda}}\left|(k\cdot q)^2\,\hat\rho(k)\right|^2d\sigma+\frac{\pi}{6}|q_1|^2\sqrt{\lambda}\Big(\frac{1}{|\beta(-\lambda+i0)|^2}+\frac{1}{|\beta(-\lambda-i0)|^2}\Big)\int\limits_{|k|=\sqrt{\lambda}}|\hat\rho(k)|^2\,d\sigma\,.\end{aligned}$$

As $\hat\rho(k)\neq0$ on the sphere $|k|=\sqrt{\lambda}=2|k_0|$, we find

$$\operatorname{Im}\Big\langle[R_A^-(\lambda)-R_A^+(\lambda)]f_1(\psi_q),f_1(\psi_q)\Big\rangle_{L^2\oplus\mathbb{C}^3}\neq0\,,$$

and therefore it is impossible to have $\delta(A-4|k_0|^2)\,f_1(\psi_q)=0$, cf. (18). This completes the proof of Theorem 1. □

5 Appendix: Existence of the dynamics in L_1^2

In this appendix we give some additional remarks concerning existence and estimates for the dynamics of (10) in weighted L^2-spaces. We mainly scetch the proofs instead of giving full details, since most of the arguments are, though lengthy, rather straightforward.

We write $\Psi=(\psi_1,\psi_2)=(\psi,\dot\psi)$, $\psi_i=(u_i,q_i)$ $(i=1,2)$, and consider (10) as

$$\frac{d}{dt}\begin{pmatrix}\psi_1\\\psi_2\end{pmatrix}=\begin{pmatrix}0&\mathrm{id}\\-\frac{1}{\omega_\varepsilon^2}A&0\end{pmatrix}\begin{pmatrix}\psi_1\\\psi_2\end{pmatrix}+\begin{pmatrix}0\\\frac{1}{\omega_\varepsilon^2}f_\varepsilon(\psi_1)\end{pmatrix}=:\mathcal{A}_\varepsilon\Psi+\mathcal{F}_\varepsilon(\Psi)\tag{A.1}$$

on

$$\mathcal{D}=D(\mathcal{A}_\varepsilon):=(H^2_{1,\Delta}\oplus\mathbb{C}^3)\oplus(H^1_1\oplus\mathbb{C}^3)\subset(H^1_1\oplus\mathbb{C}^3)\oplus(L^2_1\oplus\mathbb{C}^3)=:\mathcal{X}\,,$$

with A from (3), $f_\varepsilon(\psi)=\varepsilon^{-1}F(\varepsilon\psi)$, and F from (4). In the following, when considering (A.1), we always will include the case $\varepsilon=0$, where we let $\omega_0:=|k_0|$ and $\mathcal{F}_0:=0$. First we have

Lemma A.1 *The operator $\mathcal{A}_\varepsilon$ generates a C_0 semigroup $T_\varepsilon(t)$, $t \geq 0$, on $\mathcal{X}$. We have the estimate*

$$\sup_{t\in[0,t_*],\varepsilon\in[0,\varepsilon_0]} \|T_\varepsilon(t)\Psi\|_{\mathcal{D}} \leq C(t_*)\|\Psi\|_{\mathcal{D}} \quad \textit{for} \quad \Psi\in\mathcal{D} \quad \textit{and} \quad t_*\geq 0. \tag{A.2}$$

Proof: By classical semigroup theory, cf. [12], it is sufficient to show that for $\Psi^* \in \mathcal{X}$ and $|\lambda| < 1$, the equation $\Psi - \lambda\mathcal{A}_\varepsilon\Psi = \Psi^*$ has a solution $\Psi \in \mathcal{D}$ with $\|\Psi\|_{\mathcal{X}} \leq C(1-|\lambda|)^{-1}\|\Psi^*\|_{\mathcal{X}}$. Written in components, we obtain from (21)

$$\hat{u}_1(k) = \frac{\omega_\varepsilon^2\hat{u}^*(k) + i(k\cdot q_1)\hat{\rho}(k)}{k^2+\omega_\varepsilon^2/\lambda^2} \quad \text{with} \quad u^* = \frac{u_1^*+\lambda u_2^*}{\lambda^2} \in L_1^2, \quad \text{and} \quad \psi_2 = \frac{\psi_1-\psi_1^*}{\lambda}.$$

We first need to show $u_1 \in H^2_{1,\Delta}$, or equivalently $\hat{u}_1 \in L_2^2 \cap H^1$, $k\hat{u}_1 \in H^1$, and $k^2\hat{u}_1(k) \in H^1$, but this is straightforward from $\hat{u}^* \in H^1$, $\hat{\rho} \in \mathcal{S}$, and the additional power k^2 in the denominator. Thus also $\psi_2 \in H_1^1 \oplus \mathbb{C}^3$, since $\psi_1^* \in H_1^1 \oplus \mathbb{C}^3$, and therefore $\Psi = (\psi_1,\psi_2) \in \mathcal{D}$. Through direct calculations it is then also possible to verify $\|\Psi\|_{\mathcal{X}} \leq C(1-|\lambda|)^{-1}\|\Psi^*\|_{\mathcal{X}}$, always switching from H^1 to L_1^2 and vice versa, via Fourier transform. The estimate (A.2) uniformly in $\varepsilon \in [0,\varepsilon_0]$ also follows by direct calculation. For this, note that for $\Psi = ((u_1,q_1),(u_2,q_2)) \in \mathcal{X}$ we have

$$T_\varepsilon(t)\Psi =: ((u_{1,\varepsilon}(t,\cdot),q_{1,\varepsilon}(t)),(u_{2,\varepsilon}(t,\cdot),q_{2,\varepsilon}(t))),$$

implicitely given through

$$\begin{aligned}
\hat{u}_{1,\varepsilon}(t,k) &= \hat{u}_1(k)\cos(|k|t/\omega_\varepsilon) + \hat{u}_2(k)\frac{\omega_\varepsilon}{|k|}\sin(|k|t/\omega_\varepsilon) + \frac{i\hat{\rho}(k)}{\omega_\varepsilon|k|}\int_0^t \sin(|k|(t-s)/\omega_\varepsilon)\,k\cdot q_{1,\varepsilon}(s)\,ds,\\
q_{1,\varepsilon}(t) &= q_1\cos(\sqrt{\omega_0^2+\omega_1^2}\,t/\omega_\varepsilon) + q_2\left(\frac{\omega_\varepsilon}{\sqrt{\omega_0^2+\omega_1^2}}\right)\sin(\sqrt{\omega_0^2+\omega_1^2}\,t/\omega_\varepsilon)\\
&\quad + \frac{1}{\omega_\varepsilon\sqrt{\omega_0^2+\omega_1^2}}\int_0^t \sin(\sqrt{\omega_0^2+\omega_1^2}\,(t-s)/\omega_\varepsilon)\Big(\int dx\,\nabla\rho(x)u_{1,\varepsilon}(s,x)\Big)\,ds,\\
\hat{u}_{2,\varepsilon}(t,k) &= \frac{\partial\hat{u}_{1,\varepsilon}}{\partial t}(t,k) \quad \text{and} \quad q_{2,\varepsilon}(t) = \frac{dq_{1,\varepsilon}}{dt}(t),
\end{aligned} \tag{A.3}$$

so that a Gronwall estimate may be derived for $t \mapsto \|u_{1,\varepsilon}(t)\|_{L^2}$, which in turn yields that also $|q_{1,\varepsilon}(t)|$ is bounded on bounded t-intervals. If $\Psi \in \mathcal{D}$, then this boundeness and the regularity of Ψ can be used in the implicit representation of $T_\varepsilon(t)\Psi$ to ensure (A.2). □

By (4), since $D^2V(0) = \omega_0^2 E_3$, and because $\rho \in C_c^\infty(\mathbb{R}^3)$, it is also more or less obvious that we have the following

Lemma A.2 *The nonlinearity $\mathcal{F}_\varepsilon$ in (A.1) satisfies*

$$\mathcal{F}_\varepsilon((L^2\oplus\mathbb{C}^3)\oplus(L^2\oplus\mathbb{C}^3)) \subset \{0\}\oplus(C_c^\infty(\mathbb{R}^3)\oplus\mathbb{C}^3) \subset \mathcal{D},$$

and $\mathcal{F}_\varepsilon$ is locally Lipschitz continuous, if considered $\mathcal{X}\to\mathcal{X}$, and also, if considered $\mathcal{D}\to\mathcal{D}$, where $\mathcal{D}$ is equipped with the graph norm of $\mathcal{A}_\varepsilon$. In addition,

$$\|\mathcal{F}_\varepsilon(\Psi)\|_{\mathcal{D}} \leq C\varepsilon^{-1}\|F(\varepsilon\psi_1)\|_{H_1^1\oplus\mathbb{C}^3} \leq C\varepsilon|q_1|^2\Big(1+\varepsilon|q_1| + \sup_{\tau\in[0,1]}|D^3V(\tau\varepsilon q_1)|\Big) + C\varepsilon|q_1|\,\|u_1\|_{L^2} \tag{A.4}$$

for $\Psi = (\psi_1,\psi_2) \in (L^2\oplus\mathbb{C}^3)\oplus(L^2\oplus\mathbb{C}^3)$, with any norm for D^3V, the third derivatives of V.

According to Lemmas A.1 and A.2 we therefore find local solutions of (A.1) for $\varepsilon\in[0,\varepsilon_0]$. To be more precise, the remark following [12, Thm. 6.1.7] implies that for $\Psi_\varepsilon^0:=(\psi_\varepsilon^0,\dot\psi_\varepsilon^0)\in\mathcal{D}$ [with $(\psi_\varepsilon^0,\dot\psi_\varepsilon^0)$ from condition (11)], there exist $t_\varepsilon^{\max}\in]0,\infty]$ and a unique

$$\Psi_\varepsilon\in C^1([0,t_\varepsilon^{\max}[;\mathcal{X})\cap C([0,t_\varepsilon^{\max}[;\mathcal{D})$$

being a classical solution of (A.1) with $\Psi_\varepsilon(0)=\Psi_\varepsilon^0$. Writing $\Psi_\varepsilon=(\psi_{1,\varepsilon},\psi_{2,\varepsilon})=(\psi_\varepsilon,\dot\psi_\varepsilon)$ and $\psi_{i,\varepsilon}=(u_{i,\varepsilon},q_{i,\varepsilon})$, $i=1,2$, we therefore obtain in particular that $\psi_\varepsilon\in C^2([0,t_\varepsilon^{\max}[;L_1^2\oplus\mathbb{C}^3)$.

In fact, $t_\varepsilon^{\max}=\infty$, by the following

Lemma A.3 *We have*

$$\sup_{t\in[0,t_*],\varepsilon\in]0,\varepsilon_0]}\|\Psi_\varepsilon(t)\|_{\mathcal{D}}\le C(t_*) \tag{A.5}$$

for every $t_*\ge 0$, and hence in particular $t_\varepsilon^{\max}=\infty$ for $\varepsilon\in]0,\varepsilon_0]$. In addition,

$$\sup_{t\in[0,t_*],\varepsilon\in]0,\varepsilon_0]}\left\|\frac{d^2\psi_\varepsilon}{dt^2}(t)\right\|_{L_1^2\oplus\mathbb{C}^3}\le C(t_*). \tag{A.6}$$

This can be proved by using the variation of constants formula for Ψ_ε and by deriving a bound for $\varepsilon|q_{1,\varepsilon}(t)|$, uniformly in $\varepsilon\in]0,\varepsilon_0]$ and t in bounded intervals. The latter bound is obtain from the second-order Hamiltonian system for $\psi_{1,\varepsilon}=(u_{1,\varepsilon},q_{1,\varepsilon})$, cf. the arguments in [10, Section 2]. Here it is needed that $V(q)\to\infty$ as $|q|\to\infty$.

Identifying now $\psi_\varepsilon(t,x):=(\psi_\varepsilon(t))(x)$, we thus obtain

Corollary A.1 *For $\varepsilon\in]0,\varepsilon_0]$,*

$$\psi_\varepsilon,\, I_\varepsilon(\psi_\varepsilon)=\Big[(\omega_\varepsilon^2-|k_0|^2)\frac{\partial^2}{\partial t^2}\psi_\varepsilon+\varepsilon^{-1}F(\varepsilon\psi_\varepsilon)\Big]\in\mathcal{H}_1^2=L_1^2(S^1\oplus\mathbb{C}^3)\oplus L^2(\mathbb{C}^3).$$

Additionally, we find from (A.6) and condition (11)(b)

Corollary A.2 *As $\varepsilon\to 0^+$,*

$$\frac{(\omega_\varepsilon^2-|k_0|^2)}{\varepsilon}\Big(\frac{\partial^2}{\partial t^2}\psi_\varepsilon\Big)\to 0\quad in\quad \mathcal{H}_1^2=L_1^2(S^1\times\mathbb{R}^3)\oplus L^2(\mathbb{C}^3).$$

Having obtained the necessary bounds, next we investigate the convergence of Ψ_ε as $\varepsilon\to 0^+$. For $\varepsilon=0$ and $\psi_q\in\mathcal{E}$ we have $\Psi_{0,q}(t)=(\sin t,\cos t)\psi_q$ as global solution with initial value $\Psi_{0,q}(0)=\Psi_{0,q}^0:=(0,\psi_q)$. Thus condition (11)(a) means

$$\|\Psi_\varepsilon^0-\Psi_{0,q}^0\|_{(L^2\oplus\mathbb{C}^3)\oplus(L^2\oplus\mathbb{C}^3)}\to 0\quad\text{as}\quad\varepsilon\to 0^+. \tag{A.7}$$

Let

$$\mathcal{F}_1(\Psi)=\begin{pmatrix}0\\ \frac{1}{|k_0|^2}f_1(\psi_1)\end{pmatrix}\in\{0\}\oplus(C_c^\infty(\mathbb{R}^3)\oplus\mathbb{C}^3)\subset\mathcal{D},\quad \Psi=(\psi_1,\psi_2)\in(L^2\oplus\mathbb{C}^3)\oplus(L^2\oplus\mathbb{C}^3), \tag{A.8}$$

with f_1 from (27) above. By Taylor expansion up to order three the following result can be shown, where $\Psi_{0,q}(t)=(\sin t,\cos t)\psi_q=T_0(t)(0,\psi_q)$,with the semigroup $T_0(t)$ generated by $\mathcal{A}_0$.

Lemma A.4 *For $t_* \geq 0$,*

$$\sup_{t\in[0,t_*]} \|\varepsilon^{-1}\mathcal{F}_\varepsilon(\Psi_\varepsilon(t)) - \mathcal{F}_1(\Psi_{0,q}(t))\|_{\mathcal{D}} \to 0 \quad as \quad \varepsilon \to 0^+.$$

Recalling that $\Psi_\varepsilon(t) = (\psi_\varepsilon(t), \dot{\psi}_\varepsilon(t))$, by definition of $\mathcal{F}_\varepsilon$ and $\mathcal{F}_1$, cf. (A.1) resp. (A.8), this particularly implies

Corollary A.3 *As $\varepsilon \to 0^+$,*

$$\varepsilon^{-2}F(\varepsilon\psi_\varepsilon) \to f_1(\psi_q) \quad in \quad \mathcal{H}_1^2 = L_1^2(S^1 \times \mathrm{I\!R}^3) \oplus L^2(\mathbb{C}^3),$$

where $f_1(\psi_q)(t,x) := f_1(\sin t\psi_q)(x)$, with the first component of $f_1(\sin t\psi_q)$ being applied to x, and similarly $F(\varepsilon\psi_\varepsilon)(t,x) := F(\varepsilon\psi_\varepsilon(t))(x)$.

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