# Unbounded Solutions in Non-Smooth Dynamical Systems at Resonance

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**Abstract**

We consider periodically forced non-smooth dynamical systems at resonance, described by differential inclusions, and we show that analogously to the case of ODEs all solutions are unbounded in the $(x,\dot{x})$-phase plane, if the first Fourier coeffcient of the forcing is large compared to a certain quantity related to the nonlinearity.

## 1 Introduction

Several non-smooth problems in e.g. mechanics can be described by differential inclusion of the form

$$\ddot{x}+x+F(x,\dot{x})\ni p(t) \quad \text{a.e.}, \tag{1}$$

where $p$ is a continuous and $2\pi$-periodic forcing, whereas the multivalued operator $F:\mathbb{R}\times\mathbb{R}\to 2^{\mathbb{R}}\setminus\{\emptyset\}$ represents the nonlinearity. These differential inclusions are a generalization of the usual ODEs, suitable for the description of mechanical effects like stick-slip motions caused by Coulomb friction, cf. [6] for modelling and [2] or [4] for theory. In ODE theory one is interested in the question whether there are unbounded solutions of the equations, and by adopting a method developed in [1], which rests on the use of a suitable Lyapunov function, we shall show that one can also give a criterion for all solutions to be unbounded in the case of (1).

## 2 Unbounded Solutions

We start with some preliminaries. For simplicity we assume that for every $\xi_0=(x_0,v_0)\in\mathbb{R}^2$ and $t_0\in[0,2\pi]$ there exists a unique global solution $x(\cdot;\xi_0,t_0)$ of (1) on $[t_0,\infty[$ with initial values $x(t_0;\xi_0,t_0)=x_0$ and $\dot{x}(t_0;\xi_0,t_0)=v_0$, i.e. (1) generates a well-defined semiflow. To state our result, we let $\mathcal{E}_p(t;\xi_0,t_0)=x^2(t;\xi_0,t_0)+\dot{x}^2(t;\xi_0,t_0)$ for $t\in[t_0,\infty[$ denote the energy of the solution, where the subscript $p$ refers to the forcing $p$ in (1). We also define $\sup(F)=\sup\{z: z\in F(x,v)\,,\ x,v\in\mathbb{R}\}$ and $\inf(F)=\inf\{z: z\in F(x,v)\,,\ x,v\in\mathbb{R}\}$ for a multivalued map $F$ as above.

**Theorem 1** *Let $F:\mathbb{R}\times\mathbb{R}\to 2^{\mathbb{R}}\setminus\{\emptyset\}$ be a multivalued map with $\sup(F)<\infty$ and $\inf(F)>-\infty$, let $p$ be continuous and periodic with period $2\pi$, and suppose that*

$$\left|\int_0^{2\pi} p(t)\,e^{it}\,dt\right| > 2\,(\sup(F)-\inf(F))\,. \tag{2}$$

*Then for every $R>0$, $\inf_{|\xi_0|\le R,t_0\in[0,2\pi]}\mathcal{E}_p(t;\xi_0,t_0)\to\infty$ as $t\to\infty$. In particular, every solution of (1) is unbounded in the $(x,\dot{x})$-phase plane.*

Before going on to the proof, we need an auxiliary lemma.

**Lemma 1** *In the situation described above,*

*(a)* $\mathcal{E}_p(t+t_0;\xi_0,t_0)=\mathcal{E}_{p(\cdot+t_0)}(t;\xi_0,0)$ *for* $t\in[0,\infty[$, $t_0\in[0,2\pi]$, *and* $\xi_0\in\mathbb{R}^2$.

*(b)* $\mathcal{E}_{p(\cdot+t_0)}(2\pi n+t;\xi_0,0)=\mathcal{E}_{p(\cdot+t_0)}(t;(x(2\pi n;\xi_0,0),\dot{x}(2\pi n;\xi_0,0)),0)$ *for* $n\in\mathbb{N}_0$, $t\in[0,2\pi]$, $t_0\in[0,2\pi]$, *and* $\xi_0\in\mathbb{R}^2$.

*(c) There exist constants $c_1,c_2>0$ such that for $t\in[0,\infty[$, $t_0\in[0,2\pi]$, and $\xi_0=(x_0,v_0)\in\mathbb{R}^2$*

$$\mathcal{E}_{p(\cdot+t_0)}(t;\xi_0,0)\ge c_1(x_0^2+v_0^2)-c_2\,.$$

**Proof:** Ad (a) &(b): This follows from the assumed uniqueness of solutions to the right. Ad (c): The statement is the same as [1; Lemma 3.3]. By (1) we have $\ddot{x}+x+w=p(t+t_0)$ a.e. in $[0,\infty[$ for some function $w$ satisfying $w(t)\in F(x(t),\dot{x}(t))$ a.e., and therefore $x(t)=x_0\cos t+v_0\sin t+\int_0^t[p(\sigma+t_0)-w(\sigma)]\sin(t-\sigma)d\sigma$ for $t\in[0,\infty[$. Using $|w(\sigma)|\le\max\{|\sup(F)|,|\inf(F)|\}$ and inserting this in the definition of $\mathcal{E}_{p(\cdot+t_0)}(t;\xi_0,0)$, the claimed esimate follows easily. $\square$

Now we can proceed to show Theorem 1.

**Proof:** By Lemma 1(a) it is sufficient to show that $\inf_{|\xi_0|\le R,t_0\in[0,2\pi]}\mathcal{E}_{p(\cdot+t_0)}(t;\xi_0,0)\to\infty$ as $t\to\infty$, and Lemma 1(b) & (c) imply that for this it is enough to prove $\inf_{|\xi_0|\le R,t_0\in[0,2\pi]}\mathcal{E}_{p(\cdot+t_0)}(2\pi n;\xi_0,0)\to\infty$ as $n\to\infty$. For this, we fix $t_0\in[0,2\pi]$ and define $f_{t_0}(\xi_0)=(x(2\pi;\xi_0,0),\dot{x}(2\pi;\xi_0,0))$, with the solution $x$ corresponding to the forcing $p(\cdot+t_0)$. Hence $\mathcal{E}_{p(\cdot+t_0)}(2\pi n;\xi_0,0)=|f_{t_0}^n(\xi_0)|_{\mathbb{R}^2}^2$ by Lemma 1(b) and the uniqueness of solutions to the right. We choose $\phi\in[0,2\pi[$ in such a way that $\int_0^{2\pi}p(t+t_0)\sin(t+\phi)\,dt=|\int_0^{2\pi}p(t+t_0)\,e^{it}\,dt|=|\int_0^{2\pi}p(t)\,e^{it}\,dt|$. Next we define the continuous Lyapunov function $V_{t_0}:\mathbb{R}^2\to\mathbb{R}$ through $V_{t_0}(x,v)=v\sin\phi-x\cos\phi$ for $(x,v)\in\mathbb{R}^2$. By (1) we have $\ddot{x}+x+w=p(t+t_0)$ a.e. on $[0,\infty[$ for some function $w$ satisfying $w(t)\in F(x(t),\dot{x}(t))$ a.e. Therefore integration $\int_0^{2\pi}\ddot{x}(t;\xi_0,0)\sin(t+\phi)dt=V_{t_0}(f_{t_0}(\xi_0))-V_{t_0}(\xi_0)-\int_0^{2\pi}x(t;\xi_0,0)\sin(t+\phi)dt$, and consequently by choice of $\phi$, denoting by $\sin^+$ resp. $\sin^-$ the positive resp. negative part of sin, for every $\xi_0\in\mathbb{R}^2$

$$\begin{aligned}V_{t_0}(f_{t_0}(\xi_0)) &\ge V_{t_0}(\xi_0)+\left|\int_0^{2\pi}p(t)\,e^{it}\,dt\right|+\inf(F)\int_0^{2\pi}\sin^-(t+\phi)dt-\sup(F)\int_0^{2\pi}\sin^+(t+\phi)dt\\ &= V_{t_0}(\xi_0)+\left|\int_0^{2\pi}p(t)\,e^{it}\,dt\right|-2\,(\sup(F)-\inf(F))=:V_{t_0}(\xi_0)+\delta_0 \qquad (3)\end{aligned}$$

with $\delta_0>0$ being independent of $t_0$ by assumption (2). We find a constant $c(R)>0$ such that $|V_{t_0}(\xi_0)|\le c(R)$ for all $t_0\in[0,2\pi]$ and $|\xi_0|\le R$. Hence iteration of (3) implies $V_{t_0}(f_{t_0}^n(\xi_0))\ge V_{t_0}(\xi_0)+n\delta_0\ge -c(R)+n\delta_0$ for all $n\in\mathbb{N}$, $t_0\in[0,2\pi]$, and $|\xi_0|\le R$. Thus $\inf_{|\xi_0|\le R,t_0\in[0,2\pi]}|f_{t_0}^n(\xi_0)|_{\mathbb{R}^2}^2\to\infty$ as $n\to\infty$ as was to be shown. $\square$

**Example 1** We first consider a pendulum with dry friction at resonance, i.e. $\ddot{x}+x+\mathrm{Sgn}(\dot{x})\ni\gamma\sin t$ a.e. with $\gamma\in[0,\infty[$, cf. [2]. Since $F(\dot{x})=\mathrm{Sgn}(\dot{x})$ has $\sup(F)=1$ and $\inf(F)=-1$, and $|\int_0^{2\pi}\sin t\,e^{it}\,dt|=\pi$, condition (2) becomes $\gamma\in]4/\pi,\infty[$. Hence for those $\gamma$ Theorem 1 applies, with $p=\gamma\sin(\cdot)$.

In particular, for $\gamma > 4/\pi$ the solutions are uniformly unbounded in the $(x,\dot{x})$-phase plane. This generalizes a result of [3; Prop. 1] where it was shown that there are no $2\pi$-periodic solutions for $\gamma > 4/\pi$. We also note that this example enlightens a difference between the ODEs considered in [1] and the inclusions analyzed above: while in condition (3.2) of [1] the equality corresponding to equality in condition (2) is also sufficient to guarantee the existence of unbounded solutions, the same does not hold for the above example, and thus also not for (1) in general: in case that $\gamma = 4/\pi$ here there exist (infinitely many explicitely known) periodic solutions, cf. [3; Prop. 1], i.e. there are bounded solutions.

**Example 2** In [5] it was shown by means of KAM theory that for $\ddot{x} + x + a\,\mathrm{Sgn}(x) \ni p(t)$ a.e. with $p \in C^6(\mathbb{R})$ being $2\pi$-periodic every solution is bounded in the $(x,\dot{x})$-phase plane for parameters $a > 0$ sufficiently large. Theorem 1 shows that some condition on the size of $a$ is necessary, since for $|\int_0^{2\pi} p(t)\,e^{it}\,dt| > 4a$ the solutions will be uniformly unbounded.

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