Degenerate Sweeping Processes

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1 Introduction

The aim of this paper is to summarize some recent results concerning the evolutionary differential inclusions

$$-u'(t) \in \partial\delta_{C(t)}(Au(t)) \quad \text{a.e. in} \quad [0,T], \quad u(0) = u_0 \in D(A), \tag{1}$$

where A is a maximal monotone and strongly monotone operator in a real Hilbert space H, and $t \mapsto C(t)$ is a set-valued mapping, cf. the precise assumptions below. Moreover, as usually, $\partial\delta_{C(t)}(v)$ denotes the cone of normals to the closed convex set $C(t)$ at the point $v \in C(t)$.

The investigation of inclusions of type (1) is motivated by problems in quasistatic elastoplasticity as follows: in some plastic flow problems one has to solve

$$-\frac{\partial q}{\partial t}(t,x) \in \partial\delta_{E(x)}\Big(\frac{\partial W}{\partial q}(t,x,q(t,x))\Big), \tag{2}$$

cf. e.g. [3]. In this formulation, every $E(x) \subset \mathbb{R}^n$ is a "rigidity set", $t \geq 0$ denotes process time, $x \in \Omega \subset \mathbb{R}^n$ is the material point, $q \in \mathbb{R}^n$ is an internal variable (like plastic strain or hardening), and W denotes a (modified) stored energy function. For every fixed $x \in \Omega$, (2) has the general form

$$-q'(t) \in \partial\delta_\Gamma\Big(g(t,q(t))\Big) \tag{3}$$

with $\Gamma = E(x)$ and $g(t,q) = \frac{\partial W}{\partial q}(t,x,q)$, and in fact (3) may easily be transformed into an inclusion of type (1), as explained in the following lemma.

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Lemma 1 *If $q:[0,T]\to H$ is a solution of (3) with initial value $q(0)=q_0$, then $u:[0,T]\to \mathbb{R}\times H=:\mathcal{H}$, defined by $u(t)=(t,q(t))$, is a solution of (1) with initial value $u_0=(0,q_0)$ in the Hilbert space $\mathcal{H}$, where $C(t):=[t,\infty[\times\Gamma$ for $t\in[0,T]$ and $A(t,x):=(t,g(t,x))$ for $(t,x)\in\mathcal{H}$. Conversely, if $u=(\theta,q):[0,T]\to\mathcal{H}$ solves (1), then $q:[0,T]\to H$ solves (3).*

Thus the study of inclusions such as (1) is of use in order to solve problems of type (3).

It is also worth mentioning that problems similar to (1) arise in the study of elliptic-parabolic variational (in-) equalities as were studied in [5] or [6], cf. also the references therein for additional information. To explain this, consider e.g. an elliptic-parabolic BVP arising in the modelling of flows in porous media, with obstacle at the boundary, which may depend on time, cf. [6]. For this, let $\Omega\subset\mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$, and suppose that for each t the boundary may be decomposed disjointly into $\partial\Omega=\overline{\Gamma_D(t)}\cup\overline{\Gamma_N(t)}\cup\overline{\Gamma_U(t)}$. Let $\rho:\mathbb{R}\to\mathbb{R}$ be Lipschitz continuous and nondecreasing, and let functions r_D, q_N, r_U, and q_U on $\partial\Omega$ be given. Then the problem is to find a function $\theta=\theta(t,x)$ on $[0,T]\times\Omega$ such that

$$\begin{gathered}\tfrac{\partial}{\partial t}[\rho(\theta)]-\Delta\theta=-f \quad \text{in} \quad]0,T[\times\Omega, \quad \theta(0,\cdot)=\theta_0, \qquad (4)\\ \theta=r_D \text{ on } \Sigma_D, \quad \tfrac{\partial}{\partial n}\theta=q_N \text{ on } \Sigma_N, \quad \theta\geq r_U, \ \tfrac{\partial}{\partial n}\theta\geq q_U, \ (\theta-r_U)(\tfrac{\partial}{\partial n}\theta-q_U)=0 \text{ on } \Sigma_U,\end{gathered}$$

with $\Sigma_D=\bigcup_{0<t<T}(\{t\}\times\Gamma_D(t))$, and analogous for Σ_N and Σ_U, cf. also [1, 7, 8]. Problem (4) may be reformulated in a suitable Hilbert space H as

$$-\frac{d}{dt}[B\theta(t)]\in\partial\varphi^t(\theta(t))+f(t),$$

where every φ^t is a convex and lower semicontinuous function, depending somehow regularly on $t\in[0,T]$. With the multivalued inverse $A=B^{-1}$, we arrive at

$$-u'(t)\in\partial\varphi^t(Au(t))+f(t). \qquad (5)$$

Notice that we can obtain (1) from (5), by taking $f=0$ (which is not important), and $\varphi^t=\delta_{C(t)}$, the indicator function of the set $C(t)$, i.e. $\delta_{C(t)}(x)=0$ for $x\in C(t)$ and $\delta_{C(t)}(x)=\infty$ for $x\in H\setminus C(t)$.

Despite the similarity of (1) and (5), the problems are quite different, since in general with $\varphi^t=\delta_{C(t)}$ neither the sublevel sets $\{x\in H:\varphi^t(x)\leq r\}$ are compact, nor $\varphi^t(x)\geq c|x|^2$ for some constant $c>0$, and these were basic assumptions in [6], cf. (a1) and (a2) in that paper.

After this more or less heuristical introduction, we are going to explain the results we obtained for (1) resp. (3) in the following section. We are going to use only standard notations concerning lower semicontinuous function or maximal monotone operators, cf. e.g. [2, 4].

2 Results

We start with the definition of solutions of (1) resp. (3).

Definition 1 *(a) Let $u_0\in D(A)$ be given such that $Au_0\cap C(0)\neq\emptyset$. We call a pair (u,v) a solution of (1), if $u\in W^{1,2}([0,T];H)$, $v\in L^\infty([0,T];H)$, $v(t)\in Au(t)\cap C(t)$ for a.e. $t\in[0,T]$, and if (1) is satisfied (with $v(t)$ inserted instead of $Au(t)$).*

(b) Let $q_0\in H$ be given such that $g(0,q_0)\in\Gamma$. A function $q\in W^{1,2}([0,T];H)$ is called a solution of (3), if $g(t,q(t))\in\Gamma$ for a.e. $t\in[0,T]$, and if (3) is satisfied.

Our general assumptions to prove the existence of solutions of (1) are:

(H1) $A: D(A) \to 2^H \setminus \{\emptyset\}$ is a maximal monotone operator (mmop, for short) such that $A = \partial\psi$ for some lsc, convex and proper $\psi: H \to \mathbb{R} \cup \{\infty\}$, and there exists $\beta > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \beta |x - y|^2 \quad \text{for} \quad x, y \in D(A). \tag{6}$$

(H2) For every $t \in [0,T]$, $\emptyset \neq C(t) \subset H$ is a closed convex set, and $t \mapsto C(t)$ is Lipschitz continuous, i.e. for some $L \geq 0$

$$\mathrm{d_H}(C(t), C(s)) \leq L|t - s| \quad \text{for} \quad t, s \in [0,T], \tag{7}$$

with $\mathrm{d_H}$ being the Hausdorff-distance.

The following result was obtained in [10].

Theorem 1 *Let (H1) and (H2) be satisfied. If in addition*

(H3a) $C(0)$ *is bounded, or*

(H3b) *there exists a function* $M: [0,\infty[\to [0,\infty[$ *which maps bounded sets into bounded sets such that*

$$\|Ax\| = \sup\{|y| : y \in Ax\} \leq M(|x|) \quad \textit{for} \quad x \in D(A), \tag{8}$$

and

(H4a) $D(A) \cap \overline{B}_R(0)$ *is relatively compact for every* $R > 0$*, or*

(H4b) $C(t) \cap \overline{B}_R(0)$ *is compact for every* $t \in [0,T]$ *and* $R > 0$*,*

then (1) has a Lipschitz continuous solution, for every $u_0 \in D(A)$ *with* $Au_0 \cap C(0) \neq \emptyset$*.*

In the bounded linear case we can do much better. Here we have

Theorem 2 *Let* $A: H \to H$ *be linear, bounded and selfadjoint such that* $\langle Ax, x \rangle \geq \beta |x|^2$ *for* $x \in H$*. If (H2) holds for* $C(\cdot)$ *and if* $Au_0 \in C(0)$*, then (1) has a unique solution, and this solution is Lipschitz continuous.*

Simple examples show that this Theorem 2 (and hence also Theorem 1) might be wrong if $\beta = 0$ in (6).

In both theorems we had to assume that A be a subdifferential, cf. $(H1)$. From [9] there is also available a version of Theorem 1 which exchanges this condition against a certain compactness assumption, formulated in terms of the Yosida-approximants A_μ of the mmop A.

Theorem 3 *Let (H1) be satisfied, without the assumption on* A *to be a subdifferential, and let (H2) hold. Assume in addition that (H3b) and (H4b) are fulfilled and also that*

(H5) If $\mu_n \to 0^+$*,* $u_n \to u$ *in* $C([0,T];H)$ *and* $v_n = A_{\mu_n} u_n \to v$ *weakly in* $L^2([0,T];H)$*, then even* $v_n \to v$ *in* $L^2([0,T];H)$*.*

Then (1) has a Lipschitz continuous solution, for every $u_0 \in D(A)$ *such that* $A^0 u_0 \in C(0)$, *where* A^0 *is the minimal-norm selection of* A.

In particular this theorem is taylored to treat finite-dimensional problems, since both $(H4b)$ and $(H5)$ are compactness conditions. Via Lemma 1, Theorem 3 may be used to prove existence theorems for problems of the form (3), as shows the following result from [9].

Theorem 4 *Let* $g \in C(\mathrm{IR} \times \mathrm{IR}^n, \mathrm{IR}^n)$ *and* $\Gamma \subset \mathrm{IR}^n$ *be closed and convex. Suppose that for some* $\alpha > 0$ *and* $\beta \in]-\infty, 1[$

$$\langle g(t,x) - g(s,y), x-y \rangle \geq \alpha |x-y|^2 - \beta (t-s)^2 \quad \text{for} \quad t,s \in \mathrm{IR},\, x,y \in \mathrm{IR}^n. \tag{9}$$

Then (3) has a solution in the sense of Definition 1 on every $[0,T]$ *with* $T>0$.

In the remaining part of this paper we will sketch the proof of Theorem 1 by means of a suitable discretization, following [10]. We remark that an alternative proof can be based on a double Yosida-Moreau regularization, by first replacing A through A_μ, and then also approximating the lower semicontinuous $\partial\varphi^t = \partial\delta_{C(t)}$ by $(\partial\varphi^t)_\lambda$. This very technical approach was followed in [6] for (5), and also in [9], where Theorem 1 (with assumptions $(H3b)$, $(H4)$ and $A^0 u_0 \in C(0)$) and Theorem 3 were proved by this method.

To describe the (implicit) discretization of (1), we note that for given u_i this problem somehow consists in finding $u_{i+1} \in D(A)$ with $Au_{i+1} \in C(t_i)$ such that

$$-(u_{i+1} - u_i) = u_i - u_{i+1} \in \partial\delta_{C(t_{i+1})}(Au_{i+1}), \tag{10}$$

with a partition $0 = t_0 < t_1 < \ldots < t_k = T$ of $[0,T]$. This motivates the following definition.

Definition 2 *Let* A *be a mmop in* H *such that* (6) *holds, and let* $C \subset H$ *be nonempty, closed, and convex. A map* $D(A) \ni u \mapsto P_{A,C}(u) \in D(A)$ *is called the approximation operator, if for every* $u \in D(A)$ *there exists a* $P_{A,C}(u) = w \in D(A)$ *such that* $u - w \in N_C(Aw)$, *i.e.* $u - w \in N_C(v)$ *for some* $v \in Aw \cap C$.

Because of (6), the element $P_{A,C}(u)$ is unique. It now turns out that the approximation operator $P_{A,C}$ exists, if C is bounded. Moreover, there is also a good dependence of $P_{A,C}(\cdot)$ on C.

Lemma 2 *Let* A *be a mmop such that (6) holds. If* $C \subset H$ *is nonempty, closed, convex, and bounded, then* $P_{A,C}(\cdot)$ *exists. Moreover, if* $C_1, C_2 \subset H$ *are such that* $P_{A,C_2}(\cdot)$ *exists, and if* $u_1 \in D(A)$ *with* $Au_1 \cap C_1 \neq \emptyset$, *then*

$$|u_1 - P_{A,C_2}(u_1)| \leq \frac{1}{\beta} \mathrm{d_H}(C_1, C_2). \tag{11}$$

Although the sets $C(t)$ need not be bounded in case that $(H3b)$ holds in Theorem 1, we continue to argue with this additional assumption, since by intersection with increasing large balls this boundedness may be incorporated artificially.

If every $C(t)$ is bounded, in particular all the approximation operators $P_{A,C(t_{i+1})}(\cdot)$ exist, i.e. by definition of these operators we may solve (10) to obtain $u_{i+1}\in D(A)$ and some $v_{i+1}\in Au_{i+1}\cap C(t_{i+1})$. In addition, by (11) and (7) in $(H2)$ we find

$$|u_i - u_{i+1}| \le \beta^{-1} \mathrm{d}_{\mathrm{H}}(C(t_i), C(t_{i+1})) \le \frac{L}{\beta} |t_i - t_{i+1}|. \tag{12}$$

Thus we can define a piecewise constant right-continuous approximation u with values $u_1,\ldots,u_k$. Consequently, by (12) the variation of u is bounded by LT/β. If we let shrink $\sup_i |t_i - t_{i+1}|$ to zero by inserting more and more partition points, we thus get a sequence of right-continuous approximants u_n which are uniformly bounded in variation by LT/β, and they are also uniformly bounded in norm, since $u_n(0)=u_0$, the given initial value, for all $n\in\mathbb{N}$. This is enough to find a right-continuous $u:[0,T]\to H$ and a subsequence (again denoted by u_n) such that u has variation $\le LT/\beta$, and such that $u_n(t)\to u(t)$ weakly in H for every $t\in[0,T]$, cf. [11, Theorem 0.2.1].

Thus it remains to show that u is a solution of (1). For this the additional assumptions $(H3)$ and $(H4)$ will be used, whereas for the construction of the approximants only $(H1)$ and $(H2)$ were needed. Up to now we have obtained functions u_n and v_n such that $u_n(t)\in D(A)$, $v_n(t)\in Au_n(t)$ and $v_n(t)$ "almost in" $C(t)$ for $t\in[0,T]$. Then $(H3)$ is used to ensure that also $(v_n)_{n\in\mathbb{N}}$ is uniformly bounded, and thus w.l.o.g. $u_n\to u$ and $v_n\to v$ weakly in $L^2([0,T];H)$ for some function v. But to conclude $v(t)\in Au(t)$ a.e., one of the weak convergences has to be improved to a strong convergence, and for this $(H4)$ is needed. Finally, $-u'(t)\in \partial\delta_{C(t)}(v(t))$ a.e. also follows from this improved convergence, and the Lipschitz continuity of u is established by deriving suitable estimates of the type $|u_n(t)-u_n(s)|\le c(|t-s|+1/n)$ for the approximants, with $c>0$ being independent of $n\in\mathbb{N}$. Hence the outline of the proof of Theorem 1 is complete, cf. [10] for the details. □

We also remark that our results remain true, if $t\mapsto C(t)$ is only assumed to be absolutely continuous, i.e. $\mathrm{d}_{\mathrm{H}}(C(t),C(s))\le|a(t)-a(s)|$ for some absolutely continuous (a.c.) real function a, the only difference being that the obtained solution is then a.c. instead of Lipschitz continuous.

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