Periodic Solutions of Nonlinear Kinematic Hardening Models

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Abstract

We show the existence of periodic solutions to models in elastoplasticity, like the Chaboche model.

1 Introduction

In this paper we consider the problem of proving the existence of a periodic solution to models which arise in elastoplasticity. These models are characterized by imposing certain stress-strain (or strain-stress) laws describing the relation between load and deformation of the solid body under consideration. It is shown in [4] that many of these models, in particular the basic linear kinematic hardening model of Melan [10] and Prager [12], but also the more refined nonlinear models of Armstrong and Frederick [1], Chaboche [8], Bower [2], and Mróz [11], can be cast in the common form

$$\dot{u}(t) = \dot{\theta}(t) + \mathcal{M}(\theta,u,p)(t)|\dot{\xi}(t)| \quad \text{a.e. in} \quad [0,T], \quad \xi = \mathcal{P}(u). \tag{1.1}$$

Here the notation $\xi = \mathcal{P}(u)$ (the play operator) is an abbreviation for requiring that ξ be a solution to the "sweeping process"

$$-\dot{\xi}(t) \in N_{C_u(t)}(\xi(t)) \quad \text{a.e. in} \quad [0,T], \quad C_u(t) = u(t) - Z. \tag{1.2}$$

Note that, since the normal cone is $N_C(\xi) = \{\tau : \langle \tau, \xi - c\rangle \geq 0 \; \forall\, c \in C\}$, with $x(t) = u(t) - \xi(t)$ this means

$$\langle \dot{u}(t) - \dot{x}(t), x(t) - z\rangle \geq 0 \quad \text{for all} \quad z \in Z, \quad x(t) \in Z, \quad t \in [0,T].$$

Hence (1.2) is just a reformulation of the normality rule, also called the principle of maximal dissipation or the principle of maximal plastic work rate. In (1.2), Z is the elastic region in stress space (the Hilbert space $\mathcal{H} = \mathbb{T}_d$ of $N \times N$-tensors with zero trace; see (2.7) below), with boundary ∂Z, termed the yield surface. We choose $Z = \overline{B}_r(0) = \{\tau \in \mathbb{T}_d : |\tau| \leq r\}$, which is the so-called von Mieses yield condition. Additionally, in (1.1), θ stands for either the stress σ or the strain ε, considered as an input function, whereas $\mathcal{M}$ is a certain operator (determined by the specific model) which additionally may depend on some parameter p.

The aim of this paper is to prove the existence of a T-periodic solution to (1.1), provided the input function θ is T-periodic and Lipschitz continuous. The latter assumption simplifies the presentation and can pressumably be relaxed to absolute continuity of θ, by the same method. To emphasize the relation of (1.1) to models in elastoplasticity, we decided to carry out the proof for a specific example, namely the Chaboche model, which is introduced in detail in Section 2 and enjoys great popularity among engineers. Again we expect that our method of proof will work for different choices of the model, and another argument for not treating the general form (1.1) is that this would lead to artificially complicated periodicity assumptions on a general $\mathcal{M}$.

2 The Chaboche model

In this section we are going to introduce, following [3, 4, 5, 6], the Chaboche model [8]. The governing relations are

$$\sigma = \sigma^b + \sigma^p, \quad \varepsilon = \varepsilon^e + \varepsilon^p, \tag{2.1}$$

$$\sigma = A\varepsilon^e, \tag{2.2}$$

$$\varepsilon^p \in \mathbb{T}_d, \quad |\sigma^p_d| \leq r, \quad \langle \dot{\varepsilon}^p, \sigma^p_d - \tau\rangle \geq 0 \quad \text{for all} \quad \tau \in \mathbb{T}_d, |\tau| \leq r, \tag{2.3}$$

$$\sigma^b = \int_I \sigma_k^b \, d\nu(k) + \nu^l \sigma^l \,, \tag{2.4}$$

$$\dot\sigma_k^b = \gamma(k)(R(k)\dot\varepsilon^p - \sigma_k^b |\dot\varepsilon^p|) \quad \text{for all} \quad k \in I \,, \tag{2.5}$$

$$\sigma^l = C^l \varepsilon^p \,, \tag{2.6}$$

where σ and ε denote strain and stress, respectively, and σ^b resp. σ^p are the backstress resp. the plastic stress. Moreover, ε^e and ε^p denote elastic strain and plastic strain, with plastic strain rate $\dot\varepsilon^p$. To introduce further notation, let $\mathbb{T}$ be the symmetric $N \times N$-tensors (matrices) with the usual inner product and norm in $\mathbb{R}^{N\times N}$, and define the space of deviators

$$\mathbb{T}_d = \{\tau \in \mathbb{T} : \mathrm{trace}(\tau) = 0\} \,, \quad \text{where} \quad \mathrm{trace}(\tau) = \sum_{i=1}^N \tau_{ii} \,, \quad \tau \in \mathbb{T} \,. \tag{2.7}$$

In addition, let $\tau_d = \tau - \dfrac{\mathrm{trace}(\tau)}{N}\delta \in \mathbb{T}_d$ denote the trace-free part of $\tau \in \mathbb{T}$, with $\delta \in \mathbb{T}$ being the $N \times N$-unit matrix. As linear elastic law $A : \mathbb{T} \to \mathbb{T}$ in (2.2) we choose Hooke's law

$$A\varepsilon = 2\mu\varepsilon + \lambda \,\mathrm{trace}(\varepsilon)\delta \,, \tag{2.8}$$

with $\lambda, \mu > 0$ the Lamé constants.

Next, the σ^l are determined by (2.6) through ε^p and some fixed constants $C^l > 0$. They enter the definition of σ^b in (2.4) with fixed constants $\nu^l \geq 0$. In the standard Chaboche model with K limiting surfaces,

$$\sigma^b = \sum_{k=1}^K \sigma_k^b \,, \tag{2.9}$$

but as in [5, 6] it poses here no additional problems to generalize this to (2.4), allowing a finite non-negative measure $d\nu = d\nu(k)$ on some measure space I instead of counting measure corresponding to (2.9). Accordingly, we assume for the k-dependent parameters in (2.5)

$$R \in L^1(I; d\nu) \,, \; \gamma \in L^\infty(I; d\nu) \,, \; R(k) \geq 0 \; d\nu - \text{a.e.} \,, \; \int_I R(k) d\nu(k) > 0 \,, \text{ and } \gamma(k) \geq 0 \; d\nu - \text{a.e.} \,,$$

and also

$$0 < \gamma_{\min} \leq \gamma(k) \leq \gamma_{\max} \quad d\nu - \text{a.e.}$$

Note that (2.5) is the nonlinear kinematic hardening rule which characterizes the Chaboche model; it reduces to the Armstrong-Frederick model in case that $d\nu$ is a point-mass, i.e., that only $\dot\sigma^b = \gamma(R\dot\varepsilon^p - \sigma^b|\dot\varepsilon^p|)$ is required in (2.5), for constants $\gamma, R > 0$.

To write the problem as in (1.1), we restrict our attention to the strain-controlled case and again follow [3, 4, 5, 6] to make the ansatz

$$u = \alpha\varepsilon^p + \sigma_d^p = \xi + \sigma_d^p \,, \quad \alpha = \int_I \gamma(k) R(k) \, d\nu(k) + \nu^l C^l + 2\mu > 0 \,. \tag{2.10}$$

Then $u \in \mathbb{T}_d$ by (2.3), and also

$$-\dot\xi(t) \in N_{C_u(t)}(\xi(t)) \,, \quad \text{with} \quad C_u(t) = u(t) - Z = u(t) - \{\tau \in \mathbb{T}_d : |\tau| \leq r\}$$

by (2.3), i.e., $\xi = \mathcal{P}(u)$; see (1.2).

Now taking the deviatoric part of $\varepsilon = \varepsilon^e + \varepsilon^p$ we obtain $\varepsilon_d = \varepsilon_d^e + \varepsilon^p$, again from (2.3). Next, with (2.5) it follows that

$$\sigma_k^b(t) = \exp\Big(-\frac{\gamma(k)}{\alpha}\int_0^t |\dot\xi(s)|\,ds\Big)\Big(\sigma_k^b(0) + \frac{\gamma(k)}{\alpha}R(k)\int_0^t \exp\Big(\frac{\gamma(k)}{\alpha}\int_0^s |\dot\xi(\tau)|\,d\tau\Big)\dot\xi(s)\,ds\Big), \tag{2.11}$$

cf. (5.11) in Lemma 8. Hence also $\sigma_k^b(t) \in \mathbb{T}_d$ for $k \in I$ and $t \in [0,T]$, provided that for the initial values $\sigma_k^b(0) \in \mathbb{T}_d$, since $\dot\xi(t) \in \mathbb{T}_d$. In this case, $\sigma^b = \sigma - \sigma^p = A\varepsilon^e - \sigma^p = 2\mu\varepsilon^e + \lambda\,\mathrm{trace}(\varepsilon^e)\delta - \sigma^p$, see (2.1), (2.2), (2.8), yields, taking again the deviatoric part, $\sigma^b = \sigma_d^b = 2\mu\varepsilon_d^e - \sigma_d^p$, because $\delta_d = 0$. To summarize, we have

$$\varepsilon_d^e = \varepsilon_d - \varepsilon^p, \quad \sigma_k^b(0) \in \mathbb{T}_d \;\Rightarrow\; \sigma^b + \sigma_d^p = 2\mu\varepsilon_d^e. \tag{2.12}$$

Next, observe $\dot\sigma^b = \int_I \dot\sigma_k^b\,d\nu(k) + \nu^l\dot\sigma^l = \int_I \gamma(k)[R(k)\dot\varepsilon^p - \sigma_k^b|\dot\varepsilon^p|]\,d\nu(k) + C^l\nu^l\dot\varepsilon^p$ by (2.4), (2.5), and (2.6). Consequently, by (2.10) and (2.12),

$$\begin{aligned}
\dot u &= \Big(\int_I \gamma(k)R(k)\,d\nu(k) + \nu^l C^l + 2\mu\Big)\dot\varepsilon^p + 2\mu(\dot\varepsilon_d - \dot\varepsilon^p) - \dot\sigma^b \\
&= \Big(\int_I \gamma(k)R(k)\,d\nu(k) + \nu^l C^l\Big)\dot\varepsilon^p + 2\mu\dot\varepsilon_d - \dot\sigma^b \\
&= 2\mu\dot\varepsilon_d + \int_I \gamma(k)\sigma_k^b\,d\nu(k)\,|\dot\varepsilon^p| = 2\mu\dot\varepsilon_d + \frac{1}{\alpha}\int_I \gamma(k)\sigma_k^b\,d\nu(k)\,|\dot\xi|
\end{aligned}$$

under the assumption $\sigma_k^b(0) \in \mathbb{T}_d$ for $k \in I$. Therefore we can choose $\theta(t) = 2\mu\varepsilon_d$ and

$$\mathcal{M}(u,\sigma^b(0))(t) = \frac{1}{\alpha}\int_I \gamma(k)\sigma_k^b(t)\,d\nu(k), \quad \sigma^b(0) = (\sigma_k^b(0))_{k\in I}, \tag{2.13}$$

to cast the problem in the form (1.1), if $\sigma_k^b(0) \in \mathbb{T}_d$ for $k \in I$. Note that parameters p in (1.1) do correspond to $\sigma^b(0)$ here, and $\mathcal{M}$ is independent of θ.

We are going to prove the following result for this strain-controlled Chaboche model.

Theorem 1 *For every Lipschitz continuous T-periodic input function $\theta = 2\mu\varepsilon_d : [0,T] \to \mathbb{T}_d$ there exists a T-periodic solution of (1.1) with $\xi = \mathcal{P}(u)$ also being T-periodic. Here the operator $\mathcal{M}(u,\sigma^b(0))$ is defined through (2.13), and the initial values $\sigma_k^b(0) \in \mathbb{T}_d$ will be choosen appropriately for $k \in I$.*

We start with some preparations for the proof. Given a finite-dimensional Hilbert space $\mathcal{H}$ and fixed $K, L > 0$ let

$$\mathcal{D} = \Big\{u \in C_{\mathrm{per}}([0,\infty[;\mathcal{H}) : |u|_\infty = \sup_{t\in[0,T]} |u(t)| \le K, \quad |u(t) - u(s)| \le L|t-s|,\, t,s \in [0,T]\Big\}, \tag{2.14}$$

where $C_{\mathrm{per}}([0,\infty[;\mathcal{H}) = \{u \in C([0,\infty[;\mathcal{H}) : u(t+T) = u(t),\, t \in [0,\infty[\}$ are the T-periodic $\mathcal{H}$-valued functions. Then $\mathcal{D}$ is convex, and since $\mathcal{H}$ is finite-dimensional, due to the Arzelà-Ascoli theorem, also compact; in the sequel we will choose $\mathcal{H} = \mathbb{T}_d$, see (2.7).

Now we fix a T-periodic $\mathcal{H}$-valued input function $\theta = 2\mu\varepsilon_d \in \mathcal{H}$ being Lipschitz of constant M, and we choose

$$L = \Big(\frac{2\alpha}{\nu^l C^l + 2\mu}\Big)M \quad \text{and} \quad K = LT$$

in the definition of $\mathcal{D}$ in (2.14), with α from (2.10).

The strategy of proof is as follows. Since we intend to apply Schauder's fixed point theorem to $\mathcal{D}$, we need to construct a continuous self-map $\mathcal{T}:\mathcal{D}\to\mathcal{D}$, and the natural idea would be like this: For $u\in\mathcal{D}$ given, let $C_u(t)=u(t)-Z=u(t)-\overline{B}_r(0)$ and solve $-\dot{\xi}(t)\in N_{C_u(t)}(\xi(t))$ for a T-periodic ξ. With this ξ, then search initial values $\sigma_k^b(0)$, $k\in I$, such that $\sigma_k^b(t)$ from (2.11) is T-periodic. Afterwards, define $\mathcal{M}(u,\sigma^b(0))(t)$ by (2.13), and finally $u_1=\mathcal{T}u$ by $\dot{u}_1(t)=\dot{\theta}(t)+\mathcal{M}(u,\sigma^b(0))(t)|\dot{\xi}(t)|$.

Unfortunately, there are several difficulties which lead to additional technicalities. First, the T-periodic solution to $-\dot{\xi}(t)\in N_{C_u(t)}(\xi(t))$ needs not be unique, as may already be seen with $u=0$, since then all $z_0\in Z$ are equilibria, hence periodic solutions. This problem can be overcome by perturbing the sweeping process a bit and considering instead $-\dot{\xi}(t)\in N_{C_u(t)}(\xi(t))+\delta\xi(t)$ with small $\delta>0$. Then it is known from results in [7] that there exists a single T-periodic solution ξ^δ. Thus we have to go first for fixed points of an auxiliary map $\mathcal{T}^\delta:\mathcal{D}\to\mathcal{D}$, and then let $\delta\to 0^+$. The second difficulty arises, as a consequence of introducing the δ-problems, when proving the continuity of $\mathcal{T}^\delta$. To explain this, assume for the moment that $\sigma_k^b(t)$ would be defined through (2.11), with $\dot{\xi}^\delta$ instead of $\dot{\xi}$ at all three places where $\dot{\xi}$ appears. Then if $u_n\to u$ implies $\xi_n^\delta\to\xi^\delta$ (in a suitable sense), we certainly will have to prove $\int_0^t|\dot{\xi}_n^\delta(s)|\,ds\to\int_0^t|\dot{\xi}^\delta(s)|\,ds$ as $n\to\infty$ to obtain concergence of the σ's. Now the problem is that the sweeping process with δ only yields information on $|\dot{\xi}_n^\delta+\delta\xi_n^\delta|$, but not on $|\dot{\xi}_n^\delta|$ itself. It therefore turned out to be more convenient to modify the definition of $\sigma_k^b(t)$ by replacing the terms $|\dot{\xi}|$ through $|\dot{\xi}^\delta+\delta\xi^\delta|$, and only the last $\dot{\xi}(s)$ directly through $\dot{\xi}^\delta(s)$; to avoid confusion, we will introduce $\eta_k^b(t)$ as a new name for the resulting term.

With these two main modifications, the problems described can be overcome, and for δ sufficiently small, a fixed point of $\mathcal{T}^\delta$ can be found; this is the contents of Section 3. Finally, in Section 4 we will let δ tend to zero to conclude the proof of Theorem 1.

3 Approximation of periodic solutions

For $\delta\le\min\{\delta_0,1\}$, with δ_0 determined in Lemma 2 below (depending only on input parameters), consider the map

$$\mathcal{T}^\delta:\quad \mathcal{D}\ni u\mapsto\xi^\delta\mapsto\eta^b(0)\mapsto u_1$$

defined as follows.

Step 1: $u\mapsto\xi^\delta$. Given $u\in\mathcal{D}$, first let $C_u(t)=u(t)-Z$ and choose $\xi=\xi^\delta$ as the unique T-periodic solution to

$$-\dot{\xi}(t)\in N_{C_u(t)}(\xi(t))+\delta\xi(t)\quad\text{a.e.}\tag{3.1}$$

which exists according to Lemma 6 in the appendix.

Step 2: $\xi^\delta\mapsto\eta^b(0)$. With this $\xi=\xi^\delta$ we define $\zeta=\zeta^\delta=\dot{\xi}+\delta\xi\in L^\infty([0,T];\mathcal{H})$ (see below), and for $k\in I$

$$\begin{aligned}\eta_k^b(0)=&\left(1-\exp\left(-\frac{\gamma(k)}{\alpha}\int_0^T|\zeta(s)|\,ds\right)\right)^{-1}\left(\frac{\gamma(k)}{\alpha}R(k)\right)\exp\left(-\frac{\gamma(k)}{\alpha}\int_0^T|\zeta(s)|\,ds\right)\\&\times\int_0^T\exp\left(\frac{\gamma(k)}{\alpha}\int_0^s|\zeta(\tau)|\,d\tau\right)\dot{\xi}(s)\,ds.\end{aligned}\tag{3.2}$$

Since $\dot\xi$ and ζ are T-periodic, by Lemma 8 in the appendix, taking there $a(t)=\frac{\gamma(k)}{\alpha}R(k)\dot\xi(t)$ and $b(t)=-\frac{\gamma(k)}{\alpha}|\zeta(t)|$, the solution to the equation

$$\dot\eta_k^b(t)=\frac{\gamma(k)}{\alpha}\Big(R(k)\dot\xi(t)-|\zeta(t)|\eta_k^b(t)\Big) \tag{3.3}$$

with initial value $\eta_k^b(0)$ from (3.8) is also T-periodic and given by

$$\eta_k^b(t)=\exp\Big(-\frac{\gamma(k)}{\alpha}\int_0^t|\zeta(s)|\,ds\Big)\Big(\eta_k^b(0)+\frac{\gamma(k)}{\alpha}R(k)\int_0^t\exp\Big(\frac{\gamma(k)}{\alpha}\int_0^s|\zeta(\tau)|\,d\tau\Big)\dot\xi(s)\,ds\Big), \tag{3.4}$$

cf. (5.11); we assumed w.l.o.g. that $\int_0^T|\zeta(t)|\,dt>0$, recall that ξ is T-periodic.

<u>Step 3: $\eta^b(0)\mapsto u_1$.</u> With $\eta_k^b(t)$ from the previous step, we define

$$\mathcal{N}(u,\eta^b(0))(t)=\frac{1}{\alpha}\int_I\gamma(k)\eta_k^b(t)\,d\nu(k),\quad \eta^b(0)=(\eta_k^b(0))_{k\in I}. \tag{3.5}$$

Then $\mathcal{N}(u,\eta^b(0))$ is T-periodic, too. Finally we let

$$u_1(t)=\int_0^t[\dot\theta(s)+\mathcal{N}(u,\eta^b(0))(s)|\zeta(s)|]\,ds,\quad t\in[0,T]. \tag{3.6}$$

Next we summarize some estimates for the various maps. To prove these estimates, it will be useful to introduce also the solution $\sigma_k^b(\cdot)$ of (2.5) for $k\in I$, i.e.,

$$\dot\sigma_k^b(t)=\gamma(k)(R(k)\dot\varepsilon^p(t)-|\dot\varepsilon^p(t)|\sigma_k^b(t))=\frac{\gamma(k)}{\alpha}\Big(R(k)\dot\xi(t)-|\dot\xi(t)|\sigma_k^b(t)\Big), \tag{3.7}$$

with initial values

$$\begin{aligned}\sigma_k^b(0)=&\Big(1-\exp\Big(-\frac{\gamma(k)}{\alpha}\int_0^T|\dot\xi(s)|\,ds\Big)\Big)^{-1}\Big(\frac{\gamma(k)}{\alpha}R(k)\Big)\exp\Big(-\frac{\gamma(k)}{\alpha}\int_0^T|\dot\xi(s)|\,ds\Big)\\ &\times\int_0^T\exp\Big(\frac{\gamma(k)}{\alpha}\int_0^s|\dot\xi(\tau)|\,d\tau\Big)\dot\xi(s)\,ds.\end{aligned} \tag{3.8}$$

Then Lemma 8, with $a(t)=\frac{\gamma(k)}{\alpha}R(k)\dot\xi(t)$ and $b(t)=-\frac{\gamma(k)}{\alpha}|\dot\xi(t)|$, shows that σ_k^b is T-periodic and given through (2.11). Hence $\mathcal{M}(u,\sigma^b(0))$ defined by (2.13) is T-periodic, too. In addition,

$$|\sigma_k^b(t)|\le R(k)\quad\text{for}\quad t\in[0,T],\,k\in I, \tag{3.9}$$

as a consequence of Lemma 8, because we have $|a(t)|=-R(k)b(t)$.

Now we can proceed to the estimates. In the sequel, c will denote a positive constant which depends only on the input parameters, but not on δ.

Lemma 1 *(i) (Estimates for Step 1). For $\xi=\xi^\delta$,*

$$|\xi|_\infty\le K+r\quad\textit{and}\quad|\dot\xi|_{L^\infty([0,T])}\le L+2\delta(1+[r+2(1+L)T]e^{2T})=:L_1^\delta.$$

(ii) (Estimates for Step 2). We have

$$|\zeta|_{L^\infty([0,T])}\le L_1^\delta+\delta(K+r)=:L_2^\delta\quad\textit{and}\quad|\eta_k^b(t)|\le R(k)+c\delta\quad\textit{for}\quad t\in[0,T],\,k\in I.$$

(iii) (Estimates for Step 3). With $\kappa = \frac{1}{\alpha}(\nu^l C^l + 2\mu) \in]0,1]$,

$$|\mathcal{N}(u,\sigma^b(0))|_\infty \le (1-\kappa) + c\delta.$$

Proof: The claim of (i) follows from Lemma 6 in the appendix, note $\delta \le 1$ and $K = LT$. The first part of (ii) is clear by (i), since $\zeta = \dot{\xi} + \delta\xi$. Concerning the second part of (ii), by (i) for $t \in [0,T]$,

$$\Big||\zeta(t)| - |\dot{\xi}(t)|\Big| = \Big||\dot{\xi}(t) + \delta\xi(t)| - |\dot{\xi}(t)|\Big| \le (L + 2(1 + [r + 2(1+L)T]e^{2T})\delta = c\delta. \tag{3.10}$$

Using the bounds already obtained, a straightforward calculation hence shows

$$|\eta_k^b(t) - \sigma_k^b(t)| \le c\delta, \quad t \in [0,T],\, k \in I; \tag{3.11}$$

whence $|\eta_k^b(t)| \le R(k) + c\delta$ follows from (3.9). For (iii), by definition of $\mathcal{N}(u,\eta^b(0))$ in (3.5) and by (ii),

$$|\mathcal{N}(u,\eta^b(0))(t)| \le \frac{1}{\alpha}\int_I \gamma(k)[R(k) + c\delta]\, d\nu(k) \le \frac{1}{\alpha}\int_I \gamma(k) R(k)\, d\nu(k) + c\delta = (1-\kappa) + c\delta$$

for $t \in [0,T]$, cf. the definition of α in (2.10), and this gives (iii). □

These estimates first will be used to prove that $\mathcal{T}^\delta$ is well-defined.

Lemma 2 $\mathcal{T}^\delta : \mathcal{D} \to \mathcal{D}$, $u \mapsto u_1$, *is well-defined.*

Proof: We have to verify $\mathcal{T}^\delta(\mathcal{D}) \subset \mathcal{D}$. Let $\Gamma(t) = \dot{\theta}(t) + \mathcal{N}(u,\eta^b(0))(t)|\zeta(t)|$. First, since θ is Lipschitz with constant M, $|\dot{\theta}(t)| \le M$ a.e. Thus by Lemma 1(iii) and (ii), $|\Gamma(t)| \le M + [(1-\kappa) + c_*\delta]L_2^\delta$ for a.e. $t \in [0,T]$ and some constant c_* being defined only through the input parameters. As $L_2^\delta \to L$ as $\delta \to 0^+$, we can find $\delta_0 > 0$ such that $[(1-\kappa) + c_*\delta]L_2^\delta \le (1-\kappa/2)L$ for $\delta \le \delta_0$. Moreover, also δ_0 is determined solely through input parameters. Then a.e. $|\Gamma(t)| \le M + (1-\kappa/2)L = L$, the latter by choice of L, as $\kappa L = 2M$. Thus u_1 is Lipschitz with constant L. Additionally, $|u_1|_\infty \le TL = K$.

To show that u_1 is T-periodic, observe that for $t \ge 0$, by periodicity of θ, (3.5), and (3.3),

$$\begin{aligned}\int_t^{t+T} \Gamma(s)\, ds &= \int_t^{t+T} [\dot{\theta}(s) + \mathcal{N}(u,\eta^b(0))(s)|\zeta(s)|]\, ds \\ &= \frac{1}{\alpha}\int_I d\nu(k) \int_t^{t+T} ds\, \gamma(k)\eta_k^b(s)|\zeta(s)| \\ &= \int_I d\nu(k) \int_t^{t+T} ds \left[\frac{\gamma(k)}{\alpha} R(k)\dot{\xi}(s) - \dot{\eta}_k^b(s)\right] = 0,\end{aligned}$$

because ξ and the η_k^b are T-periodic. Thus for $t \ge 0$

$$u_1(t+T) = \int_0^{t+T} \Gamma(s)\, ds = \int_0^t \Gamma(s)\, ds + \int_t^{t+T} \Gamma(s)\, ds = \int_0^t \Gamma(s)\, ds = u_1(t),$$

i.e., u_1 is T-periodic, and therefore $u_1 = \mathcal{T}^\delta u \in \mathcal{D}$. □

In order to apply Schauder's fixed point theorem, $\mathcal{T}^\delta$ also has to be continuous. This will be verified next.

Lemma 3 *$\mathcal{T}^\delta : \mathcal{D} \to \mathcal{D}$ is continuous.*

Proof: Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{D}$ converge to $u \in \mathcal{D}$ uniformly on $[0,T]$. Then $\xi_n \to \xi$ uniformly on $[0,T]$ for the corresponding solutions of (3.1), by Lemma 6 in the appendix. With $x_n(t) = u_n(t) - \xi_n(t)$ we also have $x_n \to u - \xi = x$ uniformly on $[0,T]$. We need to show $u_{1,n} \to u_1$ uniformly on $[0,T]$, where $u_{1,n} = \mathcal{T}^\delta u_n$ and $u_1 = \mathcal{T}^\delta u$. By (3.6), (3.5), and (3.3), letting $\zeta_n = \dot{\xi}_n + \delta \xi_n$,

$$\begin{aligned} u_{1,n}(t) &= \int_0^t [\dot{\theta}(s) + \mathcal{N}(u_n, \eta_n^b(0))(s)|\zeta_n(s)|]\, ds \\ &= [\theta(t) - \theta(0)] + \frac{1}{\alpha}\int_I d\nu(k)\gamma(k)\int_0^t ds\, \eta_{k,n}^b(s)|\zeta_n(s)| \\ &= [\theta(t) - \theta(0)] + \int_I d\nu(k)\int_0^t ds\Big[\frac{\gamma(k)}{\alpha}R(k)\dot{\xi}_n(s) - \dot{\eta}_{k,n}^b(s)\Big] \\ &= [\theta(t) - \theta(0)] + \frac{1}{\alpha}\Big(\int_I \gamma(k)R(k)d\nu(k)\Big)[\xi_n(t) - \xi_n(0)] - \int_I d\nu(k)[\eta_{k,n}^b(t) - \eta_{k,n}^b(0)] \end{aligned} \tag{3.12}$$

for $t \in [0,T]$ and an analogous representation for $u_1(t)$. By the preceeding remarks it is thus sufficient to prove $\eta_{k,n}^b(t) \to \eta_k^b(t)$ as $n \to \infty$ uniformly in $t \in [0,T]$ and $k \in I$. Assume for the moment that

$$\int_0^t |\zeta_n(s)|\, ds \to \int_0^t |\zeta(s)|\, ds \quad \text{as} \quad n \to \infty \quad \text{for} \quad t \in [0,T]. \tag{3.13}$$

Let $F_n(t) = \int_0^t |\zeta_n(s)|\, ds$, $t \in [0,T]$. By the bounds obtained in Lemma 1, $\sup_{n\in\mathbb{N}} |F_n|_\infty \le c$ as well as $\sup_{n\in\mathbb{N}} |F_n(t) - F_n(s)| \le c|t-s|$. Thus the Arzelà-Ascoli theorem applies, and (3.13) yields

$$\int_0^t |\zeta_n(s)|\, ds \to \int_0^t |\zeta(s)|\, ds \quad \text{as} \quad n \to \infty \quad \text{uniformly in} \quad t \in [0,T]. \tag{3.14}$$

Moreover, by the bounds from Lemma 1,

$$\begin{aligned} \Delta_{k,n}(t) &:= \Big|\int_0^t \exp\Big(\frac{\gamma(k)}{\alpha}\int_0^s |\zeta_n(\tau)|\, d\tau\Big)\dot{\xi}_n(s)\, ds - \int_0^t \exp\Big(\frac{\gamma(k)}{\alpha}\int_0^s |\zeta(\tau)|\, d\tau\Big)\dot{\xi}(s)\, ds\Big| \\ &\le c\int_0^T \Big|\int_0^s |\zeta_n(\tau)|\, d\tau - \int_0^s |\zeta(\tau)|\, d\tau\Big| ds + \Big|\int_0^t \exp\Big(\frac{\gamma(k)}{\alpha}\int_0^s |\zeta(\tau)|\, d\tau\Big)[\dot{\xi}_n(s) - \dot{\xi}(s)]\, ds\Big|. \end{aligned}$$

The first term is independent of $t \in [0,T]$ and $k \in I$ and tends to zero as $n \to \infty$ by (3.14). Concerning the second term, through integration by parts we obtain

$$\begin{aligned} &\Big|\int_0^t \exp\Big(\frac{\gamma(k)}{\alpha}\int_0^s |\zeta(\tau)|\, d\tau\Big)[\dot{\xi}_n(s) - \dot{\xi}(s)]\, ds\Big| \\ &\quad = \Big|\exp\Big(\frac{\gamma(k)}{\alpha}\int_0^t |\zeta(\tau)|\, d\tau\Big)[\xi_n(t) - \xi(t)] - [\xi_n(0) - \xi(0)] \\ &\qquad - \frac{\gamma(k)}{\alpha}\int_0^t \exp\Big(\frac{\gamma(k)}{\alpha}\int_0^s |\zeta(\tau)|\, d\tau\Big)|\zeta(s)|[\xi_n(s) - \xi(s)]\, ds\Big| \\ &\quad \le c|\xi_n - \xi|_\infty. \end{aligned}$$

Therefore $\Delta_{k,n}(t) \to 0$ as $n \to \infty$ uniformly in $t \in [0,T]$ and $k \in I$, provided that (3.13) holds. By the form of $\eta_{k,n}^b(0), \eta_k^b(0), \eta_{k,n}^b(t)$, and $\eta_k^b(t)$, see (3.2) and (3.4), we thus obtain $\eta_{k,n}^b(t) \to$

$\eta_k^b(t)$ uniformly in $t\in[0,T]$ and $k\in I$ from (3.14), and since $\Delta_{k,n}(t)\to 0$ uniformly in $t\in[0,T]$ and $k\in I$.

Hence it remains to verify (3.13). Because ξ_n and ξ are solutions of (3.1) with moving sets $C_{u_n}(t)$ and $C_u(t)$, respectively, it follows from (5.4) in Lemma 6 below that

$$\int_0^t |\zeta_n(s)|\,ds = \int_0^t |\dot{\xi}_n(s)+\delta\xi_n(s)|\,ds = \frac{1}{r}\int_0^t \langle \dot{\xi}_n(s)+\delta\xi_n(s), x_n(s)\rangle\,ds \tag{3.15}$$

and

$$\int_0^t |\zeta(s)|\,ds = \frac{1}{r}\int_0^t \langle \dot{\xi}(s)+\delta\xi(s), x(s)\rangle\,ds. \tag{3.16}$$

Since $\xi_n\to\xi$ uniformly and $(\dot{\xi}_n)_{n\in\mathbb{N}}\subset L^2([0,T];\mathcal{H})$ is bounded by Lemma 1(i), we obtain $\dot{\xi}_n\rightharpoonup\dot{\xi}$ in $L^2([0,T];\mathcal{H})$ as $n\to\infty$. Therefore (3.13) is a consequence of $x_n\to x$ uniformly, (3.15), and (3.16). □

Because $\mathcal{D}\subset C_{\mathrm{per}}([0,T];\mathcal{H})$ is convex and compact, Lemma 2 and Lemma 3 show that Schauder's fixed point theorem can be applied to yield

Lemma 4 *For every $\delta\le\min\{\delta_0,1\}$ the map $\mathcal{T}^\delta:\mathcal{D}\to\mathcal{D}$ has a fixed point $u^\delta\in\mathcal{D}$, i.e., u^δ satisfies*

$$u^\delta(t) = \int_0^t [\dot{\theta}(s)+\mathcal{N}(u^\delta,\eta^{b,\delta}(0))(s)|\zeta^\delta(s)|]\,ds,\quad t\in[0,T], \tag{3.17}$$

where $\zeta^\delta=\dot{\xi}^\delta+\delta\xi^\delta$, ξ^δ is T-periodic and solves (3.1) with $C_{u^\delta}(t)=u^\delta(t)-Z$, and $\eta^{b,\delta}(0)$ [resp. $\eta^{b,\delta}(t)$] are accordingly given by (3.2) [resp. (3.4)].

4 The limit $\delta\to 0^+$

In this section we will conclude the proof of Theorem 1. Since $u^\delta\in\mathcal{D}$ for the approximate solutions from Lemma 4, and as $\mathcal{D}$ is compact, we can assume w.l.o.g. that $u^\delta\to u$ uniformly on $[0,T]$ for some $u\in\mathcal{D}$; to simplify notation, we just write u^δ instead of using subsequences. In addition,

$$|\xi^\delta|_\infty\le K+r=c \quad\text{and}\quad |\dot{\xi}^\delta|_{L^\infty([0,T])}\le L_1^\delta\le c$$

independently of $\delta\le 1$ by Lemma 1(i). Thus also (w.l.o.g.) $\xi^\delta\to\xi$ uniformly on $[0,T]$ for some T-periodic and Lipschitz continuous function $\xi:[0,T]\to\mathcal{H}$, and $\dot{\xi}^\delta\to\dot{\xi}$ weakly in $L^2([0,T];\mathcal{H})$. By Lemma 7 in the appendix, this implies ξ solves (1.2), i.e., $\xi=\mathcal{P}(u)$.

Letting $x^\delta(t)=u^\delta(t)-\xi^\delta(t)$, we have $x^\delta\to u-\xi=x$ uniformly on $[0,T]$, and by (5.4) in Lemma 6 and (5.9) in Lemma 7 below,

$$\begin{aligned}\int_0^t |\zeta^\delta(s)|\,ds &= \int_0^t |\dot{\xi}^\delta(s)+\delta\xi^\delta(s)|\,ds = \frac{1}{r}\int_0^t \langle\dot{\xi}^\delta(s)+\delta\xi^\delta(s), x^\delta(s)\rangle\,ds\\ &\to \frac{1}{r}\int_0^t\langle\dot{\xi}(s),x(s)\rangle\,ds = \int_0^t|\dot{\xi}(s)|\,ds\end{aligned} \tag{4.1}$$

as $\delta\to 0^+$ for all $t\in[0,T]$. By an argument similar to the one with $\Delta_{k,n}(t)$ in the proof to Lemma 3, this yields as $\delta\to 0^+$

$$\Big|\int_0^t \exp\Big(\frac{\gamma(k)}{\alpha}\int_0^s|\zeta^\delta(\tau)|\,d\tau\Big)\dot{\xi}^\delta(s)\,ds - \int_0^t \exp\Big(\frac{\gamma(k)}{\alpha}\int_0^s|\dot{\xi}(\tau)|\,d\tau\Big)\dot{\xi}(s)\,ds\Big|\to 0,\quad t\in[0,T]. \tag{4.2}$$

With the choice $t=T$ in (4.2) we obtain, see (3.2),

$$\begin{aligned}\eta_k^{b,\delta}(0) \to & \Big(1-\exp\Big(-\frac{\gamma(k)}{\alpha}\int_0^T|\dot\xi(s)|\,ds\Big)\Big)^{-1}\Big(\frac{\gamma(k)}{\alpha}R(k)\Big)\exp\Big(-\frac{\gamma(k)}{\alpha}\int_0^T|\dot\xi(s)|\,ds\Big)\\ &\times\int_0^T\exp\Big(\frac{\gamma(k)}{\alpha}\int_0^s|\dot\xi(\tau)|\,d\tau\Big)\dot\xi(s)\,ds\\ &=:\sigma_k^b(0)\in X=\mathbb{T}_d \end{aligned}\tag{4.3}$$

for $k\in I$ as $\delta\to 0^+$. This defines $\sigma^b(0)=(\sigma_k^b(0))_{k\in I}$, as required in Theorem 1, and then $\sigma_k^b(t)$ by means of (2.11). Hence (4.1), (4.3), and (4.2) yield

$$\eta_k^{b,\delta}(t)\to\sigma_k^b(t),\quad t\in[0,T],\quad k\in I,\tag{4.4}$$

as $\delta\to 0^+$; see (3.4) and (2.11). The bounds from Lemma 1 then establish for $k\in I$

$$\eta_k^{b,\delta}x^\delta\to\sigma_k^b x\quad\text{in}\quad L^2([0,T];\mathcal{H})\quad\text{as}\quad\delta\to 0^+.\tag{4.5}$$

Now another application of (5.4) in Lemma 6 and (5.9) in Lemma 7 below implies for $t\in[0,T]$ and $k\in I$

$$\begin{aligned}&\Big|\int_0^t\eta_k^{b,\delta}(s)|\zeta^\delta(s)|\,ds-\int_0^t\sigma_k^b(s)|\dot\xi(s)|\,ds\Big|\\ &=\frac{1}{r}\Big|\int_0^t\eta_k^{b,\delta}(s)\langle\dot\xi^\delta(s)+\delta\xi^\delta(s),x^\delta(s)\rangle\,ds-\int_0^t\sigma_k^b(s)\langle\dot\xi(s),x(s)\rangle\,ds\Big|\\ &\le c\delta+\frac{1}{r}\Big|\int_0^t\langle\dot\xi^\delta(s),\eta_k^{b,\delta}(s)x^\delta(s)\rangle\,ds-\int_0^t\langle\dot\xi(s),\sigma_k^b(s)x(s)\rangle\,ds\Big|\to 0,\quad\delta\to 0^+,\end{aligned}$$

by (4.5), and since $\dot\xi^\delta\rightharpoonup\dot\xi$ in $L^2([0,T];\mathcal{H})$. This yields, see (3.17) and (3.5), by Lebesgue's dominated convergence theorem,

$$\begin{aligned}u^\delta(t)&=\int_0^t[\dot\theta(s)+\mathcal{N}(u^\delta,\eta^{b,\delta}(0))(s)|\zeta^\delta(s)|]\,ds\\ &=\int_0^t\Big[\dot\theta(s)+\Big(\frac{1}{\alpha}\int_I\gamma(k)\eta_k^{b,\delta}(s)\,d\nu(k)\Big)|\zeta^\delta(s)|\Big]\,ds\\ &\to\int_0^t\Big[\dot\theta(s)+\Big(\frac{1}{\alpha}\int_I\gamma(k)\sigma_k^b(s)\,d\nu(k)\Big)|\dot\xi(s)|\Big]\,ds\\ &=\int_0^t[\dot\theta(s)+\mathcal{M}(u,\sigma^b(0))(s)|\dot\xi(s)|]\,ds;\end{aligned}$$

see (2.13). Since also $u^\delta(t)\to u(t)$, this finally finally shows

$$u(t)=\int_0^t[\dot\theta(s)+\mathcal{M}(u,\sigma^b(0))(s)|\dot\xi(s)|]\,ds,$$

and therefore u is a solution to (1.1); whence the proof of Theorem 1 is complete.

5 Appendix

In this appendix we collect some technical results, mainly on sweeping processes (i.e., on the play operator) which have been used before. Recall that $\mathcal{D}$ is defined in (2.14).

Lemma 5 *For $j=1,2$ let $u_j \in \mathcal{D}$, $C_j(t)=u_j(t)-Z=u_j(t)-\overline{B}_r(0)$, $0<\delta_j\le 1$, and ξ_j be a solution of*

$$-\dot{\xi}_j(t) \in N_{C_j(t)}(\xi_j(t))+\delta_j\,\xi_j(t) \quad a.e. \tag{5.1}$$

with initial value $\xi_j(0)=\xi_{j,0}$. Then

$$|\xi_1(t)-\xi_2(t)|^2 \le |\xi_{1,0}-\xi_{2,0}|^2+c\left(|\delta_1-\delta_2|+\int_0^t |u_1(s)-u_2(s)|\,ds\right), \quad t\in[0,T],$$

for some constant c depending only on K, L, T, and r.

Proof: A straightforward calculation shows, using the definition of the normal cone and the fact $\xi_1(t)\in C_1(t)\subset C_2(t)+\overline{B}_{\mathrm{d_H}(C_1(t),C_2(t))}(0)$ and the analogue for $\xi_2(t)$, that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\xi_1(t)-\xi_2(t)|^2 &\le -\langle\dot{\xi}_1(t)+\delta_1\xi_1(t), r_1(t)\rangle-\langle\dot{\xi}_2(t)+\delta_2\xi_2(t), r_2(t)\rangle\\ &\quad+\langle(\delta_2-\delta_1)\xi_2(t), \xi_1(t)-\xi_2(t)\rangle\end{aligned}$$

for measurable functions r_1, r_2 with $|r_1(t)|, |r_2(t)|\le \mathrm{d_H}(C_1(t),C_2(t))$ a.e. in $[0,T]$. Thus the claim follows from the bounds derived in Lemma 6 below, because of $\mathrm{d_H}(C_1(t),C_2(t))\le |u_1(t)-u_2(t)|$. □

Lemma 6 *For $u\in\mathcal{D}$ let $C_u(t)=u(t)-Z=u(t)-\overline{B}_r(0)$. Then for every $0<\delta\le 1$ there exists a unique T-periodic solution of*

$$-\dot{\xi}(t)\in N_{C_u(t)}(\xi(t))+\delta\,\xi(t) \quad a.e. \tag{5.2}$$

In addition, $|\xi|_\infty\le K+r$, and ξ is Lipschitz continuous with

$$|\dot{\xi}|_{L^\infty([0,T])}\le L+2\delta(1+[K+r+LT+2\delta T]e^{2\delta T}). \tag{5.3}$$

Moreover, with $x(t)=u(t)-\xi(t)\in Z=\overline{B}_r(0)$,

$$|\dot{\xi}(t)+\delta\xi(t)|=\frac{1}{r}\langle\dot{\xi}(t)+\delta\xi(t), x(t)\rangle \quad a.e.\ in \quad [0,T]. \tag{5.4}$$

If $(u_n)_{n\in\mathbb{N}}\subset\mathcal{D}$ and $u_n\to u$ in $L^1([0,T];\mathcal{H})$, then $\xi_n\to\xi$ uniformly on $[0,T]$ for the corresponding solutions of (5.2).

Proof: The existence of ξ follows from [7, Thm. 4.1], because $t\mapsto C_u(t)$ is L-Lipschitz w.r. to $\mathrm{d_H}$ and $C_u(0)=C_u(T)$. Also $|\xi|_\infty\le K+r$ since $\xi(t)\in C_u(t)$ for $t\in[0,T]$. Additionally, [7, Thm. 4.4, (4.4.2)] implies (5.3), as we have here $M=\sup\{|C_u(t)|: t\in[0,T]\}\le K+r$.

To show the last claim, i.e., that $\xi_n\to\xi$ uniformly, we have to go into some more details with the proof of [7, Thm. 4.1]. According to [7, Thm. 4.1, (4.1.7)],

$$|\xi_n(T;p)-\xi_n(T;q)|\le e^{-\delta T}|p-q|, \quad p,q\in C_{u_n}(0),$$

where $\xi_n(\cdot;p)$ is the unique solution to (5.2) with u_n instead of u and initial value $\xi_n(0;p)=p$. Hence the Poincaré-map $p\mapsto\xi_n(T;p)\in C_{u_n}(T)=C_{u_n}(0)$ has a unique fixed point $p_n\in C_{u_n}(0)$, and $\xi_n(t)=\xi_n(t;p_n)$ is the corresponding unique periodic solution. Thus

$$\begin{aligned}|\xi_n(0)-\xi(0)| &= |\xi_n(T;p_n)-\xi(T;p)|\le|\xi_n(T;p_n)-\xi_n(T;p)|+|\xi_n(T;p)-\xi(T;p)|\\ &\le e^{-\delta T}|\xi_n(0)-\xi(0)|+|\xi_n(T;p)-\xi(T;p)|;\end{aligned}$$

whence Lemma 5 gives

$$|\xi_n(0)-\xi(0)| \le (1-e^{-\delta T})^{-1}\left(c\int_0^T |u_n(s)-u(s)|\,ds\right)^{1/2},$$

yielding $\xi_n(0)\to\xi(0)$. By Lemma 5, this implies $\xi_n\to\xi$ uniformly on $[0,T]$.

Finally, concerning (5.4), let $\phi(t)=e^{\delta t}\xi(t)=e^{\delta t}[u(t)-x(t)]$ and $D(t)=e^{\delta t}C_u(t)=e^{\delta t}u(t)-\overline{B}_{re^{\delta t}}(0)$. Then $\dot\phi(t)=e^{\delta t}[\dot\xi(t)+\delta\xi(t)]$, and thus

$$-\dot\phi(t)\in N_{D(t)}(\phi(t)) \tag{5.5}$$

by (5.2) and definition of the normal cone. Moreover, $\phi(t)\in\partial D(t)$ iff $|x(t)|=r$, and then

$$N_{D(t)}(\phi(t))=\{\lambda[-e^{\delta t}u(t)+\phi(t)]:\lambda\ge 0\}=\{\lambda x(t):\lambda\le 0\}$$

because Z is a ball. Hence for a.e. t with $|x(t)|=r$, $\dot\phi(t)=\lambda x(t)$ for some $\lambda\ge 0$ by (5.5). Taking norms on both sides and solving for λ we find $\dot\phi(t)=\frac{1}{r}|\dot\phi(t)|x(t)$. Scalar multiplication with $x(t)$ yields, since $|x(t)|=r$,

$$\frac{1}{r}\langle\dot\phi(t),x(t)\rangle=|\dot\phi(t)|. \tag{5.6}$$

Because $\dot\phi(t)=0$ a.e. in $\{t\in[0,T]:|x(t)|<r\}$, (5.6) in fact holds a.e. in $[0,T]$, and thus inserting $\dot\phi(t)$ gives (5.4). □

Lemma 7 *Let $\delta_j\to 0^+$, and for $j\in\mathbb{N}$ let $u_j\in\mathcal{D}$ and $\xi=\xi_j$ be a solution to*

$$-\dot\xi(t)\in N_{C_{u_j}(t)}(\xi(t))+\delta_j\xi(t)\quad \text{a.e. in}\quad [0,T], \tag{5.7}$$

with $C_{u_j}(t)=u_j(t)-Z=u_j(t)-\overline{B}_r(0)$. Additionally, assume that $|\dot\xi_j|_{L^\infty([0,T])}\le c$ for $j\in\mathbb{N}$, $u_j\to u$ uniformly on $[0,T]$, $\xi_j\to\xi$ uniformly on $[0,T]$, as well as $\dot\xi_j\to\dot\xi$ weakly in $L^2([0,T];\mathcal{H})$. Then ξ is a solution to

$$-\dot\xi(t)\in N_{C_u(t)}(\xi(t))\quad \text{a.e. in}\quad [0,T], \tag{5.8}$$

with $C_u(t)=u(t)-Z$. Moreover, letting $x(t)=u(t)-\xi(t)$,

$$|\dot\xi(t)|=\frac{1}{r}\langle\dot\xi(t),x(t)\rangle\quad \text{a.e. in}\quad [0,T]. \tag{5.9}$$

Proof: Since $\xi_j(t)\in C_{u_j}(t)=u_j(t)-Z$ for $t\in[0,T]$, we obtain $\xi(t)\in C_u(t)$ for $t\in[0,T]$. To verify (5.8), note that for $y\in C_u(t)$ we find $y_j\in C_{u_j}(t)$ and r_j with $y=y_j+r_j$ and $|r_j|\le|u_j-u|_\infty$. Hence (5.7) yields for $y\in C_u(t)$ and $t\in[0,T]$, by the bounds on $|\dot\xi_j|_{L^\infty([0,T])}$ and since $|u_j|_\infty\le K$ according to the definition of $\mathcal{D}$,

$$\begin{aligned}
\langle -\dot\xi_j(t),y\rangle &= \langle -\dot\xi_j(t),y_j+r_j\rangle \le \langle -\dot\xi_j(t),y_j\rangle + c|u_j-u|_\infty\\
&= \langle -\dot\xi_j(t)-\delta_j\xi_j(t),y_j-\xi_j(t)\rangle - \langle\dot\xi_j(t),\xi_j(t)\rangle + \delta_j\langle\xi_j(t),y_j-\xi_j(t)\rangle + c|u_j-u|_\infty\\
&\le -\langle\dot\xi_j(t),\xi_j(t)\rangle + 2\delta_j(K+r)^2 + c|u_j-u|_\infty .
\end{aligned}$$

Introducing the support functional of $C_u(t)$, i.e., $\delta^*(x,C_u(t))=\sup_{y\in C_u(t)}\langle x,y\rangle$, we thus find

$$\int_0^T \Big[\delta^*(-\dot\xi_j(t),C_u(t))+\langle\dot\xi_j(t),\xi_j(t)\rangle\Big]\,dt \le c(\delta_j+|u_j-u|_\infty). \tag{5.10}$$

By assumption, $\int_0^T\langle\dot\xi_j(t),\xi_j(t)\rangle\,dt\to\int_0^T\langle\dot\xi(t),\xi(t)\rangle\,dt$ as $j\to\infty$. Standard results from convex analysis on the lower semicontinuity of integral functionals, cf. e.g. the arguments used in [9, Lemma 3.5], imply

$$\int_0^T \delta^*(-\dot\xi(t),C_u(t))\,dt \le \liminf_{j\to\infty}\int_0^T \delta^*(-\dot\xi_j(t),C_u(t))\,dt.$$

Therefore by (5.10),

$$\int_0^T \Big[\delta^*(-\dot\xi(t),C_u(t))+\langle\dot\xi(t),\xi(t)\rangle\Big]\,dt \le 0.$$

Since $\xi(t)\in C_u(t)$, the integrand is ≥ 0 a.e. on $[0,T]$, and hence $\delta^*(-\dot\xi(t),C_u(t))+\langle\dot\xi(t),\xi(t)\rangle=0$ a.e. on $[0,T]$. By definition of the normal cone, this implies (5.8). Finally, (5.9) follows analogously to (5.4) in Lemma 6; see also [3, 4, 5, 6]. □

Lemma 8 *Let $a\in L^\infty([0,T];\mathcal{H})$ and $b\in L^\infty([0,T])$ be T-periodic with $\int_0^T b(t)\,dt<0$. Then the solution to $\dot y=a(t)+b(t)y$, $y(0)=y_0$, is T-periodic iff*

$$y_0=\left(1-\exp\left(\int_0^T b(s)\,ds\right)\right)^{-1}\int_0^T \exp\left(\int_s^T b(\tau)\,d\tau\right)a(s)\,ds.$$

If additionally $|a(t)|=-Rb(t)$, $t\in[0,T]$, for some $R\ge 0$, then $|y(t)|\le R$, $t\in[0,T]$.

Proof: The unique solution with initial value $y(0)=y_0$ is

$$y(t)=\exp\left(\int_0^t b(s)\,ds\right)\left(y_0+\int_0^t \exp\Big(-\int_0^s b(\tau)\,d\tau\Big)a(s)\,ds\right), \tag{5.11}$$

and due to uniqueness, y is T-periodic iff $y(0)=y(T)$.

For the second part, we have

$$\frac{1}{2}\frac{d}{dt}|y(t)|^2=\langle a(t)+b(t)y(t),y(t)\rangle\le b(t)|y(t)|(|y(t)|-R)\quad\text{a.e.} \tag{5.12}$$

Since $b(t)\le 0$ a.e., $|y(t)|$ is decreasing as long as $|y(t)|>R$, and hence $|y(t)|\le\max\{R,|y(t_1)|\}$ for all t_1 and $t\ge t_1$. Suppose $|y(t_0)|>R$ for some t_0. Then it is impossible to have $|y(t_1)|<|y(t_0)|$ for some $t_1>t_0$, since then $|y(t)|\le\max\{R,|y(t_1)|\}<|y(t_0)|$ for all $t\ge t_1$ contradicting the fact that $|y(t)|$ as a periodic function will assume the value $|y(t_0)|$ again at a later time. Hence $|y(t_0)|>R$ for some t_0 implies $|y(t)|=|y(t_0)|>R$ for all $t\ge t_0$. Thus (5.12) gives $b(t)|y(t)|=0$ for $t\ge t_0$, and because of $|y(t)|>R\ge 0$, $b(t)=0$ for $t\ge t_0$. This is a contradiction to $\int_0^T b(t)\,dt<0$, as b is T-periodic. □

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