Nonlinear Solitary Waves with Gaussian Tails

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Abstract

We study analytically and numerically the stationary localized solutions of the nonlinear Schrödinger equation (NLSE) with an additional parabolic potential. Such a model occurs in a wide range of physical applications, including plasma physics and nonlinear optics. Bound states with Gaussian tails (these tails appear due to the parabolic linear potential) play an important role in the dynamics of the systems modeled by this equation. We prove the existence of the bound states and describe their properties.

1 Introduction

In this paper we study structure and properties of solitary waves with Gaussian tails in the NLS equation with an additional parabolic potential of trapping type. This nonlinear model occurs in different physical contexts and is of interest for practical applications. In plasma physics, this equation describes a pulse propagation in a preformed plasma channel (see e.g. [2, 1] and the references therein). In this paper we discuss some optical applications and put the main emphasis to mathematical aspects of the problem. It has been shown recently in [12] (see also [19]) that such a model describes a chirped quasi-soliton propagation along the transmission line with a special dispersion profile. One important application of the model considered is a soliton generation in actively and passively mode-locked erbium fibre lasers (see, e.g., [6, 11, 10, 4] and the references therein). The interplay between dispersion, Kerr effect and phase modulation leads to a rich variety of features of generated pulses. At low peak pulse power, an actively mode-locked fibre laser system generates the Gaussian pulse, see [11, 4]. While increasing the pulse power, the generating pulse approaches a sech (hyperbolic secant), because Kerr nonlinearity and dispersion are going to dominate. As a further practical application of the NLSE with additional parabolic potential, we want to mention here optical pulse propagation in transmission systems with in-line phase modulators. The use of in-line electro-optic phase modulators is known as a rather simple and effective way to control soliton transmissions, cf. [17, 18, 20]. An advantage of this control scheme is that no net gain has to be introduced into the transmission system to maintain the soliton power. Therefore, and in contrast to the amplitude modulation or the bandwidth filtering, this approach allows to avoid the introduction of an additional excess gain balancing average energy flow into the system. Moreover, drawbacks like the growth of the CW component and noise are eliminated. Solitons are stably guided at their temporal positions by the effective potential created due to phase modulation. Insertion of the modulators with properly designed filters results in a substantial suppression of the Gordon-Haus jitter.

It is desirable for an increase of the transmission capacity of a communication system to place carrier pulses as close to each other as possible. However, pulses cannot be placed too close because of mutual interaction. Interaction between two neighboring solitons is one of the major factors limiting transmission capacities of modern optical soliton-based communication systems. This interaction occurs due to an overlap of the exponential tails of the closely spaced fundamental solitons, leading to the reduction of the transmission rate. To provide a stable transmission, a separation between two neighboring fundamental solitons should be not less than at least five soliton widths. This is a fundamental limitation for a transmission based on the soliton with sech shape described by the NLS equation. One possible way to increase the transmission capacity is to use as an information carrier a solitary wave with tails decaying faster than the exponential tails of the NLSE soliton. This would result in a substantial suppression of the soliton interaction, and consequently, in the possibility of a more dense information packing. It has been shown in [20, 18] that soliton interaction can be effectively suppressed by a periodic phase modulation. This effect can be interpreted in the following way. The periodic phase modulation imposes a perturbation such that a carrier pulse is no longer a NLSE soliton in this case. Tails of the carrier pulse are Gaussian due to the effective parabolic potential created by the modulation. As a matter of fact, the change of the properties of a carrier pulse by the transmission control elements (for instance, by the sliding filters) is rather well-known. Mathematically, the effect of filtering can be considered as formation of the autosoliton (solitary wave with parameters determined by the system parameters rather than by input pulse characteristics) with a chirp from the input pulse in form of the NLSE soliton. This change in the features of the carrier pulse can be used for further improvement of the transmission capacities of the soliton-based communication systems. Solitons with Gaussian tails have many attractive features and can occur in a wide range of physical and optical applications. Note that phase filtering in the general formulation of the problem leads to a sinusoidal linear potential instead of a parabolic one which approximates the general case for short pulses.

In this paper we study systematically structure and properties of a soliton with Gaussian tails in the general model described by the NLS equation with additional parabolic potential. An outline of the paper is as follows. In Section 2, the model equations are formulated and general integral properties are established. Section 3 is devoted to the study of the stationary solutions. The latter are found numerically. The existence of the stationary solutions is shown in Section 4, and their stability is treated in Section 5. Finally, our results are summarized in Section 6, and we added an appendix Section 7, where some technical results are proved.

2 Basic equations

We start by discussing some general properties of the basic model

$$iQ_t + \frac{1}{2}Q_{xx} + |Q|^2 Q - ax^2 Q = 0. \tag{1}$$

Here $Q = Q(x,t)$ is a function of $x \in \mathbb{R}$ and $t \geq 0$, and $a > 0$ is a parameter. This equation (1) includes the most important effects, namely dispersion (or averaged dispersion), nonlinearity, and an effective parabolic potential which can result from different physical mechanisms, cf. the introduction. Applications which lead to problems of type (1) are dealt with more thoroughly in Section 6 below.

Equation (1) can be written in the Hamiltonian form

$$\frac{\partial Q}{\partial t} = -\frac{i}{2}\frac{\delta H}{\delta Q}, \tag{2}$$

with Hamiltonian

$$H(Q) = \frac{1}{2}\int |\, Q_x\,|^2\, dx - \frac{1}{2}\int |\, Q\,|^4\, dx + a\int x^2\, |\, Q\,|^2\, dx\,.$$

The integral

$$E(Q) = \int |Q|^2 dx \tag{3}$$

is an additional conserved quantity, typically having the meaning of energy.

One can find some integral condition on the characteristics of an input signal which provides that a maximum of the peak power will be bounded from below by a constant determined by the input signal parameters. First, using the inequality

$$\sqrt{a}E = -\sqrt{a}\int x(QQ_x^* + Q^* Q_x)dx \leq \int |\, Q_x\,|^2\, dx + a\int x^2\, |\, Q\,|^2\, dx\,,$$

we can estimate the Hamiltonian by

$$2H = \int |\, Q_x\,|^2\, dx - \int |\, Q\,|^4\, dx + 2a\int x^2\, |\, Q\,|^2\, dx \geq \sqrt{2a}E - \max|\, Q\,|^2\, E\,.$$

Because H and E are conserved quantities, we hence can estimate the maximum of a pulse peak power at every $t > 0$ from below as

$$\max|\, Q\,|^2 \geq \sqrt{2a} - \frac{2H}{E}\,. \tag{4}$$

Thus, if an input pulse satisfies the condition $2H \leq \sqrt{2a}E$, then with pulse evolution, the maximal peak power cannot decrease below some constant value which can be estimated by the right-hand side of (4). This indicates that the energy cannot be dispersed among linear modes in a way that a peak power decreases below some constant value. The numerical computations have shown that the inequality $2H \leq \sqrt{2a}E$ holds for the stationary solutions discussed in Sections 3 and 4. Hence the lower bound in relation (4) is strictly positive for the stationary solutions and their basin of attraction.

Additional information about the pulse dynamics in (1) can be obtained by considering the evolution of the average square of the pulse width. Define $R = \int x^2 |Q|^2 dx / \int |Q|^2 dx$ to be this average square of a pulse width. Then

$$\frac{d^2 R}{dt^2} = 2\frac{H}{E} + 2\frac{I_1}{E} - 6aR \geq 2\frac{H}{E} + \frac{1}{2R} - 6aR = -\frac{\partial W}{\partial R}.$$

The latter representation allows to use the analogy with the motion of a particle in the effective potential $W(R) = 3aR^2 - 2HR/E - \frac{1}{2}\ln(R)$. One observation that can be made from this analogy is that an average pulse width cannot reach neither zero nor infinity. Of course this does not exclude compression of the central peak on the broadening background.

3 Stationary solutions: Numerical results

In this section, we present numerical results about stationary solutions of (1), whereas in the next section we will give a strict analytical proof of the existence of the nonlinear bound states for (1).

We consider steady-state soliton solutions of (1) of the form

$$Q(x,t) = F(x)\exp(ikt). \tag{5}$$

The stationary solution $F = F(x)$ has to satisfy the nonlinear eigenvalue problem

$$kF = \frac{1}{2}F_{xx} + |F|^2 F - ax^2 F \tag{6}$$

with boundary conditions $F(x) \to 0$ as $|x| \to \infty$. Note that (6) for a soliton shape has been derived in [8] by a variational approach in the context of optical pulse transmission for the system with strong dispersion management.

In contrast to the NLS equation, the nonlinear eigenvalue k can be negative. This can be seen from the fact that the linear limit of (6) recovers the linear eigenvalue problem for the quantum oscillator with negative (in our notation) equidistant eigenvalues $k_n = -k_*(n+1/2)$, $n \in \mathbb{N}_0$, where $k_* = -\sqrt{a/2}$. The corresponding limits of the eigenfunctions are

$$F_n(a, k \to k_n, x) \cong e^{-\sqrt{a/2}x^2} H_n(x),$$

where $n \in \mathbb{N}_0$ enumerates the solutions, and $H_n(x)$ is the Hermite orthogonal polynomial of order n; see (12) below.

One can expect that in case $a > 0$ the shape of the ground state solution is an intermediate state between a sech-type profile of the NLSE soliton as $k \to \infty$ and the Gaussian pulse as $k \to k_*$. Indeed, if the nonlinear term in (6) dominates (i.e., $|F|^2 >> a$), then the solitons are almost fundamental NLS solitons, because in the limit $a \to 0$ the solution is a soliton $F(x) = \sqrt{2k}/\cosh(\sqrt{2k}x)$ of the NLSE. In the opposite limit case $a \to \infty$, one has $k \to k_*$, and the solution is close to a Gaussian pulse $F(x) = \exp(-\sqrt{a/2}x^2)$. The latter also describes an asymptotic decrease of the function $F(x)$ for large x, because for the localized solutions the asymptotic behavior of the solution of (6) is dominated by its linear part. The next section is devoted to the rigorous proof of the existence of families of solutions to the nonlinear eigenvalue problem (6).

It is clear that the tails of the localized pulse with $a \neq 0$ decay much faster than the exponential wings of the fundamental soliton, when $a=0$. As a consequence, solitons with Gaussian tails can be spaced much closer to each other, still suppressing interaction between neighboring pulses. Obviously, this allows a more dense packing of the information in optical applications.

Equation (6) can be rewritten in a variational form $\delta(H+kE)=0$. In particular, this means that any stationary solution of (1) realizes a critical point of H for a fixed E. This variational form can be used for the approximation of stationary solutions of (6) by trial functions.

In the context of this paper we only consider the practically relevant ground state solutions which correspond to the lowest eigenvalue in the linear limit. The boundary value problem (6) was solved numerically. In contrast to the NLSE limit, parameters k and a cannot be removed simultaneously from the equation by a simple scaling. Therefore, to describe the properties of such pulses, one has to study a whole family of the solutions depending on the two parameters k and a; in particular we obtain countably many families of bound states.

The soliton shapes for $a=2$ and different values of k are shown in Fig. 1. It is seen that that the soliton width is decreasing as a is decreasing, and the soliton shape changes from the almost Gaussian for $k=-0.99$ to the almost sech-like for $k=10$. Note that solutions for other positive values of the parameter a can be found by a simple scaling transform as it is shown below. To illustrate an asymptotic behavior of the soliton tails, the soliton shapes are plotted in Fig. 2 in logarithmic scale. A transition from Gaussian to exponential decay of the soliton tails can be observed there. Note that for large values of the parameter $k >> a$ there is an intermediate spatial scale $x_e = \sqrt{k/a}$, which separates a region with exponential decay for $x << x_e$ and the far region for $x >> x_e$, where Gaussian decay takes place.

Another illustration of the transition from Gaussian shape to the sech-like shape is shown in Fig. 3. A soliton energy E is plotted against the nonlinear eigenvalue k. It is seen that energy is approaching to that corresponding to fundamental NLSE soliton as $k \to +\infty$ for any value of the parameter a. Indeed, the relative influence of the parabolic potential in Eq. (6) is reduced with the increasing of eigenvalue k since soliton width gets smaller and its amplitude becomes higher. Similarly, the Hamiltonian is plotted in Fig. 4 against k for different values of the parameter a. Note that in the vicinity of k_* the Hamiltonian is positive in contrast to the pure NLS when $a=0$.

4 Stationary Solutions: Analytical considerations

In this section we turn to the analytical treatment of (6). We explain the functional analytic setting and the results obtained, whereas their detailed proofs have been postponed to the appendix to keep the paper readable, but nevertheless to make it self-contained.

To transform (6) to a standard bifurcation problem, we scale $F(x)=u(\gamma x)$ with $\gamma = (2a)^{1/4}$ to obtain

$$-u_{\xi\xi} + \xi^2 u - \sqrt{\frac{2}{a}}|u|^2 u = -\sqrt{\frac{2}{a}} k u, \tag{7}$$

where $\xi = \gamma x$. Writing again x for ξ, we find that (7) is a special case of equations in the general form

$$-u_{xx}+x^2u+f(x,u)=\lambda u \quad \text{for} \quad x\in \mathbb{R}. \tag{8}$$

For this equation we suppose that

$$\begin{aligned} &f:\mathbb{R}^2\to\mathbb{R} \text{ is a Carathéodory function}, \quad \text{and} \\ &|f(x,u)|\le c(1+|x|)|u|^\alpha, \quad x,u\in\mathbb{R}, \end{aligned} \tag{9}$$

for some constants $c\ge 0$ and $\alpha>1$. Here the assumption on f means that $f(x,\cdot)$ is continuous for $x\in\mathbb{R}$ and $f(\cdot,u)$ is measurable for $u\in\mathbb{R}$. Note also that, contrary to (6) resp. (7), we consider real-valued nonlinearities, but this is only for simplicity.

We will also investigate (8) under the stronger condition

$$\begin{aligned} &f(x,u)=ug(x,u), \quad g:\mathbb{R}^2\to\mathbb{R} \text{ is locally Lipschitz continuous}, \quad \text{and} \\ &|f(x,u)-f(x,\bar u)|\le c(1+|x|)^\beta|u-\bar u|[|u|^{\alpha-1}+|\bar u|^{\alpha-1}], \quad x,u,\bar u\in\mathbb{R}, \end{aligned} \tag{10}$$

for constants $c\ge 0$, $\beta\in[0,1]$ and $\alpha\ge 1+2\beta/3$, and $\alpha>1$, if $\beta=0$. In particular the nonlinearities $f(u)=c|u|^{\alpha-1}u$ with $c\in\mathbb{R}$ and $\alpha>1$ are admissible in both cases, hence our results will apply to (7) with $\lambda=-\sqrt{\frac{2}{a}}k$.

4.1 Preliminaries on the linearized problem

We start by analyzing the linear problem corresponding to (8), i.e.,

$$-u_{xx}+(1+x^2)u=\mu u, \tag{11}$$

where we shifted for later notational simplicity λ to $\mu=1+\lambda$. The natural choice of state space in which (8) resp. (11) will be considered is the finite-energy space

$$X=\{u\in H^1: \int x^2u^2(x)\,dx<\infty\}$$

which is a Hilbert space with inner product

$$\langle u,v\rangle_X=\int(1+x^2)\,u(x)v(x)\,dx+\int u'(x)v'(x)\,dx$$

and energy norm

$$|u|_X^2=\int(1+x^2)u^2(x)\,dx+\int u'(x)^2\,dx.$$

Here, as usual, $H^1=H^1(\mathbb{R})=W^{1,2}(\mathbb{R})$. Note that a key property of X is that it is compactly embedded in $L^2=L^2(\mathbb{R})$; see Lemma A.1 below.

Local bifurcation can only be expected at the eigenvalues of (11). Here we have

Lemma 1 *Let $\mu_n=2n+2$, $n\in\mathbb{N}_0$, and $u_n(x)=e^{-x^2/2}H_n(x)$, where H_n is the n^{th} Hermite polynomial*

$$H_n(x)=(-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}. \tag{12}$$

Then the μ_n are simple eigenvalues of (11) with corresponding eigenfunctions $u_n\in X$. Every u_n has exactly n zeroes in $\mathbb{R}$, and these zeroes are simple.

Proof: Cf. [3, Ch. 5, Paragraph 15.1]. □

4.2 Global branches of nonlinear bound states

Since $\lambda_n = 2n+1$, $n \in \mathbb{N}_0$, are the eigenvalues of the linearized equation corresponding to (8), we will prove that there are branches of solutions of the nonlinear equation which are bifurcating from these eigenvalues. Moreover, these branches are unbounded in $\mathbb{R} \times X$.

To see this, it will be convenient to transform (8) to a standard bifurcation problem in L^2 by means of a Green's function $g : \mathbb{R}^2 \to \mathbb{R}^2$ in order to apply the global bifurcation theorem of Rabinowitz [14, Thm. 1.3]. The appropriate "inverse" integral operator A is

$$(Av)(x) = \int g(x,\xi)v(\xi)\,d\xi \quad \text{for} \quad v \in L^2. \tag{13}$$

It may be shown that $A : L^2 \to X$ is well-defined, linear, and continuous, and that $u = Av$ satisfies $u_{xx} = (1+x^2)u - v$ in the distributional sense; see Lemma A.3 below for more information. In particular, by Lemma 1, $A : L^2 \to L^2$ has exactly the characteristic values μ_n, $n \in \mathbb{N}_0$, and they are simple. Moreover, $A : X \subset L^2 \to X$ is compact, and $\langle Av, w\rangle_X = (v,w)_{L^2}$ for $v \in L^2$ and $w \in X$; note that $w(x) \to 0$ as $|x| \to \infty$, cf. e.g. [15, Remark 5.4, p. 73].

As a compact, selfadjoint, and non-negative operator, $A : X \to X$ has a square-root $A^{1/2}$ which continuously extends to $S : L^2 \to X$ with $S = A^{1/2}$ on X and $|Sv|_X = |v|_{L^2}$ for $v \in L^2$.

Then (8) transforms to

$$v = \mu Av - Sf(\cdot, Sv) \tag{14}$$

in L^2, with the substitution $u = A^{1/2}v$. Let $Nv = Sf(\cdot, Sv)$ be the nonlinear part. Under assumption (9) we obtain that $N : L^2 \to X$ is continuous, and $\dfrac{|Nv|_X}{|v|_{L^2}} \to 0$ as $v \to 0$ in L^2. In particular, $N : L^2 \to L^2$ is compact with $\dfrac{|Nv|_{L^2}}{|v|_{L^2}} \to 0$ as $v \to 0$ in L^2; see Lemma A.5 below. Hence all assumptions of the global bifurcation theorem of Rabinowitz are satisfied. The resulting global branches in the $(\mathbb{R} \times L^2)$-closure of the set of non-trivial solutions to (14) can be transformed back to global branches of solutions to (8) by means of the isometric homeomorphism

$$\Phi : \mathbb{R} \times L^2 \to \mathbb{R} \times X, \quad \Phi(\mu, v) = (\mu - 1, Sv); \tag{15}$$

see Lemma A.7 for more details. Letting $\lambda_n = 2n+1 = \mu_n - 1$ for $n \in \mathbb{N}_0$, we hence obtain the following result on the existence of bound states for (8), and thus in particular for (7) and (6), respectively. Here $\mathcal{S}$ denotes the $(\mathbb{R} \times X)$-closure of the set of non-trivial weak solutions to (8).

Theorem 1 *Let (9) be satisfied for f. For every $n \in \mathbb{N}_0$ let $\mathcal{C}_n$ denote the component of $\mathcal{S}$ with $(\lambda_n, 0) \in \mathcal{C}_n$. Then the following alternative holds. Either*

(i) $\mathcal{C}_n$ is unbounded in $\mathbb{R} \times X$, or

(ii) $\mathcal{C}_n$ is compact, and there exists an $m \neq n$ such that $(\lambda_m, 0) \in \mathcal{C}_n$.

Our next aim is to exclude possibility (ii) in the above theorem, under more restrictive assumptions on f. This will be obtained, as usual, by investigating nodal properties of u for $(\lambda, u) \in \mathcal{C}_n$. Since the underlying interval $\mathbb{R}$ is unbounded, a key point is to ensure that zeroes of functions u with (λ, u) lying in a bounded subset of $\mathbb{R} \times X$ cannot escape to infinity. This can be guaranteed through estimates based on the presence of the confining potential $V(x) = x^2$. From now on we assume that (10) holds for f.

Lemma 2 *For all $R>0$ there is an $d_R>0$ such that $(\lambda,u)\in\mathcal{S}$, $u\neq 0$, and $|\lambda|+|u|_X\leq R$ implies: $u(x)=0\Rightarrow |x|\leq d_R$.*

Proof: Cf. appendix. □

Up to some technicalities (elaborated in the appendix), this is already enough to obtain the main result of this section.

Theorem 2 *Let (10) hold for f. Then for every $n\in\mathbb{N}_0$ the component $\mathcal{C}_n$ of $\mathcal{S}$ with $(\lambda_n,0)\in\mathcal{C}_n$ is unbounded in $\mathbb{R}\times X$.*

Proof: Cf. appendix. □

Again, Theorem 2 particularly applies to (7) and hence to (6). We formulate the result as a separate theorem.

Theorem 3 *For (6) we find countably many branches of bound states emanating off $F=0$, namely at parameter values (a,k) belonging to*

$$P_n=\{(a,k)\in[0,\infty[\times]-\infty,0]:-\sqrt{\frac{2}{a}}k=\lambda_n=2n+1\} \tag{16}$$

for some $n\in\mathbb{N}_0$. Each of the branches is unbounded in $(\lambda=-\sqrt{\frac{2}{a}}k)$-direction or in the norm of X.

Remark 1 Instead of (8), more general problems

$$-u_{xx}+V(x)u+f(x,u)=\lambda u$$

could be treated in a similar way, under appropriate conditions on the potential V. The essential conditions are $V(x)\geq 0$ and $V(x)\to\infty$ as $|x|\to\infty$. In order to keep the presentation simple, we refrained from this generalization.

Remark 2 The case that $V(x)$ is periodic will be pursued in a continuation to this paper, the situation being different inasmuch bands of continuous spectrum appear for the linearized equation.

4.3 On the local bifurcation behaviour

In this section we will derive an asymptotic formula for the bound states near the bifurcation points. The formula will then be useful in Section 5 when discussing the stability of the bound states, and it turns out to be in perfect agreement with the results obtained numerically.

The local solution branch of (8) emerging from $(\lambda_n,0)$ is given by

$$(\lambda(s),u(s))=(\lambda_n+\mu(s),s[u_n+Sv(s)]),\quad s\in(-\gamma,\gamma),$$

for some $\gamma>0$, with $(\mu_n+\mu(s),s[v_n+v(s)])$ being the local solution branch of (14), and $Sv_n=u_n$, the n^{th} eigenfunction. By local bifurcation theory, $\mu(0)=0$ and $v(0)=0$, hence $u(s)\cong su_n$, and we have to calculate $\mu'(0)$ and $\mu''(0)$. For a fixed pair $(a,k)\in P_n$, see (16), the following formulas can be derived.

Lemma 3 *We have* $\mu'(0)=0$ *and*

$$\mu''(0) = -4\sqrt{\frac{2}{a}}\,(n+1)\,\frac{|u_n|_{L^4}^4}{|u_n|_X^2} =: -\eta_n < 0\,.$$

Proof: Cf. appendix. □

Consequently, $\lambda(s) \cong \lambda_n - \frac{\eta_n}{2}s^2$. Hence in particular $\lambda(s) \le \lambda_n = -\sqrt{\frac{2}{a}}k$ around $s=0$, i.e., the bifurcation for (8) is subcritical, and also $|s| \cong \sqrt{2(\lambda_n - \lambda(s))/\eta_n}$. Thus, eliminating $s \ge 0$,

$$u_\lambda \cong s u_n \cong \sqrt{2(\lambda_n - \lambda)/\eta_n}\, u_n$$

locally for $\lambda \le \lambda_n$; whence we obtain for (6)

$$F_\lambda(x) = u_\lambda(\gamma x) \cong \sqrt{2(\lambda_n - \lambda)/\eta_n}\, u_n(\gamma x)\,, \quad \gamma = (2a)^{1/4}\,, \tag{17}$$

cf. the rescaling at the beginning of Section 4. Therefore we find for the energies $E_\lambda = \int_{\mathbb{R}} F_\lambda^2\, dx$ that

$$\frac{d}{d\lambda^-}E_\lambda\Big|_{\lambda=\lambda_n} = -\left(\frac{a}{8}\right)^{1/4} \frac{|u_n|_{L^2}^2\,|u_n|_X^2}{2(n+1)\,|u_n|_{L^4}^4}\,.$$

As an example, we consider $n=0$ and $a=2$, hence forcing $k=-1$ to obtain $(a,k) \in P_0$, since $\lambda_0 = 1$. Then $u_0(x) = e^{-x^2/2}H_0(x) = e^{-x^2/2}$, $|u_0|_{L^2}^2 = \sqrt{\pi}$, $|u_0|_{L^4}^4 = \sqrt{\pi/2}$, and also $|u_0|_X^2 = \int_{\mathbb{R}}(1+2x^2)e^{-x^2}\,dx = 2\sqrt{\pi}$. Consequently,

$$\frac{d}{d\lambda^-}E_\lambda\Big|_{\lambda=\lambda_0} = -\sqrt{\pi} \cong -1.77\,. \tag{18}$$

Hence, the local dependence of the energy E versus the nonlinear eigenvalue k in the vicinity of $k_* = -1$ is described by

$$E(k) = \sqrt{\pi}(k+1) + O((k+1)^2)\,; \tag{19}$$

note again that k and λ have opposite signs. A straight line corresponding to the leading term in (19) is plotted in the inset in Fig. 3. It is seen that the numerically obtained dependence $E(k)$ for $a=2$ shows at $k=k_*=-1$ exactly the same slope as is predicted by (19).

5 Stability of soliton solutions

In this section we will study the stability of a solitary pulse $Q_0(x,t) = F(x)\exp(ikt)$, cf. (5), with a ground state F. Writing $Q(x,t) = (F(x)+G(x,t))\exp(ikt)$ and taking into account (6) for F, linearization of (1) in the soliton solution gives

$$iG_t + \frac{1}{2}G_{xx} - kG + F^2(2G+G^*) - ax^2G = 0\,.$$

Note that F is real and non-negative. Decomposing $G = f+ig$ into real and imaginary part, we arrive at $f_t = L_- g$ and $g_t = -L_+ f$, with

$$L_- = k - \frac{1}{2}\frac{\partial^2}{\partial x^2} - F^2 + ax^2 \quad \text{and} \quad L_+ = L_- - 2F^2 = k - \frac{1}{2}\frac{\partial^2}{\partial x^2} - 3F^2 + ax^2\,, \tag{20}$$

and hence $-f_{tt}=L_-L_+f$. Thus the corresponding spectral problem is

$$L_-L_+f=\omega^2 f\,. \tag{21}$$

Since (1) is a Hamiltonian system (2), and since the energy E from (3) is also a conserved quantity, a criterion for the stability of the pulse $Q_0(x,t)$ in terms of L_- and L_+ can be obtained directly from the general results in [9, Thm. 3]. As (1) is invariant under the group action $T(s)Q=e^{is}Q$, "stability" here means "orbital stability", i.e., stability of the group orbit; see the definition on p. 167 of [9].

We treat (6) with a fixed $a>0$ and let F_k denote the ground state of (6) for $k\geq k_0:=-\sqrt{\frac{a}{2}}\lambda_0=-\sqrt{\frac{a}{2}}$, with k close to k_0, see Section 4.3. To investigate the stability of these ground states, we also need to introduce the real-valued

$$d(k)=H(F_k)+kE(F_k)\,.$$

Exactly as in Example 6C [9, p. 188 ff.], stability will be a consequence of [9, Thm. 3] (see also [16, Thm. 3.1]), provided the following points can be verified:

(P1) zero is not an eigenvalue of L_+;

(P2) L_+ has exactly one negative (simple) eigenvalue;

(P3) the kernel of L_- is spanned by F_k;

(P4) $d''(k)>0$;

(P5) $\frac{d}{dk}E(F_k)>0$.

Here (P3) is readily obtained, since it is enough to observe that $L_-F_k=0$ by (6), and that F_k is positive, hence it is the first (simple) eigenfunction of the non-negative operator L_-. Moreover, (P1), (P4), and (P5) can be verified analytically only for k close to k_0, see also the discussion in [16, p. 212]. E.g. (P1) follows from a perturbation analysis of the spectrum, and (P5) is a consequence of the calculations in Section 4.3, cf. (19) for the special case $a=2$. Then to verify (P4) we can write, analogously to [9, (2.21)], $d''(k)=(E'(F_k),\frac{d}{dk}F_k)=2\int F_k(\frac{d}{dk}F_k)\,dx$, and this has the sign of $-\int F_\lambda(\frac{d}{d\lambda}F_\lambda)\,dx=(1/\eta_n)\int|u_n(\gamma x)|^2\,dx>0$, using (17).

Therefore it remains to prove $(P2)$. In the case without potential ($a=0$) one obtains $L_+\frac{dF_k}{dx}=0$, i.e., $\frac{dF_k}{dx}$ is an eigenfunction of L_+ with zero eigenvalue. Now $\frac{dF_k}{dx}$ vanishes at exactly one point, thus zero has to be the second eigenvalue. For $a>0$, however, $L_+\frac{dF_k}{dx}=-2axF_k$, and we have to stick to a different argument, similarly to Appendix A and B from [16], relying on a variational characterization of the ground states with k close to k_0. Since $(L_-u,u)=(L_+u,u)+2\int F_k^2u^2\,dx$, it is impossible to have $(L_+u,u)\geq 0$ for all $u\in X$, as $L_-F_k=0$. Hence L_+ must have at least one negative eigenvalue. To finally show that L_+ has at most one negative eigenvalue, consider the functional

$$J^k[u]=\frac{\int\left(\frac{1}{2}u_x^2+ax^2u^2+ku^2\right)dx}{|u|_{L^4}^2},\quad 0\neq u\in X,\quad\text{and}\quad I^k=\inf\{J^k[u]:0\neq u\in X\}\,.$$

Along the lines of [16, Thm. 2.2] we will show

Lemma 4 *Let $k > k_0$. Then $I^k > 0$, and the infimum is realized by some function u_k which, after a multiplicative scaling to a function $\tilde{F}_k$, solves (6). Moreover, $\tilde{F}_k \to 0$ in X as $k \to k_0$.*

Proof: Cf. appendix. □

Consequently, $F_k = \tilde{F}_k$ for k close to k_0 implies $\delta^2 J^k[F_k](u,u) \geq 0$ for all $u \in X$. But through a straightforward calculation it can be shown that $L_+ u = \delta^2 J^k[F_k](u,u) + r_1(u)$, with a rank-one-operator r_1. Therefore L_+ can have at most one negative eigenvalue.

Hence $(P1)$-$(P5)$ are satisfied close to k_0, and can summarize our results on stability as

Theorem 4 *For $a > 0$ fixed, $k \geq k_0 = -\sqrt{\frac{a}{2}}$, and k close to k_0, the ground states F_k of (6) are orbitally stable.*

Another, more heuristically, evidence of the soliton stability can be obtained from the examination of the relation between the conserved quantities, based on the predictions of catastrophe theory (see e.g. the review [13]). Since the soliton solution corresponds to a critical point of the Hamiltonian H for a fixed energy E, the existence of both stable and unstable solutions of nonlinear evolution equations is reflected in the occurrence of singularities of the Hamiltonian as a function of the energy. In turn, the absence of singularities means either stability or instability of the whole branch of solutions. Fig. 5 shows a plot of $H(E)$ for numerically obtained soliton solutions. There is, however, no indication of any singularity seen. From this one can make a conclusion about stability of these solutions, since in the limit of high energies ($E \to +\infty$ and $H \to -\infty$) the soliton solutions correspond to the fundamental NLS solitons which are known to be stable.

6 Applications and Conclusions

In this section we present an example of the occurrence of the basic equation (1) in a practical problem. Consider an average optical pulse propagation down the cascaded transmission system with in-line phase modulators, described by the equation

$$i\frac{\partial E}{\partial Z} - \frac{\beta_2}{2}\frac{\partial^2 E}{\partial T^2} + \eta\sigma |E|^2 E = \mp [\nu_0 - \nu_2 T^2] E\,. \tag{22}$$

Here Z is the propagation distance measured in [km], T is the retarded time in [ps], $|E|^2 = P$ is the optical power in [W], β_2 is the first order group velocity dispersion measured in [ps^2/km], and $\sigma = (2\pi n_2)/(\lambda_0 \cdot A_{\text{eff}})$ is a coefficient of the nonlinearity, with n_2 being the nonlinear refractive index, $\lambda_0 = 1.55\mu$m the carrier wavelength, and A_{eff} the effective fiber area. Moreover, in (22), the enhancement factor (guiding-center factor) η is obtained through $\eta = (G-1)/(G \ln G)$ from the amplifier gain G. We assume that amplifiers and modulators are spaced with the same period Z_a. The terms on the right-hand side of (22) account for the average effect of the in-line phase modulation. The transfer function of an electro-optical modulator is $E_{\text{out}} = E_{\text{in}} \exp(i\Phi \cos\omega_m t)$, where Φ is the peak phase excursion, ω_m the angular drive frequency of the modulator, and, in (22), $\nu_0 = \Phi/Z_a$ as well as $\nu_2 = \Phi\omega_m^2/(2Z_a)$. The $\mp$ signs correspond to the up-chirped resp. down-chirped positions of the modulator cycle, cf. [17]. Equation (22) has been obtained by averaging over one amplification (modulation) period. After trivial scaling, $Z = Z_a t$, $T = \sqrt{|\beta_2| Z_a} x$

and $E(Z,T)=(\eta\sigma Z_a)^{-1/2}\exp(\mp i\nu_0 Z_a)Q(x,t)$, the basic dimensionless model reads exactly as (1) for negative β_2, which corresponds to an anomalous fiber dispersion. The parameter $a=\pm\nu_2|\beta_2|Z_a^2$ can have both a negative or a positive sign, though we focused on the positive sign, because in that case the localized solitary waves considered in this paper are supported.

In conclusion, we investigate analytically and numerically the stationary localized solutions of the nonlinear Schrödinger equation (NLSE) with an additional parabolic potential. Such a model occurs in a wide range of physical applications, including plasma physics and nonlinear optics. Bound states with Gaussian tails (these tails appear due to the parabolic linear potential) play an important role in the dynamics of the systems modeled by this equation. Gaussian tails reduce substantially the soliton interaction, thus allowing for a more dense information packing. Such pulses due to suppressed interaction can be very useful for ultra-high-bite rate optical data transmissions. We prove the existence and stability of the bound states and describe their properties.

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7 Appendix

In this appendix we add some technical lemmas and the proofs of some results from Sections 4 and 5, which have been omitted there.

Lemma A.1 *The embedding $X\subset L^2$ is compact, and $X\subset L^2$ is dense. Moreover,*

$$|u(x)|\le\sqrt{2}|u|_{H^1}\le\sqrt{2}|u|_X \text{ in } \mathbb{R} \quad for \quad u\in X. \tag{A.1}$$

Proof: Fix $u\in X$. For $h\in]0,1]$ and $R>1$ we have

$$\begin{aligned}|u(\cdot+h)-u|_{L^2}^2 &= \int_{-R}^{R}|u(x+h)-u(x)|^2\,dx+\int_{|x|>R}|u(x+h)-u(x)|^2\,dx\\ &\le \int_{-R}^{R}\left|\int_x^{x+h}u'(t)\,dt\right|^2 dx+4\int_{|x|>R-1}u^2(x)\,dx\\ &\le \int_{-R}^{R}h\left[\int_x^{x+h}u'(t)^2\,dt\right]dx+\frac{4}{(R-1)^2}\int x^2u^2(x)\,dx\le[2Rh+\frac{4}{(R-1)^2}]\,|u|_X^2\end{aligned}$$

as well as

$$|u|^2_{L^2(]-\infty,-R]\cup[R,\infty[)}=\int_{|x|>R}u^2(x)\,dx\le\frac{1}{R^2}|u|_X^2.$$

This shows that for every $M>0$ we have $\sup_{|u|_X\le M}|u(\cdot+h)-u|_{L^2}\to 0$ as $|h|\to 0$ as well as $\sup_{|u|_X\le M}|u|_{L^2(]-\infty,-R]\cup[R,\infty[)}\to 0$ as $R\to\infty$. Hence by the Fréchet-Kolmogorov theorem, cf. [21, Sect. X.1], the compactness of the embedding $X\subset L^2$ follows. The density of $X\subset L^2$ is immediate, and concerning (A.1) we can use [15, Prop. 5.3, p. 73] with $\varepsilon=1$, $p=p'=2$, $b\to\infty$, and $a\to-\infty$ to find $|u(x)|\le|u|_{L^2}+|u'|_{L^2}$, and hence the claim. □

Recall that we let

$$(Av)(x)=\int g(x,\xi)v(\xi)\,d\xi \quad \text{for} \quad v\in L^2,$$

see (13), where the explicit form of the Green's function is

$$g(x,\xi)=\begin{cases}\dfrac{1}{\sqrt{\pi}}e^{\frac{1}{2}(x^2+\xi^2)}\Big(\displaystyle\int_{-\infty}^{\xi}e^{-y^2}\,dy\Big)\Big(\displaystyle\int_{x}^{\infty}e^{-y^2}\,dy\Big), & \xi\le x,\\ \dfrac{1}{\sqrt{\pi}}e^{\frac{1}{2}(x^2+\xi^2)}\Big(\displaystyle\int_{-\infty}^{x}e^{-y^2}\,dy\Big)\Big(\displaystyle\int_{\xi}^{\infty}e^{-y^2}\,dy\Big), & x\le\xi,\end{cases}\qquad\text{(A.2)}$$

cf. [3, Ch. 5, Paragraph 15.1].

We first need an auxiliary lemma.

Lemma A.2 *There exists a constant $c>0$ such that for $x\in[0,\infty[$ and $k=0,1,2$*

$$e^{x^2}\int_x^\infty\frac{e^{-y^2}}{(1+y)^k}\,dy\le\frac{c}{(1+x)^{k+1}}\quad\textit{and}\quad e^{-x^2}\int_{-x}^{x}e^{y^2}\,dy\le\frac{c}{1+x}.$$

Proof: The first estimate follows from $(1+y)^k\ge(1+x)^k$ and from substituting $s=x(y-x)$ in $\int_x^\infty e^{x^2-y^2}\,dy$, whereas for the second one we note that via $s=x^2-y^2$

$$\begin{aligned}\int_0^x e^{y^2-x^2}\,dy&=\frac{1}{2}\Big[\int_0^{x^2/2}e^{-s}\frac{ds}{\sqrt{x^2-s}}+\int_{x^2/2}^{x^2}e^{-s}\frac{ds}{\sqrt{x^2-s}}\Big]\\&\le\frac{1}{2}\Big[\frac{\sqrt{2}}{x}\int_0^{x^2/2}e^{-s}\,ds+e^{-x^2/2}\int_{x^2/2}^{x^2}\frac{ds}{\sqrt{x^2-s}}\Big]\le c\Big[\frac{1}{x}+xe^{-x^2/2}\Big],\end{aligned}$$

and this gives the claim. □

Lemma A.3 *For $v\in L^2$ let $u=Av$. Then u is well-defined, $u\in C^1(\mathbb{R})$, and there is a constant $c>0$ independent of v such that*

$$|u(x)|\le c\frac{|v|_{L^2}}{(1+|x|)^{3/2}}\quad\textit{and}\quad|u'(x)|\le c\frac{|v|_{L^2}}{(1+|x|)^{1/2}}\quad\textit{for}\quad x\in\mathbb{R},\qquad\text{(A.3)}$$

In particular $u(x)\to0$ and $u'(x)\to0$ as $|x|\to\infty$. Additionally, $u''=(1+x^2)u-v$ holds in the distributional sense, and $A:L^2\to X$ is well-defined, linear, and continuous. If $v\in L^2\cap C(\mathbb{R})$, then $u\in C^2(\mathbb{R})$ with $u''=(1+x^2)u-v$ in $\mathbb{R}$.

Proof: By Lemma A.2 and Hölder's inequality for $x\in[0,\infty[$

$$\begin{aligned}|u(x)|&\le\Big(\int_{-\infty}^{-x}+\int_{-x}^{x}+\int_x^\infty\Big)g(x,\xi)|v(\xi)|\,d\xi\\&\le ce^{\frac{1}{2}x^2}\Big(\int_x^\infty e^{-y^2}\,dy\Big)\Big(\int_{-\infty}^{-x}+\int_{-x}^{x}\Big)e^{\frac{1}{2}\xi^2}\Big(\int_{-\infty}^{\xi}e^{-y^2}\,dy\Big)|v(\xi)|\,d\xi\\&\quad+ce^{\frac{1}{2}x^2}\int_x^\infty e^{\frac{1}{2}\xi^2}\Big(\int_\xi^\infty e^{-y^2}\,dy\Big)|v(\xi)|\,d\xi\\&\le c\frac{e^{-\frac{1}{2}x^2}}{1+x}\Big[\int_{-\infty}^{-x}e^{\frac{1}{2}\xi^2}\Big(\int_{-\infty}^{\xi}e^{-y^2}\,dy\Big)|v(\xi)|\,d\xi+\int_{-x}^{x}e^{\frac{1}{2}\xi^2}|v(\xi)|\,d\xi\Big]\\&\quad+ce^{\frac{1}{2}x^2}\int_x^\infty\frac{e^{-\frac{1}{2}\xi^2}}{1+\xi}|v(\xi)|\,d\xi\end{aligned}$$

$$
\begin{aligned}
&\le c\frac{e^{-\frac12 x^2}}{1+x}\left[\int_x^\infty \frac{e^{-\frac12\xi^2}}{1+\xi}|v(-\xi)|\,d\xi+|v|_{L^2}\left(\int_{-x}^x e^{\xi^2}\,d\xi\right)^{1/2}\right]+ce^{\frac12 x^2}\int_x^\infty\frac{e^{-\frac12\xi^2}}{1+\xi}|v(\xi)|\,d\xi\\
&\le c\frac{e^{-\frac12 x^2}}{1+x}\left[1+\left(\int_{-x}^x e^{\xi^2}\,d\xi\right)^{1/2}\right]|v|_{L^2}+ce^{\frac12 x^2}\left(\int_x^\infty\frac{e^{-\xi^2}}{(1+\xi)^2}\,d\xi\right)^{1/2}|v|_{L^2}\\
&\le c\frac{|v|_{L^2}}{(1+x)^{3/2}},
\end{aligned}
$$

and since it may be argued analogously for $x\in]-\infty,0]$, we have shown the first part of (A.3). Hence in particular $u\in L^2$. As in the case of a finite interval it is seen that $u\in C^1(\mathbb{R})$ with

$$
(Av)'(x)=u'(x)=\int\partial_x g(x,\xi)v(\xi)\,d\xi\,. \tag{A.4}
$$

Because of

$$
\partial_x g(x,\xi)=\begin{cases}\dfrac{1}{\sqrt{\pi}}e^{\frac12(x^2+\xi^2)}\left(\displaystyle\int_{-\infty}^{\xi}e^{-y^2}\,dy\right)\left(x\displaystyle\int_x^\infty e^{-y^2}\,dy-e^{-x^2}\right), & \xi<x,\\ \dfrac{1}{\sqrt{\pi}}e^{\frac12(x^2+\xi^2)}\left(\displaystyle\int_{\xi}^{\infty}e^{-y^2}\,dy\right)\left(x\displaystyle\int_{-\infty}^x e^{-y^2}\,dy+e^{-x^2}\right), & x<\xi,\end{cases},
$$

we clearly obtain $|u'(x)|\le c(1+|x|)|u(x)|$, and hence

$$
|u'(x)|\le c\frac{|v|_{L^2}}{(1+|x|)^{1/2}}\,.
$$

Since $\lim_{\xi\to x^-}\partial_x g(x,\xi)-\lim_{\xi\to x^+}\partial_x g(x,\xi)=-1$ and $\partial_x^2 g(x,\xi)=(1+x^2)g(x,\xi)$ for $x\neq\xi$, it follows by direct calculation that $u''=(1+x^2)u-v$ in the distributional sense. Moreover, as $u\in C^1(\mathbb{R})$, this relation can be multiplied by u and integrated by parts to yield for every $R>0$

$$
u'(-R)u(R)-u'(R)u(R)+\int_{-R}^R(1+x^2)u^2(x)\,dx+\int_{-R}^R u'(x)^2\,dx=\int_{-R}^R v(x)u(x)\,dx\le|v|_{L^2}|u|_{L^2}\,. \tag{A.5}
$$

Thus by (A.3) as $R\to\infty$ we get $|u|_X^2\le|v|_{L^2}|u|_{L^2}$, and this proves $u\in X$ and $|u|_X\le|v|_{L^2}$, i.e., $A:L^2\to X$ is continuous. Since the last claim is straightforward, we are done. □

To transfer (8) from X to a nonlinear eigenvalue problem in L^2, we needed the square-root $A^{1/2}:X\to X$ of A and its extension $S:L^2\to X$. Additionally we obtain

$$
|A^{1/2}v|_X^2=\langle Av,v\rangle_X=|v|_{L^2}^2\,,\quad\text{and}\quad A^{1/2}(Av)=A(A^{1/2}v)\,,\quad v\in X\,, \tag{A.6}
$$

and hence by approximation

$$
S^2v=A^{1/2}(Sv)=Av\quad\text{and}\quad S(Av)=A^{1/2}(Av)=A(Sv)\,,\quad v\in L^2\,. \tag{A.7}
$$

In Section 4 we had to see that the nonlinearity $Nv=Sf(\cdot,Sv)$ is compact and of higher order in L^2. We start with a lemma about the Nemytskii operator generated by f.

Lemma A.4 *Let (9) hold for f, and define $F:X\to L^2$, $(Fu)(x)=f(x,u(x))$. Then $|Fu|_{L^2}\le c|u|_X^\alpha$ for $u\in X$, and F is continuous. If (10) is satisfied by f, then in addition*

$$
\forall\,R>0\ \exists\,c_R>0:\quad |u|_X\le R,\ |\bar u|_X\le R\quad\Rightarrow\quad |Fu-F\bar u|_{L^2}\le c_R|u-\bar u|_X\,. \tag{A.8}
$$

Proof: For $u, \bar{u} \in X$ it follows from (9) and (A.1) that

$$
\begin{aligned}
|Fu - F\bar{u}|_{L^2}^2 &= \int |f(x,u(x)) - f(x,\bar{u}(x))|^2 \, dx && \text{(A.9)} \\
&\leq 2 \int [|f(x,u(x))|^2 + |f(x,\bar{u}(x))|^2] \, dx \\
&\leq c \int (1+|x|)^2 [|u(x)|^{2\alpha} + |\bar{u}(x)|^{2\alpha}] \, dx \\
&\leq c |u|_X^{2\alpha-2} \int (1+|x|)^2 u^2(x) \, dx + c |\bar{u}|_X^{2\alpha-2} \int (1+|x|)^2 \bar{u}^2(x) \, dx \\
&\leq c [|u|_X^{2\alpha} + |\bar{u}|_X^{2\alpha}]. && \text{(A.10)}
\end{aligned}
$$

Because of $f(x,0)=0$ for $x \in \mathbb{R}$ we have $F(0)=0$. Thus taking $\bar{u}=0$ we obtain in particular that $Fu \in L^2$ and $|Fu|_{L^2} \leq c|u|_X^{\alpha}$. Moreover, (A.10) also implies that $F: X \to L^2$ is continuous, since $u_n \to u$ in X yields w.l.o.g. $u_n(x) \to u(x)$ a.e. as $n \to \infty$. By assumption on f thus $f(x,u_n(x)) \to f(x,u(x))$ a.e. Hence Lebesgue's (generalized) dominated convergence theorem applies to (A.9) with $\bar{u} = u_n$.

Finally, if (10) holds, then analogously to (A.10), since $(1+|x|)^{2\beta} \leq (1+|x|)^2$,

$$
|Fu - F\bar{u}|_{L^2}^2 \leq c[|u|_X^{2(\alpha-1)} + |\bar{u}|_X^{2(\alpha-1)}] \int (1+|x|)^2 |u(x) - \bar{u}(x)|^2 \, dx,
$$

so that we obtain (A.8). $\square$

Next we show that the nonlinearity is of higher order in L^2.

Lemma A.5 *Let (9) be satisfied by f, and define $N: L^2 \to X$, $Nv = (S \circ F \circ S)(v)$ for $v \in L^2$. Then N is continuous, and $\dfrac{|Nv|_X}{|v|_{L^2}} \to 0$ as $v \to 0$ in L^2. In particular, $N: L^2 \to L^2$ is compact with $\dfrac{|Nv|_{L^2}}{|v|_{L^2}} \to 0$ as $v \to 0$ in L^2.*

Proof: By Lemma A.4, $N: L^2 \to X$ is well-defined and continuous. Moreover, for $v \in L^2$

$$
|Nv|_X = |S(FSv)|_X = |F(Sv)|_{L^2} \leq c|Sv|_X^{\alpha} = c|v|_{L^2}^{\alpha},
$$

and since $\alpha > 1$ this gives the claimed estimates. $\square$

To give some more details concerning the global branches of solutions to (8) and (14), let

$$
\Sigma_0 = \{(\mu, v) \in \mathbb{R} \times L^2 : v \neq 0, v = \mu A v - N v\}
$$

denote the set of nontrivial solutions of (14), and $\Sigma = \overline{\Sigma_0}^{\mathbb{R} \times L^2}$ its closure. We can use the global bifurcation theorem of Rabinowitz, cf. [14, Thm. 1.3], and also [22, Thm. 15.C] and [5, Cor. 29.1], in L^2 to obtain

Lemma A.6 *Assume (9) for f. For every $n \in \mathbb{N}_0$ let Γ_n denote the component of Σ with $(\mu_n, 0) = (2n+2, 0) \in \Gamma_n$. Then the following alternative holds. Either*

(i) Γ_n is unbounded in $\mathbb{R} \times L^2$, or

(ii) Γ_n *is compact, and there exists an* $m \neq n$ *such that* $(\mu_m, 0) \in \Gamma_n$.

This can be transferred back to (8) as follows. Let

$$\mathcal{S}_0 = \{(\lambda, u) \in \mathbb{R} \times X : u \neq 0 \text{ is a weak solution of } -u_{xx} + x^2 u + f(x,u) = \lambda u\}$$

denote the corresponding solution set, and let $\mathcal{S} = \overline{\mathcal{S}_0}^{\mathbb{R}\times X}$. Define $\Phi : \mathbb{R} \times L^2 \to \mathbb{R} \times X$, $\Phi(\mu, v) = (\mu - 1, Sv)$, cf. (15). Then

$$|\Phi(\mu,v) - \Phi(\bar\mu, \bar v)|_{\mathbb{R}\times X} = |(\mu - \bar\mu, v - \bar v)|_{\mathbb{R}\times L^2} \quad \text{for} \quad (\mu,v), (\bar\mu,\bar v) \in \mathbb{R}\times L^2.$$

Lemma A.7 $\Phi(\Sigma) = \mathcal{S}$, *i.e.,* $\Phi : \Sigma \to \mathcal{S}$ *is an isometric homeomorphism.*

Proof: If $(\mu, v) \in \Sigma_0$, then $v \neq 0$ and $v = \mu A v - N v$, hence $(\lambda, u) = \Phi(\mu, v)$ has $u \neq 0$. Moreover, $A(L^2) \subset X$ and $N(L^2) \subset X$ imply $v \in X$, and thus by (A.6), and (A.7),

$$u = Sv = A^{1/2} v = \mu A^{1/2}(Av) - A^{1/2}(S[Fu]) = \mu A(A^{1/2} v) - A(Fu) = A(\mu u - Fu).$$

Since $\mu u - Fu \in L^2$ it follows from Lemma A.3 that $u'' = (1+x^2)u - [\mu u - Fu]$ in the distributional sense, i.e., u is a weak solution of $-u'' + x^2 u + f(x,u) = \lambda u$. Consequently we have shown $\Phi(\Sigma_0) \subset \mathcal{S}_0$, and therefore $\Phi(\Sigma) \subset \mathcal{S}$ by continuity of Φ.

Conversely, if $(\lambda, u) \in \mathcal{S}_0$, then $\mu u - Fu \in L^2$ by Lemma A.4, with $\mu = \lambda + 1$. Hence by Lemma A.3 and (A.7)

$$u = A(\mu u - Fu) = Sv, \quad \text{where} \quad v = S(\mu u - Fu) \in X. \tag{A.11}$$

Moreover, by (A.7), $v = \mu S u - SFu = \mu S^2 v - SFSv = \mu A v - N v$, and thus $\mathcal{S}_0 \subset \Phi(\Sigma_0)$. Since Φ is isometric, also $\mathcal{S} \subset \Phi(\Sigma)$. □

We also need to carry out the

Proof of Lemma 2: Let $(\lambda, u) \in \mathcal{S}$, $u \neq 0$, and $u(x_0) = 0$ for some x_0, w.l.o.g. $x_0 \in [0, \infty[$. Since $u(x) \to 0$ as $x \to \infty$, there must be an $x_1 > x_0$ with $u'(x_1) = 0$. Then $u(x_1) \neq 0$ by Lemma A.9, and we may assume that $u(x_1) > 0$, since in case that $u(x_1) < 0$ the same argument applies with $\tilde u = -u$, $\tilde f(x,u) = -f(x,-u)$, and $\tilde g(x,u) = g(x,-u)$. Note that $\tilde f$ may be estimated in the same way as is f, cf. (10). <u>Case 1.</u> If $u''(x_1) > 0$, then $u''(x_2) = 0$ for a first $x_2 > x_1$, because $u'(x) \to 0$ as $x \to \infty$ by (A.3). Hence $u''(x) > 0$ in $[x_1, x_2[$ particularly yields $u(x_2) \geq u(x_1) > 0$. It follows from Lemma A.8 that

$$x_0^2 u(x_2) \leq x_2^2 u(x_2) = \lambda u(x_2) + u''(x_2) - f(x_2, u(x_2)) \leq [|\lambda| + |g(x_2, u(x_2))|]\, u(x_2).$$

Taking $\bar u = 0$ in (10) yields $|g(x,u)| \leq c(1+x)^\beta |u|^{\alpha-1}$ for $(x,u) \in [0,\infty[\times\mathbb{R}$, and thus by (A.3) for $|\lambda| + |u|_X \leq R$, since $u = A[(\lambda+1)u - Fu]$,

$$x_0^2 \leq R + c(1+x_2)^\beta |u(x_2)|^{\alpha-1} \leq R + c(1+x_2)^\beta \frac{|(\lambda+1)u - Fu|_{L^2}^{\alpha-1}}{(1+x_2)^{3(\alpha-1)/2}} \leq d_R,$$

the latter because $\beta - 3(\alpha-1)/2 \leq 0$ and $|u|_{L^2} \leq |u|_X$ as well as $|Fu|_{L^2} \leq c|u|_X^\alpha$, cf. Lemma A.4. <u>Case 2.</u> If $u''(x_1) \leq 0$, then

$$x_0^2 u(x_1) \leq x_1^2 u(x_1) = \lambda u(x_1) + u''(x_1) - f(x_1, u(x_1)) \leq [|\lambda| + |g(x_1, u(x_1))|]\, u(x_1),$$

so that the same argument as in Case 1 applies. □

Next we will proceed to the proof of Theorem 2. For this, we start with some auxiliary results.

Lemma A.8 *If $(\lambda,u)\in\mathcal{S}$, then $u=A[(\lambda+1)u-Fu]$, $u\in C^2(\mathbb{R})$ and $-u''+x^2u+f(x,u)=\lambda u$ in $\mathbb{R}$. In addition, for every $R>0$ there exists a constant $c_R>0$ such that for $(\lambda,u),(\bar\lambda,\bar u)\in\mathcal{S}$ with $|\lambda|+|u|_X\le R$ and $|\bar\lambda|+|\bar u|_X\le R$*

$$|u-\bar u|_{C_b^1(\mathbb{R})}\le c_R[\,|\lambda-\bar\lambda|+|u-\bar u|_X\,]. \tag{A.12}$$

Proof: By (A.11) we have $u=A[(\lambda+1)u-Fu]$ for $(\lambda,u)\in\mathcal{S}_0$, and this carries over to $\mathcal{S}$ by approximation. Thus $u\in C^1(\mathbb{R})$ by Lemma A.3, and hence $(\lambda+1)u-Fu\in L^2\cap C(\mathbb{R})$, because $f:\mathbb{R}^2\to\mathbb{R}$ is continuous. This in turn implies, again by Lemma A.3, that $u\in C^2(\mathbb{R})$ with $-u''+x^2u+f(x,u)=\lambda u$ for $x\in\mathbb{R}$. Concerning the second claim, first note that

$$|u(x)-\bar u(x)|\le\sqrt{2}\,|u-\bar u|_X,\quad x\in\mathbb{R}, \tag{A.13}$$

by (A.1). Moreover, with $v=(\lambda+1)u-Fu$ and $\bar v=(\bar\lambda+1)\bar u-F\bar u$, by (A.4), (A.3), and (A.8), for $x\in\mathbb{R}$

$$\begin{aligned}|u'(x)-\bar u'(x)| &= |(Av)'(x)-(A\bar v)'(x)|=|(A[v-\bar v])'(x)|\le c|v-\bar v|_{L^2}\\ &\le c[\,|u|_{L^2}|\lambda-\bar\lambda|+|\bar\lambda+1||u-\bar u|_{L^2}+|Fu-F\bar u|_{L^2}]\le c_R[\,|\lambda-\bar\lambda|+|u-\bar u|_X\,].\end{aligned}$$

Taking into account (A.13) we have shown (A.12). □

Lemma A.9 *Let $(\lambda,u)\in\mathcal{S}$. If $x_0\in\mathbb{R}$ with $u(x_0)=u'(x_0)=0$, then $u=0$. Moreover, if $u\neq0$, then u can have only finitely many zeroes in every interval $[-d,d]$, $d>0$, and each zero is simple.*

Proof: The first claim follows from the unique solvability of $-u''+x^2u+f(x,u)=\lambda u$, $u(x_0)=u'(x_0)=0$, cf. (10) and Lemma A.8. Thus every zero of $u\neq0$ must be simple. In addition, if there would be infinitely many different zeroes x_n in some $[-d,d]$, then w.l.o.g. $x_n\to x_0$ for some $x_0\in[-d,d]$ with $u(x_0)=0$. If e.g. $x_n>x_0$, then $u'(y_n)=0$ for some $y_n\in[x_0,x_n]$, and thus also $u'(x_0)=0$, a contradiction to $u\neq0$. □

Thus as an immediate consequence of (A.12) in Lemma A.8 we obtain the following corollary, where for $(\lambda,u)\in\mathcal{S}$ with $u\neq0$ and $d>0$ we denote by $\mathcal{N}((\lambda,u);d)$ the finite number of simple zeroes of u in $]-d,d[$.

Corollary A.1 *For $n\in\mathbb{N}_0$ and $d>0$ define*

$$\mathcal{Z}_n^d=\{(\lambda,u)\in\mathcal{S}:\mathcal{N}((\lambda,u);d)=n\}.$$

Then $\mathcal{Z}_n^d\subset\mathcal{S}$ is open.

Next, for $R\ge3$ define the finite

$$d_R^0=\max\{|x|:x\in\mathbb{R}\text{ is a zero of an eigenfunction }u_n\text{ for some }n\in\mathbb{N}_0\text{ with }\lambda_n\le R\}, \tag{A.14}$$

with $\lambda_n=\mu_n-1=2n+1$, $n\in\mathbb{N}_0$, and u_n from Lemma 1. Note that $u_0(x)=e^{-x^2/2}$ has no zeroes.

Lemma A.10 *For $R>0$ and $n\in\mathbb{N}_0$ with $\lambda_n\le R$ there is $\delta>0$ such that $(\lambda,u)\in\mathcal{S}$, $u\neq 0$, and $|\lambda-\lambda_n|+|u|_X\le\delta$ implies*

$$\mathcal{N}((\lambda,u);d_{R+1}+d_R^0+1)=n,$$

where d_{R+1} is from Lemma 2 with $R+1$.

Proof: If the claim were wrong, then there would be $(\lambda^j,u^j)\in\mathcal{S}$ with $\lambda^j\to\lambda_n$ in $\mathbb{R}$ and $0\neq u^j\to 0$ in X as $j\to\infty$ such that $\mathcal{N}^j=\mathcal{N}((\lambda^j,u^j);d_{R+1}+d_R^0+1)\neq n$ for all $j\in\mathbb{N}$. Let $w^j=v^j/|v^j|_{L^2}\in X\subset L^2$, with $v^j=S(\mu^j u^j-Fu^j)\in X$, where $\mu^j=\lambda^j+1\to\lambda_n+1$ as $j\to\infty$. These w^j are well-defined, since by (A.11) we have $u^j=Sv^j$, and hence $0\neq|u_j|_X=|Sv_j|_X=|v_j|_{L^2}$. Moreover, $w^j=\mu^j Aw^j-(Nv^j)/|v^j|_{L^2}$, cf. the proof of Lemma A.7. According to Lemma A.5 we have $(Nv^j)/|v^j|_{L^2}\to 0$ in L^2 as $j\to\infty$. Since $|w^j|_{L^2}=1$ and $A:L^2\to L^2$ is compact, thus w.l.o.g. $w^j\to w$ in L^2 as $j\to\infty$, with $w=(\lambda_n+1)Aw=\mu_n Aw\in X$. Define $u=Sw\in X$. Then $|u|_X=|Sw|_X=|w|_{L^2}=1$ and $u=\mu_n Au$ by (A.7), therefore $u=\alpha u_n$ for some $\alpha\neq 0$ by Lemma 1. In particular, $u\in C^2(\mathbb{R})$, u has exactly n zeroes in $\mathbb{R}$, they are all simple, and also $|x|\le d_R^0$ for these zeroes x of u by definition of d_R^0, cf. (A.14). Additionally, $u^j/|u^j|_X=(Sv^j)/|v^j|_{L^2}=Sw^j\to Sw=u$ in X as $j\to\infty$ by continuity of $S:L^2\to X$.

We claim that $|u^j/|u^j|_X-u|_{C_b^1(\mathbb{R})}\to 0$ as $j\to\infty$. This can be shown similar to (A.12) in Lemma A.8, since $u=Az$ with $z=\mu_n u$ and $u^j/|u^j|_X=Az^j$ with $z^j=(\lambda^j+1)u^j/|u^j|_X-(Fu^j)/|u^j|_X$ by Lemma A.8, and because also, by Lemma A.4,

$$\begin{aligned}|z^j-\bar{v}|_{L^2} &\le |\lambda^j+1-\mu_n|\,|u^j|_{L^2}/|u^j|_X+\mu_n|u^j/|u^j|_X-u|_{L^2}+|Fu^j|_{L^2}/|u^j|_X\\ &\le|\lambda^j+1-\mu_n|+\mu_n|u^j/|u^j|_X-u|_X+c|u^j|_X^{\alpha-1}\to 0\quad\text{as}\quad j\to\infty.\end{aligned}$$

Consequently, $|u^j/|u^j|_X-u|_{C_b^1(\mathbb{R})}\to 0$ as $j\to\infty$, and this in turn implies that $u^j/|u^j|_X$ also has exactly n simple zeroes, and all these zeroes are in $\{x\in\mathbb{R}:|x|\le d_{R+1}+d_R^0+1\}$ for large j, cf. Lemma 2 and Corollary A.1. Thus $\mathcal{N}^j=n$ for large j, a contradiction. □

Now we are in the position to give the

Proof of Theorem 2: Suppose on the contrary that alternative (ii) of Theorem 1 is satisfied. Then in particular $\mathcal{C}_n\subset\overline{B}_R^{\mathcal{S}}:=\{(\lambda,u)\in\mathcal{S}:|\lambda|+|u|_X\le R\}$ for some $R>0$ sufficiently large. We choose d_R with the property stated in Lemma 2, and w.l.o.g. we may assume that $d_{R+1}\ge d_R$. Then

$$\mathcal{C}_n\subset\overline{B}_R^{\mathcal{S}}=\overline{B}_R^{\mathcal{S}}\cap\Big[\bigcup_{j=0}^{\infty}\mathcal{Z}_j^{d_{R+1}+1}\cup\{(\lambda_j,0):\lambda_j\le R\}\Big]\tag{A.15}$$

with $\mathcal{Z}_j^{d_{R+1}+1}$ from Corollary A.1. Indeed, if $(\lambda,u)\in\overline{B}_R^{\mathcal{S}}\subset\mathcal{S}$ and $u=0$, then $|\lambda|\le R$ and $\lambda=\lambda_j$ for some $j\in\mathbb{N}_0$. On the other hand, if $u\neq 0$, then by Lemma 2 all zeroes of u are in $\{x\in\mathbb{R}:|x|\le d_R\}$, there are only $j\in\mathbb{N}_0$ (i.e., finitely many) of these zeroes, and all are simple, by Lemma A.9. In particular, $u(\pm[d_{R+1}+1])\neq 0$ and $\mathcal{N}((\lambda,u);d_{R+1}+1)=j$, hence $(\lambda,u)\in\mathcal{Z}_j^{d_{R+1}+1}$ by definition. This establishes (A.15).

Define $\tilde{\mathcal{N}}:\overline{B}_R^{\mathcal{S}}\to\mathbb{N}_0$ through $\tilde{\mathcal{N}}(\lambda,u)=\mathcal{N}((\lambda,u);d_{R+1}+d_R^0+1)$ for $u\neq 0$, and $\tilde{\mathcal{N}}(\lambda_j,0)=j$ for $j\in\mathbb{N}_0$ such that $\lambda_j\le R$, where d_R^0 is from (A.14). We claim that $\tilde{\mathcal{N}}$ is locally (w.r. to the $\mathcal{S}$-topology) constant on $\overline{B}_R^{\mathcal{S}}$. In fact, if $(\lambda_j,0)\in\overline{B}_R^{\mathcal{S}}$ with $\lambda_j\le R$, then in an $\mathcal{S}$-neighborhood $\tilde{\mathcal{N}}(\lambda,u)=j=\tilde{\mathcal{N}}(\lambda_j,0)$ by Lemma A.10. Note that $(\lambda,u)\in\mathcal{S}$ sufficiently close to $(\lambda_j,0)$ with

$u=0$ must already have $\lambda=\lambda_j$. On the other hand, if $(\lambda,u)\in\overline{B}_R^{\mathcal{S}}\cap\mathcal{Z}_j^{d_{R+1}+1}$ for some $j\in\mathbb{N}_0$, then $u\neq 0$, and by Corollary A.1 there is $\delta>0$ such that $(\bar\lambda,\bar u)\in\mathcal{S}$ with $|\bar\lambda-\lambda|+|\bar u-u|_X\le\delta$ implies $(\bar\lambda,\bar u)\in\mathcal{Z}_j^{d_{R+1}+1}$, and thus $\mathcal{N}((\bar\lambda,\bar u);d_{R+1}+1)=\mathcal{N}((\lambda,u);d_{R+1}+1)$ by definition. But $\delta\le 1$ yields $|\bar\lambda|+|\bar u|_X\le\delta+|\lambda|+|u|_X\le 1+R$, and therefore the zeroes of both u and $\bar u$ are contained in $\{x\in\mathbb{R}:|x|\le d_{R+1}\}$, by Lemma 2. Consequently for small $\delta>0$ and $(\bar\lambda,\bar u)\in\mathcal{S}$ with $|\bar\lambda-\lambda|+|\bar u-u|_X\le\delta$

$$\begin{aligned}\tilde{\mathcal{N}}(\bar\lambda,\bar u)&=\mathcal{N}((\bar\lambda,\bar u);d_{R+1}+d_R^0+1)=\mathcal{N}((\bar\lambda,\bar u);d_{R+1}+1)=\mathcal{N}((\lambda,u);d_{R+1}+1)\\&=\mathcal{N}((\lambda,u);d_{R+1}+d_R^0+1)=\tilde{\mathcal{N}}(\lambda,u).\end{aligned}$$

Therefore $\tilde{\mathcal{N}}$ is locally constant on $\overline{B}_R^{\mathcal{S}}$, and thus also on the compact connected $\mathcal{C}_n\subset\mathcal{S}$, cf. (A.15) and assumption (ii) of Theorem 1. This in turn implies that $\tilde{\mathcal{N}}$ is constant on $\mathcal{C}_n$, and since by assumption $(\lambda_m,0)\in\mathcal{C}_n$ for some $m\neq n$, we obtain the contradiction $m=\tilde{\mathcal{N}}(\lambda_m,0)=\tilde{\mathcal{N}}(\lambda_n,0)=n$. □

Next we carry out the

Proof of Lemma 3: We first consider the general problem (8) under assumption (10) on f, and we additionally suppose that $F:X\to L^2$ is C^3 with $\delta F(0)=0$, $\delta^2F(0)=0$, but $\delta^3F(0)\neq 0$, where F is from Lemma A.4. These assumptions will hold if f is a cubic nonlinearity, as is the case for (7) resp. (6).

Let $v_n=\mu_nSu_n\in X$ with u_n from Lemma 1. Then

$$Sv_n=\mu_nS^2u_n=\mu_nAu_n=u_n\quad\text{and}\quad\mu_nAv_n=\mu_n^2A(Su_n)=\mu_n^2S(Au_n)=\mu_nSu_n=v_n$$

by (A.7) and Lemma 1. Hence in particular the one-dimensional nullspace of $I-\mu_nA$ in L^2 is generated by v_n. Since the algebraic multiplicity of μ_n for A is one, a standard local bifurcation theorem (cf., e.g., [5, Thm. 28.3]) can be applied to yield a $\gamma>0$ and C^2-functions $\mu:(-\gamma,\gamma)\to\mathbb{R}$ and $v:(-\gamma,\gamma)\to(\mathbb{R}v_n)^\perp$ such that $\mu(0)=0, v(0)=0$, and $G(\mu_n+\mu(s),s[v_n+v(s)])=0$ for $s\in(-\gamma,\gamma)$, with $G(\mu,v)=v-\mu Av-Nv=v-\mu Av-(SFS)(v)$, cf. (14) and Lemma A.5. As $\delta^2F(0)=0$, we obtain $\mu'(0)=0$, cf. [5, p. 385] and the arguments to follow. Since S is linear, we have $\delta N(0)=0$ and $\delta^2N(0)=0$ by assumption, and $\delta^3N(v)(w_1,w_2,w_3)=S\delta^3F(Sv)(Sw_1,Sw_2,Sw_3)$ as a three-form on L^2. Thus

$$\delta^3N(0)[v_n]^3=S\delta^3F(0)[Sv_n]^3=S\delta^3F(0)[u_n]^3.$$

Let $Pv=\dfrac{(v_n,v)_{L^2}}{|v_n|_{L^2}^2}v_n$ denote the orthogonal projection onto $\mathbb{R}v_n$ in L^2. By differentiating $G(\mu_n+\mu(s),s[v_n+v(s)])=0$ w.r.t. to s and projection onto $\mathbb{R}v_n$ one finds

$$\mu''(0)v_n=\frac{1}{3}\mu_nP\delta^3N(0)[v_n]^3=\frac{1}{3}\mu_n\frac{(v_n,S\delta^3F(0)[u_n]^3)_{L^2}}{|v_n|_{L^2}^2}v_n,$$

and therefore

$$\mu''(0)=\frac{1}{3}\mu_n\frac{(v_n,S\delta^3F(0)[u_n]^3)_{L^2}}{|v_n|_{L^2}^2}.$$

Next recall S is selfadjoint in L^2 and $Sv_n=u_n$. Moreover, $|v_n|_{L^2}=|Sv_n|_X=|u_n|_X$, and hence

$$\mu''(0)=\frac{1}{3}\mu_n\frac{(u_n,\delta^3F(0)[u_n]^3)_{L^2}}{|u_n|_X^2}\quad\text{with}\quad\mu_n=2n+2.$$

Now we specialize this to (the real-valued analogon of) (7), i.e., we take $f(x,u) = -\sqrt{2/a}u^3$, and we fix $(a,k) \in P_n$, see (16). Since the Nemytskii operator $Fu = f(\cdot, u(\cdot))$ has $\delta^3 F(0)[u_n]^3 = -6\sqrt{2/a}u_n^3$, we obtain the claim. $\square$

Finally we turn to the variational characterization of the ground states and give, following [16, Thm. 2.2], the

Proof of Lemma 4: For notational simplicity, we consider the case $a = 2$ and transform (6) again to (7), and also $\lambda = -\sqrt{\frac{2}{a}}k = -k$ with $k > k_0$ to $\mu = 1 + \lambda$ with $\mu < \mu_0 = 2$. Thus the non-linear equation is

$$-u_{xx} + (1 + x^2)u - |u|^2 u = \mu u, \tag{A.16}$$

with corresponding functional

$$J^\mu[u] = \frac{|u|_X^2 - \mu |u|_{L^2}^2}{|u|_{L^4}^2}, \quad 0 \neq u \in X, \quad \text{and} \quad I^\mu = \inf\{J^\mu[u] : 0 \neq u \in X\}.$$

Observe first that $|u|_X^2 \geq \mu_0 |u|_{L^2}^2$, as μ_0 is the smallest eigenvalue. Consider a minimizing sequence $J^\mu[u_n] \to I^\mu$ with $|u_n|_{L^4} = 1$. Let $\delta_0 = 1 - \mu/\mu_0 > 0$ to obtain with $\mathcal{F}^\mu[u] = |u|_X^2 - \mu|u|_{L^2}^2$ the estimate $|u|_X^2 = \mathcal{F}^\mu[u] + (1-\delta_0)\mu_0|u|_{L^2}^2 \leq \mathcal{F}^\mu[u] + (1-\delta_0)|u|_X^2$. Hence $(u_n) \subset X$ is bounded, and due to Lemma A.1, w.l.o.g. $u_n \to u_\infty$ strongly in L^2 and weakly in X for some $u_\infty \in X$. By the Gagliardo-Nirenberg estimates [7, Thm. I.9.3], $|u|_{L^4} \leq C |u_x|_{L^2}^{3/4} |u|_{L^2}^{1/4}$, therefore also $u_n \to u_\infty$ strongly in L^4. Thus u_∞ is a minimizer, $J^\mu[u_\infty] = I^\mu$. As $\delta J^\mu[u_\infty] = 0$, it follows that the scaled $\tilde{u}_\infty = \sqrt{\alpha_\infty} u_\infty$ is a solution to (A.16) with $J^\mu[\tilde{u}_\infty] = I^\mu$, where $\alpha_\infty = \mathcal{F}^\mu[u_\infty]$. We now indicate the dependence on μ by writing $\tilde{u}_\infty = \tilde{u}_\infty^\mu$ to show $\tilde{u}_\infty^\mu \to 0$ in X as $\mu \uparrow \mu_0$. The proof that $\tilde{u}_\infty^\mu \to 0$ in L^2 runs exactly as in Appendix A of [16]. Let u_0 be the eigenfunction of (11) for μ_0. Then $I^\mu \leq J^\mu[u_0] = (\mu - \mu_0)|u_0|_{L^4}^{-2}|u|_{L^2}^2 = C(\mu - \mu_0)$. Consequently, $|\tilde{u}_\infty^\mu|_X^2 = \mu|\tilde{u}_\infty^\mu|_{L^2}^2 + |\tilde{u}_\infty^\mu|_{L^4}^2 I^\mu \leq \mu_0 |\tilde{u}_\infty^\mu|_{L^2}^2 + C(\mu_0 - \mu) \to 0$ as $\mu \uparrow \mu_0$, thus $\tilde{u}_\infty^\mu \to 0$ in X. $\square$

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FIGURE CAPTIONS

1. Soliton shapes for $a=2$ and different values of the parameter k. $k=10$ – solid line, $k=5$ – dashed line, $k=0.5$ – dotted line, $k=-0.5$ – dashed-dotted line, $a=-0.99$ – thin line.

2. Soliton shapes in logarithmic scale. Parameters are identical to those in Fig. 1.

3. Energy E as a function of the nonlinear eigenvalue k for different values of parameter a. $a=10$ – solid line, $a=5$ – dashed line, $a=2$ – dotted line, $a=0.1$ – dashed-dotted line, $a=0$ – thin line. A comparison of the local behavior in the vicinity of the bifurcation point as predicted by Eq. (18) is shown in inset for $a=2$.

4. Energy E as a function of the nonlinear eigenvalue k for different values of parameter a. Parameters as in Fig. 3.

5. Hamiltonian H as a function of energy E for different values of the parameter a. Parameters are identical to those in Fig. 3. Note the absence of singularities which indicates either stability or instability of the whole branch of solutions.