A note on Lipschitz continuous solutions of a parabolic quasi-variational inequality

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1 Introduction

We consider the evolutionary quasi-variational inequality

$$\text{find} \quad u(t) \in C(u(t)): \quad \langle u'(t)+F(t), v-u(t)\rangle \geq 0 \quad \text{for all} \quad v \in C(u(t)) \tag{1}$$

for a.e. $t\in[0,T]$, with initial value $u(0)=u_0\in C(u_0)$, and

$$C(u)=\{\phi\in H_0^1(0,1):|\phi'(x)|\leq\Phi(x,u(x)) \text{ for a.e. } x\in(0,1)\}, \quad u\in L^2(0,1), \tag{2}$$

in the Hilbert space $\mathcal{H}=L^2(0,1)$, where $F\in L^\infty(0,T;\mathcal{H})$ (for simplicity). The study of problems of type (1), (2), is motivated through quasi-variational inequalities arising e.g. in the evolution of sandpiles; see [3]. In this example (where one has $x\in\Omega\subset\mathbb{R}^2$ rather than $x\in(0,1)$ as we are going to treat here), the function Φ is discontinuous, posing serious technical problems to showing the existence of solutions.

In this paper, our aim is more moderate. We assume that

$$0<m\leq\Phi(x,u)\leq M\,, \quad |\Phi(x,u)-\Phi(x,v)|\leq K|u-v|, \quad \text{a.e. in} \quad (0,1), \quad u,v\in\mathbb{R}. \tag{3}$$

Under these hypotheses, applying [1, Thm. 3.5], we will obtain the existence of a solution to (1) for K small. In addition, the solution will be quite regular.

Theorem 1.1 *Let (3) be satisfied for Φ and assume that $K<\left(\frac{\sqrt{2}m}{2M+m}\right)$. Then for every $u_0\in L^2(0,1)$ such that $u_0\in C(u_0)$, there exists a Lipschitz continuous solution $u:[0,T]\to L^2(0,1)$ of (1) with initial value $u(0)=u_0$.*

We add some remarks. Since [1, Thm. 3.5] is sharp w.r.t. to the Lipschitz constant, we expect that Theorem 1.1 cannot be improved substantially, without further assumptions. In [4], results will be derived under more general conditions. The model is still one dimensional, Φ is bounded above and below, but the solutions are less regular. In [2], several time-independent models with gradient constraints are treated.

2 Proof of Theorem 1.1

We intend to apply the following

Lemma 2.1 *Let $\mathcal{H}$ be a Hilbert space, and $C(u)\subset\mathcal{H}$ be nonempty, compact, and convex for $u\in\mathcal{H}$. Assume $F\in L^\infty(0,T;\mathcal{H})$, and for some $0\le L<1$,*

$$\mathrm{d_H}(C(u),C(v))\le L|u-v|,\quad u,v\in\mathcal{H}. \tag{4}$$

Then for every $u_0\in C(u_0)$ the state-dependent sweeping process

$$-u'(t)\in N_{C(u(t))}(u(t))+F(t)\quad a.e.\ in\quad [0,T],\quad u(0)=u_0, \tag{5}$$

has a Lipschitz continuous solution $u:[0,T]\to\mathcal{H}$.

Proof: This is a special case of [1, Thm. 3.5], since the compactness of every $C(u)$ implies the compactness condition required there. □

Since $N_C(u)=\{v\in\mathcal{H}:\langle v,c-u\rangle\le 0\,\forall c\in C\}$ (the normal cone at $u\in C$), (5) is just a reformulation of (1). Note that

$$\mathrm{d_H}(C_1,C_2)=\max\Big\{\sup_{u\in C_2}\operatorname{dist}(u,C_1),\ \sup_{u\in C_1}\operatorname{dist}(u,C_2)\Big\},$$

where $\operatorname{dist}(u,C_1)=\inf\{|u-v|:v\in C_1\}$, is the Hausdorff distance between $C_1,C_2\subset\mathcal{H}$. To fix further notation, for a closed convex $C\subset\mathcal{H}$ and $u\in\mathcal{H}$, $v=\operatorname{proj}(u,C)$ will be the unique element of $\mathcal{H}$ such that $|v-u|=\operatorname{dist}(u,C)$; $\mathrm{d_H}$ and proj will always refer to $\mathcal{H}=L^2(0,1)$.

Because Φ is bounded from above and the embedding $H_0^1(0,1)\subset L^2(0,1)=\mathcal{H}$ is compact, in particular we find that $\bigcup_{u\in\mathcal{H}}C(u)\subset\mathcal{H}$ is relatively compact. Consequently, we only have to verify (4) for $C(\cdot)$ defined by (2). To prove this, observe first that $|\phi|_{L^2(0,1)}\le\frac{1}{\sqrt{2}}|\phi'|_{L^2(0,1)}$ for $\phi\in H_0^1(0,1)$. Let

$$C'(u)=\{\varphi\in L^2(0,1):\int_0^1\varphi\,dx=0,\ |\varphi(x)|\le\Phi(x,u(x))\text{ a.e.}\},\quad u\in L^2(0,1).$$

Then the aforementioned observation and $\{\phi':\phi\in C(u)\}=C'(u)$ yields

$$\mathrm{d_H}(C(u),C(v))\le\frac{1}{\sqrt{2}}\mathrm{d_H}(C'(u),C'(v)),\quad u,v\in\mathcal{H}. \tag{6}$$

Suppose for the moment that we can show the following key lemma.

Lemma 2.2 *Let $f,g:(0,1)\to\mathbb{R}$ be measurable such that*

$$0<m\le f(x),g(x)\le M\text{ a.e. }x\in(0,1), \tag{7}$$

and define

$$C'(f)=\{\varphi\in L^2(0,1):\int_0^1\varphi\,dx=0,\ |\varphi(x)|\le f(x)\text{ a.e.}\}. \tag{8}$$

Let $v\in C'(f)$ and $w=\operatorname{proj}(v,C'(g))$. Then

$$|v-w|_{L^2(0,1)}\le\left(\frac{2M+m}{m}\right)|f-g|_{L^2(0,1)}. \tag{9}$$

Then (6) and (3) give the claim of Theorem 1.1, because for $u, v \in \mathcal{H}$

$$\begin{aligned} \mathrm{d_H}(C(u), C(v)) &\le \frac{1}{\sqrt{2}} \mathrm{d_H}(C'(u), C'(v)) \le \left(\frac{2M+m}{\sqrt{2}m}\right) |\Phi(\cdot, u(\cdot)) - \Phi(\cdot, v(\cdot))|_{L^2(0,1)} \\ &\le \left(\frac{2M+m}{\sqrt{2}m}\right) K\, |u-v|_{L^2(0,1)}, \end{aligned}$$

so that we can choose $L = \left(\frac{2M+m}{\sqrt{2}m}\right) K < 1$ in (4). $\square$

Proof of Lemma 2.2 : 1. Step: Let $d = |f-g|_{L^1(0,1)}$. From (7) we obtain $|f(x) - g(x)| \le M - m$ a.e., and thus

$$d \le M - m. \tag{10}$$

W.l.o.g. we assume $d > 0$. Now define

$$v_1(x) = \begin{cases} v(x) & : \quad |v(x)| < g(x) \\ g(x) & : \quad v(x) \ge g(x) \\ -g(x) & : \quad v(x) \le -g(x) \end{cases}.$$

Since $\int_0^1 v\, dx = 0$ and $|v(x)| \le f(x)$ a.e., we find

$$\begin{aligned} \left| \int_0^1 v_1\, dx \right| &= \left| \int_{\{x : |v(x)| < g(x)\}} v(x)\, dx + \int_{\{x : v(x) \ge g(x)\}} g(x)\, dx - \int_{\{x : v(x) \le -g(x)\}} g(x)\, dx \right| \\ &= \left| \int_{\{x : v(x) \ge g(x)\}} (g(x) - v(x))\, dx - \int_{\{x : v(x) \le -g(x)\}} (g(x) + v(x))\, dx \right| \\ &\le \int_{\{x : v(x) \ge g(x)\}} |f(x) - g(x)|\, dx + \int_{\{x : v(x) \le -g(x)\}} |g(x) - f(x)|\, dx \le d. \end{aligned}$$

Moreover,

$$|v(x) - v_1(x)| = \begin{cases} 0 & : \quad |v(x)| < g(x) \\ v(x) - g(x) & : \quad v(x) \ge g(x) \\ v(x) + g(x) & : \quad v(x) \le -g(x) \end{cases} \quad \le \quad |f(x) - g(x)|,$$

and therefore, to summarize the properties of v_1,

$$|v_1(x)| \le g(x) \text{ a.e.}, \quad \left| \int_0^1 v_1\, dx \right| \le d, \quad |v - v_1|_{L^2(0,1)} \le |f - g|_{L^2(0,1)}. \tag{11}$$

2. Step: Define $v_2(x) = v_1(x) - \int_0^1 v_1(y)\, dy$. Then by (11), $|v_2(x) - v_1(x)| \le d$, and hence

$$|v_2(x)| \le g(x) + d =: f_2(x) \text{ a.e.}, \quad \int_0^1 v_2\, dx = 0, \quad |v_2 - v_1|_{L^2(0,1)} \le d. \tag{12}$$

Note that by (7) and (10), $m \le g(x) \le g(x) + d = f_2(x) \le 2M - m$, hence

$$m \le f_2(x), g(x) \le 2M - m =: M_2 \text{ a.e.} \tag{13}$$

This means that the problem has been reduced to a fixed distance (namely d) between all $f(x)$ and $g(x)$. For every $\varphi \in C'(g)$ we have from (11) and (12), using $d \le |f-g|_{L^2(0,1)}$,

$$\begin{aligned} |v - w|_{L^2(0,1)} &\le |v - \varphi|_{L^2(0,1)} \le |v - v_1|_{L^2(0,1)} + |v_1 - v_2|_{L^2(0,1)} + |v_2 - \varphi|_{L^2(0,1)} \\ &\le 2\,|f - g|_{L^2(0,1)} + |v_2 - \varphi|_{L^2(0,1)}. \end{aligned} \tag{14}$$

Our aim is to find a suitable function $\varphi \in C'(g)$ to apply this estimate.

3. Step: Let $\alpha = \int_0^1 v_2(x)\,\frac{g(x)}{f_2(x)}\,dx$.

Case 1: $\alpha = 0$. Then $\varphi(x) = v_2(x)g(x)/f_2(x)$ has $\varphi \in C'(g)$. Indeed, we have $\int_0^1 \varphi\,dx = 0$ and $|\varphi(x)| \le g(x)$ a.e. by (12). In addition, (12) also implies that

$$|\varphi(x) - v_2(x)| = \frac{|v_2(x)|}{f_2(x)}\,|f_2(x) - g(x)| \le d, \quad \text{thus} \quad |\varphi - v_2|_{L^2(0,1)} \le d.$$

This in turn yields, by (14),

$$|v - w|_{L^2(0,1)} \le 3|f - g|_{L^2(0,1)} \le \left(\frac{2M + m}{m}\right) |f - g|_{L^2(0,1)}.$$

Case 2: $\alpha < 0$. Then $\bar{v}_2(x) = -v_2(x)$ has $\int_0^1 \bar{v}_2(x)\,\frac{g(x)}{f_2(x)}\,dx > 0$, $|\bar{v}_2(x)| \le f_2(x)$ a.e., $\int_0^1 \bar{v}_2\,dx = 0$, and moreover $|v - w|_{L^2(0,1)} \le 2|f - g|_{L^2(0,1)} + |\bar{v}_2 - \varphi|_{L^2(0,1)}$ for all $\varphi \in C'(g)$, since $\varphi \in C'(g)$ iff $-\varphi \in C'(g)$. This shows that we can consider w.l.o.g. the

Case 3: $\alpha > 0$. Here let $\lambda(\beta) = \beta/(\beta + d)$ for $\beta \ge 0$. Then $0 \le \lambda(\beta) \le 1$, and $\lambda(\cdot)$ is strictly increasing. Define

$$v_3^\beta(x) = \begin{cases} \lambda(\beta)\,v_2(x)\,\dfrac{g(x)}{f_2(x)} & : \quad v_2(x) \ge 0 \\ v_2(x)\,\dfrac{g(x)}{f_2(x)} & : \quad v_2(x) < 0 \end{cases}$$

as well as

$$I(\beta) = \int_0^1 v_3^\beta\,dx = \lambda(\beta)\int_{\{x:v_2(x)\ge 0\}} v_2(x)\,\frac{g(x)}{f_2(x)}\,dx + \int_{\{x:v_2(x)<0\}} v_2(x)\,\frac{g(x)}{f_2(x)}\,dx.$$

W.l.o.g. we may assume the factor of $\lambda(\beta)$ to be positive, since otherwise $\int_0^1 v_2\,dx = 0$ implies $v_2 = 0$ a.e., and thus it is enough to take $\varphi = 0 \in C'(g)$ in (14). Then $\beta \mapsto I(\beta)$ is strictly increasing with $I(0) \le 0$ and $\lim_{\beta\to\infty} I(\beta) = \alpha > 0$. Hence there is a unique $\beta_0 \in [0, \infty)$ with $I(\beta_0) = 0$, and we claim $\varphi = v_3^{\beta_0} \in C'(g)$. We have already seen $\int_0^1 \varphi\,dx = I(\beta_0) = 0$. Additionally, $0 \le \lambda(\beta_0) \le 1$ and $|v_2(x)| \le f_2(x)$ a.e. yield $|\varphi(x)| \le g(x)$ a.e., thus $\varphi \in C'(g)$ is admissible in (14).

We first derive an estimate for β_0. By the defining property, $I(\beta_0) = 0$, and hence $f_2(x) = g(x) + d$, the monotonicity of $z \mapsto z/(z + d)$, and $m \le g(x) \le M_2$ a.e. imply

$$\frac{\beta_0}{\beta_0 + d} = \frac{-\displaystyle\int_{\{x:v_2(x)<0\}} v_2(x)\,\frac{g(x)}{g(x)+d}\,dx}{\displaystyle\int_{\{x:v_2(x)\ge 0\}} v_2(x)\,\frac{g(x)}{g(x)+d}\,dx}$$

$$\ge \frac{\left(\dfrac{m}{m+d}\right)\displaystyle\int_{\{x:v_2(x)<0\}} [-v_2(x)]\,dx}{\left(\dfrac{M_2}{M_2+d}\right)\displaystyle\int_{\{x:v_2(x)\ge 0\}} v_2(x)\,dx} = \frac{m\,(M_2 + d)}{M_2\,(m + d)},$$

the latter because $\int_0^1 v_2\,dx = 0$. Therefore $\beta_0 M_2(m + d) \ge (\beta_0 + d)m(M_2 + d)$ gives

$$\beta_0 \ge \frac{m(M_2 + d)}{M_2 - m} \ge \frac{m M_2}{M_2 - m}. \tag{15}$$

Now by definition and (12),

$$|v_2(x)-\varphi(x)| = \begin{cases} \frac{|v_2(x)|}{f_2(x)}\,|\lambda(\beta_0)g(x)-f_2(x)| \le |\lambda(\beta_0)g(x)-f_2(x)| & : \quad v_2(x)\ge 0 \\ \frac{|v_2(x)|}{f_2(x)}\,|g(x)-f_2(x)| \le d & : \quad v_2(x)<0 \end{cases}, \tag{16}$$

and by (15),

$$\begin{aligned} |\lambda(\beta_0)g(x)-f_2(x)| &\le \frac{d}{\beta_0+d}(\beta_0+f_2(x)) \le d(1+M_2/\beta_0) \le d\Big(1+\frac{M_2(M_2-m)}{mM_2}\Big) \\ &= \Big(\frac{2M-m}{m}\Big)\,d, \end{aligned}$$

recall $M_2 = 2M-m$, cf. (13). Since $(2M-m)/m \ge 1$, (16) implies

$$|v_2(x)-\varphi(x)| \le \Big(\frac{2M-m}{m}\Big)\,d \quad \text{a.e.},$$

and therefore also $|v_2-\varphi|_{L^2(0,1)} \le ((2M-m)/m)\,d$. Since $d \le |f-g|_{L^2(0,1)}$, this estimate may be used in (14) to yield (9), thus completing the proof of Lemma 2.2. □

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