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On Lyapunov Exponents for Non-Smooth Dynamical Systems with an Application to a Pendulum with Dry Friction

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Abstract

We consider dynamical systems from mechanics for which, due to some non-smooth friction effects, Oseledets' Multiplicative Ergodic Theorem cannot be applied canonically to define Lyapunov exponents. For general non-smooth systems which fit into a natural formal framework, we construct a suitable cocycle which lives on a "good" invariant set of full Lebesgue measure. Afterwards, this construction is applied to investigate a pendulum with dry friction, described through the equation $\ddot{x}+x+\mathrm{sgn}\,\dot{x}=\gamma\sin(\eta t)$. The Lyapunov exponents obtained by our construction show a good agreement with the dynamical behaviour of the system, and since we will prove that these Lyapunov exponents are always non-positive, we conclude that the system does not show chaotic behaviour.

1 Introduction

There are many concrete problems in mechanics where effects have to be taken into account which result from non-smooth phenomena: one might think of e.g. the rattling of gear boxes, the noise of squeaking chalk on a blackboard, or the chattering of machine tools, cf. [28] for more examples. Physically speaking, these effects are due to the fact that there are rigid bodies which are in contact (they "stick"), but this contact phases are interrupted by "slip" phases during which one body moves relative to another. One of the simplest examples for such "slip-stick motions" is provided by the pendulum with dry friction shown in Fig. 1(a) below, cf. e.g. [8, p. 193] or older books on mechanics like [2, 17].

Fig. 1(a): A pendulum with dry friction 1(b): Phase portrait of (1.1) with $\gamma=0$

The pendulum consists of a mass being attached to a spring, and the mass, which is sinusoidally forced, moves in a straight tube and has contact to the wall of the tube. Depending on the size of the dry friction between mass and wall, and the strength of the forcing, the mass moves up or down (slip phase"), or it sticks to the wall. By Coulomb's law, the friction force equals the negative of the constant normal force to the wall multiplied by the friction coefficient and by the sign of the velocity $\dot{x}$ of the mass. Assuming for simplicity that the friction coefficient is constant, after a suitable renormalizing the governing equation of the pendulum system is

$$\ddot{x}(\tau)+x(\tau)+\operatorname{sgn}\dot{x}(\tau)=\gamma\sin(\eta\tau) \tag{1.1}$$

with time τ and parameters $\gamma,\eta\geq 0$. Here we also assumed that on the left-hand side there is no additional viscous damping term $\delta\dot{x}(\tau)$ (e.g. through a fluid in the tube), but again this is only for simplicity.

Note that (1.1) only makes sense if $\dot{x}(t)\neq 0$, and $\operatorname{sgn}(0):=0$ is not the right choice. To see this, consider first the unforced equation with $\gamma=0$. The corresponding phase portrait is given in Fig. 1(b) above. It can be seen that all $(x_0,0)$ with $x_0\in[-1,1]$ are equilibria, but this effect is lost by defining $\operatorname{sgn}(0):=0$. This choice of $\operatorname{sgn}(0)$ is also meaningless in the presence of a forcing $(\gamma>0)$, since a solution can have dead-zones, i.e. $\dot{x}(\tau)=0$ for $\tau\in[\tau_1,\tau_2]$ with $\tau_2>\tau_1$. During these dead-zones, which correspond to a sticking of the mass at the wall, $\operatorname{sgn}(0):=0$ in (1.1) would give $x(\tau_1)=\gamma\sin(\eta\tau)$ for $\tau\in[\tau_1,\tau_2]$, which is impossible. So a common device (cf. [8, 13]) is to replace (1.1) by its multivalued extension

$$\ddot{x}(\tau)+x(\tau)\in\gamma\sin(\eta\tau)-\operatorname{Sgn}\dot{x}(\tau)\quad\text{a.e.} \tag{1.2}$$

with

$$\operatorname{Sgn}(v)=\begin{cases}\{v/|v|\} & : \quad v\neq 0\\ [-1,1] & : \quad v=0\end{cases},$$

i.e. every $\operatorname{Sgn}(v)$ is now a subset of $\mathbb{R}$. Note that by replacement of (1.1) through (1.2), in particular all $(x_0,0)$ with $x_0\in[-1,1]$ have become equilibria for (1.2) with $\gamma=0$.

By standard theory of differential inclusions, (1.2), considered as the autonomous system $\dot{y}\in F(y)$ a.e. with

$$\frac{d}{dt}\begin{pmatrix}x\\ v\\ \tau\end{pmatrix}\in F(x,v,\tau):=\{v\}\times\{\gamma\sin(\eta\tau)-x-w:w\in\operatorname{Sgn}(v)\}\times\{1\}\text{ a.e.},\quad y=(x,v,\tau)\in\mathbb{R}^3, \tag{1.3}$$

generates a continuous semiflow $\varphi:[0,\infty[\times\mathbb{R}^3\to\mathbb{R}^3$, where $\varphi(\cdot,y_0):[0,\infty[\to\mathbb{R}^3$ is the unique solution of (1.3) on $[0,\infty[$ with initial value $\varphi(0,y_0)=y_0=(x_0,v_0,\tau_0)\in\mathbb{R}^3$, i.e. (1.3) gives rise to a (semi)-dynamical system. It is for non-smooth systems of this type that we want to define Lyapunov exponents in this paper.

For smooth dynamical systems $\dot{y}=f(y)$ with $f\in C^1(\mathbb{R}^d,\mathbb{R}^d)$, Lyapunov exponents are a well-established tool for making predictions about the long-term behaviour of the system. Defining $T_f(t,y)=\partial_y\varphi_f(t,y)$, with φ_f being the respective flow, this linearization $T_f:[0,\infty[\times\mathbb{R}^d\to\mathcal{L}(\mathbb{R}^d)$ defines what is called a cocycle for $(\varphi_f^t)_{t\geq 0}$, that is

$$T_f(t+s,y)=T_f(t,\varphi_f^s(y))\,T_f(s,y)\quad\text{for}\quad t,s\in[0,\infty[\quad\text{and}\quad y\in\mathbb{R}^3, \tag{1.4}$$

and the corresponding Lyapunov exponents are $\lambda(y_0,y)=\lim_{t\to\infty}\left(\frac{1}{t}\ln|T(t,y_0)y|\right)$ for $y_0,y\in\mathbb{R}^d$, if the limit $t\to\infty$ exists. The central result concerning this existence of Lyapunov exponents is given by Oseledets' Multiplicative Ergodic Theorem [26] (henceforth abbreviated MET), which we state her for the present purpose of motivation only in a quite reduced form. In Theorem 4.3 of the appendix we will give the full theorem and also the definitions of the relevant terms from ergodic theory.

Theorem. *Let φ_f be a measurable semiflow on $\mathbb{R}^d$, T_f be a measurable cocycle for $(\varphi_f^t)_{t\geq 0}$ as above, and let μ be an ergodic probability measure which is invariant w.r.t. the semiflow. Then, if two integrability conditions are satisfied, there exists an invariant set $\Gamma \subset \mathbb{R}^d$ with $\mu(\Gamma)=1$, a $k\in\{1,\ldots,d\}$, and distinct numbers $\lambda^{(1)},\ldots,\lambda^{(k)}\in\mathbb{R}$ (where $\lambda^{(1)}$ may be $-\infty$) such that for every $y_0\in\Gamma$ and $y\in\mathbb{R}^d$ one has $\lambda(y_0,y)=\lambda^{(i)}$ for some $i\in\{1,\ldots,k\}$, and thus in particular the limit $t\to\infty$ exists.*

Because in non-smooth systems like the pendulum with dry friction (1.2), the semiflow may no longer be expected to be smooth w.r.t. y, the basic question to obtain Lyapunov exponents by application of the MET for such systems is

How to define a canonical cocycle for the non-smooth system ?

The construction of this cocycle will be carried out in Section 2 and described there in greater detail, so that we restrict ourselves here to a sketch. In principle, we rely on partitioning $\mathbb{R}^d$ in suitable manifolds, the validity regions of different smooth systems, which yield the complete non-smooth system when being pieced together. For example, denoting by $y=(x,v,\tau)$ a typical point in $\mathbb{R}^3$, (1.3) means by definition of Sgn that $\dot{y}=f_+(y)=(v,-x+b^-(\tau),1)$ in $\{y\in\mathbb{R}^3 : v>0\}$ and $\dot{y}=f_-(y)=(v,-x+b^+(\tau),1)$ in $\{y\in\mathbb{R}^3 : v<0\}$ with $b^\pm(\tau)=\gamma\sin(\eta\tau)\pm 1$; the situation is more complicated along the surface $\{y\in\mathbb{R}^3 : v=0\}$.

For the construction of the cocycle assume for the moment that every initial value $y_0\in\mathbb{R}^d$ generates a sequence of switching times $(t_i(y_0))_{i\in\mathbb{N}}$ from one manifold to the next, and so forth, along the trajectory starting in y_0. Then, on a formal level, $\varphi^t(\cdot)$ may be expected to be C^1 for $t\in]t_i(y_0),t_{i+1}(y_0)[$ resp. for $t\in]t_{i+1}(y_0),t_{i+2}(y_0)[$. Hence $T(t,y_0)=\partial_y\varphi^t(y_0)$ is well-defined for $t\in]t_i(y_0),t_{i+1}(y_0)[$ and for $t\in]t_{i+1}(y_0),t_{i+2}(y_0)[$, so that we have to investigate the transition of the linearization at the switching time $t_{i+1}(y_0)$. A formula of the type $\lim_{t\to t_{i+1}(y_0)^+}\partial_y\varphi^t(y_0)= A_{i+1}(y_0)\left(\lim_{t\to t_{i+1}(y_0)^-}\partial_y\varphi^t(y_0)\right)$ for this transition will be derived in Section 2.2, and under suitable additional assumptions the transition operator $A_{i+1}(y_0)$ can be given in a very concrete form. Thus we will obtain in Section 2.3 a canonical cocycle for the underlying non-smooth system by defining $T(t,y_0)=\partial_y\varphi^t(y_0)$ for $t\in]t_{i+1}(y_0),t_{i+2}(y_0)[$, and at the transition time $t=t_{i+1}(y_0)$, $T(t_{i+1}(y_0),y_0)=A_{i+1}(y_0)\left(\lim_{t\to t_{i+1}(y_0)^-}\partial_y\varphi^t(y_0)\right)$.

In fact matters are much more technical, and in particular it is not the complete truth that $\varphi^t(\cdot)$ is C^1 for fixed $t\in]t_i(y_0),t_{i+1}(y_0)[$, as may already be seen through the argument that for y near to y_0 then also $t\in]t_i(y),t_{i+1}(y)[$ is required in order to be in the same manifold of the partition, and hence to have the same governing right-hand side. This amounts in imposing additional assumptions which will be made precise in Section 2.1. These assumptions will lead to a restriction of admissible initial values to some forward invariant good" set $G\subset\mathbb{R}^d$. To have $y_0\in G$ essentially means that the generated trajectory should never meet some exceptional sets in the different manifolds, and those exceptional sets will be part of the respective lower-dimensional boundaries. Therefore it will be possible in Section 2.4 to prove that $\lambda^d(\mathbb{R}^d\setminus G)=0$, with λ^d being Lebesgue's measure in $\mathbb{R}^d$.

Hence we obtain a well-defined cocycle for every non-smooth system which satisfies several natural assumptions described in Section 2. Assuming the existence of an (physically relevant ergodic) invariant measure, we thus can apply the MET to obtain Lyapunov exponents for the corresponding non-smooth system. Since for complicated systems with practical relevance it is in general almost impossible to calculate or estimate the Lyapunov exponents directly, they often have to be calculated numerically along the lines described above, cf. [4, 6, 19, 23, 38] for examples. So the next basic question is:

Do these exponents correctly reflect the dynamical behaviour of the non-smooth system ?

It is the purpose of Section 3 to give a positive answer to this question in case of the pendulum with dry friction introduced above, so that one can also be confident for more complicated systems. We start in Section 3.1 by explaining where the semiflow for (1.2) comes from, whereas in Section 3.2 we verify the general assumptions imposed in Section 2 in case of the pendulum with dry friction. In Section 3.3 we first collect known results from [9, 10] concerning the existence of periodic solutions of (1.2) which we also want to summarize here briefly. In the most interesting resonant case $\eta=1$ the situation is as follows: for $\gamma\in[0,1]$ exactly the equilibria $x_0\in[\gamma-1,-\gamma+1]$ of (1.2) are the periodic solutions. (Note that it is not possible for a forced smooth equation $\ddot{x}+g(x,\dot{x})=p(t)$ to have a fixed point, whereas here we have a whole interval.) If $\gamma\in]1,4/\pi[$, then a unique periodic solution x_γ appears, it is non-trivial and stable, and of a very special form: there are $\tau_1^\gamma\in]0,\pi/2[$ and $\tau_2^\gamma\in]\pi/2,\pi[$ such that x_γ is given through $x_\gamma(\tau)=-1+\gamma\sin\tau_1^\gamma$ for $\tau\in[0,\tau_1^\gamma]$, $\dot{x}_\gamma(\tau)>0$ for $\tau\in]\tau_1^\gamma,\tau_2^\gamma[$, $x_\gamma(\tau)=1-\gamma\sin\tau_1^\gamma$ for $\tau\in[\tau_2^\gamma,\pi]$, and $x_\gamma(\tau)=-x_\gamma(\tau+\pi)$ for $\tau\in\mathbb{R}$. Therefore the existence of x_γ is a related to the non-smoothness of the system, since x_γ has dead-zones in $[0,\tau_1^\gamma]$, $[\tau_2^\gamma,\pi+\tau_1^\gamma]$, $[\pi+\tau_2^\gamma,2\pi]$, and so on. Moreover, it is shown in [10] that no periodic solutions of (1.2) can exist for $\gamma\in]4/\pi,\infty[$. These results are extended in Section 3.3 by proving that in fact the interval of fixed points $[\gamma-1,-\gamma+1]$ for $\gamma\in[0,1]$ and the periodic solution x_γ for $\gamma\in]1,4/\pi[$ are asymptotically stable *uniformly* w.r.t. initial values in bounded sets and also uniformly w.r.t phase shifts of the forcing. Moreover, we know from [18] that for $\gamma\in]4/\pi,\infty[$ all solutions are unbounded, again uniformly w.r.t. initial values in bounded sets and phase shifts. So the dynamical behaviour of the resonant system is completely understood, and to answer the above question we have to see how this is reflected in the corresponding Lyapunov exponents.

As will be explained in Sections 3.4.2 and 3.4.3, it is a consequence of the stability results just mentioned that for $\dot{y}\in F(y)$ from (1.3), considered on $\mathbb{R}^2\times\mathbb{S}^1$, the ergodic measures are as follows: if $\gamma\in[0,1]$, then the only ergodic measures are the Dirac measures δ_{x_0} with $x_0\in[\gamma-1,-\gamma+1]$, whereas it is uniform distribution on the graph of x_γ in case that $\gamma\in]1,4/\pi[$. Moreover, there are no invariant measures at all for $\gamma\in]4/\pi,\infty[$. Therefore we apply the MET with the cocycle constructed and these ergodic measures to obtain as Lyapunov exponents $-\infty,0,0$ for $\gamma\in[0,1]$ resp. $-\infty,\pi^{-1}\ln|\cos(\tau_2^\gamma-\tau_1^\gamma)|,0$ for $\gamma\in]1,4/\pi[$, whereas for $\gamma\in]4/\pi,\infty[$ the MET is not applicable. It is clear that for $\gamma\in[0,4/\pi[$ these exponents correctly reflect the dynamical behaviour of the underlying resonant system, and they also agree with the values which are numerically found, cf. [23]. Moreover, it should be remarked that the exponents are continuous at $\gamma=1$, cf. Remark 3.41.

It is of course true that in fact this gives nothing new, since the respective long-term behaviour of the system is already know. But at least it serves as a hint that for predictions of the dynamical behaviour one can rely on Lyapunov exponents also in non-smooth systems like (1.2).

In the context of (1.2) another related basic question is

Does system (1.2) show "chaotic behaviour" ?

In smooth systems it is argued that positive Lyapunov exponents indicate chaotic behaviour" (we don't make this term more precise), so that we are interested whether such positive exponents will also appear for (1.2), not especially at resonance. By analysis of the cocycle we will show in Section 3.4.1 that in fact all upper Lyapunov exponents $\lambda^+(y_0,y)=\limsup_{t\to\infty}\left(\frac{1}{t}\ln|T(t,y_0)y|\right)$ are *non-positive*, i.e. whenever the MET is applicable, none of the exponents will be larger than zero. Since we argued above that application of the MET to the constructed cocycle will yield Lyapunov exponents which correctly reflect the dynamical behaviour, we thus conclude that the pendulum with dry friction (1.2) *does not* show chaotic behaviour. This result is not restricted to the special forcing $\gamma\sin(\eta\tau)$, but carries over to all forcings which lead to a well-defined cocycle.

The last Section 4 is an appendix, where we first collect some results from ergodic theory in Section 4.1, whereas in Section 4.2 we investigate the boundedness of solutions to differential inclusions. Finally, Section 4.3 contains some purely technical lemmas.

Of course there are several questions left open concerning Lyapunov exponents for non-smooth systems, and also concerning (1.2). It is a main point to clarify the existence of invariant measures under some general assumptions, whereas in concrete examples this problem often will reduce to finding forward invariant sets, since the semiflow induced by differential inclusion often is continuous in "dissipative" cases. A much more difficult task is then so single out a "physically relevant" ergodic measure. Moreover, some relation should be made in the general context between Lyapunov exponents and dynamical properties of the system, like those mentioned in [11, p. 632] for smooth systems. With regard to (1.2) it has to be mentioned that it is known from [10] that a periodic solution exists also in the non-resonant case $\eta \neq 1$, but it is not known whether this solution is unique or uniformly attracting, or whether it has to have a dead zone.

To put our work into more context w.r.t. existing literature, we remark that in [20] there was already introduced a "formal system" like here in Section 2.1 to describe non-smooth systems as (1.2) and to define the corresponding Lyapunov exponents. Although correct, this approach had the drawback that some assumptions imposed cannot be verified for concrete systems, cf. the comments about this in front of Corollary 2.13 in Section 2.1 below. Besides [9, 10] already cited above, there are several papers by Reissig [30, 31, 32, 33, 34], which investigate dynamical properties of sundry friction-oscillators or non-smooth pendulums, cf. also [5, 14, 21, 24, 25, 29, 36]. These works often take a numerical or more heuristical point of view, or the dynamical analysis is restricted to the proof of the existence of periodic solutions. An interesting new direction can be found in the recent paper [12] where the existence of periodic solutions with arbitrary large period ("chaos") is proved in particular for non-smooth systems of the type $\ddot{x}+g(x,\dot{x})+\varepsilon_1 \mathrm{Sgn}\dot{x} \ni \varepsilon_2 p(\tau)$, provided that $\varepsilon_1, \varepsilon_2$ are small and that the unperturbed ODE $\ddot{x}+g(x,\dot{x})=0$ has a homoclinic orbit. Note that this result does not apply to (1.2), cf. Fig. 1(b) above. In addition, we don't want to consider the non-smooth effect as being a small perturbation.

Finally we have to apologize that for people working in ergodic theory perhaps the expression "non-smooth" is somehow misleading, since in the present paper we don't investigate return maps which are differentiable outside some exceptional set, cf. the standard reference [16]. In fact one can try to reduce systems like (1.2) to investigating Poincaré-maps, but most probably it will turn out be hard to verify rigorously the required properties of the return maps in more complicated examples. The paper is organized as follows:

2 Lyapunov Exponents – General Theory

In this section we will define a canonical cocycle for non-smooth systems which satisfy several assumptions. By application of the MET Theorem 4.3 we thus will obtain Lyapunov exponents for such systems.

To motivate the assumptions which will be needed, we first have a look back on (1.3) from the introduction. We already noted there that (1.3) implies $\dot{y}=f_+(y)$ in $\{y\in\mathbb{R}^3: v>0\}$ and $\dot{y}=f_-(y)$ in $\{y\in\mathbb{R}^3: v<0\}$, and we will see later that also the surface $\{y\in\mathbb{R}^3: v=0\}$ may be partitioned into finitely many parts, according to which ODE $\dot{y}=f(y)$ is governing the flow of the inclusion $\dot{y}\in F(y)$, at least for some time, when starting in this surface. So we may summarize what has to be expected for a non-smooth system like (1.3):

There are manifolds $M_k\subset\mathbb{R}^d$ for $k\in I$, I being some finite index set, and a manifold $M_\infty\subset\mathbb{R}^d$ together with C^1-functions f_k and f_∞ acting as right-hand sides, such that the semiflow is $\varphi^t(y_0)=\Phi_{f_k}(t,y_0)$ in $[0,t_1(y_0)[$, where for $g\in C^1(\mathbb{R}^d,\mathbb{R}^d)$, $\Phi_g(\cdot,y_0)$ denotes the global flow generated by $\dot{y}=g(y)$. In addition, $t_1(y_0)>0$ is the first switching time to a different manifold after starting in y_0. The manifold M_∞ will play a special rôle: it cannot be left if once reached, and thus $t_1(y_0)=\infty$ for $y_0\in M_\infty$. In the example of the pendulum with dry friction we have $M_\infty=\{y\in\mathbb{R}^3: v=0, \gamma-1\le x\le 1-\gamma\}$ for $\gamma\in[0,1]$, corresponding to the interval $[\gamma-1,1-\gamma]$ of stationary solutions, and we let $M_\infty=\emptyset$ for $\gamma\in]1,\infty[$. There are several natural assumptions to be made: if $y_0\in M_k$, then the trajectory should not leave M_k until the first switching time, i.e. $\varphi^t(y_0)\in M_k$ for $t\in[0,t_1(y_0)[$, and also the consistency condition $t_1(\varphi^t(y_0))=t_1(y_0)-t$ for $t\in[0,t_1(y_0)[$ should hold. Moreover, $\varphi(\cdot,y_0)$ can be expected to be continuous.

Since we want define a cocycle for non-smooth systems by imitating $T(t,y_0)=\partial_y\varphi^t(y_0)$ from the smooth case, we have to remark at this point that the requirements mentioned so far are not enough to advance further in this direction. To explain this, note that if e.g. $y_0\in M_k$ and $\xi_1(y_0)=\varphi^{t_1(y_0)}(y_0)\in M_l$, then $\varphi^t(y_0)=\Phi_{f_l}(t-t_1(y_0),\xi_1(y_0))$ for $t\in[t_1(y_0),t_1(y_0)+t_1(y_1)[$ by the above assumptions, because we have to solve $\dot{y}=f_l(y)$ and start in $\xi_1(y_0)$ at time $t_1(y_0)$. As we don't know anything so far about differentiability of $t_1(\cdot)$ and $\xi_1(\cdot)$, we have to include further restrictions.

In concrete non-smooth examples it cannot be expected to have $t_1\in C^1(\mathbb{R}^d)$, as may be seen from the following Fig. 2, where typical example trajectories of the autonomous system (1.3) of the pendulum with dry friction for $\gamma\in]0,1[$ are shown.

Fig. 2: Example trajectories of (1.3) for $\gamma\in]0,1[$.

In Fig. 2, τ resp. x are displayed in vertical resp. horizontal direction, whereas the v-axis is perpendicular, i.e. the plane $\{v=0\}$ coincides with the plane of the paper with $\{v>0\}$ lying behind. Since the right-hand side of (1.3) has a 1 in the third line, the semiflow is of the form $\varphi^t(y_0)=(*,*,\tau_0+t)$ for $y_0=(\tau_0,x_0,v_0)\in\mathbb{R}^3$. Note that we don't take time $\mathrm{mod}(2\pi/\eta)$ for the moment being, because this simplifies notation and diagrams. Although we didn't specify precisely the different manifolds for the system (1.3) up to now, comparing both displayed trajectories it is clear that $t_1(\cdot)$ is discontinuous at y_0, since some y close to y_0 will have a considerably smaller first switching time than y_0. The above figure also makes clear that we can do better if the reference trajectory of y_0 does not meet some "exceptional sets" E_k, which for the present purpose of motivation may be thought of to be both boundary curves $\{y\in\mathbb{R}^3: v=0, x=b^{\pm}(\tau)\}$; in fact these exceptional sets are smaller. The corresponding requirement for the first arrival point $\xi_1(y_0)$ at a different manifold M_l after start in $y_0\in M_k$ is that there should be a M_k-neighborhood of y_0 which is sent to M_l by $\xi_1(\cdot)$. Therefore, on every

$$\begin{aligned} D_k := \{\, & y\in M_k : \varphi^t(y)\notin E_k\ \forall t\in[0,t_1(y)], \text{ and} \\ & l\in I\cup\{\infty\}, \xi_1(y)\in M_l \Rightarrow \forall\, t\in[0,t_1(y)[\ \exists\,\varepsilon>0 : \xi_1(B_\varepsilon(\varphi^t(y))\cap M_k)\subset M_l\}, \end{aligned} \tag{2.1}$$

it makes sense to assume $t_1(\cdot)$ to be differentiable.

Here are some additional remarks related to the definition of D_k: we want to find a "good" set $G\subset\mathbb{R}^d$ of initial values y_0 such that for trajectories starting in y_0 the cocycle $T(t,y_0)$ can be constructed meaningfully. So an additional requirement is to have $t_i(y_0)\to\infty$ as $i\to\infty$ (where the $t_i(y_0)$ are the consecutive switching times) in case that the trajectory does not arrive in M_∞, since the positive real axis has to be partitioned by the switching times to define $T(t,y_0)$ for all $t\in[0,\infty[$. In case that a trajectory arrives at M_∞, no further switching will take place. The set G will simply be obtained through the requirement that if a trajectory arrives at the manifold M_k at some time $t_i(y_0)$, then the arrival point should be already in D_k. Since we need to have G forward invariant in order to verify the cocycle property (1.4), it is thus not enough to write, in the definition of D_k, "$\varphi^{t_1(y_0)}(y_0)\notin E_k$" instead of "$\varphi^t(y)\notin E_k$ for all $t\in[0,t_1(y)]$", and for the same reason we have the second condition instead of just "$\exists\,\varepsilon>0:\xi_1(B_\varepsilon(y)\cap M_k)\subset M_l$". In concrete examples, both the condition on $\xi_1(\cdot)$ not to separate neighborhoods and the condition on the semiflow not to meet E_k in (2.1) will be related somehow, but in general they are not the same as it may be seen in the example of the pendulum with dry friction in Section 3.

2.1 Preliminaries

After the preceding motivation, we will impose the following hypotheses: We are given a partition $\mathbb{R}^d=\left(\bigcup_{k=1}^K M_k\right)\cup M_\infty$ of $\mathbb{R}^d$ in mutually disjoint C^1-manifolds. Moreover, we suppose that we have an exceptional set $E_k\subset\overline{M_k}^{\mathbb{R}^d}$ and a given function $f_k\in C^1(\mathbb{R}^d,\mathbb{R}^d)$ for every $k\in I\cup\{\infty\}:=\{1,\ldots,K\}\cup\{\infty\}$. Let $\varphi:[0,\infty[\times\mathbb{R}^d\to\mathbb{R}^d$ be a measurable semiflow, i.e. $(t,y)\mapsto\varphi^t(y)=\varphi(t,y)$ is (Borel-) measurable, $\varphi^0=\mathrm{id}_{\mathbb{R}^d}$ and $\varphi^{t+s}=\varphi^t\circ\varphi^s$ for all $t,s\in[0,\infty[$. Let the function $t_1(\cdot):\mathbb{R}^d\to\,]0,\infty]$ be given. In the case that $M_\infty\neq\emptyset$ we fix some $y_\infty\in M_\infty$ and define

$$\xi_0(y)=y\,,\ y\in\mathbb{R}^d,\quad \xi_1(y)=\begin{cases}\varphi^{t_1(y)}(y) & : \ y\in\mathbb{R}^d\setminus M_\infty\\ y_\infty & : \ y\in M_\infty\end{cases}\quad\text{and}$$
$$\xi_{i+1}(y)=\xi_1(\xi_i(y))\quad\text{for}\quad i\in\mathbb{N}, y\in\mathbb{R}^d \tag{2.2}$$

as well as

$$t_0(y)=0,\quad t_{i+1}(y)=t_1(\xi_i(y))+t_i(y)\geq t_i(y)\quad\text{for}\quad i\in\mathbb{N}, y\in\mathbb{R}^d. \tag{2.3}$$

We assume that

(A1) $M_\infty = \{y \in \mathbb{R}^d : t_1(y) = \infty\}$,

hence $t_{i+1}(y) = \infty$ iff $\xi_i(y) \in M_\infty$ or $t_i(y) = \infty$. We also assume that

(A2) $y \in \mathbb{R}^d \setminus M_\infty$, $t \in [0, t_1(y)[\Rightarrow t_1(\varphi^t(y)) + t = t_1(y)$,

(A3) $k \in I \cup \{\infty\}$, $y \in M_k \Rightarrow \varphi^t(y) \in M_k$ for $t \in [0, t_1(y)[$,

(A4) $\varphi(\cdot, y)$ is continuous for $y \in \mathbb{R}^d$,

(A5) $\varphi^t(y) = \Phi_{f(\cdot,y)}(t, y)$ for $y \in \mathbb{R}^d$ and $t \in [0, t_1(y)[$,

(A6) $y \in \mathbb{R}^d$, $t_i(y) < \infty$ for all $i \in \mathbb{N} \Rightarrow 0 = t_0(y) < t_1(y) < t_2(y) \ldots \to \infty$,

(A7) $D_\infty := \{y \in M_\infty : \varphi^t(y) \notin E_\infty \; \forall t \in [0, \infty[\} = M_\infty \setminus E_\infty \neq \emptyset$, if $M_\infty \neq \emptyset$, and

(A8) $t_1 \in C^1(D_k)$ for $k \in I$, with D_k from (2.1).

We start with some elementary relationships between these objects.

Lemma 2.1 *Let $(A1) - (A7)$ be satisfied. Then the following assertions hold.*

(a) For $i, j \in \mathbb{N}_0$ and $y \in \mathbb{R}^d$ we have $\xi_{i+j}(y) = \xi_i(\xi_j(y))$.

(b) For $i \in \mathbb{N}_0$, $j \in \mathbb{N}$ and $y \in \mathbb{R}^d$ we have $t_j(\xi_i(y)) + t_i(y) = t_{i+j}(y)$.

(c) For $i \in \mathbb{N}_0$ and $y \in \mathbb{R}^d$ we have

$$\xi_i(y) = \begin{cases} \varphi^{t_i(y)}(y) & : \quad t_i(y) < \infty \\ y_\infty & : \quad t_i(y) = \infty \end{cases}.$$

(d) For $i \in \mathbb{N}$, $j \in \mathbb{N}_0$, $y \in \mathbb{R}^d$ with $t_j(y) < \infty$ and $t \in [t_j(y), t_{j+1}(y)[$ we have $\xi_i(\varphi^t(y)) = \xi_{i+j}(y)$.

(e) For $i \in \mathbb{N}_0$, $y \in \mathbb{R}^d$ with $t_i(y) < \infty$ and $t \in [t_i(y), t_{i+1}(y)[$ we have $t_j(\varphi^t(y)) + t = t_{i+j}(y)$ for $j \in \mathbb{N}$.

(f) For $i \in \mathbb{N}_0$ and $y \in \mathbb{R}^d$ with $t_i(y) < \infty$ there exists $k \in I \cup \{\infty\}$ such that $\varphi([t_i(y), t_{i+1}(y)[, y) \subset M_k$.

(g) For $i \in \mathbb{N}_0$, $y \in \mathbb{R}^d$ with $t_i(y) < \infty$ and $t \in [t_i(y), t_{i+1}(y)[$

$$\varphi^t(y) = \Phi_{f(\cdot, \varphi^{t_i(y)}(y))}\left(t - t_i(y), \varphi^{t_i(y)}(y)\right). \tag{2.4}$$

If also $t_{i+1}(y) < \infty$, then (2.4) holds for $t = t_{i+1}(y)$, too.

(h) For $i \in \mathbb{N}$ and $y \in \mathbb{R}^d$: $t_i(y) < \infty \iff t_j(y) < \infty$ for all $j = 1, \ldots, i \iff \xi_j(y) \notin M_\infty$ for all $j = 0, \ldots, i-1$.

Proof: Ad (a): This is obvious by (2.2). Ad (b): For $j = 1$ this is just (2.3). If (b) holds for j, then by (a) $t_{j+1}(\xi_i(y)) + t_i(y) = t_1(\xi_j(\xi_i(y))) + t_j(\xi_i(y)) + t_i(y) = t_1(\xi_{i+j}(y)) + t_{i+j}(y) = t_{i+j+1}(y)$. Ad (c): For $i = 0, 1$ the claim holds by (2.2) and (2.3). For $i \Rightarrow i+1$ we note that in the case $t_{i+1}(y) < \infty$ we obtain due to (2.2), (c) for i and (2.3), $\xi_{i+1}(y) = \xi_1(\xi_i(y)) = \varphi^{t_1(\xi_i(y))}(\varphi^{t_i(y)}(y)) = \varphi^{t_{i+1}(y) - t_i(y)}(\varphi^{t_i(y)}(y)) = \varphi^{t_{i+1}(y)}(y)$. On the other hand, if $t_{i+1}(y) = \infty$, then $\xi_i(y) \in M_\infty$ or $t_i(y) = \infty$, and the latter also implies $\xi_i(y) = y_\infty \in M_\infty$ by induction hypotheses. Thus in every case $\xi_{i+1}(y) = \xi_1(\xi_i(y)) = y_\infty$ by definition. Ad (d): We first consider the case $j = 0$, i.e. $t \in [0, t_1(y)[$. Suppose first that $t_1(y) < \infty$. Then by $(A2)$ we have $\xi_1(\varphi^t(y)) = \varphi^{t_1(\varphi^t(y))}(\varphi^t(y)) = \varphi^{t_1(y)-t}(\varphi^t(y)) =$

$\varphi^{t_1(y)}(y)=\xi_1(y)$, and hence $\xi_i(\varphi^t(y))=\xi_i(y)$ for $i\in\mathbb{N}$ by (a). In case that $t_1(y)=\infty$ we have $y\in M_\infty$, hence $\xi_1(y)=y_\infty$, and also $\varphi^t(y)\in M_\infty$ by $(A3)$, and thus $\xi_1(\varphi^t(y))=y_\infty=\xi_1(y)$, and again the claim for arbitrary $i\in\mathbb{N}$ follows from (a). So (d) holds for $j=0$ and all $i\in\mathbb{N}$. If $j\in\mathbb{N}$ is arbitrary and $t\in[t_j(y),t_{j+1}(y)[$ with $t_j(y)<\infty$, we first consider the case $t_{j+1}(y)<\infty$. Then $\bar{y}=\xi_j(y)$ has $t_1(\bar{y})=t_{j+1}(y)-t_j(y)$ by (b), hence $s=t-t_j(y)\in[0,t_1(\bar{y})[$. So we obtain by the part already proven and by (a), $\xi_i(\varphi^t(y))=\xi_i(\varphi^s(\bar{y}))=\xi_i(\bar{y})=\xi_{i+j}(y)$. On the other hand, if $t_{j+1}(y)=\infty$ and $\bar{y}$ is defined as before, then $t_j(y)<\infty$ implies $\bar{y}\in M_\infty$. Since by (c) and $(A3)$ also $\varphi^t(y)=\varphi^{t-t_j(y)}(\bar{y})\in M_\infty$, we again arrive at $\xi_1(\varphi^t(y))=y_\infty=\xi_1(\bar{y})=\xi_{j+1}(y)$, and the claim follows from (a). Ad (e): We first prove the claim for $i=0$ and all $j\in\mathbb{N}$. For that, the case $j=1$ is just $(A2)$. To obtain $j\Rightarrow j+1$ we first assume $t_1(y)<\infty$. Then (d) and the assumption for j imply $t_{j+1}(\varphi^t(y))+t=t_1(\xi_j(\varphi^t(y)))+t_j(\varphi^t(y))+t=t_1(\xi_j(y))+t_j(y)=t_{j+1}(y)$. If $t_1(y)=\infty$, then $t_j(y)=\infty$ for all $j\geq 1$, and also $y\in M_\infty$, hence $\varphi^t(y)\in M_\infty$ for $t\in[0,\infty[$, and thus $t_j(\varphi^t(y))=\infty$ for $j\in\mathbb{N}$. Hence (e) holds for $i=0$ and all $j\in\mathbb{N}$. If $i\in\mathbb{N}_0$, $j\in\mathbb{N}$, $y\in\mathbb{R}^d$ with $t_i(y)<\infty$ and $t\in[t_i(y),t_{i+1}(y)[$ are arbitrary, we first consider the case $t_{i+1}(y)<\infty$. Then $\bar{y}=\xi_i(y)$ has $t_1(\bar{y})=t_{i+1}(y)-t_i(y)$ by (2.3), hence $\bar{t}=t-t_i(y)\in[0,t_1(\bar{y})[$ and $\varphi^{\bar{t}}(\bar{y})=\varphi^t(y)$ by (c). So by the part already proven and (b), $t_j(\varphi^t(y))+t=t_j(\varphi^{\bar{t}}(\bar{y}))+t=t_j(\bar{y})-\bar{t}+t=t_j(\xi_i(y))+t_i(y)=t_{i+j}(y)$. On the other hand, if $t_{i+1}(y)=\infty$, then $t_{i+j}(y)=\infty$ for all $j\in\mathbb{N}$, and as in (d), $\varphi^t(y)\in M_\infty$, hence $t_j(\varphi^t(y))=\infty$ for $j\in\mathbb{N}$. Ad (f) & (g): Again we first assume that $t_{i+1}(y)<\infty$. With $\bar{y}=\xi_i(y)$ and $\bar{t}=t-t_i(y)$ we have $[t_i(y),t_{i+1}(y)[=[0,t_1(\bar{y})[+t_i(y)$. Hence $\varphi([t_i(y),t_{i+1}(y)[,y)=\varphi([0,t_1(\bar{y})[,\bar{y})$, and thus (f) is a consequence of $(A3)$. This also yields, because of $(A5)$,

$$\varphi^t(y)=\varphi^{\bar{t}}(\bar{y})=\Phi_{f(\cdot,\bar{y})}(\bar{t},\bar{y})=\Phi_{f(\cdot,\varphi^{t_i(y)}(y))}\Big(t-t_i(y),\varphi^{t_i(y)}(y)\Big).$$

Moreover, $(A4)$ and the continuity of every $\Phi_g(\cdot,z)$ show that (2.4) also holds for $t=t_{i+1}(y)$. In the case that $t_{i+1}(y)=\infty$, again $\bar{y}=\xi_i(y)=\varphi^{t_i(y)}(y)\in M_\infty$, hence $\varphi^t(y)=\varphi^{t-t_i(y)}(\bar{y})\in M_\infty$ for $t\in[t_i(y),\infty[$ by $(A3)$, so that we can choose $k=\infty$ in (f). Concerning (g), we have $\varphi^s(\bar{y})=\Phi_{f_\infty}(s,\bar{y})$ for $s\in[0,\infty[$ by $(A5)$. Consequently we take $s=t-t_i(y)$ to obtain the claim of (g) for $t_{i+1}(y)=\infty$. Ad (h): The first equivalence is because of $t_{p+1}(y)\geq t_p(y)$ for all $p\in\mathbb{N}_0$. Concerning the second, "$\Rightarrow$" follows from (2.3). "$\Leftarrow$": If $t_{j+1}(y)=\infty$ for $j\in\{0,\ldots,i-1\}$, then also $t_j(y)=\infty$ by assumption and (2.3). Repeating this argument, we arrive at $t_1(y)=\infty$, contradicting $y=\xi_0(y)\notin M_\infty$. Therefore the proof of Lemma 2.1 is complete. □

For later reference we also state explicitly the obvious

Lemma 2.2 *Let $(A8)$ hold. Then for every $k\in I$ and $R>0$ there exists $T=T(k,R)>0$ such that $t_1(y)\leq T$ for $y\in\overline{B}_R(0)\cap D_k$.*

Now we make the following

General Hypothesis: Throughout the rest of Section 2 we always will assume that $(A1)-(A8)$ from above are satisfied. Only additional assumptions will be mentioned in the results.

Our next step is to introduce the "good" set G of initial values y_0 for which the cocycle $T(t,y_0)$ can be defined. As already mentioned in the introductory paragraph before Section 2.1, G will be obtained by taking those y_0 for which $\xi_i(y_0)\in M_k$ already implies $\xi_i(y_0)\in D_k$. Hence we have to distinguish between initial values y_0 which lead to trajectories never reaching $D_\infty\subset M_\infty$, cf. Lemma 2.3, and those whose trajectory arrive at D_∞, cf. Lemma 2.4. In the latter case we also have to pay attention at which step i we first have $\xi_i(y_0)\in D_\infty$.

Lemma 2.3 *Let $G^0_{<\infty}=\mathbb{R}^d$, and for $i\in\mathbb{N}_0$*

$$G^{i+1}_{<\infty}=G^i_{<\infty}\cap\{y\in\mathbb{R}^d:t_{i+1}(y)<\infty\,\text{and}\,\xi_i(y)\in M_k\,\text{for}\,k\in I\Rightarrow\xi_i(y)\in D_k\}.$$

Then $G^{i+1}_{<\infty} \subset G^i_{<\infty}$ *and*

$$\begin{aligned} G^i_{<\infty} = \{ & y \in \mathbb{R}^d : t_i(y) < \infty, \text{and} \\ & 0 \le j \le i-1, \xi_j(y) \in M_k \text{ for } k \in I \Rightarrow \varphi^t(y) \notin E_k \text{ for } t \in [t_j(y), t_{j+1}(y)] \text{ and} \\ & \xi_{j+1}(y) \in M_l \text{ for } l \in I \cup \{\infty\} \Rightarrow \forall\, t \in [t_j(y), t_{j+1}(y)[\ \exists\, \varepsilon > 0 : \xi_1(B_\varepsilon(\varphi^t(y)) \cap M_k) \subset M_l \}. \end{aligned}$$

Moreover,

$$\begin{aligned} G_{<\infty} &= \bigcap_{i=0}^{\infty} G^i_{<\infty} \\ &= \{ y \in \mathbb{R}^d : t_i(y) < \infty \text{ for all } i \in \mathbb{N}, \text{and} \\ &\quad\ i \in \mathbb{N}_0, k \in I, \xi_i(y) \in M_k \Rightarrow \varphi^t(y) \notin E_k \text{ for } t \in [t_i(y), t_{i+1}(y)] \text{ and} \\ &\quad\ l \in I, \xi_{i+1}(y) \in M_l \Rightarrow \forall\, t \in [t_i(y), t_{i+1}(y)[\ \exists\, \varepsilon > 0 : \xi_1(B_\varepsilon(\varphi^t(y)) \cap M_k) \subset M_l \} \end{aligned}$$

is forward invariant, i.e. $\varphi^t(G_{<\infty}) \subset G_{<\infty}$ *for all* $t \in [0, \infty[$.

Proof: The characterizations of $G^i_{<\infty}$ and $G_{<\infty}$ are immediate. To show the invariance of the latter, let $y \in G_{<\infty}$ and $\bar{y} = \varphi^t(y)$. Then $t \in [t_j(y), t_{j+1}(y)[$ for some $j \in \mathbb{N}_0$ by (*A6*). Hence by Lemma 2.1(e), $t_p(\bar{y}) = t_{j+p}(y) - t < \infty$ for $p \in \mathbb{N}$. If $i \in \mathbb{N}_0$, then choose $k \in I$ with $\xi_i(\bar{y}) \in M_k$ and fix $s \in [t_i(\bar{y}), t_{i+1}(\bar{y})]$. In case that $i = 0$ we have $s + t \in [t, t_{j+1}(y)] \subset [t_j(y), t_{j+1}(y)]$. Since $\varphi^t(y) = \xi_0(\bar{y}) \in M_k$, Lemma 2.1(f) and the mutual disjointness of the manifolds implies that also $\xi_j(y) \in M_k$, and hence $\varphi^s(\bar{y}) = \varphi^{s+t}(y) \notin E_k$, because of $y \in G_{<\infty}$. On the other hand, if $i \in \mathbb{N}$ we obtain $s \in [t_{i+j}(y), t_{i+j+1}(y)] - t$ and by means of Lemma 2.1(d), $\xi_{i+j}(y) = \xi_i(\bar{y}) \in M_k$. Since $y \in G_{<\infty}$ we find again $\varphi^s(\bar{y}) = \varphi^{s+t}(y) \notin E_k$. The second condition is checked in the same way, and thus we have shown $\bar{y} \in G_{<\infty}$. □

Lemma 2.4 *For* $i \in \mathbb{N}_0$ *define*

$$\begin{aligned} G^i_\infty &= G^i_{<\infty} \cap \{ y \in \mathbb{R}^d : \xi_i(y) \in D_\infty \} \\ &= \{ y \in \mathbb{R}^d : t_i(y) < \infty, \text{and} \\ &\quad\ 0 \le j \le i-1, \xi_j(y) \in M_k \text{ for } k \in I \Rightarrow \varphi^t(y) \notin E_k \text{ for } t \in [t_j(y), t_{j+1}(y)] \text{ and} \\ &\quad\ \xi_{j+1}(y) \in M_l \text{ for } l \in I \cup \{\infty\} \Rightarrow \forall\, t \in [t_j(y), t_{j+1}(y)[\ \exists\, \varepsilon > 0 : \xi_1(B_\varepsilon(\varphi^t(y)) \cap M_k) \subset M_l, \text{and} \\ &\quad\ \varphi^t(y) \in D_\infty \text{ for } t \in [t_i(y), \infty[\}. \end{aligned}$$

Then $G^i_\infty \cap G^p_\infty = \emptyset$ *for* $i \ne p$, *and* $G_\infty = \bigcup_{i \in \mathbb{N}_0} G^i_\infty$ *is forward invariant.*

Proof: Similar to the proof of Lemma 2.3. □

We also formulate explicitly two facts which were obtained in the above proof.

Corollary 2.5 *For* $i \in \mathbb{N}$

$$\xi_{i-1}(G^i_{<\infty}) \subset G^1_{<\infty}.$$

Proof: This follows from an inspection of the proof to Lemma 2.4: For $y \in G^i_{<\infty}$ and $t = t_{i-1}(y)$ in this proof, we have $\bar{y} = \xi_{i-1}(y)$ and $p = i-1$, hence $\bar{y} \in G^{i-p}_{<\infty} = G^1_{<\infty}$. □

Corollary 2.6 *For* $i \in \mathbb{N}_0$, $y \in G^i_\infty$, $p \in \{0, \dots, i\}$, *and* $t \in [t_p(y), t_{p+1}(y)[$

$$\varphi^t(y) \in G^{i-p}_\infty.$$

Proof: Again this follows from the proof of Lemma 2.4: If $p=i$, then $t \geq t_i(y)$, and it was shown that in this case $\varphi^t(y) \in G_\infty^0$. On the other hand, if $p \in \{0,\ldots,i-1\}$, then we obtained above $\varphi^t(y) = \bar{y} \in G_\infty^{i-p}$. $\square$

The next corollary serves to introduce the set G of "good" initial values, for which later in Theorem 2.14 the cocycle can be defined.

Corollary 2.7 *The set*

$$G = G_{<\infty} \cup G_\infty$$

is forward invariant, with $G_{<\infty}$ from Lemma 2.3 and G_∞ from Lemma 2.4.

Now we turn to the investigation of the differentiability of $t_i(\cdot)$ and $\xi_i(\cdot)$ on $G_{<\infty}^i$.

Lemma 2.8 *For $i \in \mathbb{N}$ and $k \in I$*

$$t_i \in C^1(G_{<\infty}^i \cap M_k, \mathbb{R}) \quad \text{and} \quad \xi_i \in C^1(G_{<\infty}^i \cap M_k, \mathbb{R}^d). \tag{2.5}$$

In particular, $t_i \in C^1(G_\infty^i \cap M_k, \mathbb{R})$ and $\xi_i \in C^1(G_\infty^i \cap M_k, \mathbb{R}^d)$.

Proof: The claim will be proved by induction. For that, we will also show that

$$\begin{aligned} & i \in \mathbb{N}, \quad y_0 \in G_{<\infty}^i \cap M_k \text{ for } k \in I, \quad \xi_i(y_0) \in M_l \text{ for } l \in I \cup \{\infty\}, \quad \eta > 0 \\ & \Longrightarrow \exists \rho > 0 : \xi_i(B_\rho(y_0) \cap G_{<\infty}^i \cap M_k) \subset B_\eta(\xi_i(y_0)) \cap M_l \end{aligned} \tag{2.6}$$

is valid, and we will use the abbreviation $\Sigma_{i,k} = G_{<\infty}^i \cap M_k$. If $i=1$, then we have $\Sigma_{1,k} = D_k$ for $k \in I$ by definition, and hence $t_1 \in C^1(\Sigma_{1,k}, \mathbb{R})$ by $(A7)$. Thus for $y_0 \in \Sigma_{1,k}$ we find $\delta_1 > 0$ and $\tau_1 \in C^1(B_{\delta_1}(y_0), \mathbb{R})$ such that $\tau_1(y) = t_1(y)$ for $y \in B_{\delta_1}(y_0) \cap \Sigma_{1,k}$. Hence $\phi_1 \in C^1(B_{\delta_1}(y_0), \mathbb{R}^d)$, where $\phi_1(y) = \Phi_{f_k}(\tau_1(y), y)$. Moreover, by definition of $\Sigma_{1,k}$ and by (2.4) we have $\phi_1(y) = \xi_1(y)$ for $y \in B_{\delta_1}(y_0) \cap \Sigma_{1,k}$, and thus $\xi_1 \in C^1(\Sigma_{1,k}, \mathbb{R}^d)$. In particular, if $\eta > 0$ is given, then $\phi_1(B_{\rho_1}(y_0)) \subset B_\eta(\phi_1(y_0))$ for some $\rho_1 \in (0, \delta_1]$. If $l \in I \cup \{\infty\}$ is such that $\xi_1(y_0) \in M_l$, then by definition of $G_{<\infty}^1$ there exists $\rho_2 > 0$ with $\xi_1(B_{\rho_2}(y_0) \cap M_k) \subset M_l$. Hence with $\rho = \rho_1 \wedge \rho_2$ we obtain $\xi_1(B_\rho(y_0) \cap \Sigma_{1,k}) \subset B_\eta(\xi_1(y_0)) \cap M_l$, and therefore (2.5) and (2.6) for $i=1$.

If (2.5) and (2.6) already hold for i and $y_0 \in \Sigma_{i+1,k} \subset \Sigma_{i,k}$ as well as $\eta > 0$ are fixed, then by Corollary 2.5, $\bar{y}_0 := \xi_i(y_0) \in G_{<\infty}^1 \cap M_l = \Sigma_{1,l}$ for some $l \in I$ and $\xi_1(\bar{y}_0) = \xi_{i+1}(y_0) \in M_p$ for some $p \in I \cup \{\infty\}$. Hence, by assumption, there are $\rho_1 > 0$, $\bar{\tau}_1 \in C^1(B_{\rho_1}(\bar{y}_0), \mathbb{R})$ and $\bar{\phi}_1 \in C^1(B_{\rho_1}(\bar{y}_0), \mathbb{R}^d)$ such that $\bar{\tau}_1(\bar{y}) = t_1(\bar{y})$ and $\bar{\phi}_1(\bar{y}) = \xi_1(\bar{y})$ for $\bar{y} \in B_{\rho_1}(\bar{y}_0) \cap \Sigma_{1,l}$. Moreover, by (2.6) applied to $i=1$ and $\bar{y}_0$, we may suppose that ρ_1 is chosen small enough to ensure also $\xi_1(B_{\rho_1}(\bar{y}_0) \cap \Sigma_{1,l}) \subset B_\eta(\xi_1(\bar{y}_0)) \cap M_p$. Next, since $y_0 \in \Sigma_{i,k}$, by assumption we find $\rho_2 > 0$, $\tau_i \in C^1(B_{\rho_2}(y_0), \mathbb{R})$ and $\phi_i \in C^1(B_{\rho_2}(y_0), \mathbb{R}^d)$ satisfying $\tau_i(y) = t_i(y)$ and $\phi_i(y) = \xi_i(y)$ for $y \in B_{\rho_2}(y_0) \cap \Sigma_{i,k}$, and we can assume that $\phi_i(y) \in B_{\rho_1}(\phi_i(y_0)) = B_{\rho_1}(\bar{y}_0)$ for $y \in B_{\rho_2}(y_0)$. By (2.6) for i we find a $\rho_3 > 0$ with $\xi_i(B_{\rho_3}(y_0) \cap \Sigma_{i,k}) \subset B_{\rho_1}(\bar{y}_0) \cap M_l$. Let $\delta_{i+1} = \rho_2 \wedge \rho_3 > 0$, $\tau_{i+1}(y) = \bar{\tau}_1(\phi_i(y)) + \tau_i(y)$, and $\phi_{i+1}(y) = \bar{\phi}_1(\phi_i(y))$ for $y \in B_{\delta_{i+1}}(y_0)$. Then $\tau_{i+1}(y) = t_{i+1}(y)$ and $\phi_{i+1}(y) = \xi_{i+1}(y)$ in $B_{\delta_{i+1}}(y_0) \cap \Sigma_{i+1,k}$. Furthermore, $\tau_{i+1} \in C^1(B_{\delta_{i+1}}(y_0), \mathbb{R})$ and $\phi_{i+1} \in C^1(B_{\delta_{i+1}}(y_0), \mathbb{R}^d)$. Finally, let $\rho = \rho_3 > 0$. By Corollary 2.5 we have

$$\xi_i(B_\rho(y_0) \cap \Sigma_{i+1,k}) \subset \xi_i(G_{<\infty}^{i+1}) \subset G_{<\infty}^1,$$

and we also obtain

$$\xi_i(B_\rho(y_0) \cap \Sigma_{i+1,k}) \subset \xi_i(B_{\rho_3}(y_0) \cap \Sigma_{i,k}) \subset B_{\rho_1}(\bar{y}_0) \cap M_l,$$

and thus $\xi_i(B_\rho(y_0) \cap \Sigma_{i+1,k}) \subset B_{\rho_1}(\bar{y}_0) \cap \Sigma_{1,l}$ what in turn yields

$$\xi_{i+1}(B_\rho(y_0) \cap \Sigma_{i+1,k}) = \xi_1(\xi_i(\ldots)) \subset \xi_1(B_{\rho_1}(\bar{y}_0) \cap \Sigma_{1,l}) \subset B_\eta(\xi_{i+1}(y_0)) \cap M_p,$$

i.e. (2.6) for $i+1$. $\square$

Corollary 2.9 *For $i\in\mathbb{N}_0$, $k\in I$, $y_0\in G^{i+1}_{<\infty}\cap M_k$, and $t\in]t_i(y_0),t_{i+1}(y_0)[$ there exists $r>0$ such that the following holds:*

(a) If $y\in B_r(y_0)\cap G^{i+1}_{<\infty}\cap M_k$, then $t\in]t_i(y),t_{i+1}(y)[$.

(b) $\varphi^t(\cdot)\in C^1(B_r(y_0)\cap G^{i+1}_{<\infty}\cap M_k,\mathbb{R}^d)$.

Proof: Since $y_0\in\Sigma_{i+1,k}\subset\Sigma_{i,k}$, by Lemma 2.8 we find $\delta>0$, $\tau_i,\tau_{i+1}\in C^1(B_\delta(y_0),\mathbb{R})$ and $\phi_i\in C^1(B_\delta(y_0),\mathbb{R}^d)$ such that $\tau_i(y)=t_i(y)$ and $\phi_i(y)=\xi_i(y)$ for $y\in B_\delta(y_0)\cap\Sigma_{i,k}$ as well as $\tau_{i+1}(y)=t_{i+1}(y)$ for $y\in B_\delta(y_0)\cap\Sigma_{i+1,k}$. Moreover, if $\bar y_0=\xi_i(y_0)\in M_l$ for some $l\in I$, we may also assume that $\xi_i(B_\delta(y_0)\cap\Sigma_{i,k})\subset M_l$ by (2.6). Then by continuity of τ_i and τ_{i+1} there exists $r\in]0,\delta]$ with $|\tau_i(y)-t_i(y_0)|\le(t-t_i(y_0))/2$ and $|\tau_{i+1}(y)-t_{i+1}(y_0)|\le(t_{i+1}(y_0)-t)/2$ for $y\in B_r(y_0)$. Therefore (a) holds. If we define $\theta(y)=\Phi_{f_l}(t-\tau_i(y),\phi_i(y))$, then we obtain $\theta\in C^1(B_r(y_0),\mathbb{R}^d)$, and by Lemma 2.1 (g) & (c),

$$\varphi^t(y)=\Phi_{f(\cdot,\xi_i(y))}(t-t_i(y),\xi_i(y))=\Phi_{f_l}(t-\tau_i(y),\phi_i(y))=\theta(y)$$

for $y\in B_r(y_0)\cap\Sigma_{i+1,k}\subset B_r(y_0)\cap\Sigma_{i,k}$. □

Next we state a similar corollary for the case of $y_0\in G^i_\infty$, i.e. for the transition from $\mathbb{R}^3\setminus M_\infty$ to $D_\infty\subset M_\infty$ at time $t_i(y_0)$. Note in particular that only $y_0\in G^i_\infty$ is required here, compared to the assumption $y_0\in G^{i+1}_{<\infty}$ from Corollary 2.9, since we know that $t_{i+1}(y)=\infty$ for $y\in G^i_\infty$, and we thus only have to ensure that $t_i(\cdot)$ varies continuously.

Corollary 2.10 *For $i\in\mathbb{N}_0$, $k\in I\cup\{\infty\}$, $y_0\in G^i_\infty\cap M_k$, and $t\in]t_p(y_0),t_{p+1}(y_0)[$ for a suitable $p\in\{0,\dots,i\}$, there exists $r>0$ such that the following holds:*

(a) If $y\in B_r(y_0)\cap G^i_\infty\cap M_k$, then $t\in]t_p(y),t_{p+1}(y)[$.

(b) $\varphi^t(\cdot)\in C^1(B_r(y_0)\cap G^i_\infty\cap M_k,\mathbb{R}^d)$.

Proof: Similar to the proof of Corollary 2.9, by Lemma 2.8. □

2.2 Transition of the linearization at switching times

In this section we derive the form of the transition operators $A_{i+1}(y_0):\mathcal{L}(\mathbb{R}^d)\to\mathcal{L}(\mathbb{R}^d)$ of the linearized flow at switching times $t_{i+1}(y_0)$ for suitable initial values y_0. An analogous formula was also obtained in [24] by formal investigations of nearby trajectories. We remark that in particular in the case of a smooth underlying system, where all f_k's are the same, formula (2.9) below yields $A_{i+1}(y_0)=\mathrm{id}_{\mathcal{L}(\mathbb{R}^d)}$, i.e. the transition is smooth.

We start with transitions from $\mathbb{R}^3\setminus M_\infty$ to $\mathbb{R}^3\setminus M_\infty$ at some time $t_{i+1}(y_0)$.

Lemma 2.11 *For $i\in\mathbb{N}_0$ and $y_0\in G^{i+2}_{<\infty}$ choose $l,p\in I$ with $\xi_i(y_0)\in M_l$ and $\xi_{i+1}(y_0)\in M_p$. Let*

$$\Gamma^i_l(t,y)=\Phi_{f_l}(t-t_i(y),\xi_i(y))\quad\textit{for}\quad(t,y)\in\mathbb{R}\times\mathbb{R}^d. \tag{2.7}$$

Then for $t\in[t_{i+1}(y_0),t_{i+2}(y_0)[$

$$\begin{aligned}&\partial_y\varphi^t(y_0)\\&=[\partial_y\Phi_{f_p}(t-t_{i+1}(y_0),\xi_{i+1}(y_0))\,f_l(\xi_{i+1}(y_0))-f_p(\Phi_{f_p}(t-t_{i+1}(y_0),\xi_{i+1}(y_0)))]\,\langle\nabla t_{i+1}(y_0),\cdot\rangle\\&\quad+\partial_y\Phi_{f_p}(t-t_{i+1}(y_0),\xi_{i+1}(y_0))\,\partial_y\Gamma^i_l(t_{i+1}(y_0),y_0)\,.\end{aligned} \tag{2.8}$$

In addition, define $A_{i+1}(y_0):\mathcal{L}(\mathbb{R}^d)\to\mathcal{L}(\mathbb{R}^d)$ through

$$A_{i+1}(y_0)(B)=[f_l(\xi_{i+1}(y_0))-f_p(\xi_{i+1}(y_0))]\,\langle\nabla t_{i+1}(y_0),\cdot\rangle+B\,,\quad B\in\mathcal{L}(\mathbb{R}^d)\,. \tag{2.9}$$

Then

$$\lim_{t\to t_{i+1}(y_0)^+} \partial_y \varphi^t(y_0) = A_{i+1}(y_0) \left(\lim_{t\to t_{i+1}(y_0)^-} \partial_y \varphi^t(y_0) \right).$$

Proof: The following formal calculations are justified by Lemma 2.8, (2.6), and Corollary 2.9(a), (b). By Lemma 2.1(g) we obtain for $t \in]t_{i+1}(y_0), t_{i+2}(y_0)[$

$$\begin{aligned} \partial_y \varphi^t(y_0) &= \partial_t \Phi_{f_p}(t - t_{i+1}(y_0), \xi_{i+1}(y_0)) \, (-\nabla t_{i+1}(y_0)) + \partial_y \Phi_{f_p}(t - t_{i+1}(y_0), \xi_{i+1}(y_0)) \, \xi'_{i+1}(y_0) \\ &= -f_p(\Phi_{f_p}(t - t_{i+1}(y_0), \xi_{i+1}(y_0))) \, \nabla t_{i+1}(y_0) + \partial_y \Phi_{f_p}(t - t_{i+1}(y_0), \xi_{i+1}(y_0)) \, \xi'_{i+1}(y_0). \end{aligned} \tag{2.10}$$

Thus, since $\Phi_{f_p}(0,y) = y$ and $\partial_y \Phi_{f_p}(0,y) = \mathrm{id}_{\mathbb{R}^d}$ for all $y \in \mathbb{R}^d$,

$$\lim_{t\to t_{i+1}(y_0)^+} \partial_y \varphi^t(y_0) = -f_p(\xi_{i+1}(y_0)) \, \langle \nabla t_{i+1}(y_0), \cdot \rangle + \xi'_{i+1}(y_0). \tag{2.11}$$

We have $\partial_t \Gamma_l^i(t,y) = f_l(\Gamma_l^i(t,y))$. Also, for $t \in]t_i(y_0), t_{i+1}(y_0)[$ and y in a suitable $G^1_{<\infty} \cap M_k$ – neighborhood of y_0, it follows from (2.6) and Lemma 2.1(g) that $\varphi^t(y) = \Gamma_l^i(t,y)$ as well as

$$\begin{aligned} \xi_{i+1}(y) &= \varphi^{t_{i+1}(y)}(y) = \Phi_{f(\cdot,\varphi^{t_i(y)}(y))}(t_{i+1}(y) - t_i(y), \varphi^{t_i(y)}(y)) \\ &= \Phi_{f_l}(t_{i+1}(y) - t_i(y), \xi_i(y)) = \Gamma_l^i(t_{i+1}(y), y). \end{aligned}$$

Therefore

$$\begin{aligned} \xi'_{i+1}(y_0) &= \partial_t \Gamma_l^i(t_{i+1}(y_0), y_0) \, \nabla t_{i+1}(y_0) + \partial_y \Gamma_l^i(t_{i+1}(y_0), y_0) \\ &= f_l(\xi_{i+1}(y_0)) \, \langle \nabla t_{i+1}(y_0), \cdot \rangle + \partial_y \Gamma_l^i(t_{i+1}(y_0), y_0). \end{aligned} \tag{2.12}$$

Inserting this into (2.10) we obtain (2.8). In addition, taking into account that

$$\partial_y \varphi^t(y_0) = \partial_y \Gamma_l^i(t, y_0) \to \partial_y \Gamma_l^i(t_{i+1}(y_0), y_0) \quad \text{as} \quad t \to t_{i+1}(y_0)^-, \tag{2.13}$$

the claimed form of the transition operator $A_{i+1}(y_0)$ follows from (2.11) and (2.12). □

Next we consider transitions from $\mathbb{R}^3 \setminus M_\infty$ to $D_\infty \subset M_\infty$ at time $t_{i+1}(y_0)$.

Lemma 2.12 *For $i \in \mathbb{N}_0$ and $y_0 \in G_\infty^{i+1}$ choose $l \in I$ such that $\xi_i(y_0) \in M_l$. Then all assertions of Lemma 2.11 hold with $p = \infty$.*

Proof: Because of Corollary 2.10, the proof of Lemma 2.11 carries over. Note that here $\xi_{i+1}(y_0) \in M_\infty$, i.e. $p = \infty$ in the above notation, and $t_{i+2}(y_0) = \infty$. □

Sometimes it is possible to find a form of $\partial_y \varphi^t(y_0)$ and of the transition operators in Lemma 2.11 resp. Lemma 2.12 which is easier to handle. It will turn out that for this the following additional assumption is sufficient.

(A9) For every $k \in I$ there exist $h_k \in C^1(\mathbb{R}^d, \mathbb{R})$ such that

$$y \in M_k, \quad \xi_1(y) \notin E_k \quad \Longrightarrow \quad h_k(\xi_1(y)) = 0, \quad \langle \nabla h_k(\xi_1(y)), f_k(\xi_1(y)) \rangle \neq 0. \tag{2.14}$$

So condition (2.14) means that for a trajectory starting in $y \in M_k$ the function h_k being zero indicates the next switching to a different manifold. In addition, the trajectory should not arrive tangentially at $\{h_k = 0\}$.

This is the right place to comment on the assumptions imposed in [20] to obtain a "formal system" like in Section 2.1 to describe non-smooth systems appropriately. In condition $(A3)$ of this paper it was required that for $y \in M_k$, $h_k(\Phi_{f_k}(t,y))=0$ if and only if $t=t_1(y)$. This cannot be verified in concrete systems, as may be seen from the example of the pendulum with dry friction (1.3). In this case, the function $h_k(x,v,\tau):=v$ being zero means that a trajectory starting e.g. in $y_0 \in M_k=\{y \in \mathbb{R}^3 : v<0\}$ reaches the surface $\{y \in \mathbb{R}^3 : v=0\}$. Of course this will happen at time $t=t_1(y_0)$, but since here the flow $\Phi_{f_k}(\cdot,y_0)$ induced by the right-hand side f_k on M_k is oscillatory, we will find (infinitely many) $t_*>t_1(y_0)$ such that also $\Phi_{f_k}(t_*,y_0) \in \{y \in \mathbb{R}^3 : v=0\}$. Thus the "only if" in the former $(A3)$ does not hold, and it was exactly this condition which was needed to show the differentiability of $t_1(\cdot)$ in this paper. To summarize, although the results in [20] were correct, the developed "formal system" was not appropriate for applications.

After this short interlude, we turn back to calculate a simpler form of the transition operators $A_{i+1}(y_0)$ from Lemma 2.11 under assumption $(A9)$. In the following we identify $\mathcal{L}(\mathbb{R}^d)$ with $\mathbb{R}^{d\times d}$. We will find that now $A_{i+1}(y_0): \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ is linear, i.e. given through a matrix $A_{i+1}(y_0) \in \mathbb{R}^{d\times d}$.

Corollary 2.13 *Let, in addition to the general assumptions $(A1)-(A8)$, also $(A9)$ be satisfied. Fix $i \in \mathbb{N}_0$ and $y_0 \in G^{i+2}_{<\infty} \cup G^{i+1}_{\infty}$, and choose $l \in I$ and $p \in I \cup \{\infty\}$ such that $\xi_i(y_0) \in M_l$ and $\xi_{i+1}(y_0) \in M_p$. Then for $t \in [t_{i+1}(y_0), t_{i+2}(y_0)[$*

$$\begin{aligned}\partial_y \varphi^t(y_0) = \Bigg(&\frac{1}{\langle \nabla h_l, f_l\rangle(\xi_{i+1}(y_0))} \Big([f_p(\Phi_{f_p}(t-t_{i+1}(y_0),\xi_{i+1}(y_0))) \\ &\quad - \partial_y \Phi_{f_p}(t-t_{i+1}(y_0),\xi_{i+1}(y_0))\, f_l(\xi_{i+1}(y_0))]_\kappa \cdot [\nabla h_l(\xi_{i+1}(y_0))]_\nu \Big)_{1\le\kappa,\nu\le d} \\ &+ \partial_y \Phi_{f_p}(t-t_{i+1}(y_0),\xi_{i+1}(y_0)) \Bigg) \circ \Big(\lim_{s\to t_{i+1}(y_0)^-} \partial_y \varphi^s(y_0) \Big). \qquad (2.15)\end{aligned}$$

Here $\partial_y \varphi^t(y_0)$ at $t=t_{i+1}(y_0)$ is understood to be the limit from the right, and $[\ldots]_\kappa$ and $[\ldots]_\nu$ are the κ^{th} and ν^{th} component of the respective d-vectors.

In particular, we also find that $A_{i+1}(y_0) \in \mathbb{R}^{d\times d}$ is the matrix

$$\begin{aligned}&A_{i+1}(y_0) \qquad (2.16)\\ &= \frac{1}{\langle \nabla h_l, f_l\rangle(\xi_{i+1}(y_0))} \Big([f_p(\xi_{i+1}(y_0)) - f_l(\xi_{i+1}(y_0))]_\kappa \cdot [\nabla h_l(\xi_{i+1}(y_0))]_\nu \Big)_{1\le\kappa,\nu\le d} + \mathrm{id}_{\mathbb{R}^d}.\end{aligned}$$

Proof: Since $G^{i+2}_{<\infty} \cup G^{i+1}_{\infty} \subset G^{i+1}_{<\infty}$, in fact we find $l \in I$ with $\bar{y}_0 = \xi_i(y_0) \in M_l$. In addition, $\xi_1(\bar{y}_0) = \xi_{i+1}(y_0) \notin E_l$ by definition of $G^{i+1}_{<\infty}$, cf. Lemma 2.3. In particular the denominators in (2.15) and (2.16) are not zero, by the second condition in (2.14) of $(A9)$. We continue now to argue formally, which is again justified by Lemma 2.8, (2.6), Corollary 2.9, and Corollary 2.10, in both of the respective cases $y_0 \in G^{i+2}_{<\infty}$ and $y_0 \in G^{i+1}_{\infty}$. For y in a suitable $G^{i+2}_{<\infty}$-neighborhood resp. G^{i+1}_{∞}-neighborhood U of y_0 we also have $\xi_i(y) \in M_l$, $t_{i+1}(y)<\infty$, and $\xi_{i+1}(y) \notin E_l$. Then $\Phi_{f_l}(t_{i+1}(y)-t_i(y),\xi_i(y)) = \varphi(t_{i+1}(y),y) = \xi_{i+1}(y) = \xi_1(\xi_i(y))$ by Lemma 2.1(g) and (c). Let $g(t,y) = h_l(\Phi_{f_l}(t-t_i(y),\xi_i(y))) = h_l(\Gamma^i_l(t,y))$ for $t \in \mathbb{R}$ and $y \in \mathbb{R}^d$, cf. (2.7). We conclude that

$$\partial_t g(t,y) = \langle \nabla h_l, f_l\rangle(\Phi_{f_l}(t-t_i(y),\xi_i(y))) \quad \text{and} \quad \partial_y g(t,y) = \nabla h_l(\Gamma^i_l(t,y))\, \partial_y \Gamma^i_l(t,y).$$

From the first condition of (2.14) in assumption $(A9)$ applied to $\xi_i(y)$ we obtain that $g(t_{i+1}(y),y) = h_l(\xi_1(\xi_i(y))=0$ for $y \in U$, hence $\nabla t_{i+1}(y) = -(\partial_t g(t_{i+1}(y),y))^{-1} \partial_y g(t_{i+1}(y),y)$. This implies

$$\nabla t_{i+1}(y_0) = -\frac{\nabla h_l(\Gamma^i_l(t_{i+1}(y_0),y_0))\, \partial_y \Gamma^i_l(t_{i+1}(y_0),y_0)}{\langle h_l, f_l\rangle(\xi_{i+1}(y_0))} = -\frac{\nabla h_l(\xi_{i+1}(y_0))\, \partial_y \Gamma^i_l(t_{i+1}(y_0),y_0)}{\langle h_l, f_l\rangle(\xi_{i+1}(y_0))}.$$

This can be inserted into (2.8) resp. (2.9) to yield (2.15) resp. (2.16), since $\lim_{s\to t_{i+1}(y_0)^-} \partial_y \varphi^s(y_0) = \partial_y \Gamma^i_l(t_{i+1}(y_0),y_0)$ by (2.13). □

2.3 Construction of a cocycle

By means of Lemma 2.11 and Lemma 2.12 we can now construct a canonical cocycle. Instead of defining $T:[0,\infty[\times\mathbb{R}^d\to\mathcal{L}(\mathbb{R}^d)$ we could also restrict T to $[0,\infty[\times G$, since G is forward invariant.

Theorem 2.14 *Let the forward invariant G be as in Corollary 2.7. Define $T:[0,\infty[\times\mathbb{R}^d\to\mathcal{L}(\mathbb{R}^d)$ as follows*

$$T(t,y)=\begin{cases}\partial_y\varphi^t(y) & : \quad y\in G_{<\infty},\,\forall\, i\in\mathbb{N}: t\neq t_i(y)\\ A_{i+1}(y)\Big(\partial_y\Gamma_l^i(t_{i+1}(y),y)\Big) & : \quad y\in G_{<\infty},\,\exists i\in\mathbb{N}_0: t=t_{i+1}(y)\\ & \quad\;\; [\text{with}\, l\in I \text{ such that}\, \xi_i(y)\in M_l]\\ \partial_y\varphi^t(y) & : \quad \exists i_0\in\mathbb{N}_0: y\in G_\infty^{i_0},\, t\neq t_1(y),\dots,t_{i_0}(y)\\ A_{i+1}(y)\Big(\partial_y\Gamma_l^i(t_{i+1}(y),y)\Big) & : \quad \exists i_0\in\mathbb{N}_0: y\in G_\infty^{i_0}\,\exists i\in\{0,\dots,i_0-1\}: t=t_{i+1}(y)\\ & \quad\;\; [\text{with}\, l\in I \text{ such that}\, \xi_i(y)\in M_l]\\ 0 & : \quad y\notin G\end{cases},$$

with Γ_l^i from (2.7). Then T is a cocycle for $(\varphi^t)_{t\geq 0}$.

Proof: We first note that T is well-defined, because $G_\infty^i\cap G_\infty^j=\emptyset$ for $i\neq j$, by Lemma 2.4, and also $G_\infty^{i_0}\subset G_{<\infty}^{i_0}$. The latter implies that $A_1(y),\dots,A_{i_0-1}(y)$ are defined for $y\in G_\infty^{i_0}$ through Lemma 2.11, and $A_{i_0}(y)$ is defined through Lemma 2.12.

We have to show that

$$T(t+s,y)=T(t,\varphi^s(y))\,T(s,y)\quad\text{for}\quad t,s\in[0,\infty[\quad\text{and}\quad y\in\mathbb{R}^d,\tag{2.17}$$

and again we argue formally. First we remark that (2.17) is true for $y\notin G$. Note also that (2.17) holds if $t=0$ or if $s=0$, since G is forward invariant and $T(0,y)=\mathrm{id}_{\mathbb{R}^d}$ for all $y\in G$, by the 1st and 3rd line of the definition. So fix $t,s>0$ and $y\in G$. According to the definition of G we distinguish cases.
<u>Case 1:</u> $y\in G_{<\infty}$. Choose $i,j\in\mathbb{N}$ such that $s\in[t_i(y),t_{i+1}(y)[$ and $t+s\in[t_j(y),t_{j+1}(y)[$. Assume first that in addition $s\neq t_i(y)$ and $t+s\neq t_j(y)$, i.e. $s\neq t_k(y)$ and $t+s\neq t_k(y)$ for all $k\in\mathbb{N}$. Then also $t\neq t_k(\varphi^s(y))$ for all $k\in\mathbb{N}$ by Lemma 2.1(e), and hence the invariance of $G_{<\infty}$ and Corollary 2.9 imply that we can differentiate $\varphi^{t+s}(y)=\varphi^t(\varphi^s(y))$ to obtain (2.17) by definition of T for such s and $t+s$. Next, if $s=t_i(y)$ or $t+s=t_j(y)$, then in all cases $s+\tau\in]t_i(y),t_{i+1}(y)[$ and $t+(s+\tau)\in]t_j(y),t_{j+1}(y)[$ for $\tau>0$ sufficiently small. Consequently, $T(t+s+\tau,y)=T(t,\varphi^{s+\tau}(y))T(s+\tau,y)$ by the part already shown. Since $T(\cdot,y)$ is right-continuous by definition of T and by Lemma 2.11, cf. (2.13), it is hence sufficient to prove that for fixed $t>0$ and $y_0\in G_{<\infty}$ the function $T(t,\cdot)$ is continuous at y_0 on the set $D_{y_0}=\{z\in G_{<\infty}: t_k(z)\leq t_k(y_0)\text{ for all } k\in\mathbb{N}\}$. Indeed, if this is shown, then we can take, due to the invariance of $G_{<\infty}$, $y_0=\varphi^s(y)\in G_{<\infty}$ and $z_\tau=\varphi^{s+\tau}(y)\in G_{<\infty}$, because $t_k(z_\tau)=t_k(\varphi^{s+\tau}(y))=t_{i+k}(y)-(s+\tau)\leq t_{i+k}(y)-s=t_k(\varphi^s(y))=t_k(y_0)$ for $k\in\mathbb{N}$ by Lemma 2.1(e). The continuity of $\varphi(\cdot,y)$, cf. (*A*4), yields $z_\tau\to\varphi^s(y)$ as $\tau\to 0^+$, and therefore $T(t,\varphi^{s+\tau}(y))\to T(t,\varphi^s(y))$ as $\tau\to 0^+$. This gives (2.17) in the limit.

To prove the continuity of $T(t,\cdot)$ at $y_0\in G_{<\infty}$ on D_{y_0}, we may assume that $t=t_{i+1}(y_0)$ for some $i\in\mathbb{N}_0$, since otherwise Corollary 2.9 implies that $T(t,\cdot)=\partial_y\varphi^t(\cdot)$ is continuous even in a whole $G_{<\infty}$-neighborhood (not restricted to D_{y_0}) of y_0. So let $t=t_{i+1}(y_0)$ and choose $l,p\in I$ such that $\xi_i(y_0)\in M_l$ and $\xi_{i+1}(y_0)\in M_p$. Hence by definition and by (2.11)

$$T(t,y_0)=A_{i+1}(y_0)\Big(\partial_y\Gamma_l^i(t_{i+1}(y_0),y_0)\Big)=-f_p(\xi_{i+1}(y_0))\,\langle\nabla t_{i+1}(y_0),\cdot\rangle+\xi_{i+1}'(y_0).\tag{2.18}$$

Because of $t_i(y_0)<t=t_{i+1}(y_0)<t_{i+2}(y_0)$, we have $t\in[t_{i+1}(z),t_{i+2}(z)[$ for $z\in D_{y_0}$ sufficiently near to y_0 by definition of D_{y_0} and by Lemma 2.8. For $z\in G_{<\infty}$ near to y_0 we also have $\xi_i(z)\in M_l$

and $\xi_{i+1}(z) \in M_p$ by (2.6). Since thus either $T(t,z) = A_{i+1}(z)\Big(\partial_y \Gamma_l^i(t_{i+1}(z),z)\Big)$ if $t = t_{i+1}(z)$ or, cf. (2.10),

$$T(t,z) = \partial_y \varphi^t(z) = -f_p(\Phi_{f_p}(t_{i+1}(y_0) - t_{i+1}(z),\xi_{i+1}(z)))\,\langle \nabla t_{i+1}(z),\cdot\rangle \\ + \partial_y \Phi_{f_p}(t_{i+1}(y_0) - t_{i+1}(z),\xi_{i+1}(z))\,\xi'_{i+1}(z)$$

if $t \in]t_{i+1}(z),t_{i+2}(z)[$, the claim hence follows from (2.18), the regularity of the functions involved, from $\Phi_{f_p}(0,\xi_{i+1}(y_0)) = \xi_{i+1}(y_0)$ and $\partial_y \Phi_{f_p}(0,\xi_{i+1}(y_0)) = \mathrm{id}_{\mathbb{R}^d}$.

Case 2: $y \in G_\infty$. Then $y \in G_\infty^{i_0}$ for a unique $i_0 \in \mathbb{N}_0$ by definition of G_∞, and since $t_{i_0+1}(y) = \infty$ we can choose $i,j \in \{0,\ldots,i_0\}$ with $s \in [t_i(y),t_{i+1}(y)[$ and $t+s \in [t_j(y),t_{j+1}(y)[$. In case that $s \neq t_i(y)$ and $t+s \neq t_j(y)$ we find again by means of Lemma 2.1(e) that also $t \neq t_k(y)$ for $k \in \{0,\ldots,i_0\}$. Hence Corollary 2.10 implies that once more $\varphi^{t+s}(y) = \varphi^t(\varphi^s(y))$ may be differentiated to yield (2.17). Now we can continue to argue as in Case 1, and we thus only have to show that

$$T(t,\varphi^{s+\tau}(y)) \to T(t,\varphi^s(y)) \quad \text{as} \quad \tau \to 0^+, \tag{2.19}$$

since again $T(\cdot,y)$ is right-continuous, by Lemma 2.11 and Lemma 2.12. To prove (2.19) we may assume in addition that $i \neq i_0$. Indeed, if $i = i_0$, then $\varphi^s(y) \in G_\infty^0 = D_\infty$ by Corollary 2.6, and therefore also $\varphi^{s+\tau}(y) \in D_\infty$ for $\tau \geq 0$ by definition of D_∞ and $(A3)$. Since $T(t,z) = \partial_y \varphi(t,z) = \partial_y \Phi_{f_\infty}(t,z)$ for $t \in [0,\infty[$ and $z \in G_\infty^0 = D_\infty$, (2.19) follows from the continuity of $\varphi(\cdot,y)$ in case that $i = i_0$. So we suppose $i \leq i_0 - 1$ and let $y_0 = \varphi^s(y)$, as well as $D_{y_0} = \{z \in G_\infty^{i_0-i} : t_k(z) \leq t_k(y_0) \text{ for } k \in \{0,\ldots,i_0-i\}\}$. Note that $y_0 \in D_{y_0}$ and $\varphi^{s+\tau}(y) \in G_\infty^{i_0-i}$ for $\tau > 0$ small, by Corollary 2.6. Also $t_k(\varphi^{s+\tau}(y)) = t_{i+k}(y) - (s+\tau) \leq t_{i+k}(y) - s = t_k(\varphi^s(y)) = t_k(y_0)$ for $\tau > 0$ small and $k \in \{0,\ldots,i_0-i\}$ by Lemma 2.1(e), i.e. $\varphi^{s+\tau}(y) \in D_{y_0}$. Hence, to show (2.19), it is again enough to prove the continuity of $T(t,\cdot)$ at y_0 on D_{y_0}. Because $t_{i_0-i+1}(y_0) = t_{i_0+1}(y) = \infty$, and as a consequence of Corollary 2.10, we may assume that $t = t_{k_0+1}(y_0)$ for some $k_0 \in \{0,\ldots,i_0-i-1\}$. Since we can repeat the arguments from Case 1 above, by means of Lemma 2.11 and Lemma 2.12, we only have to show that

$$z \in D_{y_0}, \quad z \text{ sufficiently close to } y_0 \quad \Longrightarrow \quad t \in [t_{k_0+1}(z),t_{k_0+2}(z)[. \tag{2.20}$$

For this, we first remark that $k_0 + 1 \leq i_0 - i$, and thus $t < t_{k_0+1}(z)$ is impossible, because then the definition of D_{y_0} would imply $t < t_{k_0+1}(z) \leq t_{k_0+1}(y_0) = t$. To prove $t < t_{k_0+2}(z)$ for $z \in D_{y_0}$ sufficiently close to y_0, we start with the case $k_0 \leq i_0 - i - 2$. Then (2.20) follows from Lemma 2.8: Since $t_{k_0+2}(\cdot)$ in particular is continuous on a $G_{<\infty}^{k_0+2}$-neighborhood of y_0, and since $k_0 + 2 \leq i_0 - i$, we have $G_{<\infty}^{k_0+2} \supset G_{<\infty}^{i_0-i} \supset G_\infty^{i_0-i} \supset D_{y_0}$. Therefore $t_{k_0+2}(\cdot)$ is also continuous on a D_{y_0}-neighborhood of y_0. Thus we have to consider finally the case that $k_0 = i_0 - i - 1$. Then $D_{y_0} \subset G_\infty^{i_0-i}$ implies that for $z \in D_{y_0}$ we must have $\xi_{i_0-i}(z) \in D_\infty \subset M_\infty$, and therefore $t_{k_0+2}(z) = t_{i_0-i+1}(z) = t_1(\xi_{i_0-i}(z)) + t_{i_0-i}(z) = \infty$, cf. (2.3). This completes the proof of (2.20), and hence also the proof of the theorem. □

Remark 2.15 By (2.13) and Theorem 2.14 in fact we have

$$T(t,y) = \begin{cases} \partial_y \varphi^t(y) & : \quad y \in G_{<\infty},\, \forall\, i \in \mathbb{N} : t \neq t_i(y) \\ A_{i+1}(y)\Big(\lim\limits_{s \to t_{i+1}(y)^-} T(s,y)\Big) & : \quad y \in G_{<\infty},\, \exists i \in \mathbb{N}_0 : t = t_{i+1}(y) \\ & \quad\;\; [\text{with } l \in I \text{ such that } \xi_i(y) \in M_l] \\ \partial_y \varphi^t(y) & : \quad \exists i_0 \in \mathbb{N}_0 : y \in G_\infty^{i_0},\, t \neq t_1(y),\ldots,t_{i_0}(y) \\ A_{i+1}(y)\Big(\lim\limits_{s \to t_{i+1}(y)^-} T(s,y)\Big) & : \quad \exists i_0 \in \mathbb{N}_0 : y \in G_\infty^{i_0}\, \exists i \in \{0,\ldots,i_0-1\} : t = t_{i+1}(y) \\ & \quad\;\; [\text{with } l \in I \text{ such that } \xi_i(y) \in M_l] \\ 0 & : \quad y \notin G \end{cases},$$

and both the the $\partial_y \varphi^t(y)$-terms and the matrices $A_{i+1}(y)$ can be calculated by means of Corollary 2.13 in case that additionally $(A9)$ holds. So we have obtained a complete description of the cocycle.

Next we show that the cocycle T satisfies a certain integral equation. This integral representation may be used in certain cases to verify the integrability conditions (4.1) from the MET Theorem 4.3, cf. Corollary 2.17 below.

Corollary 2.16 *For $y \in G$ and $t \in [t_i(y), t_{i+1}(y)[$,*

$$T(t,y) = Q_i(y) + \int_0^t P_i(s,y)\, F(s,y)\, T(s,y)\, ds. \tag{2.21}$$

Here $i \in \mathbb{N}_0$, if $y \in G_{<\infty}$, and $i \in \{0,\dots,i_0\}$ for $y \in G_\infty^{i_0} \subset G_\infty$. Moreover, we define $Q_i(y) = (A_i(y) \cdot \ldots \cdot A_1(y))\,(\mathrm{id}_{\mathbb{R}^d}) \in \mathcal{L}(\mathbb{R}^d)$, $P_i(s,y) = A_i(y) \cdot \ldots \cdot A_{j+2}(y) \cdot A_{j+1}(y) : \mathcal{L}(\mathbb{R}^d) \to \mathcal{L}(\mathbb{R}^d)$ in case that $s \in [t_j(y), t_{j+1}(y)[$ and $P_i(s,y) = \mathrm{id}_{\mathcal{L}(\mathbb{R}^d)}$ for $j \geq i$. In addition, we let $F(s,y) = f'(\varphi^s(y), \xi_j(y))$ if $s \in [t_j(y), t_{j+1}(y)[$.

Proof: We first consider the case $i=0$, i.e. $t \in [0, t_1(y)[$ for some fixed $y \in G$. Then $Q_0(y) := \mathrm{id}_{\mathbb{R}^d}$ and $F(s,y) = f'(\varphi^s(y), y)$ for $s \in [0, t_1(y)[$. If we let $f_k = f(\cdot, y)$, $k \in I \cup \{\infty\}$, we obtain by $(A5)$ and (2.7), $\varphi^s(y) = \Phi_{f_k}(s,y) = \Gamma_k^0(s,y)$ for $s \in [0, t_1(y)[$. Thus,

$$\begin{aligned} \partial_t T(s,y) &= \partial_t\,(\partial_y \Gamma_k^0(s,y)) = \partial_y\,(\partial_t \Gamma_k^0(s,y)) = \partial_y\,(f_k(\Gamma_k^0(s,y))) \\ &= f_k'(\Gamma_k^0(s,y))\,\partial_y \Gamma_k^0(s,y) = F(s,y)\,T(s,y) \in \mathcal{L}(\mathbb{R}^d). \end{aligned} \tag{2.22}$$

Since $T(0,y) = \mathrm{id}_{\mathbb{R}^d}$ this may be integrated to give (2.21) for $i=0$.

If (2.21) already holds for some $i \in \mathbb{N}$, then in particular by definition of $T(t_i(y), y)$ and by assumption

$$T(t_i(y),y) = A_i(y)\Big(\lim_{t \to t_i(y)^-} T(t,y)\Big) = A_i(y)\,Q_{i-1}(y) + \int_0^{t_i(y)} A_i(y) P_{i-1}(s,y) F(s,y) T(s,y)\, ds. \tag{2.23}$$

Moreover, if $f_l = f(\cdot, \xi_i(y))$, $l \in I \cup \{\infty\}$, then $\varphi^s(y) = \Gamma_l^i(s,y)$ for $s \in [t_i(y), t_{i+1}(y)[$, and hence the differential equation for T from (2.22) is also valid for $s \in [t_i(y), t_{i+1}(y)[$. Integrating this from $t_i(y)$ to t and using (2.23) we thus obtain for $t \in [t_i(y), t_{i+1}(y)[$

$$\begin{aligned} T(t,y) &= A_i(y)\,Q_{i-1}(y) + \int_0^{t_i(y)} A_i(y) P_{i-1}(s,y) F(s,y) T(s,y)\, ds + \int_{t_i(y)}^t F(s,y)\,T(s,y)\, ds \\ &= Q_i(y) + \int_0^t P_i(s,y) F(s,y) T(s,y)\, ds, \end{aligned}$$

by definition of Q_i, P_i and F. □

We remark that (2.21) is similar to formulas for solutions of certain impulsive equations, cf. e.g. [3]. The following corollary is analogous to [20, Cor. 3].

Corollary 2.17 *Suppose that $(A1)-(A9)$ hold. If*

$$\exists c > 0 : |f_k'|_\infty \leq c, \quad k \in I \cup \{\infty\},$$

and if $\|A_{i+1}(y)\|_d \leq 1$ for all $y \in G \setminus G_\infty^0$ and i, where $i \in \mathbb{N}_0$ for $y \in G_{<\infty}$, and $i \in \{0,\dots,i_0\}$ for $y \in G_\infty^{i_0+1}$ with $i_0 \in \mathbb{N}_0$, then for every $t_* > 0$ there exists $C(t_*) > 0$ such that

$$\sup\{\|T(t,y)\|_3 : t \in [0,t_*], y \in \mathbb{R}^d\} \leq C(t_*).$$

In particular, the integrability conditions (4.1) in the MET Theorem 4.3 are satisfied for every probability measure $\mu \in \mathcal{P}_{\mathrm{inv}}(\varphi)$.

Proof: This follows from Corollary 2.16, since $\|T(t,y)\|_3 \leq 1 + c\int_0^t \|T(s,y)\|_3\, ds$ for $t \in [0,t_*]$. Thus $C(t_*) = \exp(ct_*)$ is suitable, by Gronwall's lemma. □

2.4 Sets of measure zero

Of course one is interested to have a set G of "good" initial values as large as possible for the definition of the cocycle T in Theorem 2.14. In this section we will give a criterion to ensure that $\lambda^d(\mathbb{R}^d\setminus G)=0$, and it will turn out that this criterion is easy to satisfy in applications, cf. the example of the pendulum with dry friction in Section 3.

In addition to $(A1)-(A8)$ from Section 2.1 we consider the following assumption.

(A10) The dimension d is $d\geq 2$, and there exists a set $C_{I\cup\{\infty\}}\subset\mathbb{R}^d$ such that $C_{I\cup\{\infty\}}\subset\bigcup_{k\in I\cup\{\infty\}} C_k$. Here for $k\in I\cup\{\infty\}$, either $C_k=\emptyset$, or C_k is a manifold of dimension $\leq d-2$ and class C^1, which can be described by a single chart $\alpha_k:\mathbb{R}^{d-2}\to C_k$. In addition it is supposed that

$$D_k\supset\{y\in M_k:\varphi^t(y)\notin C_{I\cup\{\infty\}}\;\forall t\in[0,t_1(y)]\},\quad k\in I, \tag{2.24}$$

and

$$D_\infty\supset\{y\in M_\infty:\varphi^t(y)\notin C_{I\cup\{\infty\}}\;\forall t\in[0,\infty[\}. \tag{2.25}$$

The set $C_{I\cup\{\infty\}}$ may be thought of to be a union of boundary curves of the different manifolds, and the condition means that if the flow does not hit these boundary curves, then it should neither hit the exceptional set E_k, nor there should be neighborhoods separated by ξ_1, cf. the definition of D_k at (2.1) and in $(A7)$. Of course, in a concrete example the C_k will be related to the E_k, but there is no need to impose assumptions on this a priori. We also remark that we allow $\{y\in M_k:\varphi^t(y)\notin C_{I\cup\{\infty\}}\;\forall t\in[0,t_1(y)]\}$ to be empty, since in that case the flow transports M_k to the lower-dimensional $C_{I\cup\{\infty\}}$.

We start with some auxiliary results.

Lemma 2.18 *Let $(A1)-(A8)$ be satisfied. If $k\in I$ and $N\subset\mathbb{R}^d$ is such that $\lambda^d(N)=0$, then we have $\lambda^d(\{y\in D_k:\xi_1(y)\in N\})=0$.*

Proof: Fix $R>0$ and choose $T=T(k,R)$ with $y\in\overline{B}_R(0)\cap D_k\Rightarrow t_1(y)\leq T$, cf. Lemma 2.2. Since $t_1(y)<\infty$ and $f(\cdot,y)=f_k$ for $y\in D_k$, we have $\xi_1(y)=\varphi(t_1(y),y)=\Phi_{f_k}(t_1(y),y)$ by Lemma 2.1(g). Since every $\Phi_{f_k}(t,\cdot)$ is a diffeomorphism with inverse $\Phi_{f_k}(-t,\cdot)$, we obtain $\{y\in\overline{B}_R(0)\cap D_k:\xi_1(y)\in N\}\subset\Phi_{f_k}([-T,0]\times N)$. Since it follows from Lemma 4.9(b) that $\lambda^d(\Phi_{f_k}([-T,0]\times N))=0$, the claim is obtained by letting $R\to\infty$. □

Lemma 2.19 *Let $(A1)-(A8)$ and $(A10)$ be satisfied. If $k\in I$, then $\lambda^d(M_k\setminus D_k)=0$.*

Proof: By (2.24) of $(A10)$

$$M_k\setminus D_k\subset\{y\in M_k:\exists t\in[0,t_1(y)]:\varphi^t(y)\in C_{I\cup\{\infty\}}\}\subset\bigcup_{l\in I\cup\{\infty\}}\{y\in\mathbb{R}^d:\exists t\in[0,\infty[:\Phi_{f_k}(t,y)\in C_l\},$$

where the last inclusion follows from Lemma 2.1(g) and $(A10)$. Thus $\lambda^d(M_k\setminus D_k)=0$ by Lemma 4.9(a). □

Lemma 2.20 *Let $(A1)-(A8)$ and $(A10)$ be satisfied. Then*

$$\mathbb{R}^d\setminus G\subset\bigcup_{i\in\mathbb{N}_0}\{y\in\mathbb{R}^d:t_i(y)<\infty\text{ and }\exists t\in[t_i(y),t_{i+1}(y)[:\varphi^t(y)\in C_{I\cup\{\infty\}}\}.$$

Proof: If $y \in \mathbb{R}^d$ is not an element of the right-hand side, then for every $i \in \mathbb{N}_0$ the following alternative holds: either $t_i(y) = \infty$, or $t_i(y) < \infty$ and $\varphi^t(y) \notin C_{I\cup\{\infty\}}$ for all $t \in [t_i(y), t_{i+1}(y)[$. We will show $y \in G$, and for this it is not too hard to prove first by induction that

$$i_0 \in \mathbb{N}_0\,, \quad t_{i_0}(y) < \infty \quad \Longrightarrow \quad y \in G^{i_0}_{<\infty}\,. \tag{2.26}$$

Then to see $y \in G$, we first consider Case 1: $t_i(y) < \infty$ for all $i \in \mathbb{N}_0$. Then $y \in G^i_{<\infty}$ for all $i \in \mathbb{N}_0$ by (2.26), and thus $y \in G_{<\infty} \subset G$. Case 2: $t_{i_0+1}(y) = \infty$ for some first $i_0 \in \mathbb{N}_0$. Then $t_{i_0}(y) < \infty$, and therefore $y \in G^{i_0}_{<\infty}$ by (2.26). Additionally, $t_0(y) < t_1(y) < \ldots t_{i_0}(y) < \infty$, and thus $\varphi^s(y) \notin C_{I\cup\{\infty\}}$ for $s \in [t_k(y), t_{k+1}(y)[$, with $k \in \{0, \ldots, i_0\}$, by the above alternative. Since $t_{i_0+1}(y) = \infty$, this means $\varphi^s(y) \notin C_{I\cup\{\infty\}}$ for $s \in [0,\infty[$. By $(A1)$ we also must have $\xi_{i_0}(y) \in M_\infty$, because $t_1(\xi_{i_0}(y)) = t_{i_0+1}(y) - t_{i_0}(y) = \infty$, cf. (2.3). Since $\varphi^t(\xi_{i_0}(y)) = \varphi(t + t_{i_0}(y), y) \notin C_{I\cup\{\infty\}}$ for $t \in [0,\infty[$, (2.25) of assumption $(A10)$ gives $\xi_{i_0}(y) \in D_\infty$, and therefore, cf. Lemma 2.4, $y \in G^{i_0}_\infty \subset G$ as was to be shown. □

By means of the previous lemma we can prove

Theorem 2.21 *Let $(A1) - (A8)$ and $(A10)$ hold. Then $\lambda^d(\mathbb{R}^d \setminus G) = 0$.*

Proof: By Lemma 2.20 it is sufficient to show $\lambda^d(A_i) = 0$ for every $i \in \mathbb{N}_0$, with

$$A_i = \{y \in \mathbb{R}^d : t_i(y) < \infty \text{ and } \exists t \in [t_i(y), t_{i+1}(y)[: \varphi^t(y) \in C_{I\cup\{\infty\}}\}\,.$$

This will be achieved by induction, and for this we first claim that

$$A_{i+1} \subset \xi_1^{-1}(A_i) \quad \text{for} \quad i \in \mathbb{N}_0\,. \tag{2.27}$$

Indeed, if $y \in A_{i+1}$, then $t_i(\xi_1(y)) = t_{i+1}(y) - t_1(y)$ by Lemma 2.1(b), and thus $t_i(\xi_1(y)) < \infty$. By assumption we find $\bar{t} \in [t_{i+1}(y), t_{i+2}(y)[$ with $\varphi^{\bar{t}}(y) \in C_{I\cup\{\infty\}}$. Therefore $t = \bar{t} - t_1(y)$ has $t \in [t_{i+1}(y) - t_1(y), t_{i+2}(y) - t_1(y)[= [t_i(\xi_1(y)), t_{i+1}(\xi_1(y))[$ and $\varphi^t(\xi_1(y)) = \varphi^{t+t_1(y)}(y) \in C_{I\cup\{\infty\}}$. Hence (2.27) is satisfied. Next we remark that

$$A_0 \subset \bigcup_{l \in I\cup\{\infty\}} \bigcup_{k \in I\cup\{\infty\}} \{y \in \mathbb{R}^d : \exists t \in [0,\infty[: \Phi_{f_l}(t,y) \in C_k\}$$

by $(A5)$ and $(A10)$. As a consequence of $f_k \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ this yields $\lambda^d(A_0) = 0$, by Lemma 4.9(a) and assumption $(A10)$ on the C_k. Suppose now we have already shown $\lambda^d(A_i) = 0$ for some $i \in \mathbb{N}_0$. Because in particular $t_1(y) < \infty$ for $y \in A_{i+1}$, we have $A_{i+1} \cap M_\infty = \emptyset$, and thus by (2.27)

$$A_{i+1} = \bigcup_{k \in I} \{y \in A_{i+1} \cap M_k : \xi_1(y) \in A_i\} \subset \bigcup_{k \in I} \{y \in D_k : \xi_1(y) \in A_i\} \cup \Big(\bigcup_{k \in I} (M_k \setminus D_k) \Big)\,.$$

Hence also $\lambda^d(A_{i+1}) = 0$ by Lemma 2.18 and Lemma 2.19, and this completes the proof. □

2.5 Remarks on the periodic case

Up to now we always considered the semiflow φ being defined on $\mathbb{R}^d$, i.e. $\varphi : [0,\infty[\times \mathbb{R}^d \to \mathbb{R}^d$. In case that this semiflow is induced by a ω-periodically forced $(d-1)$-dimensional non-smooth system (as in the example of the pendulum with dry friction (1.2)), the state space $\mathbb{R}^d$ equivalently may be replaced by the d-dimensional metric manifold $\mathbb{R}^d_\omega := \mathbb{R}^{d-1} \oplus \mathbb{S}^1_\omega \subset \mathbb{R}^{d+1}$, where $\mathbb{S}^1_\omega$ is $\mathbb{S}^1$parameterized through $[0,\omega] \ni \tau \mapsto e^{2\pi i \tau/\omega} \in \mathbb{S}^1$. It is more or less obvious that all results from Sections 2.1-2.4 have analogs when the system is considered from this point of view, and we want to give some more comments (without proofs) concerning this subject.

Since the system was made autonomous here artificially by introducing time as a new variable additionally to the $(d-1)$ "space"-variables, every right-hand side $f_k \in C^1(\mathbb{R}^d,\mathbb{R}^d)$, $k \in I \cup \{\infty\}$, is of the form $f_k(y) = (*,\ldots,*,1)$, resulting from the additional "$\dot{t}=1$". By $(A5)$ this implies that φ: $[0,\infty[\times\mathbb{R}^d \to \mathbb{R}^d$ has to have the form $\varphi^t(y_0) = (\varphi_1(t,y_0),\ldots,\varphi_{d-1}(t,y_0),\tau_0+t)$ for $t \in [0,\infty[$ and $y_0 = (\xi_0,\tau_0) \in \mathbb{R}^{d-1}\times\mathbb{R}$. By definition of the cocycle $T:[0,\infty[\times\mathbb{R}^d \to \mathcal{L}(\mathbb{R}^d)$ in Theorem 2.14, cf. also Lemma 2.11 and Lemma 2.12, this in turn implies for $y_0 \in G$

$$T(t,y_0) = \begin{pmatrix} T_1(t,y_0) \\ \vdots \\ T_{d-1}(t,y_0) \\ 0 \ldots 0\ 1 \end{pmatrix} \in \mathbb{R}^{d\times d}, \tag{2.28}$$

with $T_j(t,y_0) \in \mathbb{R}^d$ being the j^{th} line of $T(t,y_0)$. By the assumed periodicity in the time component, in addition $\varphi^t(\xi_0,\tau_0+k\omega) = \varphi^t(\xi_0,\tau_0) + (0,\ldots,0,k\omega)$, hence also $T(t,\xi_0,\tau_0+k\omega) = T(t,\xi_0,\tau_0)$, for $k \in \mathbb{Z}$. Therefore the semiflow and the cocycle carry over well-defined to $\mathbb{R}^d_\omega$. To be more precise, let $\varphi_{\text{per}}:[0,\infty[\times\mathbb{R}^d_\omega \to \mathbb{R}^d_\omega$ be defined as

$$\varphi^t_{\text{per}}(\xi_0, e^{2\pi i\tau_0/\omega}) = (\varphi_1(t,\xi_0,\tau_0),\ldots,\varphi_{d-1}(t,\xi_0,\tau_0), e^{2\pi i[\tau_0+t]/\omega}) \tag{2.29}$$

for $t \in [0,\infty[$ and $(\xi_0, e^{2\pi i\tau_0/\omega}) \in \mathbb{R}^{d-1}\oplus\mathbb{S}^1_\omega = \mathbb{R}^d_\omega$. Then φ_{per} is a well-defined semiflow on $\mathbb{R}^d_\omega$ which inherits all properties of φ. To transfer T to $\mathbb{R}^d_\omega$, let $\mathcal{T}(\mathbb{R}^d_\omega) = \mathbb{R}^{d-1}\oplus\mathbb{S}^1$denote the tangent bundle of $\mathbb{R}^d_\omega$ with fibers $\mathcal{T}_{y_0}(\mathbb{R}^d_\omega) = \mathbb{R}^{d-1}\oplus(\mathbb{R}ie^{2\pi i\tau_0/\omega})$ for $y_0 = (\xi_0, e^{2\pi i\tau_0/\omega}) \in \mathbb{R}^d_\omega$. These fibers have a natural norm given through $\|(x_1,\ldots,x_{d-1},x_d ie^{2\pi i\tau_0/\omega})\|_{\mathcal{T}_{y_0}(\mathbb{R}^d_\omega)} := |(x_1,\ldots,x_d)|_{\mathbb{R}^d}$, and $T_{\text{per}}(t,y_0) \in \mathcal{L}(\mathcal{T}_{y_0}(\mathbb{R}^d_\omega), \mathcal{T}_{\varphi^t_{\text{per}}(y_0)}(\mathbb{R}^d_\omega))$ is defined as

$$T_{\text{per}}(t,y_0)\,(x_1,\ldots,x_{d-1},x_d ie^{2\pi i\tau_0/\omega}) := \begin{pmatrix} \langle T_1(t,\xi_0,\tau_0),(x_1,\ldots,x_d)\rangle_{\mathbb{R}^d} \\ \vdots \\ \langle T_{d-1}(t,\xi_0,\tau_0),(x_1,\ldots,x_d)\rangle_{\mathbb{R}^d} \\ x_d ie^{2\pi i[\tau_0+t]/\omega} \end{pmatrix} \in \mathcal{T}_{\varphi^t_{\text{per}}(y_0)}(\mathbb{R}^d_\omega).$$

This T_{per} is also well-defined, a cocycle for $(\varphi^t_{\text{per}})_{t\geq 0}$, and inherits the properties of T. In particular,

$$\|T_{\text{per}}(t,y_0)\,(x_1,\ldots,x_{d-1},x_d ie^{2\pi i\tau_0/\omega})\|_{\mathcal{T}_{\varphi^t_{\text{per}}(y_0)}(\mathbb{R}^d_\omega)} = |T(t,\xi_0,\tau_0)(x_1,\ldots,x_d)|_{\mathbb{R}^d} \tag{2.30}$$

by (2.28), meaning that it is equivalent to calculate Lyapunov exponents for T or for T_{per}, cf. the MET Theorem 4.3.

To summarize, also we were not too precise in this section, it should be clear that in periodically forced problems we can switch from φ and T to φ_{per} and T_{per} whenever the need arises. Since the formulation in $\mathbb{R}^d$ usually is more convenient, this will only be the case when we are looking for invariant measures w.r.t. the semiflow, since by the unboundedness of $\varphi^t(y_0) = (\varphi_1(t,y_0),\ldots,\varphi_{d-1}(t,y_0),\tau_0+t)$ in the last component, $\mathcal{P}_{\text{inv}}(\varphi)$ will always be empty, while $\mathcal{P}_{\text{inv}}(\varphi_{\text{per}})$ is the natural set to consider here.

3 An example: A pendulum with dry friction

In this section we investigate in detail the example

$$\ddot{x}(\tau) + x(\tau) + \operatorname{sgn}\dot{x}(\tau) = \gamma\sin(\eta\tau) \quad \text{for} \quad \tau \in \mathbb{R}, \tag{1.1}$$

cf. the introduction. This equation will be considered dynamically correct as the differential inclusion

$$\ddot{x}(\tau)+x(\tau)\in\gamma\sin(\eta\tau)-\mathrm{Sgn}\,\dot{x}(\tau)\quad\text{a.e.},\tag{1.2}$$

with the corresponding autonomous version $\dot{y}\in F(y)$ a.e., i.e.

$$\frac{d}{dt}\begin{pmatrix}x\\ v\\ \tau\end{pmatrix}\in F(x,v,\tau):=\{v\}\times\{\gamma\sin(\eta\tau)-x-w:w\in\mathrm{Sgn}(v)\}\times\{1\}\text{ a.e.},\quad y=(x,v,\tau)\in\mathbb{R}^3.\tag{1.3}$$

3.1 Definition of the semiflow

We start by explaining how (1.3) generates a continuous semiflow $\varphi:[0,\infty[\times\mathbb{R}^3\to\mathbb{R}^3$.

Theorem 3.1 *For all $y_0=(x_0,v_0,\tau_0)\in\mathbb{R}^3$ there exists a unique solution $\varphi(\cdot,y_0):[0,\infty[\to\mathbb{R}^3$ of (1.3) with initial conditions $\varphi(0,y_0)=y_0$, or equivalently, a unique solution $x(\cdot;y_0):[\tau_0,\infty[\to\mathbb{R}$ of (1.2) with $x(\tau_0;y_0)=x_0$ and $\dot{x}(\tau_0;y_0)=v_0$. Thus (1.3) induces a semiflow $\varphi:[0,\infty[\times\mathbb{R}^3\to\mathbb{R}^3$, where*

$$\varphi^t(y_0)=(x(\tau_0+t;y_0),\dot{x}(\tau_0+t;y_0),\tau_0+t)\quad\textit{for}\quad t\in[0,\infty[.\tag{3.1}$$

In addition, $\varphi^t(x_0,v_0,\tau_0+k(2\pi/\eta))=\varphi^t(y_0)+(0,0,k(2\pi/\eta))$ for $k\in\mathbb{N}_0$ by uniqueness of solutions. Moreover, the semiflow is locally Lipschitz continuous on $[0,\infty[\times\mathbb{R}^3$.

We remark that by definition, a solution x of (1.2) on some (finite or infinite) interval $J\subset\mathbb{R}$ is understood to be C^1 on J such that in addition $\dot{x}$ is absolutely continuous with $\ddot{x}\in L^1_{\mathrm{loc}}(J)$, cf. [8, Paragraph 5].

Proof of Theorem 3.1 : Let the multivalued map $F:\mathbb{R}^3\to 2^{\mathbb{R}^3}\setminus\{\emptyset\}$ be defined as in (1.3). Then F is upper semicontinuous, has compact convex values, and satisfies

$$\|F(y)\|\le c(1+|y|)\quad\text{for}\quad y\in\mathbb{R}^3,\quad\text{with}\quad c=2(1+\gamma).\tag{3.2}$$

Moreover,

$$\langle F(y)-F(\bar{y}),y-\bar{y}\rangle\le\frac{\gamma\eta}{2}|y-\bar{y}|^2\quad\text{for}\quad y,\bar{y}\in\mathbb{R}^3,\tag{3.3}$$

meaning that $\langle z-\bar{z},y-\bar{y}\rangle\le\frac{\gamma\eta}{2}|y-\bar{y}|^2$ for $z\in F(y)$, $\bar{z}\in F(\bar{y})$, as is straightforward to verify. Consequently, every initial value problem corresponding to (1.3) has a solution on $[0,\infty[$ by [8, Lemma 5.1]. This solution is unique by (3.3), cf. [8, p. 137]. From (3.2) together with Gronwall's lemma and (1.3) it follows that for fixed $T>0$, $t\in[0,T]$ and $y_0\in\mathbb{R}^3$

$$|\varphi(t,y_0)|\le(|y_0|+cT)e^{cT}\quad\text{and}\quad|\partial_t\varphi(t,y_0)|\le c[1+(|y_0|+cT)e^{cT}]\quad\text{a.e.}\tag{3.4}$$

Let $y(t)=\varphi(t,y_0)$ and $\bar{y}(t)=\varphi(t,\bar{y}_0)$. By differentiating $|y(t)-\bar{y}(t)|^2$ and application of (3.3) we obtain

$$|\varphi(t,y_0)-\varphi(t,\bar{y}_0)|\le|y_0-\bar{y}_0|\,e^{\gamma\eta T/2}\quad\text{for}\quad t\in[0,T].\tag{3.5}$$

Thus (3.4) and (3.5) imply that φ is locally Lipschitz continuous on $[0,\infty[\times\mathbb{R}^3$. □

Since we intend to apply the results from Section 2, we already note that in particular we have obtained in Theorem 3.1 that

$$\varphi:[0,\infty[\times\mathbb{R}^3\to\mathbb{R}^3\quad\text{is a measurable semiflow which satisfies }(A4)\text{ in Section 2.1.}\tag{3.6}$$

3.2 Verification of $(A1)-(A10)$

In this section we will show that the pendulum example (1.3) fits into the framework built up in Section 2.1. To have a non-empty and non-trivial manifold M_∞, we only consider the case

$$\gamma \in]0,1[\tag{3.7}$$

in (1.3) resp. (1.2). Then

$$\mathcal{S}_\gamma = [\gamma-1, -\gamma+1] \subset \mathbb{R} \tag{3.8}$$

is the set of stationary solutions of (1.2). We remark that $\gamma = 1$ is excluded in order not to have M_∞ only a line, cf. Fig. 2 in Section 2. Although in what follows some arguments and estimates will rely on $\gamma \le 1$, for $\gamma \in]1,\infty[$ the approach to describe the system as in Section 2.1 and the verification of assumptions $(A1)-(A10)$ are similar. In case of $\gamma \in]1,\infty[$ there are no equilibria.

Here and in the following we always let $y=(x,v,\tau)$ denote a typical point in $\mathbb{R}^3$. We let

$$b^\pm(\tau) = \gamma \sin(\eta\tau) \pm 1, \tag{3.9}$$

and define $\tau_{\mathrm{mod}} = \tau \bmod (2\pi/\eta)$ for $\tau \in \mathbb{R}$. Then $b^\pm(\tau) = b^\pm(\tau_{\mathrm{mod}})$. Let

$$\begin{aligned}
M_+ := M_1 &:= \{y\in\mathbb{R}^3 : v>0\} \cup \{y\in\mathbb{R}^3 : v=0, x<b^-(\tau)\} \\
&\quad \cup \{y\in\mathbb{R}^3 : v=0, x=b^-(\tau), \tau_{\mathrm{mod}} \in [0,\pi/2\eta[\,\cup[3\pi/2\eta, 2\pi/\eta[\,\}, \\
M_- := M_2 &:= \{y\in\mathbb{R}^3 : v<0\} \cup \{y\in\mathbb{R}^3 : v=0, x>b^+(\tau)\} \\
&\quad \cup \{y\in\mathbb{R}^3 : v=0, x=b^+(\tau), \tau_{\mathrm{mod}} \in [\pi/2\eta, 3\pi/2\eta[\,\}, \\
M_{\mathrm{left}} := M_3 &:= \{y\in\mathbb{R}^3 : v=0, x=b^-(\tau), \tau_{\mathrm{mod}} \in]\pi/2\eta, 3\pi/2\eta[\,\} \\
&\quad \cup \{y\in\mathbb{R}^3 : v=0, b^-(\tau)<x<\gamma-1\}, \\
M_{\mathrm{right}} := M_4 &:= \{y\in\mathbb{R}^3 : v=0, x=b^+(\tau), \tau_{\mathrm{mod}} \in]3\pi/2\eta, 2\pi/\eta[\,\cup[0,\pi/2\eta[\,\} \\
&\quad \cup \{y\in\mathbb{R}^3 : v=0, -\gamma+1<x<b^+(\tau)\} \\
M_\infty &:= \{y\in\mathbb{R}^3 : v=0, \gamma-1 \le x \le -\gamma+1\},
\end{aligned} \tag{3.10}$$

In particular the M_k are mutually disjoint, and also $\mathbb{R}^3 = \bigcup_{k=1}^4 M_k \cup M_\infty$. The subscripts "left" resp. "right" indicate that the respective manifold is lying to the left-hand resp. right-hand side of M_∞. In fact, M_{left} and M_{right} need not be distinguished form the dynamical point of view, but this turns out to make the description easier to follow.

Next, we define the right-hand sides f_k with $b^\pm$ from (3.9) as

$$\begin{aligned}
&f_+(x,v,\tau) = (v, -x+b^-(\tau), 1), \quad f_-(x,v,\tau) = (v, -x+b^+(\tau), 1), \quad \text{and} \\
&f_{\mathrm{left}}(x,v,\tau) = f_{\mathrm{right}}(x,v,\tau) = f_\infty(x,v,\tau) = (0,0,1) \quad \text{for} \quad (x,v,\tau)\in\mathbb{R}^3,
\end{aligned} \tag{3.11}$$

and we set $f_1 = f_+$, $f_2 = f_-$, $f_3 = f_{\mathrm{left}}$, and $f_4 = f_{\mathrm{right}}$.

Finally, for $k \in \{1,2,3,4\} \cup \{\infty\}$ the exceptional sets $E_k \subset \overline{M_k}^{\mathbb{R}^3}$ are given through

$$\begin{aligned}
E_+ := E_1 &:= \{y\in\mathbb{R}^3 : v=0, x=b^-(\tau), \tau_{\mathrm{mod}} \in [0,\pi/2\eta] \cup [3\pi/2\eta, 2\pi/\eta[\,\}, \\
E_- := E_2 &:= \{y\in\mathbb{R}^3 : v=0, x=b^+(\tau), \tau_{\mathrm{mod}} \in [\pi/2\eta, 3\pi/2\eta]\}, \\
E_{\mathrm{left}} := E_3 &:= \{y\in\mathbb{R}^3 : v=0, x=b^-(\tau), \tau_{\mathrm{mod}} \in]\pi/2\eta, 3\pi/2\eta[\,\}, \\
E_{\mathrm{right}} := E_4 &:= \{y\in\mathbb{R}^3 : v=0, x=b^+(\tau), \tau_{\mathrm{mod}} \in [0,\pi/2\eta[\,\cup]3\pi/2\eta, 2\pi/\eta[\,\}, \\
E_\infty &:= \{\gamma-1, -\gamma+1\} \times \{0\} \times \mathbb{R}.
\end{aligned} \tag{3.12}$$

We start with the investigation of M_∞.

Lemma 3.2 *Let $t_1(y)=\infty$ for $y\in M_\infty$. Then for every $y=(x,0,\tau)\in M_\infty$, $\varphi^t(y)=(x,0,\tau+t)$ for $t\in[0,t_1(y)[$. Thus in particular $\varphi^t(y)\in M_\infty$ and $\varphi^t(y)=\Phi_{f_\infty}(t,y)$ for $t\in[0,t_1(y)[$.*

Proof: This follows from the uniqueness of solutions of (1.3) to the right and from the fact that $x-\gamma\sin(\eta[\tau+t])\in[x-\gamma,x+\gamma]\subset[-1,1]$ for $t\in[0,\infty[$, because we have $\gamma-1\le x\le -\gamma+1$ by definition of M_∞. □

In the following we always consider $\arcsin:[-1,1]\to[-\pi/2,\pi/2]$.

Lemma 3.3 *For $y=(x,0,\tau)\in M_{\text{left}}$ let $\sigma_{\text{left}}(x)=\dfrac{1}{\eta}\left(\dfrac{\pi}{2}+\arcsin\left(\dfrac{x+1}{\gamma}\right)\right)$, and define*

$$t_1(y)=\begin{cases}\sigma_{\text{left}}(x)-\tau_{\text{mod}}-\pi/2\eta & : \ \tau_{\text{mod}}\in[0,\pi/2\eta[\\ \sigma_{\text{left}}(x)-\tau_{\text{mod}}+3\pi/2\eta & : \ \tau_{\text{mod}}\in]\pi/2\eta,2\pi/\eta[\end{cases}. \tag{3.13}$$

Analogously, for $y=(x,0,\tau)\in M_{\text{right}}$ we let $\sigma_{\text{right}}(x)=\dfrac{1}{\eta}\left(\dfrac{\pi}{2}-\arcsin\left(\dfrac{x-1}{\gamma}\right)\right)$, and define

$$t_1(y)=\begin{cases}\sigma_{\text{right}}(x)-\tau_{\text{mod}}+\pi/2\eta & : \ \tau_{\text{mod}}\in[0,3\pi/2\eta[\\ \sigma_{\text{right}}(x)-\tau_{\text{mod}}+5\pi/2\eta & : \ \tau_{\text{mod}}\in]3\pi/2\eta,2\pi/\eta[\end{cases}. \tag{3.14}$$

Then for $y\in M_{\text{left}}$

(a) $t_1(y)\in]0,2\pi/\eta[$,

(b) $\varphi^t(y)=(x,0,\tau+t)$ for $t\in[0,t_1(y)[$, and hence in particular for $t\in[0,t_1(y)[$ we have $\varphi^t(y)\in M_{\text{left}}$, $\varphi^t(y)=\Phi_{f_{\text{left}}}(t,y)$, and $t_1(\varphi^t(y))+t=t_1(y)$. In addition,

$$\varphi(t_1(y),y)\in\{y\in\mathbb{R}^3: v=0, x=b^-(\tau), \tau_{\text{mod}}\in[0,\pi/2\eta[\cup]3\pi/2\eta,2\pi/\eta[\}\subset M_+. \tag{3.15}$$

An analogous statement holds for $y\in M_{\text{right}}$. Also $(x,0,\tau)\in M_{\text{left}}\Leftrightarrow(-x,0,\tau+\pi/\eta)\in M_{\text{right}}$, and in this case $t_1(x,0,\tau)=t_1(-x,0,\tau+\pi/\eta)$.

Proof: We only consider $y\in M_{\text{left}}$ and remark that the function $\sigma_{\text{left}}:]-\gamma-1,\gamma-1[\to]0,\pi/\eta[$ measures the distance in (vertical) τ-direction of the boundary curve given through $x=b^-(\tau)$, $\tau_{\text{mod}}\in]3\pi/2\eta,2\pi/\eta[\cup[0,\pi/2\eta[$, to the line $\tau=3\pi/2\eta$. If we treat as an example the case $\tau_{\text{mod}}\in[0,\pi/2\eta[$, then we obtain (a), since $b^-(\tau)<x<\gamma-1\Rightarrow\sin(\eta\tau)<(x+1)/\gamma<1\Rightarrow\tau_{\text{mod}}<\eta^{-1}\arcsin((x+1)/\gamma)<\pi/2\eta\Rightarrow 0<t_1(y)<\pi/2\eta-\tau_{\text{mod}}<2\pi/\eta$. Concerning (b), it follows from the uniqueness of solutions of (1.3) that the flow moves just on a straight line upwards until it hits the curve $x=b^-(\tau)$, because, again for $\tau_{\text{mod}}\in[0,\pi/2\eta[$ and for $t\in[0,t_1(y)[$, we obtain $\tau+t\in[\tau,\sigma_{\text{left}}(x)-\pi/2\eta[=[\tau,\eta^{-1}\arcsin((x+1)/\gamma)[\Rightarrow\gamma\sin(\eta[\tau+t])-x\in[\gamma\sin(\eta\tau)-x,1]\subset[-1,1]$, the latter being a consequence of $x<\gamma-1$ and $\gamma\le 1$. Thus $\frac{d}{dt}(x,0,\tau+t)=(0,0,1)\in F(\varphi^t(y))$ for $t\in[0,t_1(y)[$, hence $\varphi^t(y)=(x,0,\tau+t)$. From this we clearly get $\varphi^t(y)\in M_{\text{left}}$ and $\varphi^t(y)=\Phi_{f_{\text{left}}}(t,y)$ in $[0,t_1(y)[$. Moreover, if $\tau_{\text{mod}}\in[0,\pi/2\eta[$ and $t\in[0,t_1(y)[$, then $(\tau+t)_{\text{mod}}=\tau_{\text{mod}}+t\in[0,\pi/2\eta[$, and therefore $t_1(x,0,\tau+t)+t=\sigma_{\text{left}}(x)-(\tau+t)_{\text{mod}}-\pi/2\eta+t=t_1(y)$. Since also (3.15) is clear from the definition of $t_1(y)$, we thus have proved (b).

To show the relation between M_{left} and M_{right}, we again consider as an example only $y=(x,0,\tau)\in M_{\text{left}}$ with $\tau_{\text{mod}}\in[0,\pi/2\eta[$. Then it follows from $b^-(\tau)<x<\gamma-1$ that $-\gamma+1<-x<b^+(\tau+\pi/\eta)$, since $b^+(\tau+\pi/\eta)=-b^-(\tau)$, and hence $(-x,0,\tau+\pi/\eta)\in M_{\text{right}}$. Also, $(\tau+\pi/\eta)_{\text{mod}}=\tau_{\text{mod}}+\pi/\eta\in[\pi/\eta,3\pi/2\eta[$, and consequently $\sigma_{\text{left}}(x)=\sigma_{\text{right}}(-x)$ yields $t_1(x,0,\tau)=\sigma_{\text{left}}(x)-\tau_{\text{mod}}-\pi/2\eta=\sigma_{\text{right}}(-x)-(\tau+\pi/\eta)_{\text{mod}}+\pi/2\eta=t_1(-x,0,\tau+\pi/\eta)$. □

The definition of $t_1(\cdot)$ on M_+ or M_- is more complicated, since trajectories starting in e.g. $y_0\in M_+$ may return to the surface $\{v=0\}$ at some time $s_1(y_0)$ without passing to a different manifold afterwards, cf. the trajectory of y_0 in Fig. 2 in Section 2. We start with a preparatory lemma.

Lemma 3.4 *For $y_0 \in M_+$ there exists $\varepsilon > 0$ such that*

$$\varphi(]0,\varepsilon] \times \{y_0\}) \subset \{y \in \mathbb{R}^3 : v > 0\} \subset M_+ . \tag{3.16}$$

Define

$$s_1(y_0) = \sup\Big\{\varepsilon > 0 : \varphi(]0,\varepsilon] \times \{y_0\}) \subset \{y \in \mathbb{R}^3 : v > 0\}\Big\}. \tag{3.17}$$

Then the following holds:

(a) $s_1(y_0) \in]0,\infty[$.

(b) $\varphi(t,y_0) \in \{y \in \mathbb{R}^3 : v > 0\}$ *for* $t \in]0, s_1(y_0)[$, *and* $\varphi(s_1(y_0), y_0) \in \{y \in \mathbb{R}^3 : v = 0, x \geq b^-(\tau)\}$.

(c) Let $y_1 = (x_1, 0, \tau_1) = \varphi(s_1(y_0), y_0)$. *If* $y_1 \in M_+$, *then*

$$y_1 \in \{y \in \mathbb{R}^3 : v = 0, x = b^-(\tau), \tau_{\mathrm{mod}} \in [0, \pi/2\eta[\cup]3\pi/2\eta, 2\pi/\eta[\}. \tag{3.18}$$

In addition, if also $y_2 = (x_2, 0, \tau_2) = \varphi(s_1(y_1), y_1) \in M_+$, *then* $s_1(y_1) \geq \pi \wedge (\pi/\eta)$.

(d) If $y_0 = (x_0, 0, \tau_0) \in \{y \in \mathbb{R}^3 : v = 0, x = b^-(\tau), \tau_{\mathrm{mod}} \in [0, \pi/2\eta[\cup]3\pi/2\eta, 2\pi/\eta[\} \subset M_+$, *then*

$$s_1(y_0) > \begin{cases} \pi \wedge \Big(\pi/2\eta - (\tau_0)_{\mathrm{mod}}\Big) & : \quad (\tau_0)_{\mathrm{mod}} \in [0, \pi/2\eta[\\ \pi \wedge \Big(5\pi/2\eta - (\tau_0)_{\mathrm{mod}}\Big) & : \quad (\tau_0)_{\mathrm{mod}} \in]3\pi/2\eta, 2\pi/\eta[\end{cases}.$$

(e) If $\gamma \in]0,1]$ *and* $\varepsilon_0 \in]0, 2\gamma[$, *then there exist* $\varepsilon_1 > 0$ *and* $\delta_1 > 0$ *with the following property:*

$$\begin{aligned} y_0 = (x_0, 0, \tau_0) \in \{y \in \mathbb{R}^3 : v = 0, x = b^-(\tau), \tau_{\mathrm{mod}} \in [0, \pi/2\eta[\cup]3\pi/2\eta, 2\pi/\eta[, x \leq \gamma - 1 - \varepsilon_0\} \\ \Longrightarrow \quad \varepsilon_1 \in]0, s_1(y_0)[, \quad \varphi(\varepsilon_1, y_0) \in \{y \in \mathbb{R}^3 : v \geq \delta_1\}. \end{aligned}$$

(f) $\varphi(s_1(y_0), y_0) \in \mathbb{R}^3 \setminus \{y \in \mathbb{R}^3 : v = 0, x = b^-(\tau), \tau_{\mathrm{mod}} \in]\pi/2\eta, 3\pi/2\eta[\}$.

(g) For $y_0 \in M_+ \setminus \{y \in \mathbb{R}^3 : v = 0, x = b^-(\tau)\}$ *there exists* $\delta > 0$ *such that* $s_1(y) > \delta$ *for every* $y \in B_\delta(y_0) \cap M_+$.

An analogous statement holds for $y_0 \in M_-$.

Proof: We consider only $y_0 = (x_0, v_0, \tau_0) \in M_+$ and prove (3.16) first. If $v_0 > 0$, then the claim follows from the continuity of $\varphi(\cdot, y_0)$. The second case to handle is $y_0 = (x_0, 0, \tau_0) \in M_+$ with $x_0 < b^-(\tau_0)$. We denote by $x(\tau)$, $\tau \in [\tau_0, \infty[$, the corresponding solution of (1.2) with $x(\tau_0) = x_0$ and $\dot{x}(\tau_0) = 0$. Define $\Delta(\tau) = b^-(\tau) - x(\tau)$. Then $\Delta \in C^1([\tau_0, \infty[)$ and $\Delta(\tau_0) > 0$. Since (1.2) is equivalent to

$$\left.\begin{array}{lll} \ddot{x}(\tau) + x(\tau) = b^-(\tau) & \text{a.e. in } \{\tau \in [\tau_0, \infty[: \dot{x}(\tau) > 0\}, \\ \ddot{x}(\tau) + x(\tau) = b^+(\tau) & \text{a.e. in } \{\tau \in [\tau_0, \infty[: \dot{x}(\tau) < 0\}, \\ \ddot{x}(\tau) + x(\tau) \in [b^-(\tau), b^+(\tau)] & \text{a.e. in } \{\tau \in [\tau_0, \infty[: \dot{x}(\tau) = 0\}, \end{array}\right\} \tag{3.19}$$

we have $\ddot{x}(\tau) \geq \Delta(\tau)$ a.e. in $[\tau_0, \infty[$, and thus $\dot{x}(\tau) = \dot{x}(\tau_0) + \int_{\tau_0}^{\tau} \ddot{x}(\sigma)\, d\sigma \geq \int_{\tau_0}^{\tau} \Delta(\sigma)\, d\sigma > 0$ locally to the right of τ_0. The last case to analyze is $y_0 = (x_0, 0, \tau_0) \in M_+$ with $x_0 = b^-(\tau_0)$ and $(\tau_0)_{\mathrm{mod}} \in [0, \pi/2\eta[\cup [3\pi/2\eta, 2\pi/\eta[$. Consider the solution $z(\cdot)$ on $[\tau_0, \infty[$ of the initial value problem

$$\ddot{z} + z = b^-(\tau), \quad z(\tau_0) = x_0, \quad \dot{z}(\tau_0) = 0, \quad \text{i.e.} \quad z(\tau) = x_0 \cos(\tau - \tau_0) + \int_{\tau_0}^{\tau} \sin(\tau - \sigma)\, b^-(\sigma)\, d\sigma.$$

Then it follows from $x_0 = b^-(\tau_0)$, by deriving a second order ODE for $\dot{z}$, that

$$\dot{z}(\tau) = \gamma\eta \int_{\tau_0}^{\tau} \sin(\tau - \sigma) \cos(\eta\sigma)\, d\sigma \quad \text{for} \quad \tau \in [\tau_0, \infty[. \tag{3.20}$$

Hence $(\tau_0)_{\rm mod}\in[0,\pi/2\eta[\cup[3\pi/2\eta,2\pi/\eta[$ implies $\dot z(\tau)>0$ in some $]\tau_0,\tau_0+\varepsilon]$. By uniqueness to the right of solutions to (1.2) we obtain $x(\tau)=z(\tau)$ until the first time $\tau_1>\tau_0$ with $z(\tau_1)=0$, so at least in $]\tau_0,\tau_0+\varepsilon]$. Transferred back to (1.3) this gives (3.16), since in $]0,\varepsilon]$ we have $\varphi(t,y_0)=(z(\tau_0+t),\dot z(\tau_0+t),\tau_0+t)$.

Therefore $s_1(y_0)>0$ is well-defined, and by definition and from the continuity of $\varphi(\cdot,y_0)$ it follows that $\varphi(t,y_0)\in\{y\in\mathbb{R}^3:v>0\}$ for $t\in]0,s_1(y_0)[$ and $\varphi(s_1(y_0),y_0)\in\{y\in\mathbb{R}^3:v=0\}$. Let $y_1=(x_1,v_1,\tau_1)=\varphi(s_1(y_0),y_0)$ denote the point of arrival at the surface $\{v=0\}$ of the trajectory at time $s_1(y_0)$. Then $v_1=0$ and $\tau_1=\tau_0+s_1(y_0)$, and also $x_1\ge b^-(\tau_1)$. To see the latter, we assume on the contrary that $x_1<b^-(\tau_1)$. The corresponding solution $x(\cdot)$ of (1.2) satisfies $\ddot x+x=b^-(\tau)$ in $]\tau_0,\tau_1[$, $\dot x(\tau)>0$ in $]\tau_0,\tau_1[$, $x(\tau_0)=x_0$, $\dot x(\tau_0)=v_0$, $x(\tau_1)=x_1$, and $\dot x(\tau_1)=v_1=0$. Then $\ddot x(\tau)=b^-(\tau)-x(\tau)\ge\delta_0>0$ in some interval $]\tau_1-\delta_1,\tau_1[\subset]\tau_0,\tau_1[$, and therefore $0=\dot x(\tau_1)=\dot x(\tau_1-\delta_1)+\int_{\tau_1-\delta_1}^{\tau_1}\ddot x(\sigma)\,d\sigma\ge\delta_0\delta_1$, a contradiction. Hence $x_1\ge b^-(\tau_1)$, and thus (b) is proved. Also $s_1(y_0)$ is finite, since otherwise, by (3.16), there would exist $\tau_1\ge\tau_0$ and a solution $x(\cdot)$ on $[\tau_1,\infty[$ of $\ddot x+x=b^-(\tau)$, $x(\tau_1)=x_1$ and $\dot x(\tau_1)=v_1>0$ with $\dot x(\tau)>0$ in $[\tau_1,\infty[$, a contradiction to Lemma 4.8. This proves (a).

Concerning (c), let $y_1=(x_1,0,\tau_1)=\varphi(s_1(y_0),y_0)$ be as above. Then $x_1\ge b^-(\tau_1)$ by (b). Suppose now in addition that $y_1\in M_+$ and assume $(\tau_1)_{\rm mod}=3\pi/2\eta$. Then $x(\tau)<x(\tau_1)=b^-(\tau_1)\le b^-(\tau)$ in $]\tau_0,\tau_1[$, and consequently we obtain the contradiction $0=\dot x(\tau_1)=\dot x(\tau_0)+\int_{\tau_0}^{\tau_1}\ddot x(\sigma)\,d\sigma>0$. Because of $y_1\in M_+$ and $v_1=0$, (3.18) follows.

To derive the last statement of (c), we suppose that also $y_2=(x_2,0,\tau_2)=\varphi(s_1(y_1),y_1)\in M_+$, and we will first show that

$$s_1(y_1)>\begin{cases}\pi\wedge\Big(\pi/2\eta-(\tau_1)_{\rm mod}\Big) & : \quad (\tau_1)_{\rm mod}\in[0,\pi/2\eta[\\ \pi\wedge\Big(5\pi/2\eta-(\tau_1)_{\rm mod}\Big) & : \quad (\tau_1)_{\rm mod}\in]3\pi/2\eta,2\pi/\eta[\end{cases}. \tag{3.21}$$

To prove this, we consider as an example the case $(\tau_1)_{\rm mod}\in[0,\pi/2\eta[$. From the first part of (c), the discussion above, and from (3.20) we know that

$$\dot x(\tau)=\gamma\eta\int_{\tau_1}^{\tau}\sin(\tau-\sigma)\cos(\eta\sigma)\,d\sigma\quad\text{for}\quad\tau\in[\tau_1,\tau_2], \tag{3.22}$$

$\dot x(\tau)>0$ in $]\tau_1,\tau_2[$, and that $\tau_2=\tau_1+s_1(y_1)$ is the first zero of $\dot x$ after the time τ_1. For $\tau\in]\tau_1,\tau_1+(\pi/2\eta-(\tau_1)_{\rm mod})]\cap]\tau_1,\tau_1+\pi]$ we clearly have $\dot x(\tau)>0$ by (3.22), and therefore we obtain (3.21). To show next $s_1(y_1)\ge\pi\wedge(\pi/\eta)$, we continue to treat the case $(\tau_1)_{\rm mod}\in[0,\pi/2\eta[$. By (3.21) we thus may assume that $\pi\wedge(\pi/2\eta-(\tau_1)_{\rm mod})=\pi/2\eta-(\tau_1)_{\rm mod}$, i.e. $s_1(y_1)>\pi/2\eta-(\tau_1)_{\rm mod}$. Choose the unique $k,l\in\mathbb{Z}$ such that $\tau_1=k(2\pi/\eta)+(\tau_1)_{\rm mod}$ and $\tau_2=l(2\pi/\eta)+(\tau_2)_{\rm mod}$. Since $\tau_2-\tau_1=s_1(y_1)>0$ we have $l\ge k$. In case that $l\ge k+1$ we find $s_1(y_1)=\tau_2-\tau_1\ge2\pi/\eta+(\tau_2)_{\rm mod}-(\tau_1)_{\rm mod}\ge 2\pi/\eta-\pi/2\eta=3\pi/2\eta$. On the other hand, in case that $k=l$ we first note that by the part of (c) already proven, $y_2\in\{y\in\mathbb{R}^3:v=0,x=b^-(\tau),\tau_{\rm mod}\in[0,\pi/2\eta[\cup]3\pi/2\eta,2\pi/\eta[\}$, and thus in particular $(\tau_2)_{\rm mod}\in[0,\pi/2\eta[\cup]3\pi/2\eta,2\pi/\eta[$. Since $(\tau_2)_{\rm mod}=\tau_2-\tau_1+(\tau_1)_{\rm mod}=s_1(y_1)+(\tau_1)_{\rm mod}>\pi/2\eta$, we hence have $(\tau_2)_{\rm mod}\in]3\pi/2\eta,2\pi/\eta[$. It follows that $s_1(y_1)=(\tau_2)_{\rm mod}-(\tau_1)_{\rm mod}\ge3\pi/2\eta-\pi/2\eta=\pi/\eta$, as was claimed.

The proof of (d) is analogous to the one given for (3.21), and to show (e) we may argue as follows: since $x\le\gamma-1-\varepsilon_0$ for all y_0 in the set in question, we find $\varepsilon_1\in]0,(\pi/2)\wedge(\pi/4\eta)[$ such that $(\tau_0)_{\rm mod}\in[0,\pi/2\eta-2\varepsilon_1[\cup]3\pi/2\eta,2\pi/\eta[$ for these $y_0=(x_0,0,\tau_0)$. In particular, by (d), $s_1(y_0)\ge2\varepsilon_1$. Let

$$\delta_1=\min_{\sigma_0\in[0,2\pi/\eta-\varepsilon_1]\cup[3\pi/2\eta,2\pi/\eta]}\left(\gamma\eta\int_{\sigma_0}^{\sigma_0+\varepsilon_1}\sin(\sigma_0+\varepsilon_1-\sigma)\cos(\eta\sigma)\,d\sigma\right).$$

Since $[\eta\sigma_0,\eta(\sigma_0+\varepsilon_1)]\subset[0,\pi/2]\cup[3\pi/2,5\pi/4]$ for $\sigma_0\in[\ldots]\cup[\ldots]$, it follows that $\delta_1>0$. Because $(x(\tau_0+t),\dot x(\tau_0+t),\tau_0+t)=\varphi^t(y_0)$ for $t\in[0,s_1(y_0)[$ with the corresponding solution x of (1.2), we obtain like in (3.20) or (3.22) for the y_0 under consideration that

$$\dot x(\tau_0+\varepsilon_1)=\gamma\eta\int_{\tau_0}^{\tau_0+\varepsilon_1}\sin(\tau_0+\varepsilon_1-\sigma)\cos(\eta\sigma)\,d\sigma$$

$$= \gamma\eta \int_{(\tau_0)_{\text{mod}}}^{(\tau_0)_{\text{mod}}+\varepsilon_1} \sin((\tau_0)_{\text{mod}} + \varepsilon_1 - \sigma)\cos(\eta\sigma)\,d\sigma \geq \delta_1\,,$$

and hence the claim of (e).

Concerning (f), if $y_1 = (x_1, 0, \tau_1) = \varphi(s_1(y_0), y_0)$ had $x_1 = b^-(\tau_1)$ and $(\tau_1)_{\text{mod}} \in]\pi/2\eta, 3\pi/2\eta[$, then for the corresponding solution $x(\cdot)$ of (1.2) we obtain $x(\tau) = x_1\cos(\tau - \tau_1) + \int_{\tau_1}^{\tau}\sin(\tau-\sigma)\,b^-(\sigma)\,d\sigma$ for $\tau \in [\tau_0, \tau_1]$, as above by uniqueness and continuity of the solutions. It follows that $0 < \dot{x}(\tau) = \gamma\eta\int_{\tau_1}^{\tau}\sin(\tau-\sigma)\cos(\eta\sigma)d\sigma = \gamma\eta\int_{\tau}^{\tau_1}\sin(\sigma-\tau)\cos(\eta\sigma)d\sigma$ for $\tau \in]\tau_0, \tau_1[$. But this formula gives the contradiction $\dot{x}(\tau) < 0$ for τ sufficiently close to τ_1, since $(\tau_1)_{\text{mod}} \in]\pi/2\eta, 3\pi/2\eta[$.

To show (g), let $y_0 = (x_0, v_0, \tau_0) \in M_+ \setminus \{y \in \mathbb{R}^3 : v = 0, x = b^-(\tau)\}$ be fixed, and assume first that $v_0 = 0$. Then $b^-(\tau_0) - x_0 = 2\delta_0 > 0$ by definition of M_+. Choose $\delta_1 > 0$ and $c_1 > 0$ such that $b^-(\bar{\tau}_0) - \bar{x}_0 \geq \delta_0$ and $|\bar{x}_0| + |\bar{v}_0| \leq c_1$ for all $\bar{y}_0 = (\bar{x}_0, \bar{v}_0, \bar{\tau}_0) \in B_{\delta_1}(y_0)$. Fix $\bar{y}_0 \in B_{\delta_1}(y_0) \cap M_+$ and let $\bar{y}_1 = (\bar{x}_1, 0, \bar{\tau}_1) = \varphi(s_1(\bar{y}_0), \bar{y}_0)$. Then $\bar{\tau}_1 = \bar{\tau}_0 + s_1(\bar{y}_0)$ and $\ddot{x} + x = b^-(\tau)$ in $]\bar{\tau}_0, \bar{\tau}_1[$ for the corresponding solution $x(\cdot)$ of (1.2). Therefore

$$\begin{aligned} |\dot{x}(\tau)| &= \left| -\bar{x}_0\sin(\tau - \bar{\tau}_0) + \bar{v}_0\cos(\tau - \bar{\tau}_0) + \int_{\bar{\tau}_0}^{\tau}\cos(\tau-\sigma)\,b^-(\sigma)\,d\sigma \right| \\ &\leq |\bar{x}_0| + |\bar{v}_0| + (1+\gamma)\,(\tau - \bar{\tau}_0) \leq c_1 + (1+\gamma)\,s_1(\bar{y}_0) \end{aligned}$$

for $\tau \in [\bar{\tau}_0, \bar{\tau}_1]$, the estimate being fairly optimal in the resonant case $\eta = 1$. By part (b) we have $\bar{x}_1 \geq b^-(\bar{\tau}_1)$, and thus it follows from $\bar{x}_0 = x(\bar{\tau}_0)$ and $\bar{x}_1 = x(\bar{\tau}_1)$ that

$$\delta_0 \leq \bar{x}_1 - \bar{x}_0 + b^-(\bar{\tau}_0) - b^-(\bar{\tau}_1) \leq s_1(\bar{y}_0)\,(c_1 + \gamma\eta + (1+\gamma)\,s_1(\bar{y}_0))\,.$$

Using

$$x > 0,\, a > 0,\, c > 0,\, b \geq 0, \quad a \leq x(b + cx) \quad \Longrightarrow \quad x \geq \frac{a}{\sqrt{b^2 + 4ac}}\,, \tag{3.23}$$

this gives the claim of (g) in case that $v_0 = 0$. So we assume next that $y_0 = (x_0, v_0, \tau_0) \in M_+ \setminus \{y \in \mathbb{R}^3 : v = 0, x = b^-(\tau)\}$ with $v_0 > 0$. If (g) would be wrong, then $y_n \to y_0$ for some sequence $(y_n) \subset M_+$ with $s_1(y_n) \to 0$. By (b) we have $\varphi(s_1(y_n), y_n) \in \{y \in \mathbb{R}^3 : v = 0\}$, and thus by continuity of φ, cf. Theorem 3.1, also $y_0 = \varphi(0, y_0) \in \{y \in \mathbb{R}^3 : v = 0\}$, a contradiction. Hence we are done. □

Thus for given $y_0 \in M_+$ there are two possibilities: either $\varphi(s_1(y_0), y_0) \in \mathbb{R}^3 \setminus M_+$, or alternatively $y_1 = \varphi(s_1(y_0), y_0) \in M_+$. In the first case, we have found a true switching point from M_+ to a different manifold, i.e. we can define $t_1(y_0) = s_1(y_0)$. On the other hand, if $y_1 \in M_+$ we can continue and let $s_2(y_0) = s_1(y_1) + s_1(y_0) > s_1(y_0)$ and investigate whether $y_2 = \varphi(s_2(y_0), y_0) = \varphi(s_1(y_1), y_1) \in \mathbb{R}^3 \setminus M_+$ or if $y_2 \in M_+$, and so on. Hence in general we can define $s_{j+1}(y_0) = s_1(y_j) + s_j(y_0) > s_j(y_0)$, whenever $y_j = \varphi(s_j(y_0), y_0) = \varphi(s_1(y_{j-1}), y_{j-1}) \in M_+$. The next lemma in particular says that at some time $s_j(y_0)$ there will be a true switching point.

Lemma 3.5 *For $y_0 \in M_+$ there exists a first $j = j(y_0) \in \mathbb{N}$ such that $\varphi(s_j(y_0), y_0) \in \mathbb{R}^3 \setminus M_+$. We define*

$$t_1(y_0) = s_j(y_0)\,.$$

Then

(a) $t_1(y_0) \in]0, \infty[$,

(b) $\varphi(t, y_0) \in \{y \in \mathbb{R}^3 : v > 0\} \subset M_+$ *for* $t \in]0, s_1(y_0)[\cup]s_1(y_0), s_2(y_0)[\cup \ldots \cup]s_{j-1}(y_0), s_j(y_0)[$ *and* $y_1, y_2, \ldots, y_{j-1} \in \{y \in \mathbb{R}^3 : v = 0, x = b^-(\tau), \tau_{\text{mod}} \in [0, \pi/2\eta[\cup]3\pi/2\eta, 2\pi/\eta[\} \subset M_+$, *in particular*

(b1) $\varphi(t, y_0) \in M_+$ *for* $t \in [0, t_1(y_0)[$, *and*

(b2) for $y_0=(x_0,v_0,\tau_0)$ *and* $t\in[0,t_1(y_0)[$

$$\varphi(t,y_0)=\Phi_{f_+}(t,y_0)=\begin{pmatrix} x_0\cos t+v_0\sin t+\int_0^t\sin(t-s)\,b^-(\tau_0+s)\,ds \\ -x_0\sin t+v_0\cos t+\int_0^t\cos(t-s)\,b^-(\tau_0+s)\,ds \\ \tau_0+t \end{pmatrix}. \tag{3.24}$$

(c) $\varphi(t_1(y_0),y_0)\in\{y\in\mathbb{R}^3:v=0\}\cap\left(\mathbb{R}^3\setminus M_+\right)$.

(d) $\varphi(t_1(y_0),y_0)\in\mathbb{R}^3\setminus\{y\in\mathbb{R}^3:v=0,x=b^-(\tau),\tau_{\rm mod}\in]\pi/2\eta,3\pi/2\eta[\}$.

(e) If $j=j(y_0)\geq 2$*, then* $t_1(y_0)\geq\pi\wedge(\pi/\eta)$.

An analogous statement holds for $y_0\in M_-$*. In particular, the formula corresponding to (3.24) is obtained by replacing* f_+ *through* f_- *and* $b^-(\tau_0+s)$ *through* $b^+(\tau_0+s)$.

Proof: By Lemma 3.4 and by (3.19) we only have to show that at some future time there will be a true switching point from M_+ to a different manifold. In the opposite case the $s_j(y_0)$ would be defined for every $j\in\mathbb{N}$, and by subsequent application of the last part of Lemma 3.4(c) we would obtain $s_{j+1}(y_0)=s_1(y_j)+s_j(y_0)\geq s_j(y_0)+(\pi\wedge(\pi/\eta))$, in particular $s_j(y_0)\uparrow\infty$ as $j\to\infty$. With $y_0=(x_0,v_0,\tau_0)$, by (b) of the present lemma we find $\tau_1\geq\tau_0$ and a solution of $\ddot{x}+x=b^-(\tau)$ on $[\tau_1,\infty[$ with $\dot{x}(\tau_1)>0$ and $\dot{x}(\tau)\geq 0$ in $[\tau_1,\infty[$, but this is a contradiction to Lemma 4.8. □

Lemma 3.6 *For* $y_0\in M_+$ *we have*

$$t_1(y_0)=\sup\{t_*>0:\varphi(t,y_0)\in M_+ \textit{ for } t\in[0,t_*[\}. \tag{3.25}$$

Hence in particular $t_1(\varphi(t,y_0))+t=t_1(y_0)$ *for* $t\in[0,t_1(y_0)[$*. An analogous statement holds for* $y_0\in M_-$*.*

Proof: If $t_*>0$ is given such that $\varphi(t,y_0)\in M_+$ for $t\in[0,t_*[$, then $t_1(y_0)\geq t_*$, since otherwise $\varphi(t_1(y_0),y_0)\in M_+$, a contradiction to Lemma 3.5(c). Thus $t_1(y_0)\geq\sup\{t_*>0:\ldots\}$. On the other hand, $t_1(y_0)\in\{t_*>0:\ldots\}$ by Lemma 3.5(b1), hence proving (3.25). Taking into account (3.25), also $t_1(\varphi(t,y_0))+t=t_1(y_0)$ for $t\in[0,t_1(y_0)[$ follows. □

To summarize, by (3.6), Lemma 3.2, Lemma 3.3, Lemma 3.5, and Lemma 3.6, so far we have defined the function $t_1(\cdot):\mathbb{R}^3\to]0,\infty[$, and we have shown that

$$(A1),\,(A2),\,(A3),\,(A4),\text{ and }(A5)\text{ in Section 2.1 are satisfied.} \tag{3.26}$$

Next we have to verify that the switching times of a trajectory which does not enter M_∞ do not have an accumulation point, i.e. $(A6)$. For this we first state some simple but useful consequences of the above definitions of $t_1(\cdot)$.

Corollary 3.7 *If* $y_0=(x_0,0,\tau_0)\in M_+\cap\{y\in\mathbb{R}^3:v=0\}$ *and* $\varphi(t_1(y_0),y_0)\in M_-$*, then*

$$t_1(y_0)\geq\frac{2}{\left((|x_0|+\gamma\eta)^2+8(1+\gamma)\right)^{1/2}}.$$

The same estimate holds for a switching from $y_0\in M_-\cap\{y\in\mathbb{R}^3:v=0\}$ *to* $\varphi(t_1(y_0),y_0)\in M_+$.

Proof: The argument is similar to Lemma 3.4(g). Let $(x_1,v_1,\tau_1)=\varphi(t_1(y_0),y_0)$, i.e. $\tau_1-\tau_0=t_1(y_0)$ by Lemma 3.5(b2) and also $v_1=0$ by Lemma 3.5(c). Hence $x_0\le b^-(\tau_0)$ and $x_1\ge b^+(\tau_1)$ by definition of M_+ and M_-. The corresponding solution $x(\cdot)$ of (1.2) satisfies $\ddot{x}+x=b^-(\tau)$ in $]\tau_0,\tau_1[$, $x(\tau_0)=x_0$, $x(\tau_1)=x_1$, and $\dot{x}(\tau_0)=0=\dot{x}(\tau_1)$. Therefore

$$|\dot{x}(\tau)|=\left|-x_0\sin(\tau-\tau_0)+\int_{\tau_0}^{\tau}\cos(\tau-\sigma)\,b^-(\sigma)\,d\sigma\right|\le|x_0|+(1+\gamma)(\tau-\tau_0)\le|x_0|+(1+\gamma)\,t_1(y_0)$$

for $\tau\in[\tau_0,\tau_1]$. Thus

$$\begin{aligned}2=b^+(\tau_1)-b^-(\tau_1)&\le b^+(\tau_1)-b^-(\tau_0)+|b^-(\tau_0)-b^-(\tau_1)|\le x(\tau_1)-x(\tau_0)+\gamma\eta\,|\tau_0-\tau_1|\\&\le t_1(y_0)\Big(|x_0|+\gamma\eta+(1+\gamma)\,t_1(y_0)\Big).\end{aligned}\tag{3.27}$$

Using estimate (3.23), this gives the claim. □

Corollary 3.8 *If $y_0\in M_+\cap\{y\in\mathbb{R}^3:v=0\}$ and $y_1=\varphi(t_1(y_0),y_0)\in M_{\rm right}$, then*

$$t_1(y_0)+t_1(y_1)\ge\min\left\{\frac{\pi}{4\eta},\,\frac{\pi^2\gamma}{16\sqrt{16|x_0|^2+\pi^2\gamma(1+\gamma)}}\right\}.$$

The same estimate holds for a switching from $y_0\in M_-\cap\{y\in\mathbb{R}^3:v=0\}$ to $\varphi(t_1(y_0),y_0)\in M_{\rm left}$.

Proof: Let $y_1=(x_1,0,\tau_1)\in M_{\rm right}$. We will distinguish cases. Case 1: $(\tau_1)_{\rm mod}\in]3\pi/2\eta,2\pi/\eta[$. Then by Lemma 3.3 we have $t_1(y_1)=\sigma_{\rm right}(x_1)-(\tau_1)_{\rm mod}+5\pi/2\eta\ge\sigma_{\rm right}(x_1)+\pi/2\eta\ge\pi/2\eta$. Thus also $t_1(y_0)+t_1(y_1)\ge\pi/2\eta$. Case 2: $(\tau_1)_{\rm mod}\in]\pi/\eta,3\pi/2\eta[$. We may write $\tau_1=k(2\pi/\eta)+(\tau_1)_{\rm mod}$ and $\tau_0=l(2\pi/\eta)+(\tau_0)_{\rm mod}$ for unique $k,l\in\mathbb{Z}$, and since $\tau_1-\tau_0=t_1(y_0)>0$ we may assume that $k=l$, because otherwise $t_1(y_0)\ge2\pi/\eta+(\tau_1)_{\rm mod}-(\tau_0)_{\rm mod}\ge\pi/\eta$. Thus $t_1(y_0)=(\tau_1)_{\rm mod}-(\tau_0)_{\rm mod}$. Case 2.1: $(\tau_0)_{\rm mod}\in[0,3\pi/4\eta]$. Then $t_1(y_0)\ge\pi/4\eta$. Case 2.2: $(\tau_0)_{\rm mod}\in]3\pi/4\eta,(\tau_1)_{\rm mod}[$. This implies, retaining the notation from Corollary 3.7, that $x_0=x(\tau_0)\le\gamma\sin(\eta\tau_0)-1\le\gamma\sin(3\pi/4)-1=2^{-1/2}\gamma-1$, and consequently as in (3.27)

$$(\pi/8)^2\gamma<2-(1+2^{-1/2})\gamma=(1-\gamma)-(2^{-1/2}\gamma-1)\le x(\tau_1)-x(\tau_0)\le t_1(y_0)\Big(|x_0|+(1+\gamma)\,t_1(y_0)\Big).$$

By (3.23) we hence find the claimed estimate. Case 3: $(\tau_1)_{\rm mod}\in[0,\pi/\eta]$. Case 3.1: $(\pi/8)^2\gamma\le t_1(y_0)\Big(|x_0|+(1+\gamma)\,t_1(y_0)\Big)$. Here we obtain the same estimate as before. Case 3.2: $(\pi/8)^2\gamma> t_1(y_0)\Big(|x_0|+(1+\gamma)\,t_1(y_0)\Big)$. In this most interesting case, $t_1(y_0)$ is somehow "small", and we will show that $t_1(y_1)$ is large enough to compensate for this. Since $x_0=x(\tau_0)\le b^-(\tau_0)\le\gamma-1\le1-\gamma$ and $x_1=x(\tau_1)\ge1-\gamma$, we have

$$0\le x_1-(1-\gamma)\le x(\tau_1)-x(\tau_0)\le t_1(y_0)\Big(|x_0|+(1+\gamma)\,t_1(y_0)\Big)=:T_1(y_0).$$

Thus $0\le\gamma^{-1}(x_1-1)+1\le\gamma^{-1}T_1(y_0)$, and hence in particular $-1\le\gamma^{-1}(x_1-1)\le(\pi/8)^2-1\le0$ by assumption. We have $t_1(y_1)=\pi/2\eta-(\tau_1)_{\rm mod}+\Big(\pi/2-\arcsin(\gamma^{-1}[x_1-1])\Big)/\eta$ by Lemma 3.3. Using the estimate $|\arcsin(x)+\pi/2|\le2\sqrt{x+1}$, which is valid for $x\in[-1,0]$, we therefore obtain

$$|\arcsin(\gamma^{-1}[x_1-1])+\pi/2|\le2\Big(\gamma^{-1}T_1(y_0)\Big)^{1/2},$$

and this in turn yields

$$t_1(y_1)=3\pi/2\eta-(\tau_1)_{\rm mod}-\Big(\arcsin(\gamma^{-1}[x_1-1])+\pi/2\Big)/\eta\ge\pi/2\eta-2\Big(\gamma^{-1}T_1(y_0)\Big)^{1/2}/\eta\ge\pi/4\eta,$$

again by assumption $(\pi/8)^2\gamma>T_1(y_0)$. □

Now can proceed to the proof of assumption $(A6)$.

Lemma 3.9 *If $y_0 \in \mathbb{R}^3$ and $t_i(y_0) < \infty$ for all $i \in \mathbb{N}$, then $0 = t_0(y_0) < t_1(y_0) < t_2(y_0) \ldots \to \infty$, and thus (A6) from Section 2.1 holds.*

Proof: Suppose on the contrary that $t_i(y_0) \uparrow t_\infty$ as $i \to \infty$ for some $t_\infty \in]0,\infty[$, and let $y_i = \xi_i(y_0) = (x_i, v_i, \tau_i) = \varphi(t_i(y_0), y_0)$ denote the switching points, see (2.2), (2.3), and Lemma 2.1(c). Since $y_i \notin M_\infty$ for $i \geq 0$, the only possible switchings are $M_+ \to M_{\text{left}}$, $M_+ \to M_{\text{right}}$, $M_+ \to M_-$, $M_{\text{left}} \to M_+$, $M_{\text{right}} \to M_-$, $M_- \to M_{\text{right}}$, $M_- \to M_{\text{left}}$, and $M_- \to M_+$, and we have $v_i = 0$ for $i \geq 1$, by (3.15) and Lemma 3.5(c). Because $\varphi(\cdot, y_0)$ is continuous by ($A4$), it follows that in particular $\sup_{i \geq 0} |x_i| \leq \max_{t \in [0, t_\infty]} |\varphi(t, y_0)| = C < \infty$. Since $t_{i+1}(y_0) = t_i(y_0) + t_1(y_i)$, we also find $t_1(y_i) \to 0$ as $i \to \infty$, and $y_{i+1} = \varphi(t_{i+1}(y_0), y_0) = \varphi(t_1(y_i), \varphi(t_i(y_0), y_0)) = \varphi(t_1(y_i), y_i)$. Hence, by Corollary 3.7, along the trajectory considered, there are only finitely many switchings from some $y_i \in M_+ \cap \{y \in \mathbb{R}^3 : v = 0\}$ to $y_{i+1} = \varphi(t_1(y_i), y_i) \in M_-$, and the same holds for switchings $M_- \to M_+$, i.e. finally there are no switchings $M_+ \to M_-$ resp. $M_- \to M_+$. Taking into account that $t_1(y_i) + t_1(y_{i+1}) \to 0$ as $i \to \infty$, Corollary 3.8 implies that also the switchings $M_+ \to M_{\text{right}}$ resp. $M_- \to M_{\text{left}}$ are not possible infinitely often. Thus for i sufficiently large, every switching $y_i \to y_{i+1}$ is of type $M_+ \to M_{\text{left}}$, $M_{\text{left}} \to M_+$, $M_{\text{right}} \to M_-$, or $M_- \to M_{\text{right}}$, i.e. we may assume w.l.o.g. a sequence of switchings

$$\begin{array}{ccccccccccc} M_+ & \to & M_{\text{left}} & \to & M_+ & \to & M_{\text{left}} & \to & M_+ & \to & \ldots \\ y_{i_0} & \to & y_{i_0+1} & \to & y_{i_0+2} & \to & y_{i_0+3} & \to & y_{i_0+4} & \to & \ldots \end{array}$$

for some $i_0 \in \mathbb{N}$. Since $t_1(y_i) \to 0$ as $i \to \infty$, we may also assume that $j(y_{i_0+2p}) = 1$ for $p \in \mathbb{N}_0$ by Lemma 3.5(e). If $p \in \mathbb{N}_0$, then $y_{i_0+1+2p} \in M_{\text{left}}$ and $y_{i_0+2(p+1)} = \varphi(t_1(y_{i_0+1+2p}), y_{i_0+1+2p})$. Therefore by Lemma 3.3(b) for $p \in \mathbb{N}_0$

$$y_{i_0+2(p+1)} \in \{y \in \mathbb{R}^3 : v = 0, x = b^-(\tau), \tau_{\text{mod}} \in [0, \pi/2\eta[\cup]3\pi/2\eta, 2\pi/\eta[\}. \tag{3.28}$$

It is impossible to have $(\tau_{i_0+2p})_{\text{mod}} \in]3\pi/2\eta, 2\pi/\eta[$ for infinitely many $p \in \mathbb{N}$, because in this case by Lemma 3.4(d)

$$0 \leftarrow t_1(y_{i_0+2p}) = s_{j(y_{i_0+2p})}(y_{i_0+2p}) = s_1(y_{i_0+2p}) > \pi \wedge \Big(5\pi/2\eta - (\tau_{i_0+2p})_{\text{mod}}\Big) \geq \pi \wedge (\pi/2\eta)$$

on a subsequence $p' \to \infty$, a contradiction. Consequently,

$$(\tau_{i_0+2p})_{\text{mod}} \in [0, \pi/2\eta[\tag{3.29}$$

for sufficiently large p. By Lemma 3.3(b) and Lemma 3.5(b2) we have $\tau_{i+1} = \tau_i + t_1(y_i)$ for $i \geq i_0$. Thus by (3.28) and Lemma 3.4(d) for $p \in \mathbb{N}$ large

$$\tau_{i_0+1+2p} - \tau_{i_0+2p} = t_1(y_{i_0+2p}) = s_1(y_{i_0+2p}) > \pi \wedge \Big(\pi/2\eta - (\tau_{i_0+2p})_{\text{mod}}\Big) = \pi/2\eta - (\tau_{i_0+2p})_{\text{mod}} > 0, \tag{3.30}$$

the last "=" following from $t_1(y_{i_0+2p}) \to 0$ as $p \to \infty$. Next we choose the unique $k_i \in \mathbb{Z}$ with $\tau_i = k_i(2\pi/\eta) + (\tau_i)_{\text{mod}}$ for $i \geq i_0$. Since $\tau_{i+1} > \tau_i$ we have $k_{i+1} \geq k_i$. Because $\tau_i = \tau_0 + \sum_{l=0}^{i-1}(\tau_{l+1} - \tau_l) = \tau_0 + \sum_{l=0}^{i-1} t_1(y_l) = \tau_0 + t_i(y_0)$, the sequence $(k_i)_{i \geq i_0}$ is bounded. Therefore $k_i \uparrow k_\infty$ for some $k_\infty \in \mathbb{Z}$, and this implies that $k_i = k_\infty$ for sufficiently large i, and hence in particular $(\tau_{i+1})_{\text{mod}} - (\tau_i)_{\text{mod}} = \tau_{i+1} - \tau_i > 0$ for sufficiently large i. Thus we obtain from (3.29) and (3.30) that $\pi/2\eta - (\tau_{i_0+2p})_{\text{mod}} > (\tau_{i_0+2(p+1)})_{\text{mod}} - (\tau_{i_0+2p})_{\text{mod}} > (\tau_{i_0+1+2p})_{\text{mod}} - (\tau_{i_0+2p})_{\text{mod}} = \tau_{i_0+1+2p} - \tau_{i_0+2p} > \pi/2\eta - (\tau_{i_0+2p})_{\text{mod}}$ for sufficiently large $p \in \mathbb{N}$, a contradiction. □

Our next step will be

Lemma 3.10 *Assumption (A9) from Section 2.2 holds.*

Proof: Let

$$h_+(y) = h_-(y) = v\,, \quad h_{\rm left}(y) = x - b^-(\tau)\,, \quad \text{and} \quad h_{\rm right}(y) = x - b^+(\tau) \quad \text{for} \quad y = (x,v,\tau) \in \mathbb{R}^3\,. \tag{3.31}$$

Then $h_+(\xi_1(y_0)) = 0$ for $y_0 \in M_+$ by Lemma 3.5(c), and $h_{\rm left}(\xi_1(y_0)) = 0$ for $y_0 \in M_{\rm left}$ by (3.15) in Lemma 3.3(b). Also, $\langle \nabla h_+, f_+\rangle(y) = -x + b^-(\tau)$ and $\langle \nabla h_{\rm left}, f_{\rm left}\rangle(y) = -\gamma\eta\cos(\eta\tau)$ for $y \in \mathbb{R}^3$, cf. (3.11). For $y_0 \in M_{\rm left}$ it follows from (3.15) in Lemma 3.3(b) that $\xi_1(y_0) \in \{y \in \mathbb{R}^3 : \tau_{\rm mod} \neq \pi/2\eta, \tau_{\rm mod} \neq 3\pi/2\eta\}$. Therefore $\langle \nabla h_{\rm left}, f_{\rm left}\rangle(\xi_1(y_0)) \neq 0$. Finally, if $y_0 \in M_+$, then $\xi_1(y_0) \notin \{y \in \mathbb{R}^3 : v = 0, x = b^-(\tau), \tau_{\rm mod} \in]\pi/2\eta, 3\pi/2\eta[\}$ by Lemma 3.5(d). Hence, if $\xi_1(y_0) \notin E_+$, then by definition of E_+, cf. (3.12), we obtain $\xi_1(y_0) \notin \{y \in \mathbb{R}^3 : v = 0, x = b^-(\tau)\}$, and thus $\langle \nabla h_+, f_+\rangle(\xi_1(y_0)) \neq 0$ by Lemma 3.5(c). The cases $y_0 \in M_-$ and $y_0 \in M_{\rm right}$ are treated analogously. $\square$

Now we turn to the verification of $(A8)$ and $(A10)$. According to (2.1), for $k \in \{1,2,3,4\}$ the sets D_k are given through

$$\begin{aligned} D_k = \{&y \in M_k : \varphi^t(y) \notin E_k \ \forall t \in [0, t_1(y)], \text{ and} \\ &l \in I \cup \{\infty\}, \xi_1(y) \in M_l \Rightarrow \forall\, t \in [0, t_1(y)[\ \exists\, \varepsilon > 0 : \xi_1(B_\varepsilon(\varphi^t(y)) \cap M_k) \subset M_l\}, \end{aligned}$$

and we let $D_+ = D_1$, $D_- = D_2$, $D_{\rm left} = D_3$, and $D_{\rm right} = D_4$, cf. the definition of the manifolds M_k in (3.10) and the definition of the exceptional sets $E_k \subset M_k$ in (3.12).

We also need to introduce

$$\begin{aligned} C_+ &:= \{y \in \mathbb{R}^3 : v = 0, x = b^-(\tau)\}\,, \quad C_- := \{y \in \mathbb{R}^3 : v = 0, x = b^+(\tau)\}\,, \\ C_{\rm left} &:= \{y \in \mathbb{R}^3 : v = 0, x = \gamma - 1\}\,, \quad C_{\rm right} := \{y \in \mathbb{R}^3 : v = 0, x = -\gamma + 1\}\,, \\ C_\infty &= \emptyset\,, \quad \text{and} \quad C_{I\cup\{\infty\}} = C_+ \cup C_- \cup C_{\rm left} \cup C_{\rm right}\,. \end{aligned} \tag{3.32}$$

Since here the dimension of the system is three, and of course C_+, C_-, $C_{\rm left}$, and $C_{\rm right}$ are C^1-manifolds of dimension 1, each of which can be described with a single global chart, in particular the first part of assumption $(A10)$ from Section 2.4 holds.

Next we recall from $(A7)$ in Section 2.1 that $D_\infty = \{y \in M_\infty : \varphi^t(y) \notin E_\infty \ \forall t \in [0,\infty[\}$, with E_∞ from (3.12). Because the flow in M_∞ is just a straight line in τ-direction by Lemma 3.2, we obtain

Lemma 3.11 *Assumption $(A7)$ from Section 2.1 is satisfied, and*

$$D_\infty \supset \{y \in M_\infty : \varphi^t(y) \notin C_{I\cup\{\infty\}} \ \forall t \in [0,\infty[\}\,.$$

Next we consider $D_{\rm left}$ and $D_{\rm right}$.

Lemma 3.12 *We have*

$$D_{\rm left} = M_{\rm left} \setminus E_{\rm left} \supset \{y \in M_{\rm left} : \varphi^t(y) \notin C_{I\cup\{\infty\}} \ \forall t \in [0, t_1(y)]\}\,, \tag{3.33}$$

and $t_1(\cdot) \in C^1(D_{\rm left})$. An analogous statement holds for $D_{\rm right}$.

Proof: We only consider the case of $D_{\rm left}$. By Lemma 3.3(b),

$$\begin{aligned} D_{\rm left} = \{&y \in M_{\rm left} : \varphi^t(y) \notin E_{\rm left} \ \forall t \in [0, t_1(y)], \text{ and} \\ &\forall\, t \in [0, t_1(y)[\ \exists\, \varepsilon > 0 : \xi_1(B_\varepsilon(\varphi^t(y)) \cap M_{\rm left}) \subset M_+\}\,, \end{aligned}$$

because $\xi_1(y) = \varphi(t_1(y), y) \in M_+$ for $y \in M_{\rm left}$. This also shows that

$$D_{\rm left} = \{y \in M_{\rm left} : \varphi^t(y) \notin E_{\rm left} \ \forall t \in [0, t_1(y)]\}\,, \tag{3.34}$$

since for $y \in M_{\rm left}$, $t \in [0, t_1(y)[$, and arbitrary $\varepsilon > 0$ we have $\xi_1(B_\varepsilon(\varphi^t(y)) \cap M_{\rm left}) \subset \xi_1(M_{\rm left}) \subset M_+$. But $\varphi^t(y) = (x, 0, \tau + t)$ for $y = (x, 0, \tau) \in M_{\rm left}$ and $t \in [0, t_1(y)[$ by Lemma 3.3(b), i.e. the flow is a straight line in τ-direction. Thus $\varphi^t(y) \notin E_{\rm left}$ for all $t \in [0, t_1(y)]$ iff $y = \varphi^0(y) \notin E_{\rm left}$, and this implies $D_{\rm left} = M_{\rm left} \setminus E_{\rm left}$. Because every trajectory starting in $y_0 \in M_{\rm left}$ will reach $C_+ \subset C_{I \cup \{\infty\}}$ at time $t = t_1(y_0)$, we have $\{y \in M_{\rm left} : \varphi^t(y) \notin C_{I \cup \{\infty\}}\ \forall t \in [0, t_1(y)]\} = \emptyset$, but this makes sense here, since $\lambda^3(M_{\rm left}) = 0$, cf. the remark after the introduction of $(A10)$ in Section 2.4. Hence (3.33) holds.

To prove the second claim, define for $k \in \mathbb{Z}$

$$\begin{aligned} U_k = \{ & y \in \mathbb{R}^3 : x \in]-\gamma-1, \gamma-1[,\ v \in]-1,1[,\ \tau \in](k-1)(2\pi/\eta) + \pi/2\eta, k(2\pi/\eta) + \pi/2\eta[, \\ & \tau < (k-1)(2\pi/\eta) + 3\pi/2\eta + \sigma_{\rm left}(x) \}, \end{aligned}$$

with $\sigma_{\rm left}$ from Lemma 3.3. Thus $U_k \subset \mathbb{R}^3$ is open and $U_k \cap U_l = \emptyset$ for $k \neq l$. Therefore

$$T_1 : \quad U := \bigcup_{k \in \mathbf{Z}} U_k \to \mathbb{R}, \quad T_1(x, v, \tau) = (k-1)(2\pi/\eta) + 3\pi/2\eta + \sigma_{\rm left}(x) - \tau, \quad (x, v, \tau) \in U_k,$$

is well-defined, and $T_1 \in C^1(U)$. Since it follows from Lemma 3.3 that $D_{\rm left} \subset M_{\rm left} \subset U$ and $T_1(y) = t_1(y)$ for $y \in U$, we are done. $\square$

Next we intend to show that $t_1(\cdot) \in C^1(D_+)$ and $t_1(\cdot) \in C^1(D_-)$, and afterwards we will give a characterization of D_+ resp. D_-.

Lemma 3.13 *Let $y_0 \in M_+$ be such that $\varphi^t(y_0) \notin E_+$ for $t \in [0, t_1(y_0)]$. Then the following assertions hold.*

(a) There exist $\delta_1 > 0$, $\delta_2 \in]0, t_1(y_0)/2[$, and $T_1 : B_{\delta_1}(y_0) \to]t_1(y_0) - \delta_2, t_1(y_0) + \delta_2[$ such that $T_1 \in C^2(B_{\delta_1}(y_0))$ and $T_1(y) = t_1(y)$ for all $y \in B_{\delta_1}(y_0) \cap M_+$. In particular, $t_1(\cdot) \in C^1(D_+)$.

(b) For every $\delta_3 > 0$ there exists $\delta_4 > 0$ such that $B_{\delta_4}(\xi_1(y_0)) \cap \{y \in \mathbb{R}^3 : v = 0\} \subset \xi_1(B_{\delta_3}(y_0) \cap M_+)$.

(c) For every $\delta_5 > 0$ and $t \in [0, t_1(y_0)[$ there exists $\delta_6 > 0$ with $\xi_1(B_{\delta_6}(\varphi^t(y_0)) \cap M_+) \subset B_{\delta_5}(\xi_1(y_0)) \cap \{y \in \mathbb{R}^3 : v = 0\}$.

An analogous statement holds for D_-.

Proof: We start with some general observations. Let $y_0 = (x_0, v_0, \tau_0)$ be as in the lemma. Since $s_1(y_0) \le t_1(y_0)$, cf. (3.17), we have $\varphi(s_1(y_0), y_0) \notin E_+$, and therefore $j(y_0) = 1$, i.e. $s_1(y_0) = t_1(y_0)$, by Lemma 3.4(c), cf. Lemma 3.5. Hence $\xi_1(y_0) = \varphi(s_1(y_0), y_0) \notin E_+$. Taking also into account Lemma 3.4(f), it follows that $y_1 = (x_1, 0, \tau_1) = \xi_1(y_0) \notin C_+$. By Lemma 3.4(b) we thus obtain

$$x_1 > b^-(\tau_1), \tag{3.35}$$

and we have $\tau_1 = \tau_0 + t_1(y_0)$. In addition, because of $\varphi^t(y_0) \in \{y \in \mathbb{R}^3 : v > 0\}$ for $t \in]0, t_1(y_0)[$ and $y_0 \in M_+ \setminus E_+$, even

$$\varphi^t(y_0) \notin C_+ \quad \text{for} \quad t \in [0, t_1(y_0)]. \tag{3.36}$$

From the continuity of φ and Lemma 3.5(b2), cf. (3.24), it follows that for

$$\varphi(t, y_0) = \Phi_{f_+}(t, y_0) = (g_1(t, y_0), g_2(t, y_0), \tau_0 + t) \quad \text{for} \quad t \in [0, t_1(y_0)], \tag{3.37}$$

with

$$g_1(t, y) = x \cos t + v \sin t + \int_0^t \sin(t-s)\, b^-(\tau + s)\, ds$$

and $g_2(t, y) = \partial_t g_1(t, y)$ for $t \in \mathbb{R}$ and $y = (x, v, \tau) \in \mathbb{R}^3$. By continuity of the global flow Φ_{f_+} we find $\varepsilon_1 > 0$ such that

$$\Phi_{f_+}([0, t_1(y_0) + \varepsilon_1[\times B_{\varepsilon_1}(y_0)) \subset \mathbb{R}^3 \setminus C_+. \tag{3.38}$$

We prove (a) and (b) together and begin with Case 1: $v_0>0$. To show (a) in this case, note that here $B_{\varepsilon_2}(y_0)\subset\{y\in\mathbb{R}^3:v>0\}\subset M_+$ for some $\varepsilon_2>0$, w.l.o.g. $\varepsilon_2<t_1(y_0)/2$. Now we consider the map $g_2:]t_1(y_0)-\varepsilon_2,t_1(y_0)+\varepsilon_2[\times B_{\varepsilon_2}(y_0)\to\mathbb{R}$. As a consequence of

$$(g_1(t_1(y_0),y_0),g_2(t_1(y_0),y_0),\tau_0+t_1(y_0))=\Phi_{f_+}(t_1(y_0),y_0)=\varphi(t_1(y_0),y_0)=y_1\in\{y\in\mathbb{R}^3:v=0\},$$

cf. Lemma 3.4(b), we obtain $g_2(t_1(y_0),y_0)=0$. In addition,

$$\partial_t g_2(t,y)=-x\cos t-v\sin t+b^-(\tau+t)-\int_0^t\sin(t-s)b^-(\tau+s)\,ds=-g_1(t,y)+b^-(\tau+t),$$

and hence $\partial_t g_2(t_1(y_0),y_0)=-g_1(t_1(y_0),y_0)+b^-(\tau_0+t_1(y_0))=-x_1+b^-(\tau_1)\neq0$ by (3.35). Hence, by the implicit function theorem there exist $0<\varepsilon_3,\varepsilon_4\le\varepsilon_2$ and a function $T_1:B_{\varepsilon_3}(y_0)\to]t_1(y_0)-\varepsilon_4,t_1(y_0)+\varepsilon_4[$ such that $g_2(T_1(y),y)=0$ for $y\in B_{\varepsilon_3}(y_0)$, and with the property

$$(t,y)\in]t_1(y_0)-\varepsilon_4,t_1(y_0)+\varepsilon_4[\times B_{\varepsilon_3}(y_0),\quad g_2(t,y)=0\quad\Longrightarrow\quad t=T_1(y).\tag{3.39}$$

In addition we may assume $T_1\in C^2(B_{\varepsilon_3}(y_0))$, and by shrinking ε_3 if necessary, also that $\varepsilon_3,\varepsilon_4<\varepsilon_1$. By keeping ε_4 fixed and reducing ε_3 further, we may also assume that

$$(t,y)\in[0,t_1(y_0)-\varepsilon_4]\times B_{\varepsilon_3}(y_0)\quad\Longrightarrow\quad g_2(t,y)\neq0,\tag{3.40}$$

since otherwise $g_2(t_*,y_0)=0$ for some $t_*\in[0,t_1(y_0)-\varepsilon_4]$. This is a contradiction, because $v_0>0$, and thus $g_2(t,y_0)>0$ for $t\in[0,s_1(y_0)[=[0,t_1(y_0)[$ by Lemma 3.4(b). Hence we also have (3.40). Now we claim that $T_1(\bar y_0)=t_1(\bar y_0)$ for all $\bar y_0\in B_{\varepsilon_3}(y_0)=B_{\varepsilon_3}(y_0)\cap M_+$. For this, fix such $\bar y_0=(\bar x_0,\bar v_0,\bar\tau_0)$. By choice of ε_2 we find $g_2(0,\bar y_0)=\bar v_0>0$, and therefore, by Lemma 3.4, $s_1(\bar y_0)$ is the unique first positive zero of $g_2(\cdot,\bar y_0)$. Thus $s_1(\bar y_0)>t_1(y_0)-\varepsilon_4$ by (3.40). Because $g_2(T_1(\bar y_0),\bar y_0)=0$, we also must have $0<s_1(\bar y_0)\le T_1(\bar y_0)<t_1(y_0)+\varepsilon_4$. Therefore $s_1(\bar y_0)\in]t_1(y_0)-\varepsilon_4,t_1(y_0)+\varepsilon_4[$, and it follows from (3.39) that $T_1(\bar y_0)=s_1(\bar y_0)$. Next, (3.38) implies

$$\varphi([0,s_1(\bar y_0)]\times\{\bar y_0\})=\Phi_{f_+}([0,s_1(\bar y_0)]\times\{\bar y_0\})\subset\mathbb{R}^3\setminus C_+.$$

If $\bar y_1:=\varphi(s_1(\bar y_0),\bar y_0)\in M_+$, then $\bar y_1\in E_+\subset C_+$ by Lemma 3.4(c), a contradiction. Therefore $\bar y_1\notin M_+$, and thus $j(\bar y_0)=1$ by Lemma 3.5, i.e. $T_1(\bar y_0)=s_1(\bar y_0)=t_1(\bar y_0)$. So (a) holds in the first case $v_0>0$.

To prove (b) for $v_0>0$, we want to apply a suitable open mapping theorem, cf. [7, Cor. 15.2]. For this we note that $\xi_1:B_{\varepsilon_3}(y_0)\to\mathbb{R}^3$ is C^2 and of the form

$$\xi_1(y)=\varphi(t_1(y),y)=\Phi_{f_+}(t_1(y),y)=(g_1(t_1(y),y),0,\tau+t_1(y))\equiv(g_1(t_1(y),y),\tau+t_1(y))$$

for $y=(x,v,\tau)\in B_{\varepsilon_3}(y_0)$, i.e. ξ_1 is in fact two-dimensional. Hence, by the cited result, we only have to check that $\xi_1'(y_0)\in\mathcal{L}(\mathbb{R}^3,\mathbb{R}^2)$ has rank two, since $\xi_1'(\cdot)$ is continuous at y_0, and since it is sufficient to show (b) for $\delta_3>0$ being sufficiently small. By the implicit function theorem,

$$\begin{aligned}\nabla t_1(y_0)&=-(\partial_t g_2(t_1(y_0),y_0))^{-1}\partial_y g_2(t_1(y_0),y_0)\\&=(x_1-b^-(\tau_1))^{-1}(-\sin(t_1(y_0)),\cos(t_1(y_0)),\partial_\tau g_2(t_1(y_0),y_0)).\end{aligned}$$

In addition,

$$\begin{aligned}\partial_y(g_1(t_1(y),y)&=\partial_t g_1(t_1(y),y)\,\nabla t_1(y_0)+\partial_y g_1(t_1(y),y)=g_2(t_1(y),y)+\partial_y g_1(t_1(y),y)\\&=\partial_y g_1(t_1(y),y)=(\cos(t_1(y_0)),\sin(t_1(y_0)),\partial_\tau g_1(t_1(y_0),y_0))\end{aligned}$$

for $y\in B_{\varepsilon_3}(y_0)$, and therefore

$$\xi_1'(y_0)=\begin{pmatrix}\cos(t_1(y_0)) & \sin(t_1(y_0)) & \partial_\tau g_1(t_1(y_0),y_0)\\ -\dfrac{\sin(t_1(y_0))}{(x_1-b^-(\tau_1))} & \dfrac{\cos(t_1(y_0))}{(x_1-b^-(\tau_1))} & 1+\dfrac{\partial_\tau g_2(t_1(y_0),y_0)}{(x_1-b^-(\tau_1))}\end{pmatrix}.$$

The left 2×2-matrix has determinant $(x_1 - b^-(\tau_1))^{-1} \neq 0$, cf. (3.35), and thus we have proved (a) and (b) in case that $v_0 > 0$.

Next we turn to Case 2: $v_0 = 0$. By (3.36) in particular $y_0 \notin C_+$. Therefore there exists $\varepsilon_5 > 0$ such that $t_1(y) \geq s_1(y) > \varepsilon_5$ for $y \in B_{\varepsilon_5}(y_0) \cap M_+$, by Lemma 3.4(g). Let $\bar{y}_0 = (\bar{x}_0, \bar{v}_0, \bar{\tau}_0) = \varphi(\varepsilon_5, y_0) = \Phi_{f_+}(\varepsilon_5, y_0)$. Then $\bar{v}_0 > 0$ and $\bar{y}_0 \in M_+$ by Lemma 3.4(b). Moreover, $t_1(\bar{y}_0) = t_1(y_0) - \varepsilon_5$ by Lemma 3.6, and by assumption this implies $\varphi(t, \bar{y}_0) = \varphi(t + \varepsilon_5, y_0) \notin E_+$ for $t \in [0, t_1(\bar{y}_0)]$. Hence the first case applies to $\bar{y}_0$, and we obtain the existence of $\varepsilon_3 > 0$, $\varepsilon_4 \in]0, t_1(\bar{y}_0)/2[$, and of a C^2-function $\bar{T}_1 : B_{\varepsilon_3}(\bar{y}_0) \to]t_1(\bar{y}_0) - \varepsilon_4, t_1(\bar{y}_0) + \varepsilon_4[$ such that $B_{\varepsilon_3}(\bar{y}_0) \subset \{y \in \mathbb{R}^3 : v > 0\} \subset M_+$ and $\bar{T}_1(y) = t_1(y)$ for $y \in B_{\varepsilon_3}(\bar{y}_0)$. Since $\Phi_{f_+}(\varepsilon_5, \cdot) : \mathbb{R}^3 \to \mathbb{R}^3$ is a diffeomorphism, we find $\varepsilon_6 \in]0, \varepsilon_5[$ with $B_{\varepsilon_6}(y_0) \subset \Phi_{f_+}(-\varepsilon_5, B_{\varepsilon_3}(\bar{y}_0))$, thus $\Phi_{f_+}(\varepsilon_5, B_{\varepsilon_6}(y_0)) \subset B_{\varepsilon_3}(\bar{y}_0)$. Define $T_1 : B_{\varepsilon_6}(y_0) \to \mathbb{R}$ through $T_1(y) = \varepsilon_5 + \bar{T}_1(\Phi_{f_+}(\varepsilon_5, y))$. Then T_1 is well-defined and of class C^2, and we have $T_1(y) = t_1(y)$ for $y \in B_{\varepsilon_6}(y_0) \cap M_+$. To see this, fix $y \in B_{\varepsilon_6}(y_0) \cap M_+$. By choice of ε_5 hence $t_1(y) \geq s_1(y) > \varepsilon_5$, and consequently Lemma 3.5 (b2) resp. Lemma 3.6 yield $\varphi(\varepsilon_5, y) = \Phi_{f_+}(\varepsilon_5, y)$ resp. $t_1(\varphi(\varepsilon_5, y)) = t_1(y) - \varepsilon_5$. Thus $T_1(y) = \varepsilon_5 + \bar{T}_1(\Phi_{f_+}(\varepsilon_5, y)) = \varepsilon_5 + t_1(\varphi(\varepsilon_5, y)) = t_1(y)$, as was to be shown for (a) in case that $v_0 = 0$.

Finally, to prove (b) for $v_0 = 0$, we first note that $\xi_1(\bar{y}_0) = \xi_1(y_0)$ by Lemma 2.1(d), take $i = 1$ and $j = 0$ there. Since $\bar{y}_0 \neq y_0$, we may assume above, by shrinking ε_6, that $|\bar{y}_0 - y_0| \geq 2\varepsilon_6$. We claim that

$$\forall\, \delta_3 \in]0, \varepsilon_6[\ \exists\, \delta > 0 : \quad B_\delta(\bar{y}_0) \subset \varphi([0, \varepsilon_5] \times (B_{\delta_3}(y_0) \cap M_+)). \tag{3.41}$$

To see this, we remark that $g_2(0, y_0) = v_0 = 0$ and $g_2(\varepsilon_5, y_0) = \bar{v}_0 > 0$, cf. (3.37). Hence for every large $n \in \mathbb{N}$ there exists $\sigma_n \in]0, \varepsilon_5[$ such that $g_2(\sigma_n, y_0) = \delta_3/n$. Assume $|\varphi(\sigma_n, y_0) - y_0| \geq \delta_3$ for large n. Then by continuity of φ we obtain $\varphi(\sigma_*, y_0) \in \{y \in \mathbb{R}^3 : v = 0\}$ for some $\sigma_* \in [0, \varepsilon_5]$, i.e. $g_2(\sigma_*, y_0) = 0$. But $\varphi(\sigma_*, y_0) \neq y_0$, and thus $\sigma_* \neq 0$. This is a contradiction, because $s_1(y_0) = t_1(y_0)$ is the first positive zero of $g_2(\cdot, y_0)$, and $0 < \sigma_* \leq \varepsilon_5 < s_1(y_0)$. Therefore we find $\varepsilon_7 \in]0, \delta_3 \wedge \bar{v}_0/2]$ and $y_1 \in \varphi(]0, \varepsilon_5[\times \{y_0\}) \cap B_{\delta_3}(y_0) \cap \{y \in \mathbb{R}^3 : v = \varepsilon_7\}$. Next, there exists $c_1 > 0$ such that $|\Phi_{f_+}(-t, \bar{y}) - \Phi_{f_+}(-t, \bar{y}_0)| \leq c_1 |\bar{y} - \bar{y}_0|$ for $t \in [0, \varepsilon_5]$ and $\bar{y} \in B_1(\bar{y}_0)$. By the properties of y_1 we thus obviously we may choose $\delta > 0$ so small that for $\bar{y} \in B_\delta(\bar{y}_0)$ there exists $t(\bar{y}) \in]0, \varepsilon_5[$ with $\Phi_{f_+}(-t(\bar{y}), \bar{y}) \in B_{\delta_3}(y_0) \cap \{y \in \mathbb{R}^3 : v = \varepsilon_7\}$. We find $\eta \in]0, \varepsilon_5[$ such that for all $\delta \in]0, \eta[$: $\bar{y} \in B_\delta(\bar{y}_0) \Rightarrow t(\bar{y}) \leq \varepsilon_5 - \eta$, since in the opposite case there would exist a sequence $\bar{y}_n \to \bar{y}_0$ with $t(\bar{y}_n) \to \varepsilon_5$ and $\Phi_{f_+}(-t(\bar{y}_n), \bar{y}_n) \in \{y \in \mathbb{R}^3 : v = \varepsilon_7\}$, yielding the contradiction $y_0 = \Phi_{f_+}(-\varepsilon_5, \bar{y}_0) \in \{y \in \mathbb{R}^3 : v = \varepsilon_7\}$. Moreover, for sufficiently small $\delta > 0$ we additionally may assume

$$\Phi_{f_+}(-[0, \varepsilon_5 - \eta] \times B_\delta(\bar{y}_0)) \subset \{y \in \mathbb{R}^3 : v > 0\}. \tag{3.42}$$

Indeed, w.l.o.g. $\Phi_{f_+}(0, B_\delta(\bar{y}_0)) = B_\delta(\bar{y}_0) \subset \{y \in \mathbb{R}^3 : v > 0\}$ and hence, if (3.42) were wrong, there would exist sequences $\sigma_n \in [0, \varepsilon_5 - \eta]$ and $\bar{y}_n \to \bar{y}_0$ such that $\Phi_{f_+}(-\sigma_n, \bar{y}_n) \in \{y \in \mathbb{R}^3 : v = 0\}$. This implies $\varphi(\varepsilon_5 - \sigma_*, y_0) = \Phi_{f_+}(\varepsilon_5 - \sigma_*, y_0) = \Phi_{f_+}(-\sigma_*, \Phi_{f_+}(\varepsilon_5, y_0)) = \Phi_{f_+}(-\sigma_*, \bar{y}_0) \in \{y \in \mathbb{R}^3 : v = 0\}$ for some $\sigma_* \in [0, \varepsilon_5 - \eta]$, a contradiction, since $\varepsilon_5 - \sigma_* \in]0, s_1(y_0)[$. Thus (3.42) holds for small $\delta > 0$, and for such $\delta > 0$ we can also prove (3.41): for $\bar{y} \in B_\delta(\bar{y}_0)$ let $\hat{y} = \Phi_{f_+}(-t(\bar{y}), \bar{y}) \in B_{\delta_3}(y_0) \cap \{y \in \mathbb{R}^3 : v = \varepsilon_7\} \subset B_{\delta_3}(y_0) \cap M_+$. Then $\bar{y} = \varphi(t(\bar{y}), \hat{y})$, because by the above reasoning $t(\bar{y}) \leq \varepsilon_5 - \eta$, and hence $\Phi_{f_+}(-[0, t(\bar{y})], \bar{y}) \subset \{y \in \mathbb{R}^3 : v > 0\}$ by (3.42). This means $\Phi_{f_+}(t, \hat{y}) \in \{y \in \mathbb{R}^3 : v > 0\}$ for $t \in [0, t(\bar{y})]$. By definition, $s_1(\hat{y})$ is the unique first time with $\Phi_{f_+}(s_1(\hat{y}), \hat{y}) \in \{y \in \mathbb{R}^3 : v = 0\}$. Thus we obtain $t(\bar{y}) < s_1(\hat{y}) \leq t_1(\hat{y})$, and hence $\varphi(t(\bar{y}), \hat{y}) = \Phi_{f_+}(t(\bar{y}), \hat{y}) = \bar{y}$, and this completes the proof of (3.41).

Now we can continue to show (b) in case that $v_0 = 0$ as follows. Fix $\delta_3 \in]0, \varepsilon_6[$ and choose $\delta > 0$ according to (3.41). Additionally we may assume that $B_\delta(\bar{y}_0) \subset \{y \in \mathbb{R}^3 : v > 0\} \subset M_+$. By the first case of (b) applied to $\bar{y}_0$ as above, we find $\delta_4 > 0$ with $B_{\delta_4}(\xi_1(\bar{y}_0)) \cap \{y \in \mathbb{R}^3 : v = 0\} \subset \xi_1(B_\delta(\bar{y}_0) \cap M_+) = \xi_1(B_\delta(\bar{y}_0))$. Therefore by (3.41)

$$B_{\delta_4}(\xi_1(y_0)) \cap \{y \in \mathbb{R}^3 : v = 0\} \subset \xi_1\Big(\varphi([0, \varepsilon_5] \times (B_{\delta_3}(y_0) \cap M_+))\Big) \subset \xi_1(B_{\delta_3}(y_0) \cap M_+). \tag{3.43}$$

To see the last inclusion, fix $t\in[0,\varepsilon_5]$ and $y\in B_{\delta_3}(y_0)\cap M_+$. Since $\delta_3<\varepsilon_6<\varepsilon_5$, we obtain $t_1(y)\geq s_1(y)>\varepsilon_5$ by choice of ε_5, and thus $t\in[0,t_1(y)[$. Since it follows from Lemma 2.1(d) that $\xi_1(\varphi(t,y))=\xi_1(y)$, we have shown (3.43). Hence the proof of (a) and (b) is complete.

Concerning (c), we may restrict ourselves to $t=0$, since for $\bar{y}_0=\varphi^{t_0}(y_0)$ with $t_0\in[0,t_1(y_0)[$ we have $\xi_1(\bar{y}_0)=\xi_1(y_0)$ resp. $t_1(\bar{y}_0)=t_1(y_0)-t_0$ by Lemma 2.1(d) resp. Lemma 3.6, and also $\varphi^t(\bar{y}_0)\in\mathbb{R}^3\setminus C_+\subset\mathbb{R}^3\setminus E_+$ for $t\in[0,t_1(\bar{y}_0)]$ by (3.36). So we consider only $t=0$. By (a) we find $\delta_1>0$, $\delta_2\in]0,t_1(y_0)/2[$, and a C^2-function $T_1:B_{\delta_1}(y_0)\to]t_1(y_0)-\delta_2,t_1(y_0)+\delta_2[$ with $T_1(y)=t_1(y)$ for $y\in B_{\delta_1}(y_0)\cap M_+$. Define $\Xi_1:B_{\delta_1}(y_0)\to\mathbb{R}^3$ through $\Xi_1(y)=\Phi_{f_+}(T_1(y),y)$. If $y\in B_{\delta_1}(y_0)\cap M_+$, then $\Xi_1(y)=\Phi_{f_+}(t_1(y),y)=\varphi(t_1(y),y)=\xi_1(y)$. In particular, $\Xi_1(y_0)=\xi_1(y_0)$, and hence, by the continuity of Ξ_1, for given $\delta_5>0$ there exists $\delta_6\in]0,\delta_1[$ such that $\Xi_1(B_{\delta_6}(y_0))\subset B_{\delta_5}(\xi_1(y_0))$. Since $\xi_1(M_+)\subset\{y\in\mathbb{R}^3:v=0\}$ by Lemma 3.5(c), thus (c) follows, and the proof of the lemma is complete. $\square$

By means of Lemma 3.12 and Lemma 3.13(a) we obtain

Lemma 3.14 *Assumption (A8) from Section 2.1 is satisfied.*

Thus is remains to show that also $(A10)$ from Section 2.4 holds.

Lemma 3.15 *We have*

$$\begin{aligned} D_+ &= \{y\in M_+:\varphi^t(y)\notin E_+\ \forall t\in[0,t_1(y)],\text{ and }\xi_1(y)\notin C_{I\cup\{\infty\}}\}\\ &\supset \{y\in M_+:\varphi^t(y)\notin C_{I\cup\{\infty\}}\ \forall t\in[0,t_1(y)]\}, \end{aligned}$$

and an analogous statement for D_-.

Proof: To show the equality, we remark that for both $y_0\in D_+$ and y_0 being an element of the right-hand side, the assumptions of Lemma 3.13 are satisfied. "$\subseteq$": If $y_0\in D_+$, then $\xi_1(y_0)\notin C_+$ by (3.36). Choose $l\in\{\text{left},\infty,\text{right},-\}$ such that $\xi_1(y_0)\in M_l$. Then $\xi_1(B_\varepsilon(y_0)\cap M_+)\subset M_l$ for some $\varepsilon>0$ by assumption. Consequently, Lemma 3.13(b) implies that $B_{\delta_4}(\xi_1(y_0))\cap\{y\in\mathbb{R}^3:v=0\}\subset M_l$ for some $\delta_4>0$. Thus by definition of C_{left}, C_{right}, and C_+ we obtain $\xi_1(y_0)\notin C_{\text{left}}\cup C_{\text{right}}\cup C_-$, since in the opposite case every neighborhood of $\xi_1(y_0)$ in $\{y\in\mathbb{R}^3:v=0\}$ would intersect at least two of the manifolds. "$\supseteq$": Let y_0 be an element of the right-hand side and choose $l\in\{\text{left},\infty,\text{right},-\}$ with $\xi_1(y_0)\in M_l$. Because $\xi_1(M_+)\subset\{y\in\mathbb{R}^3:v=0\}$, cf. Lemma 3.5(c), we find $\delta>0$ such that $B_\delta(\xi_1(y_0))\cap\{y\in\mathbb{R}^3:v=0\}\subset M_l$. Hence, for fixed $t\in[0,t_1(y_0)[$, we may apply Lemma 3.13(c) to find $\delta_6>0$ such that $\xi_1(B_{\delta_6}(\varphi^t(y_0))\cap M_+)\subset M_l$, as was to be shown for $D_+=\{y\in M_+:\ldots\}$. The missing inclusion "$\supseteq$" follows from the definition of $C_{I\cup\{\infty\}}$, cf. (3.32), since $E_+\subset C_{I\cup\{\infty\}}$ and $\xi_1(y)=\varphi(t_1(y),y)$ for $y\notin M_\infty$. $\square$

Hence Lemma 3.11, Lemma 3.12, and Lemma 3.15 imply that also $(A10)$ from Section 2.4 holds, with $C_{I\cup\{\infty\}}$ from (3.32). Thus we can summarize this, (3.26), Lemma 3.9, Lemma 3.11, Lemma 3.14, and Lemma 3.10 to

Theorem 3.16 *If $\gamma\neq 1$, then assumptions $(A1)-(A10)$ from Section 2.1 and Section 2.4 are satisfied for the pendulum with dry friction, described through (1.2) in Section 3.*

Although we have shown this explicitly only in the case $\gamma\in]0,1[$, cf. (3.7), it may be seen by a similar reasoning that Theorem 3.16 also holds for $\gamma\in]1,\infty[$, with the main difference being that then $M_\infty=\emptyset$. The case $\gamma=1$ is considered to be exceptional, since then M_∞ degenerates to a line.

3.3 Dynamical behaviour of the pendulum with dry friction

In this section we collect and extend some results concerning the existence and stability of periodic solutions of (1.2), and we are also interested in the boundedness of solutions, so that one sometimes might find it helpful to refer forward to Section 4.2.

We start by overviewing what is known from [9, 10] where is considered

$$\ddot{x}(\tau)+x(\tau)+\mu\,\mathrm{Sgn}\,\dot{x}(\tau)\ni p(\tau) \quad \text{a.e.} \tag{3.44}$$

with $\mu\geq 0$.

Theorem 3.17 (Nonresonant case) *Suppose that $\omega>0$ and $p\in C(\mathbb{R})$ is ω-periodic. If $\omega\neq 2k\pi$ for $k\in\mathbb{N}$, then there exists an ω-periodic solution of (3.44), for every $\mu\geq 0$.*

Proof: For special p this is [10, Thm. 1], while for general p cf. [9, Section 2]. □

Theorem 3.18 (Resonant case) *Let $\omega=2\pi$ and $p\in C(\mathbb{R})$ be ω-periodic.*

(a) If $\left|\int_0^\omega p(\tau)e^{i\tau}\,d\tau\right|>4\mu$, then (3.44) has no ω-periodic solution.

(b) If $\left|\int_0^\omega p(\tau)e^{i\tau}\,d\tau\right|<4\mu$, then (3.44) has an ω-periodic solution.

(c) Let $a\in[0,2\pi[$ denote the argument of $\int_0^\omega p(\tau)e^{i\tau}\,d\tau\in\mathbb{C}$. If $\left|\int_0^\omega p(\tau)e^{i\tau}\,d\tau\right|=4\mu$, then (3.44) has an ω-periodic solution if and only if $\int_0^\pi p(\tau-a-\pi/2)\cos\tau\,d\tau=0$. In this case, there are infinitely many such solutions, and they may be given in an explicit form.

For $p(\tau)=\sin\tau$, in addition the following holds.

(d) If $\mu\geq 1$, then the only ω-periodic solutions of (3.44) are the constants $x_0\in[1-\mu,1+\mu]$.

(e) If $\mu\in]\pi/4,1[$, then there is a unique ω-periodic solution $x_\mu:[0,2\pi]\to\mathbb{R}$ of (3.44). There exist $\tau_1\in]0,\pi/2[$ and $\tau_2\in]\pi/2,\pi[$ such that x_μ is of the form

$$x_\mu(\tau)=-\mu+\sin\tau_1 \ \textit{in}\ [0,\tau_1], \quad \dot{x}_\mu(\tau)>0 \ \textit{in}\ ]\tau_1,\tau_2[, \quad x_\mu(\tau)=\mu-\sin\tau_1 \ \textit{in}\ [\tau_2,\pi],$$

and $x_\mu(\tau)=-x_\mu(\tau+\pi)$ for $\tau\in\mathbb{R}$. The solution x_μ is globally asymptotically stable, in that for every other solution x of (3.44) it follows that

$$(x(\tau)-x_\mu(\tau))^2+(\dot{x}(\tau)-\dot{x}_\mu(\tau))^2\to 0 \quad \textit{as} \quad \tau\to\infty. \tag{3.45}$$

Proof: (a), (b), and (c) are from [9], while (d) and (e) can be found in [10]. □

We want to exploit and supplement these results for the case of

$$\ddot{x}(\tau)+x(\tau)+\mathrm{Sgn}\,\dot{x}(\tau)\ni\gamma\sin(\eta\tau) \quad \text{a.e.} \tag{1.2}$$

from above, with $\gamma,\eta\in[0,\infty[$. We start with

Corollary 3.19 *(a) If $\eta\neq 1$, then (1.2) has a $(2\pi/\eta)$ -periodic solution.*

(b) For $\eta=1$ the following holds.

(b1) *If* $\gamma \in [0,1]$*, then the equilibria* $x_0 \in [\gamma - 1, -\gamma + 1]$ *are the only* 2π*-periodic solutions of (1.2).*

(b2) *If* $\gamma \in]1, 4/\pi[$*, then there exists a unique* 2π*-periodic solutions* $x_\gamma : [0,2\pi] \to \mathbb{R}$ *of (1.2). In addition, there are* $\tau_1^\gamma \in]0,\pi/2[$ *and* $\tau_2^\gamma \in]\pi/2,\pi[$ *such that* x_γ *is given through*

$$\begin{aligned} x_\gamma(\tau) &= -1 + \gamma \sin\tau_1^\gamma = b^-(\tau_1^\gamma) \quad \text{for} \quad \tau \in [0,\tau_1^\gamma], \quad \dot{x}_\gamma(\tau) > 0 \quad \text{for} \quad \tau \in]\tau_1^\gamma, \tau_2^\gamma[, \\ x_\gamma(\tau) &= 1 - \gamma \sin\tau_1^\gamma = -b^-(\tau_1^\gamma) \quad \text{for} \quad \tau \in [\tau_2^\gamma, \pi], \quad x_\gamma(\tau) = -x_\gamma(\tau + \pi) \text{ for } \tau \in \mathbb{R}. \end{aligned} \tag{3.46}$$

This x_γ *is globally asymptotically stable for (1.2), i.e. for every* $\xi_0 = (x_0, v_0) \in \mathbb{R}^2$ *and* $\tau_0 \in [0,2\pi]$*, in the notation of Theorem 3.1,*

$$\Big(x(\tau_0 + t; \xi_0, \tau_0) - x_\gamma(\tau_0 + t)\Big)^2 + \Big(\dot{x}(\tau_0 + t; \xi_0, \tau_0) - \dot{x}_\gamma(\tau_0 + t)\Big)^2 \to 0 \quad \text{as} \quad t \to \infty. \tag{3.47}$$

(b3) *If* $\gamma = 4/\pi$*, then there are infinitely many* 2π*-periodic solutions of (1.2), and they may be calculated explicitly.*

(b4) *If* $\gamma \in]4/\pi, \infty[$*, then (1.2) has no* 2π*-periodic solution.*

Proof: Since $\int_0^{2\pi} \sin(\tau) e^{i\tau}\, d\tau = i\pi$, and since $\operatorname{Sgn}(\alpha v) = \operatorname{Sgn}(v)$ for $\alpha \in [0,\infty[$ and $v \in \mathbb{R}$, this follows directly from Theorem 3.18. Concerning (3.47), we note that by the proof in [10] also $\ddot{x} + x + \operatorname{Sgn}\dot{x} \ni \gamma \sin(\tau_0 + \tau)$ has a unique periodic solution x_{γ,τ_0} for every $\tau_0 \in [0,2\pi]$, and thus by uniqueness $x_{\gamma,\tau_0} = x_\gamma(\tau_0 + \cdot)$. Because $x(\tau_0 + \cdot; \xi_0, \tau_0)$ is a solution of $\ddot{x} + x + \operatorname{Sgn}\dot{x} \ni \gamma \sin(\tau_0 + \tau)$, (3.47) is just (3.45) applied to the problem with forcing $\gamma \sin(\tau_0 + \cdot)$. □

Next we summarize results concerning boundedness resp. unboundedness of solutions of (1.2), cf. also Section 4.2 below. For $\xi_0 = (x_0, v_0) \in \mathbb{R}^2$ and $\tau_0 \in \mathbb{R}$ define

$$\mathcal{E}(\tau; \xi_0, \tau_0) = x^2(\tau; \xi_0, \tau_0) + \dot{x}^2(\tau; \xi_0, \tau_0) \quad \text{for} \quad \tau \in [\tau_0, \infty[, \tag{3.48}$$

cf. Theorem 3.1.

Definition 3.20 *(a) We say that the solutions of (1.2) are uniformly bounded w.r.t. initial values and phases, if for every* $R > 0$ *there exists a constant* $c > 0$ *such that*

$$\sup \Big\{ \mathcal{E}(\tau; \xi_0, \tau_0) : |\xi_0| \le R, \tau_0 \in [0, 2\pi/\eta], \tau \in [\tau_0, \infty[\Big\} \le c. \tag{3.49}$$

(b) We call the solutions of (1.2) uniformly unbounded w.r.t initial values and phases, if for every $R > 0$

$$\inf_{|\xi_0| \le R, \tau_0 \in [0,2\pi/\eta]} \mathcal{E}(\tau; \xi_0, \tau_0) \to \infty \quad \text{as} \quad \tau \to \infty.$$

The following theorem gives an answer to the question for which parameter values the solutions of (1.2) are bounded resp. unbounded.

Theorem 3.21 *(a) If either* $\gamma \in [0,\infty[$ *and* $\eta \ne 1$*, or* $\gamma \in [0, 4/\pi]$ *and* $\eta = 1$*, then the solutions of (1.2) are uniformly bounded w.r.t initial values and phases.*

(b) If $\gamma \in]4/\pi, \infty[$ *and* $\eta = 1$*, then the solutions of (1.2) are uniformly unbounded w.r.t initial values and phases.*

Proof: Ad (a): This follows from Example 4.5 and Example 4.7 in Section 4.2 below, since by Corollary 3.19(a), (b1), (b2), and (b3), Problem (1.2) has a $(2\pi/\eta)$-periodic solution for the parameters under consideration. Ad (b): This was shown in [18]. $\square$

Note in particular that Theorem 3.21(b) implies Corollary 3.19(b4). We also call attention to the fact that it is equivalent to have "$\tau_0 \in [0, 2\pi/\eta]$" and "$\tau_0 \in [0, \infty[$" in (3.49), cf. (4.9) in Theorem 4.6, since the forcing is $(2\pi/\eta)$-periodic, and thus $\varphi^t(\xi_0, \tau_0 + k(2\pi/\eta)) = \varphi^t(\xi_0, \tau_0) + (0, 0, k(2\pi/\eta))$ for $k \in \mathbb{N}_0$, cf. Theorem 3.1. We also state a consequence of Theorem 3.21(a).

Corollary 3.22 *If either $\gamma \in [0, \infty[$ and $\eta \neq 1$, or $\gamma \in [0, 4/\pi]$ and $\eta = 1$, then for every $R > 0$ the semiflow φ from Theorem 3.1 is Lipschitz continuous on $[0, \infty[\times (\overline{B}_R(0) \times [0, 2\pi/\eta])$.*

Proof: Since the uniform boundedness of $\mathcal{E}$ from (3.49) implies via the inclusion (1.2) also the uniform boundedness of the corresponding second derivatives, for $R > 0$ we find $c_1 > 0$ such that for all $|\xi_0| \le R$, $\tau_0 \in [0, 2\pi/\eta]$, and $\tau, \bar{\tau} \in [\tau_0, \infty[$

$$|x(\tau; \xi_0, \tau_0) - x(\bar{\tau}; \xi_0, \tau_0)| + |\dot{x}(\tau; \xi_0, \tau_0) - \dot{x}(\bar{\tau}; \xi_0, \tau_0)| \le c_1 |\tau - \bar{\tau}|. \tag{3.50}$$

Because φ is Lipschitz continuous on $[0, 2\pi/\eta] \times (\overline{B}_R(0) \times [0, 2\pi/\eta])$ by Theorem 3.1, there exists $c_2 > 0$ such that for (σ, ξ, τ) and $(\bar{\sigma}, \bar{\xi}, \bar{\tau})$ in this set

$$|\varphi^\sigma(\xi, \tau) - \varphi^{\bar{\sigma}}(\bar{\xi}, \bar{\tau})| \le c_2 (|\sigma - \bar{\sigma}| + |\xi - \bar{\xi}| + |\tau - \bar{\tau}|). \tag{3.51}$$

Let $\Delta = \Delta_{\mathcal{E}_{\gamma \sin(\eta \cdot)}}$ be defined as in (4.5) and fix $t, s \in [0, \infty[$, $\xi_0, \xi_1 \in \overline{B}_R(0)$, and $\tau_0, \tau_1 \in [0, 2\pi/\eta]$ with $\tau_1 \ge \tau_0$. Then by (3.1), Hölder's inequality, and (3.50)

$$\begin{aligned}
&|\varphi^t(\xi_0, \tau_0) - \varphi^s(\xi_1, \tau_1)| \\
&\le \Big(\big(x(\tau_0 + t; \xi_0, \tau_0) - x(\tau_1 + s; \xi_1, \tau_1) \big)^2 + \big(\dot{x}(\tau_0 + t; \xi_0, \tau_0) - \dot{x}(\tau_1 + s; \xi_1, \tau_1) \big)^2 \Big)^{1/2} \\
&\quad + |\tau_0 - \tau_1| + |t - s| \qquad (3.52) \\
&\le \Big(\big(x(\tau_0 + t; \xi_0, \tau_0) - x(\tau_1 + t; \xi_0, \tau_0) \big)^2 + \big(\dot{x}(\tau_0 + t; \xi_0, \tau_0) - \dot{x}(\tau_1 + t; \xi_0, \tau_0) \big)^2 \Big)^{1/2} \\
&\quad + \Delta^{1/2}(\tau_1 + t; \xi_0, \tau_0, \xi_1, \tau_0) + \Delta^{1/2}(\tau_1 + t; \xi_1, \tau_0, \xi_1, \tau_1) + |\tau_0 - \tau_1| + |t - s| \\
&\quad + \Big(\big(x(\tau_1 + t; \xi_1, \tau_1) - x(\tau_1 + s; \xi_1, \tau_1) \big)^2 + \big(\dot{x}(\tau_1 + t; \xi_1, \tau_1) - \dot{x}(\tau_1 + s; \xi_1, \tau_1) \big)^2 \Big)^{1/2} \\
&\le \Delta^{1/2}(\tau_1 + t; \xi_0, \tau_0, \xi_1, \tau_0) + \Delta^{1/2}(\tau_1 + t; \xi_1, \tau_0, \xi_1, \tau_1) + (1 + c_1)(|\tau_0 - \tau_1| + |t - s|). \qquad (3.53)
\end{aligned}$$

By Example 4.5 in Section 4.2 we know that for the pendulum with dry friction the assumptions of Lemma 4.4 are satisfied. Thus part (a) of this lemma implies $\Delta^{1/2}(\tau_1 + t; \xi_0, \tau_0, \xi_1, \tau_0) \le |\xi_0 - \xi_1|$, whereas part (b), (3.1), and (3.51) yield

$$\Delta(\tau_1 + t; \xi_1, \tau_0, \xi_1, \tau_1) \le \Delta(\tau_1; \xi_1, \tau_0, \xi_1, \tau_1) = |\varphi^{\tau_1 - \tau_0}(\xi_1, \tau_0) - \varphi^0(\xi_1, \tau_1)|^2_{\mathbb{R}^3} \le 4c_2^2 |\tau_0 - \tau_1|^2.$$

Inserting both estimates into (3.52) gives the claim. $\square$

Next we have a closer look on some special cases. In Section 3.3.1 we consider parameters $\gamma \in [0, 1]$, $\eta \in [0, \infty[$, which we will call the "small amplitude case", whereas in Section 3.3.2 we investigate the "resonant case" $\gamma \in [0, \infty[$, $\eta = 1$.

3.3.1 The small amplitude case $\gamma\in[0,1]$, $\eta\in[0,\infty[$

Our objective is here to prove that for these parameters $\mathcal{S}_\gamma=[\gamma-1,1-\gamma]$, cf. (3.8), the set of stationary solutions of (1.2), is uniformly asymptotically stable w.r.t initial values and phases. This will imply that the only ergodic measures for the periodic flow $\varphi_{\rm per}$, cf. Section 2.5, are the Dirac measures on $\mathcal{S}_\gamma$, or, to be more precise, their counterparts in the periodic setting, see Corollary 3.35 below.

Definition 3.23 *The set $\mathcal{S}_\gamma$ is called uniformly asymptotically stable w.r.t initial values and phases, if*

$$\begin{aligned}&\forall R>0\ \forall\varepsilon>0\ \exists\tau_*>0\ \forall\tau\in[\tau_*,\infty[\ \forall|\xi_0|\le R\ \forall\tau_0\in[0,2\pi/\eta]:\\ &\operatorname{dist}_{\mathbb{R}^2}\Big((x(\tau;\xi_0,\tau_0),\dot x(\tau;\xi_0,\tau_0)),\mathcal{S}_\gamma\times\{0\}\Big)\le\varepsilon.\end{aligned}\tag{3.54}$$

Theorem 3.24 *For $\gamma\in[0,1]$ and $\eta\in]0,\infty[$, $\mathcal{S}_\gamma$ is uniformly asymptotically stable w.r.t initial values and phases.*

Proof: We will assume w.l.o.g. that $\gamma>0$. Step 1. Fix $R>0$, $\varepsilon>0$, $\xi_0\in\mathbb{R}^2$ with $|\xi_0|\le R$, and $\tau_0\in[0,2\pi/\eta]$. We show that there exists $\tau_*(\xi_0,\tau_0)>0$ such that for $\tau\in[\tau_*(\xi_0,\tau_0),\infty[$ we have $(x(\tau;\xi_0,\tau_0),\dot x(\tau;\xi_0,\tau_0))\in U_\varepsilon$, where $U_\varepsilon\subset\mathbb{R}^2$ is any ε-neighborhood of $\mathcal{S}_\gamma\times\{0\}$ in $\mathbb{R}^2$. Let $x(\tau)=x(\tau;\xi_0,\tau_0)$ and $\mathcal{E}(\tau)=\mathcal{E}(\tau;\xi_0,\tau_0)$, cf. (3.48). The first part of the argument (i.e. $\dot x(\tau)\to0$ as $\tau\to\infty$) now is similar to [8, Example 14.2]. With a suitable $w(\tau)\in\operatorname{Sgn}(\dot x(\tau))$ it follows a.e. on $[\tau_0,\infty[$ that

$$\dot{\mathcal{E}}(\tau)=2\dot x(\tau)\,[x(\tau)+\ddot x(\tau)]=2\dot x(\tau)\,[\gamma\sin(\eta\tau)-w(\tau)]\le-2(1-\gamma)|\dot x(\tau)|\le0,\tag{3.55}$$

and therefore $\mathcal{E}(\tau)\le\mathcal{E}(\tau_0)$ for $\tau\in[\tau_0,\infty[$, and $\mathcal{E}(\tau)\to\mathcal{E}_\infty$ for some $\mathcal{E}_\infty\ge0$ as $\tau\to\infty$. We claim that $\dot x(\tau)\to0$ as $\tau\to\infty$. If not, then we find $\delta_0>0$ and $\tau_{n+1}\ge\tau_n+1\ge\tau_0$ such that w.l.o.g. $\dot x(\tau_n)\ge2\delta_0$ for $n\in\mathbb{N}$. Because of the boundedness of $\mathcal{E}$ it follows from the inclusion that $\operatorname{esssup}_{\tau\in[\tau_0,\infty[}|\ddot x(\tau)|\le c_0$ for some finite constant c_0, w.l.o.g. $c_0>0$. With $\delta_1=\delta_0/c_0\wedge1>0$ we obtain for $n\in\mathbb{N}$ and $\tau\in[\tau_n,\tau_n+\delta_1]$ that $\dot x(\tau)=\dot x(\tau_n)+\int_{\tau_n}^{\tau}\ddot x(\sigma)d\sigma\ge2\delta_0-(\tau-\tau_n)c_0\ge\delta_0$. Hence (3.55) yields by the property of w

$$\begin{aligned}\mathcal{E}(\tau_{n+1})-\mathcal{E}(\tau_n)&=2\int_{\tau_n}^{\tau_{n+1}}\dot x(\tau)\,[\gamma\sin(\eta\tau)-w(\tau)]\,d\tau\le2\int_{\tau_n}^{\tau_n+\delta_1}\dot x(\tau)\,[\gamma\sin(\eta\tau)-w(\tau)]\,d\tau\\&=2\int_{\tau_n}^{\tau_n+\delta_1}\dot x(\tau)\,[\gamma\sin(\eta\tau)-1]\,d\tau\le-2\delta_0\Big(\delta_1-\gamma\int_{\tau_n}^{\tau_n+\delta_1}\sin(\eta\tau)\,d\tau\Big)\\&\le-2\delta_0\delta_1\,(1-\gamma)\le0.\end{aligned}\tag{3.56}$$

Since the left-hand side tends to zero as $n\to\infty$, this clearly is a contradiction for $\gamma\in[0,1[$. On the other hand, (3.56) implies for $\gamma=1$ that $\lim_{n\to\infty}\int_{\tau_n}^{\tau_n+\delta_1}\sin(\eta\tau)\,d\tau=\delta_1$, which is also contradictory, because with $\tau_n=k_n\,(2\pi/\eta)+r_n$, $k_n\in\mathbb{N}_0$, $r_n\in[0,2\pi/\eta[$, we then would have w.l.o.g. $r_n\to r_\infty\in[0,2\pi/\eta]$, and thus $0\leftarrow\delta_1-\int_{\tau_n}^{\tau_n+\delta_1}\sin(\eta\tau)\,d\tau=\int_{r_n}^{r_n+\delta_1}[1-\sin(\eta\tau)]\,d\tau\to\int_{r_\infty}^{r_\infty+\delta_1}[1-\sin(\eta\tau)]\,d\tau$, a contradiction. Consequently, $\dot x(\tau)\to0$ as $\tau\to\infty$, and from this also $x^2(\tau)=\mathcal{E}(\tau)-\dot x^2(\tau)\to\mathcal{E}_\infty$. Since we are done if $\mathcal{E}_\infty=0$, we can assume w.l.o.g. that $(x(\tau),\dot x(\tau))\to(x_\infty,0)$ as $\tau\to\infty$ for some $x_\infty<0$. If the claim were wrong, then $(x_\infty,0)\notin U_\varepsilon$, and thus $x_\infty\le\gamma-1-2\varepsilon_0$ for some $\varepsilon_0>0$. Hence also $x(\tau)\le\gamma-1-\varepsilon_0$ for $\tau\in[\tau_0+\tau_1,\infty[$, with a suitable $\tau_1\ge0$. Let $y_1=(x(\tau_0+\tau_1),\dot x(\tau_0+\tau_1),\tau_0+\tau_1)=\varphi^{\tau_1}(y_0)$ with $y_0=(\xi_0,\tau_0)$. In terms of the semiflow, $\varphi^{\tau}(y_0)\in\{y\in\mathbb{R}^3:x\le\gamma-1-\varepsilon_0\}$ for $\tau\in[\tau_1,\infty[$, and thus in particular $y_1\in M_+\cup M_{\rm left}\cup M_-$. Therefore we find $\tau_2\ge\tau_1$ such that $y_2=\varphi^{\tau_2}(y_0)\in M_+$, by Lemma 3.3(b) or Lemma 3.5(c) for M_-. Let $\tau_i=t_{i-2}(y_2)$ and $y_i=\varphi(\tau_i,y_0)$ for $i\ge3$. Then $\tau_{i+1}=t_1(y_i)+\tau_i$ and $y_{i+1}=\varphi(t_1(y_i),y_i)$ for $i\ge2$. Since under the present hypotheses only switchings from M_+ to $M_{\rm left}$ and vice versa are possible, it follows that $y_3\in M_{\rm left}$, $y_4\in M_+$, $y_5\in M_{\rm left}$, ... Therefore

$y_{2k}\in\{y\in\mathbb{R}^3 : v=0, x=b^-(\tau), \tau_{\rm mod}\in[0,\pi/2\eta[\cup]3\pi/2\eta,2\pi/\eta[, x\le\gamma-1-\varepsilon_0\}\subset M_+$ for $k\ge 2$ by assumption and by (3.15) in Lemma 3.3(b). Corresponding to ε_0 from above choose $\varepsilon_1>0$ and $\delta_1>0$ with the property described in Lemma 3.4(e). Then $\varphi(\varepsilon_1+\tau_{2k},y_0)=\varphi(\varepsilon_1,y_{2k})\in\{y\in\mathbb{R}^3 : v\ge\delta_1\}$, meaning that $\dot{x}(\tau_0+\varepsilon_1+\tau_{2k})\ge\delta_1$ for $k\ge 2$. Since $y_i\in\{y\in\mathbb{R}^3 : x\le\gamma-1-\varepsilon_0\}\subset\mathbb{R}^3\setminus M_\infty$, we have $t_i(y_2)<\infty$ for $i\in\mathbb{N}_0$, thus $t_i(y_2)\to\infty$ as $i\to\infty$ by Lemma 3.9, and hence also $\tau_{2k}\to\infty$ as $k\to\infty$. This contradicts $\dot{x}(\tau)\to 0$ as $\tau\to\infty$. Therefore we must have $(x_\infty,0)\in U_\varepsilon$, yielding the claim of the first step. Step 2. By compactness of $\overline{B}_R(0)\times[0,2\pi/\eta]$ it is sufficient to prove

$$\begin{aligned}&\forall R>0\ \forall\varepsilon>0\ \forall|\bar{\xi}_0|\le R\ \forall\bar{\tau}_0\in[0,2\pi/\eta]\ \exists\delta>0\ \exists\tau_*>0 :\\ &\xi_0\in\overline{B}_R(0)\cap\overline{B}_\delta(\bar{\xi}_0),\ \tau_0\in[0,2\pi/\eta]\cap[\bar{\tau}_0-\delta,\bar{\tau}_0+\delta]\quad\Longrightarrow\\ &\qquad(x(\tau;\xi_0,\tau_0),\dot{x}(\tau;\xi_0,\tau_0))\in U_\varepsilon\quad\text{for}\quad\tau\in[\tau_*,\infty[,\end{aligned}\tag{3.57}$$

where $U_\varepsilon\subset\mathbb{R}^2$ is a ε-neighborhood of $\mathcal{S}_\gamma\times\{0\}$. Let L denote the Lipschitz constant of φ on $[0,\infty[\times(\overline{B}_R(0)\times[0,2\pi/\eta])$, cf. Corollary 3.22. For $\bar{y}_0=(\bar{\xi}_0,\bar{\tau}_0)$ as above choose $\bar{\tau}_*=\tau_*(\bar{\xi}_0,\bar{\tau}_0)\ge 2\pi/\eta$ according to the first step, with $U_{\varepsilon/2}$ there. Moreover, fix $\varepsilon_1>0$ such that $U_{\varepsilon/2}+\overline{B}_{\varepsilon_1}(0)\subset U_\varepsilon$. Define $\delta=\varepsilon_1/(3L)$ and $\tau_*=\bar{\tau}_*$, and let $y_0=(\xi_0,\tau_0)$ be as in (3.57). Then for $\tau\in[\tau_*,\infty[$

$$\begin{aligned}\left|\begin{pmatrix}x(\tau;y_0)\\ \dot{x}(\tau;y_0)\end{pmatrix}-\begin{pmatrix}x(\tau;\bar{y}_0)\\ \dot{x}(\tau;\bar{y}_0)\end{pmatrix}\right|_{\mathbb{R}^2}&=|\varphi(\tau-\tau_0,y_0)-\varphi(\tau-\bar{\tau}_0,\bar{y}_0)|_{\mathbb{R}^3}\le L\,(|\tau_0-\bar{\tau}_0|+|y_0-\bar{y}_0|_{\mathbb{R}^3})\\ &\le(1+\sqrt{2})L\,\delta\le\varepsilon_1,\end{aligned}$$

and hence $(x(\tau;\xi_0,\tau_0),\dot{x}(\tau;\xi_0,\tau_0))\in U_\varepsilon$ for $\tau\in[\tau_*,\infty[$. □

Corollary 3.25 *Let $\varphi_{\rm per}$ be the periodic flow on $\mathbb{R}^3_{2\pi/\eta}=\mathbb{R}^2\oplus\mathbb{S}^1_{2\pi/\eta}$ corresponding to φ, see Section 2.5. Then*

$$\begin{aligned}&\forall R>0\ \forall\varepsilon>0\ \exists t_*>0\ \forall t\in[t_*,\infty[\ \forall|\xi_0|\le R\ \forall\tau_0\in[0,2\pi/\eta] :\\ &\mathrm{dist}_{\mathbb{R}^3_{2\pi/\eta}}\Big(\varphi^t_{\rm per}(\xi_0,e^{i\tau_0\eta}),\mathcal{S}_\gamma\times\{0\}\times\mathbb{S}^1_{2\pi/\eta}\Big)\le\varepsilon.\end{aligned}$$

Proof: By (2.29) and (3.1),

$$\varphi^t_{\rm per}(\xi_0,e^{i\tau_0\eta})=(\varphi_1(t,\xi_0,\tau_0),\varphi_2(t,\xi_0,\tau_0),e^{i[\tau_0+t]\eta})=(x(\tau_0+t;\xi_0,\tau_0),\dot{x}(\tau_0+t;\xi_0,\tau_0),e^{i[\tau_0+t]\eta}).$$

Hence we may define $t_*=\tau_*$, with τ_* from (3.54). □

3.3.2 The resonant case $\gamma\in[0,\infty[$, $\eta=1$

Since we are interested in the bifurcations when γ varies, we note that we have already obtained the following:

(a) For $\gamma\in[0,1]$, the fixed points $\mathcal{S}_\gamma$ are uniformly asymptotically stable w.r.t initial values and phases, cf. Theorem 3.24.

(b) For $\gamma\in]4/\pi,\infty[$, the solutions of (1.2) are uniformly unbounded w.r.t initial values and phases, cf. Theorem 3.21(b).

So we are left to investigate more closely the case $\gamma\in]1,4/\pi[$, where we also have already some information from Corollary 3.19(b2). For fixed $\gamma\in]1,4/\pi[$ we let

$$\mathrm{gr}(x_\gamma)=\{(x_\gamma(\tau),\dot{x}_\gamma(\tau),e^{i\tau}) : \tau\in[0,2\pi]\}\subset\mathbb{R}^3_{2\pi},$$

cf. Section 2.5, denote the graph in $\mathbb{R}^3_{2\pi}$ of the unique periodic solution x_γ.

Our aim is to prove the following uniform asymptotic stability of $\mathrm{gr}(x_\gamma)$ which will be used in Section 3.4.3 below to show that $\mathcal{P}_{\rm erg}(\varphi_{\rm per})=\mathcal{P}_{\rm inv}(\varphi_{\rm per})=\{\mu_\gamma\}$ contains exactly one probability measure μ_γ determined by x_γ.

Lemma 3.26 *Let φ_{per} be the periodic flow on $\mathbb{R}^3_{2\pi} = \mathbb{R}^2 \oplus \mathbb{S}^1_{2\pi}$ corresponding to φ, see Section 2.5. Then*

$$\forall R > 0\ \forall \varepsilon > 0\ \exists t_* > 0\ \forall t \in [t_*, \infty[\ \forall |\xi_0| \le R\ \forall \tau_0 \in [0, 2\pi]: \quad \operatorname{dist}_{\mathbb{R}^3_{2\pi}}\Big(\varphi^t_{\text{per}}(\xi_0, e^{i\tau_0}), \operatorname{gr}(x_\gamma)\Big) \le \varepsilon .$$

Proof: The proof is similar to the proof of Theorem 3.24, cf. the second step there, and therefore omitted. □

3.4 Lyapunov exponents for the pendulum with dry friction

Let us first summarize what we have got so far. In Section 3.2 we have shown that assumptions $(A1)$-$(A10)$ from Sections 2.1 and 2.4 hold for the pendulum with dry friction, cf. Theorem 3.16, if $\gamma \neq 1$. Hence Theorem 2.14 yields a canonical cocycle $T : [0, \infty[\times \mathbb{R}^3 \to \mathcal{L}(\mathbb{R}^3)$ corresponding to the semiflow $\varphi : [0, \infty[\times \mathbb{R}^3 \to \mathbb{R}^3$, cf. Section 3.1. This cocycle in fact lives on a set $G \subset \mathbb{R}^3$ of "good" initial values, cf. Corollary 2.7, which has full Lebesgue measure by Theorem 2.21. Since (1.2) is $\omega = 2\pi/\eta$-periodic, we also have a canonical periodic flow φ_{per} and cocycle T_{per} on $\mathbb{R}^3_{2\pi/\eta}$ by Section 2.5, and it is clear that the MET Theorem 4.3 also holds in the setting of $\mathbb{R}^3_{2\pi/\eta}$. Moreover, by Corollary 3.32 below, the integrability conditions (4.1) in the MET are satisfied.

In this section, we first show that for the pendulum with dry friction the upper Lyapunov exponents

$$\lambda^+(y_0, y) := \limsup_{t \to \infty} \left(\frac{1}{t} \ln |T(t, y_0) y| \right), \quad y_0, y \in \mathbb{R}^3, \tag{3.58}$$

are non-positive, cf. Theorem 3.33. In particular, this implies that no positive Lyapunov exponents are possible. Afterwards we turn to some special cases. First, in Section 3.4.2, we investigate the "small amplitude case" $\gamma \in [0, 1[$, $\eta \in [0, \infty[$, from Section 3.3.1 in greater detail, while Section 3.4.3 is devoted to the study of the "resonant case" $\gamma \in [0, \infty[$, $\gamma \neq 1$, $\eta = 1$, from Section 3.3.2. Note that in comparison to Sections 3.3.1 and 3.3.2 we have to exclude the value $\gamma = 1$ here, since we didn't construct a cocycle for this value of γ.

3.4.1 Non-positivity of the Lyapunov exponents

To estimate the upper Lyapunov exponents for (1.2) we investigate closer the form of the cocycle $T : [0, \infty[\times \mathbb{R}^3 \to \mathcal{L}(\mathbb{R}^3)$. We start with some preliminary results.

Let

$$p(\tau) = \gamma \sin(\eta \tau), \quad \tau \in \mathbb{R}, \tag{3.59}$$

denote the forcing, i.e. $b^\pm(\tau) = p(\tau) \pm 1$, cf. (3.9). Define

$$P(t, \bar{t}) = -\sin(t - \bar{t}) p(\bar{t}) + \int_{\bar{t}}^{t} \cos(t - s) p(s)\, ds, \quad t, \bar{t} \in \mathbb{R}. \tag{3.60}$$

Therefore

$$P'(t, \bar{t}) := \partial_t P(t, \bar{t}) = -\cos(t - \bar{t}) p(\bar{t}) + p(t) - \int_{\bar{t}}^{t} \sin(t - s) p(s)\, ds \quad \text{for} \quad t, \bar{t} \in \mathbb{R}.$$

We first state some elementary properties of P and P'.

Lemma 3.27 *For $0 \le \bar{t} \le t$*

$$P(t, \bar{t}) = -\sin(t - \bar{t}) b^\pm(\bar{t}) + \int_{\bar{t}}^{t} \cos(t - s) b^\pm(s)\, ds$$

and

$$P'(t, \bar{t}) = -\cos(t - \bar{t}) b^\pm(\bar{t}) + b^\pm(t) - \int_{\bar{t}}^{t} \sin(t - s) b^\pm(s)\, ds .$$

Proof: The first equality follows from inserting $b^{\pm}(\bar{t})=p(\bar{t})\pm 1$ and $b^{\pm}(s)=p(s)\pm 1$, the second by differentiating the first. $\square$

Lemma 3.28 *There exists a constant c_* such that for all $0\le \bar{t}\le t$*

$$|P(t,\bar{t})|\le c_*(t-\bar{t}) \quad \textit{and} \quad |P'(t,\bar{t})|\le c_*(t-\bar{t}).$$

Proof: Since $1-\cos(x)\le x$ for $x\in[0,\infty[$, we have e.g. $|P'(t,\bar{t})|\le(2|p|_\infty+|p'|_\infty)(t-\bar{t})$. We thus may define $c_*=2|p|_\infty+|p'|_\infty$ to obtain both claimed estimates, using $|\sin(x)|\le x$, $x\in[0,\infty[$, for the first. $\square$

Next we introduce the notation

$$A(t,\bar{t},\alpha)=\begin{pmatrix} \cos(t-\bar{t}) & \alpha\sin(t-\bar{t}) & P(t,\bar{t}) \\ -\sin(t-\bar{t}) & \alpha\cos(t-\bar{t}) & P'(t,\bar{t}) \\ 0 & 0 & 1 \end{pmatrix}, \quad t,\bar{t},\alpha\in\mathbb{R}, \tag{3.61}$$

and in particular

$$A(\alpha=0):=A(t,t,0)=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad t\in\mathbb{R}.$$

By direct calculation we obtain

Lemma 3.29 *We have $A(t,\bar{t},1)\circ A(\bar{t},s,1)=A(t,s,1)$ for $t,\bar{t},s\in\mathbb{R}$.*

Let $T:[0,\infty[\times\mathbb{R}^3\to\mathcal{L}(\mathbb{R}^3)$ denote the cocycle for the pendulum with dry friction, constructed according to Theorem 2.14 with G from Corollary 2.7. The next two lemmas describe the form of $T(\cdot,y)$ for $y\in G=G_{<\infty}\cup G_\infty=G_{<\infty}\cup\bigcup_{i\in\mathbb{N}_0}G^i_\infty$.

Lemma 3.30 *Let $y_0\in G^0_\infty=D_\infty$. Then $T(t,y_0)=A(\alpha=0)$ for $t\in[0,\infty[$. In particular, we have $\|T(t,y_0)\|_3=1$ for $t\in[0,\infty[$.*

Proof: By the 3$^{\text{rd}}$ line of the definition of T in Theorem 2.14 we have $T(t,y_0)=\partial_y\varphi^t(y_0)$ for $t\in[0,\infty[$, and $\varphi^t(y)=(x,0,\tau+t)$ if $y=(x,v,\tau)$ by Lemma 3.2. Thus the claim follows. $\square$

Next we turn to

Lemma 3.31 *For $y_0=(x_0,v_0,\tau_0)\in G\setminus G^0_\infty$ fixed let $t_i=t_i(y_0)$ be the switching times along the trajectory generated by y_0, i.e. $i\in\mathbb{N}_0$ for $y_0\in G_{<\infty}$, and $i\in\{0,\dots,i_0+1\}$ for $y_0\in G^{i_0+1}_\infty$ with $i_0\in\mathbb{N}_0$. Also, let $y_{i+1}=\xi_{i+1}(y_0)=\xi_1(y_i)=\varphi(t_{i+1},y_0)=\varphi(t_1(y_i),y_i)=(x_{i+1},v_{i+1},\tau_{i+1})$, as well as*

$$\alpha^+_{i+1}=\left[1-\frac{2}{x_{i+1}-b^-(\tau_{i+1})}\right] \quad \textit{and} \quad \alpha^-_{i+1}=\left[1-\frac{2}{b^+(\tau_{i+1})-x_{i+1}}\right], \tag{3.62}$$

with $i\in\mathbb{N}_0$ for $y_0\in G_{<\infty}$, and $i\in\{0,\dots,i_0\}$ for $y_0\in G^{i_0+1}_\infty$.

There are the following possibilities of switchings at time t_{i+1}:

(a) from $M_+\to M_-$, i.e. $y_i\in M_+$ and $y_{i+1}\in M_-$. Then for $t\in[t_{i+1},t_{i+2}[$

$$T(t,y_0)=A(\tau_0+t,\tau_0+t_{i+1},\alpha^+_{i+1})\circ\left(\lim_{s\to t^-_{i+1}}T(s,y_0)\right).$$

In this case, additionally $\alpha^+_{i+1}\in[0,1[$.

(b) switching $M_+ \to M_{\text{right}}$, i.e. $y_i \in M_+$ and $y_{i+1} \in M_{\text{right}}$. Then for $t \in [t_{i+1}, t_{i+2}[$

$$T(t,y_0) = A(\alpha=0) \circ \left(\lim_{s \to t_{i+1}^-} T(s,y_0) \right). \tag{3.63}$$

(c) switching $M_+ \to M_{\text{left}}$, i.e. $y_i \in M_+$ and $y_{i+1} \in M_{\text{left}}$. Then for $t \in [t_{i+1}, t_{i+2}[$ formula (3.63) holds for $T(t,y_0)$.

(d) switching $M_+ \to M_\infty$, i.e. $y_i \in M_+$ and $y_{i+1} \in M_\infty$. Then for $t \in [t_{i+1}, t_{i+2}[$ formula (3.63) holds for $T(t,y_0)$.

(e) switching $M_- \to M_+$, i.e. $y_i \in M_-$ and $y_{i+1} \in M_+$. Then for $t \in [t_{i+1}, t_{i+2}[$

$$T(t,y_0) = A(\tau_0 + t, \tau_0 + t_{i+1}, \alpha_{i+1}^-) \circ \left(\lim_{s \to t_{i+1}^-} T(s,y_0) \right).$$

In this case, additionally $\alpha_{i+1}^- \in [0,1[$.

(f) switching $M_- \to M_{\text{left}}$, i.e. $y_i \in M_-$ and $y_{i+1} \in M_{\text{left}}$. Then for $t \in [t_{i+1}, t_{i+2}[$ formula (3.63) holds for $T(t,y_0)$.

(g) switching $M_- \to M_{\text{right}}$, i.e. $y_i \in M_-$ and $y_{i+1} \in M_{\text{right}}$. Then for $t \in [t_{i+1}, t_{i+2}[$ formula (3.63) holds for $T(t,y_0)$.

(h) switching $M_- \to M_\infty$, i.e. $y_i \in M_-$ and $y_{i+1} \in M_\infty$. Then for $t \in [t_{i+1}, t_{i+2}[$ formula (3.63) holds for $T(t,y_0)$.

(i) switching $M_{\text{left}} \to M_+$, i.e. $y_i \in M_{\text{left}}$ and $y_{i+1} \in M_+$. Then for $t \in [t_{i+1}, t_{i+2}[$

$$T(t,y_0) = A(\tau_0 + t, \tau_0 + t_{i+1}, 1) \circ \left(\lim_{s \to t_{i+1}^-} T(s,y_0) \right).$$

(j) switching $M_{\text{right}} \to M_-$, i.e. $y_i \in M_{\text{right}}$ and $y_{i+1} \in M_-$. Then for $t \in [t_{i+1}, t_{i+2}[$

$$T(t,y_0) = A(\tau_0 + t, \tau_0 + t_{i+1}, 1) \circ \left(\lim_{s \to t_{i+1}^-} T(s,y_0) \right).$$

For the transition matrices $A_{i+1}(y_0)$ of the linearization, cf. Lemma 2.11 resp. Lemma 2.12 and Corollary 2.13, we obtain

$$\begin{aligned} &(a): \quad A_{i+1}(y_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha_{i+1}^+ & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (b),(c),(d),(f),(g),(h): \quad A_{i+1}(y_0) = A(\alpha=0) \\ &(e): \quad A_{i+1}(y_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha_{i+1}^- & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (i),(j): \quad A_{i+1}(y_0) = \mathrm{id}_{\mathbb{R}^3}. \end{aligned} \tag{3.64}$$

In particular, $\|A_{i+1}(y_0)\|_3 = 1$ for $y_0 \in G \setminus G_\infty^0$.

Proof: Ad (a): By Remark 2.15 we have $T(t,y_0) = \partial_y \varphi^t(y_0)$ for $t \in]t_{i+1}, t_{i+2}[$, and this may be calculated by means of (2.15) in Corollary 2.13. In the notation of this corollary, $f_l = f_+$, $h_l = h_+$, and $f_p = f_-$, cf. (3.11) and (3.31) for the definitions of the functions. Since $\lim_{s\to t_{i+1}^-} \partial_y \varphi^s(y_0) = \lim_{s\to t_{i+1}^-} T(s,y_0)$ by definition of T, we obtain from (2.15) for $t \in]t_{i+1}, t_{i+2}[$

$$
\begin{aligned}
T(t,y_0) = \Bigg(\frac{1}{\langle \nabla h_+, f_+\rangle(y_{i+1})} \Big([f_-(\Phi_{f_-}(t-t_{i+1}, y_{i+1})) \\
- \partial_y \Phi_{f_-}(t-t_{i+1}, y_{i+1})\, f_+(y_{i+1})]_\kappa \cdot [\nabla h_+(y_{i+1})]_\nu \Big)_{1\le\kappa,\nu\le 3} + \partial_y \Phi_{f_-}(t-t_{i+1}, y_{i+1}) \Bigg) \\
\circ \Big(\lim_{s\to t_{i+1}^-} T(s,y_0) \Big).
\end{aligned}
\tag{3.65}
$$

The right-continuity of $T(\cdot, y_0)$ at t_{i+1}, cf. Lemma 2.11, shows that (3.65) in fact holds for $t \in [t_{i+1}, t_{i+2}[$. To derive (a) from (3.65), we first remark that $\langle h_+, f_+\rangle(y_{i+1}) = -x_{i+1} + b^-(\tau_{i+1})$. Moreover, by the analog of (3.24) in Lemma 3.5, for $t \in \mathrm{I\!R}$ and $y = (x,v,\tau) \in \mathrm{I\!R}^3$

$$
\Phi_{f_-}(t,y) = \begin{pmatrix} x\cos t + v \sin t + \int_0^t \sin(t-s)\, b^+(\tau+s)\, ds \\ -x \sin t + v\cos t + \int_0^t \cos(t-s)\, b^+(\tau+s)\, ds \\ \tau + t \end{pmatrix},
\tag{3.66}
$$

and thus by changing variables $\sigma = \tau + s$ in the integrals, and by Lemma 3.27 and (3.61)

$$
\partial_y \Phi_{f_-}(t,y) = A(\tau+t, \tau, 1).
\tag{3.67}
$$

Since $y_{i+1} = \xi_1(y_i) \in \{y \in \mathrm{I\!R}^3 : v = 0\}$ by Lemma 3.5(c), we have $v_{i+1} = 0$. Therefore we obtain from Lemma 3.27 and $b^\pm(\tau) = p(\tau) \pm 1$, cf. (3.9),

$$
f_-(\Phi_{f_-}(t-t_{i+1}, y_{i+1})) - \partial_y \Phi_{f_-}(t-t_{i+1}, y_{i+1})\, f_+(y_{i+1}) = 2 \begin{pmatrix} \sin(t-t_{i+1}) \\ \cos(t-t_{i+1}) \\ 0 \end{pmatrix}.
$$

Because $\nabla h_+(y) = (0,1,0)^{\mathrm{tr}}$, we can insert into (3.65) and find by (3.61) and (3.62)

$$
\begin{aligned}
&T(t,y_0) \\
&= \left(\frac{2}{-x_{i+1} + b^-(\tau_{i+1})} \begin{pmatrix} 0 & \sin(t-t_{i+1}) & 0 \\ 0 & \cos(t-t_{i+1}) & 0 \\ 0 & 0 & 0 \end{pmatrix} + A(\tau_{i+1} + t - t_{i+1}, \tau_{i+1}, 1) \right) \circ \Big(\lim_{s\to t_{i+1}^-} T(s,y_0) \Big) \\
&= A(\tau_{i+1} + t - t_{i+1}, \tau_{i+1}, \alpha_{i+1}^+) \circ \Big(\lim_{s\to t_{i+1}^-} T(s,y_0) \Big).
\end{aligned}
$$

Since $\varphi^t(y) = (\ldots, \ldots, \tau + t)$ for every $y = (x,v,\tau) \in \mathrm{I\!R}^3$ and $t \in [0, t_1(y)[$, we have $\tau_1 = \tau_0 + t_1$, $\tau_2 = \tau_1 + t_1(y_1), \ldots, \tau_{i+1} = \tau_i + t_1(y_i)$, and thus by definition of the t_i in (2.3), $\tau_{i+1} = \tau_0 + t_{i+1}$. This gives the claimed form of $T(t,y_0)$. In addition, note that $y_{i+1} \in \{y \in \mathrm{I\!R}^3 : v = 0\} \cap M_-$ yields $x_{i+1} \ge b^+(\tau_{i+1})$ by definition of M_-, cf. (3.10), and thus $x_{i+1} \ge b^+(\tau_{i+1}) - b^-(\tau_{i+1}) + b^-(\tau_{i+1}) = 2 + b^-(\tau_{i+1})$. It follows that $\alpha_{i+1}^+ \in [0,1[$.

Ad (b): We have to replace f_- through f_{right} in (3.65). Here $\Phi_{f_{\mathrm{right}}}(t,y) = (x, v, \tau + t)$ for $t \in [0, \infty[$ and $y = (x,v,\tau) \in \mathrm{I\!R}^3$, cf. (3.11). Hence $\partial_y \Phi_{f_{\mathrm{right}}}(t,y) = \mathrm{id}_{\mathrm{I\!R}^3}$, and $v_{i+1} = 0$ yields $f_{\mathrm{right}}(\Phi_{f_{\mathrm{right}}}(t - t_{i+1}, y_{i+1})) - f_+(y_{i+1}) = (0, x_{i+1} - b^-(\tau_{i+1}), 0)^{\mathrm{tr}}$. Therefore we obtain from $\langle h_+, f_+\rangle(y_{i+1}) = -x_{i+1} + b^-(\tau_{i+1})$ that

$$
T(t,y_0) = \left((-1) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mathrm{id}_{\mathrm{I\!R}^3} \right) \circ \Big(\lim_{s\to t_{i+1}^-} T(s,y_0) \Big) = A(\alpha = 0) \circ \Big(\lim_{s\to t_{i+1}^-} T(s,y_0) \Big).
$$

Since $f_{\text{left}} = f_\infty = f_{\text{right}}$, this additionally yields (c) and (d), while (e) is analogous to (a), and (f), (g), and (h) are obtained in the same way as (b).

Ad (i): Here $f_l = f_{\text{left}}$ and $f_p = f_+$ in (2.15). By (3.24) in Lemma 3.5(b2) we know that $\Phi_{f_+}(t,y)$ is obtained by replacing $b^+(\tau+s)$ through $b^-(\tau+s)$ in (3.66). Thus Lemma 3.27 shows that also $\partial_y \Phi_{f_+}(t,y) = A(\tau+t,\tau,1)$ for $t \in \mathbb{R}$ and $y = (x,v,\tau) \in \mathbb{R}^3$, since by this lemma it plays no rôle whether one has b^+ or b^- in (3.67). Hence by Lemma 3.27

$$f_+(\Phi_{f_+}(t-t_{i+1},y_{i+1})) - \partial_y \Phi_{f_+}(t-t_{i+1},y_{i+1})\, f_{\text{left}}(y_{i+1}) = \left[b^-(\tau_{i+1}) - x_{i+1}\right] \begin{pmatrix} \sin(t-t_{i+1}) \\ \cos(t-t_{i+1}) \\ 0 \end{pmatrix}.$$

But $y_{i+1} = \xi_1(y_i)$ has $x_{i+1} = b^-(\tau_{i+1})$ by (3.15) in Lemma 3.3(b). It follows from (2.15), cf. (3.65), that

$$T(t,y_0) = \partial_y \Phi_{f_+}(t-t_{i+1},y_{i+1}) \circ \left(\lim_{s\to t_{i+1}^-} T(s,y_0)\right) = A(\tau_{i+1}+t-t_{i+1},\tau_{i+1},1) \circ \left(\lim_{s\to t_{i+1}^-} T(s,y_0)\right),$$

and thus we obtain (i), cf. (a). In the same way, (j) can be shown.

Concerning the form of the $A_{i+1}(y_0)$, we remark that $A_{i+1}(y_0)$ is obtained as the limit $t \to t_{i+1}^+$ of the respective matrices above, cf. Lemma 2.11, Lemma 2.12, and Corollary 2.13. For example, in case (a) we find

$$A_{i+1}(y_0) = \lim_{t\to t_{i+1}^+} A(\tau_0+t,\tau_0+t_{i+1},\alpha_{i+1}^+) = A(\tau_0+t_{i+1},\tau_0+t_{i+1},\alpha_{i+1}^+) = A(\tau_{i+1},\tau_{i+1},\alpha_{i+1}^+),$$

and since $P(t,t) = P'(t,t) = 0$ for $t \in \mathbb{R}$, this gives the claimed matrices. So the proof of the lemma is complete. □

Corollary 3.32 *For every probability measure μ the integrability conditions (4.1) in the MET Theorem 4.3 are satisfied.*

Proof: By definition of the f_k in (3.11) and of $b^\pm$ in (3.9) we have $|f_k'|_\infty \le c$ for some constant $c > 0$ and $k \in \{+,-,\text{left},\text{right},\infty\}$. Consequently the claim follows from $\|A_{i+1}(y_0)\|_3 = 1$ for $y_0 \in G \setminus G_\infty^0$ and the corresponding $i's$, cf. Lemma 3.31, by applying Corollary 2.17. □

Now we are going to estimate the upper Lyapunov exponents for the pendulum with dry friction.

Theorem 3.33 *For all $y_0, y \in \mathbb{R}^3$*

$$\lambda^+(y_0,y) \le 0. \tag{3.68}$$

In particular, the pendulum with dry friction has no positive Lyapunov exponent.

Proof: By definition of T, cf. Theorem 2.14, and by Lemma 3.30 we can restrict ourselves to consideration of $y_0 = (x_0,v_0,\tau_0) \in G \setminus G_\infty^0$. We retain the notation from Lemma 3.31. Case 1: $y_0 \in G_\infty^{i_0+1}$ for some $i_0 \in \mathbb{N}_0$. Then $y_{i_0+1} \in D_\infty \subset M_\infty$, $t_{i_0+1} < \infty$, and $t_{i_0+2} = \infty$. We are thus in one of the cases (d) or (h) in Lemma 3.31, and thus it follows by taking $i = i_0$ that for $t \in [t_{i_0+1},\infty[$

$$\|T(t,y_0)\|_3 = \left\| A(\alpha=0) \circ \left(\lim_{s\to t_{i_0+1}^-} T(s,y_0)\right)\right\|_3 \le c\|A(\alpha=0)\|_3 = c$$

for some constant $c>0$. This yields (3.68). Case 2: $y_0 \in G_{<\infty}$. Then $t_i<\infty$, $y_i \in \mathbb{R}^3 \setminus M_\infty$ for all $i \in \mathbb{N}_0$, and also $t_i \to \infty$ as $i \to \infty$ by Lemma 3.9. For $i \in \mathbb{N}_0$ define functions $\vartheta_{i+1}:[t_{i+1},t_{i+2}] \to [t_{i+1},t_{i+2}]$ and numbers $\beta_{i+1} \in [0,1]$ as follows:

$$\left.\begin{array}{ll} \vartheta_{i+1}(t)=t, & \beta_{i+1}=\alpha_{i+1}^+ \\ \vartheta_{i+1}(t)=t_{i+1}, & \beta_{i+1}=0 \\ & \\ \vartheta_{i+1}(t)=t, & \beta_{i+1}=\alpha_{i+1}^- \\ \vartheta_{i+1}(t)=t, & \beta_{i+1}=1 \end{array}\right\} \quad \text{for} \quad \left\{\begin{array}{l} y_i \in M_+\,,\, y_{i+1} \in M_- \\ [y_i \in M_+\,,\, y_{i+1} \in M_{\text{right}}] \text{ or } [y_i \in M_+\,,\, y_{i+1} \in M_{\text{left}}] \text{ or} \\ [y_i \in M_-\,,\, y_{i+1} \in M_{\text{left}}] \text{ or } [y_i \in M_-\,,\, y_{i+1} \in M_{\text{right}}] \\ y_i \in M_-\,,\, y_{i+1} \in M_+ \\ [y_i \in M_{\text{left}}\,,\, y_{i+1} \in M_+] \text{ or } [y_i \in M_{\text{right}}\,,\, y_{i+1} \in M_-] \end{array}\right. .$$

Since $A(\alpha=0)=A(t,t,0)$ for $t \in \mathbb{R}$, it follows from Lemma 3.31(a), (b), (c), (e), (f), (g), (i), and (j) that for $i \in \mathbb{N}_0$

$$T(t,y_0)=A(\tau_0+\vartheta_{i+1}(t),\tau_0+t_{i+1},\beta_{i+1}) \circ \left(\lim_{s \to t_{i+1}^-} T(s,y_0)\right), \quad t \in [t_{i+1},t_{i+2}[. \tag{3.69}$$

Let $T_1^-=\lim_{t \to t_1^-} T(t,y_0)$. Then (3.69) may be iterated to yield for $k \in \mathbb{N}_0$ and $t \in [t_{k+1},t_{k+2}[$

$$\begin{aligned} T(t,y_0) &= A(\tau_0+\vartheta_{k+1}(t),\tau_0+t_{k+1},\beta_{k+1}) \circ \left(\prod_{i=0}^{k-1} A(\tau_0+\vartheta_{i+1}(t_{i+2}),\tau_0+t_{i+1},\beta_{i+1})\right) \circ T_1^- \\ &=: T_k(t,y_0) \circ T_1^-\,. \end{aligned} \tag{3.70}$$

Define $e_1=(1,0,0)^{\text{tr}}$, $e_2=(0,1,0)^{\text{tr}}$, and $e_3=(0,0,1)^{\text{tr}}$. We claim that there exists a constant $c_{**}>0$ such that for $k \in \mathbb{N}_0$, $t \in [t_{k+1},t_{k+2}[$, and $j=1,2,3$

$$|T_k(t,y_0)e_j| \le 1+c_{**}t\,. \tag{3.71}$$

Indeed, let $c_{**}=\sqrt{2}c_*$, with c_* from Lemma 3.28. We are in the situation of Lemma 4.10: For $i=0,\dots,k-1$ define $\theta_i=\vartheta_{i+1}(t_{i+2})-t_{i+1}$, $\alpha_i=\beta_{i+1}$, $P_i=P(\tau_0+\vartheta_{i+1}(t_{i+2}),\tau_0+t_{i+1})$ and $P_i'=P'(\tau_0+\vartheta_{i+1}(t_{i+2}),\tau_0+t_{i+1})$. Moreover, let $\theta_k=\vartheta_{k+1}(t)-t_{k+1}$, $\alpha_k=\beta_{k+1}$, $P_k=P(\tau_0+\vartheta_{k+1}(t),\tau_0+t_{k+1})$ and $P_k'=P'(\tau_0+\vartheta_{k+1}(t),\tau_0+t_{k+1})$. Then $A(\tau_0+\vartheta_{i+1}(t_{i+2}),\tau_0+t_{i+1},\beta_{i+1})=B(\theta_i,\alpha_i,P_i,P_i')=B_i$ for $i=0,\dots,k-1$, and $A(\tau_0+\vartheta_{k+1}(t),\tau_0+t_{k+1},\beta_{k+1})=B(\theta_k,\alpha_k,P_k,P_k')=B_k$, cf. (3.61) and the definition of the B's in (4.10). Since $T_k(t,y_0)=\prod_{i=0}^k B_i$ and since $\alpha_i \in [0,1]$ we first conclude from Lemma 4.10(d) and (e) that

$$|T_k(t,y_0)e_1| \le 1 \quad \text{and} \quad |T_k(t,y_0)e_2| \le 1\,. \tag{3.72}$$

Because $\vartheta_{i+1}(\bar{t}) \le \bar{t} \le t_{i+2}$ for $\bar{t} \in [t_{i+1},t_{i+2}]$ by definition of ϑ_{i+1}, Lemma 3.28 implies that for $i=0,\dots,k-1$

$$\left|\begin{pmatrix} P_i \\ P_i' \end{pmatrix}\right|_{\mathbb{R}^2}^2 = (P_i)^2+(P_i')^2 \le 2c_*^2\,(\vartheta_{i+1}(t_{i+2})-t_{i+1})^2 \le 2c_*^2\,(t_{i+2}-t_{i+1})^2\,.$$

In the same way we obtain from $\vartheta_{k+1}(t) \le t$ that $\left|\begin{pmatrix} P_k \\ P_k' \end{pmatrix}\right|_{\mathbb{R}^2}^2 \le 2c_*^2\,(t-t_{k+1})^2$. Hence

$$|T_k(t,y_0)e_3| \le 1+\sum_{i=0}^k \left|\begin{pmatrix} P_i \\ P_i' \end{pmatrix}\right|_{\mathbb{R}^2} \le 1+\sqrt{2}c_*\,(t-t_1) \le 1+c_{**}t$$

by Lemma 4.10(f). Together with (3.72) this shows (3.71).

To prove the theorem, fix $y \in \mathbb{R}^3$ and $t \in [t_1,\infty[$, and choose $k \in \mathbb{N}_0$ with $t \in [t_{k+1},t_{k+2}[$. Then (3.70) and (3.71) imply for some constant $c(y)>0$

$$|T(t,y_0)y|=|T_k(t,y_0)(T_1^- y)| \le c(y)\,(1+c_{**}t)\,,$$

and thus $\lambda^+(y_0,y) \le 0$. □

3.4.2 The small amplitude case $\gamma \in [0,1[$, $\eta \in [0,\infty[$

We intend to show that here the Lyapunov exponents equal $-\infty,0,0$. Since we will derive that now the only ergodic measures are just the δ_{x_0} with $x_0 \in \mathcal{S}_\gamma = [\gamma-1, 1-\gamma]$, considered on $\mathbb{R}^3_{2\pi/\eta}$, cf. Section 2.5, this in fact amounts to an application of the MET Theorem 4.3 to Dirac measures. For notation used in this section cf. Sections 2.5 and 4.1. The following Lemma does not rely on the cocycle, but only on Section 3.3.1, hence the value $\gamma = 1$ is included.

Lemma 3.34 *For $\gamma \in [0,1]$ and $\eta \in [0,\infty[$ we have*

$$\mathcal{P}_{\mathrm{inv}}(\varphi_{\mathrm{per}}) = \{\mu \in \mathcal{P}(\mathbb{R}^3_{2\pi/\eta}) : \mu = \mu_1 \otimes \delta_0 \otimes \sigma, \mu_1 \in \mathcal{P}(\mathbb{R}), \mu_1(\mathcal{S}_\gamma) = 1\},$$

where σ denotes arclength on $\mathbb{S}^1_{2\pi/\eta} = \mathbb{S}^1$.

Proof: Let $\alpha : [0, 2\pi/\eta] \ni \tau \to e^{i\tau\eta} \in \mathbb{S}^1_{2\pi/\eta}$ denote the canonical map. We consider α also on $\mathbb{R}$, and for simplicity we write $[\tau_1,\tau_2]$ instead of $\alpha([\tau_1,\tau_2])$, i.e. $\sigma([\tau_1,\tau_2]) = \tau_2 - \tau_1$ if $\tau_2 - \tau_1 \in [0, 2\pi/\eta]$. Since $\mathcal{B}(\mathbb{R}^3_{2\pi/\eta})$ is generated by sets of the form $A_1 \times A_2 \times [\tau_1,\tau_2]$ with $A_1, A_2 \in \mathcal{B}(\mathbb{R})$ and $\tau_1,\tau_2 \in \mathbb{R}$, $\tau_2 - \tau_1 \in [0,2\pi/\eta]$, equality of measures only has to be checked for sets of this type. "$\subseteq$": For $\varepsilon > 0$ define $V_\varepsilon = \{(\xi, e^{i\tau\eta}) \in \mathbb{R}^3_{2\pi/\eta} : \mathrm{dist}_{\mathbb{R}^3_{2\pi/\eta}}((\xi,e^{i\tau\eta}), \mathcal{S}_\gamma \times \{0\} \times \mathbb{S}^1_{2\pi/\eta}) \le \varepsilon\}$. Fix $\mu \in \mathcal{P}_{\mathrm{inv}}(\varphi_{\mathrm{per}})$ and $R > 0$. It follows from Corollary 3.25 that $\overline{B}_R(0) \times \mathbb{S}^1_{2\pi/\eta} \subset (\varphi^t_{\mathrm{per}})^{-1}(V_\varepsilon)$ for sufficiently large t, and thus $\mu(\overline{B}_R(0) \times \mathbb{S}^1_{2\pi/\eta}) \le \mu((\varphi^t_{\mathrm{per}})^{-1}(V_\varepsilon)) = \mu(V_\varepsilon)$. With $R \to \infty$ we arrive at $\mu(V_\varepsilon) = 1$ for every $\varepsilon > 0$, hence $\varepsilon \to 0^+$ implies $\mu(\mathcal{S}_\gamma \times \{0\} \times \mathbb{S}^1_{2\pi/\eta}) = 1$. This in turn yields for $t \in [0,\infty[$ and $A \in \mathcal{B}(\mathbb{R}^2)$, because the $x_0 \in \mathcal{S}_\gamma$ are stationary solutions,

$$\begin{aligned}
&\mu(A \times [\tau_1+t, \tau_2+t]) \\
&= \mu\Big((\varphi^t_{\mathrm{per}})^{-1}(A \times [\tau_1+t,\tau_2+t]) \cap (\mathcal{S}_\gamma \times \{0\} \times \mathbb{S}^1_{2\pi/\eta})\Big) \\
&= \mu(\{(x,0,e^{i\tau\eta}) \in \mathcal{S}_\gamma \times \{0\} \times \mathbb{S}^1_{2\pi/\eta} : (x,0,e^{i[\tau+t]\eta}) = \varphi^t_{\mathrm{per}}(x,0,e^{i\tau\eta}) \in A \times [\tau_1+t,\tau_2+t]\}) \\
&= \mu(\{(x,0,e^{i\tau\eta}) \in \mathcal{S}_\gamma \times \{0\} \times \mathbb{S}^1_{2\pi/\eta} : (x,0,e^{i\tau\eta}) \in A \times [\tau_1,\tau_2]\}) = \mu(A \times [\tau_1,\tau_2]). \qquad (3.73)
\end{aligned}$$

Let $\bar{\mu}(A) = \mu(A \times \mathbb{S}^1_{2\pi/\eta})$ denote the marginal distribution of μ on $\mathcal{B}(\mathbb{R}^2)$, and for fixed A with $\bar{\mu}(A) > 0$ define $\bar{\mu}_A([\tau_1,\tau_2]) = \mu(A \times [\tau_1,\tau_2])/\bar{\mu}(A)$ on $\mathcal{B}(\mathbb{S}^1_{2\pi/\eta})$. Then (3.73) implies $\bar{\mu}_A([\tau_1,\tau_2]) = \bar{\mu}_A([\tau_1+t,\tau_2+t])$ for $t \in [0,\infty[$, and thus $\bar{\mu}_A = \sigma$, i.e. $\mu = \bar{\mu} \otimes \sigma$. Since $\mu(\mathcal{S}_\gamma \times \{0\} \times \mathbb{S}^1_{2\pi/\eta}) = 1$, we also have $\bar{\mu}(\mathcal{S}_\gamma \times \{0\}) = 1$ and $\bar{\mu}(A) = \bar{\mu}(A \cap (\mathcal{S}_\gamma \times \{0\}))$ for $A \in \mathcal{B}(\mathbb{R}^2)$. Hence with $\mu_1(A_1) = \bar{\mu}(A_1 \times \mathbb{R})$ for $A_1 \in \mathcal{B}(\mathbb{R})$ we obtain $\mu_1 \in \mathcal{P}(\mathbb{R})$ and $\bar{\mu} = \mu_1 \otimes \delta_0$, and therefore $\mu = \mu_1 \otimes \delta_0 \otimes \sigma$. "$\supseteq$": This follows directly from the fact that all $x_0 \in \mathcal{S}_\gamma$ are equilibria. $\Box$

Corollary 3.35 *For $\gamma \in [0,1]$ and $\eta \in [0,\infty[$ we have $\mathcal{P}_{\mathrm{erg}}(\varphi_{\mathrm{per}}) = \{\delta_{x_0} \otimes \delta_0 \otimes \sigma : x_0 \in S_\gamma\}$, where σ denotes arclength on $\mathbb{S}^1_{2\pi/\eta} = \mathbb{S}^1$.*

Proof: Since $\{\mu_1 \in \mathcal{P}(\mathbb{R}) : \mu_1(\mathcal{S}_\gamma) = 1\}$ may be identified with $\mathcal{P}(\mathcal{S}_\gamma) = \mathcal{P}([\gamma-1,1-\gamma])$ and since $\mathcal{P}_{\mathrm{erg}}(\varphi_{\mathrm{per}}) = \mathrm{ext}(\mathcal{P}_{\mathrm{inv}}(\varphi_{\mathrm{per}}))$, cf. Lemma 4.1, we clearly obtain from Lemma 3.34 that

$$\mathcal{P}_{\mathrm{erg}}(\varphi_{\mathrm{per}}) = \{\mu_1 \otimes \delta_0 \otimes \sigma : \mu_1 \in \mathrm{ext}(\mathcal{P}(\mathcal{S}_\gamma))\}.$$

Hence the claim follows from Example 4.2. $\Box$

Now we are going to apply the MET to the ergodic measures found in Corollary 3.35. Since the stationary solutions $x_0 = \gamma - 1$ and $x_0 = 1 - \gamma$ lie on the boundary of M_∞, in particular $y_0 = (x_0, 0, \tau_0) \notin G$ for these both x_0. Hence the cocycle $T(t, y_0)$ does not give information, and thus $x_0 = \gamma - 1$ resp. $x_0 = 1 - \gamma$ will not be taken into account.

Theorem 3.36 *In the "small amplitude case" $\gamma \in [0,1[$, $\eta \in [0,\infty[$, the Lyapunov exponents equal $-\infty,0,0$. To be more precise, we obtain these values when the MET Theorem 4.3 is applied in the periodic setting to one of the ergodic measures $\delta_{x_0} \otimes \delta_0 \otimes \sigma$ from Corollary 3.35, with $x_0 \in]\gamma-1,1-\gamma[$.*

Proof: Since the integrability conditions hold, cf. Corollary 3.32 and Section 2.5, in particular (2.30), we find $\Gamma_{2\pi/\eta} \subset \mathbb{R}^3_{2\pi/\eta}$ with $(\delta_{x_0} \otimes \delta_0 \otimes \sigma)(\Gamma_{2\pi/\eta}) = 1$ such that the conclusions of the MET are satisfied for $(\xi_0, e^{i\tau_0\eta}) \in \Gamma_{2\pi/\eta}$. In particular, there must be one $(x_0, 0, e^{i\tau_0\eta}) \in \Gamma_{2\pi/\eta}$. Let $y_0 = (x_0, 0, \tau_0) \in \mathbb{R}^3$. Then $y_0 \in D_\infty = G^0_\infty \subset G$ by definition, cf. Corollary 2.7. Hence $T(t,y_0) = A(\alpha = 0)$ by Lemma 3.30. Letting e_j, $j = 1,2,3$, denote the unit vectors in $\mathbb{R}^3$, we obtain $|T(t,y_0)e_1| = 1 = |T(t,y_0)e_3|$ and $|T(t,y_0)e_2| = 0$. Taking into account the criterion (4.2) in the MET and (2.30), we hence have the exponents $-\infty,0,0$. □

3.4.3 The resonant case $\gamma \in [0,\infty[$, $\gamma \neq 1$, $\eta = 1$

We already know from Lemma 3.34 resp. Corollary 3.35 what are the invariant resp. ergodic probability measures for φ_{per}, when $\gamma \in [0,1]$. The next lemma shows that there are no such invariant measures, if $\gamma \in]4/\pi,\infty[$. Hence the MET Theorem 4.3 may not be applied, i.e. there are no reasonably defined Lyapunov exponents in this case.

Lemma 3.37 *For $\gamma \in]4/\pi,\infty[$ and $\eta = 1$ we have $\mathcal{P}_{\text{inv}}(\varphi_{\text{per}}) = \emptyset$.*

Proof: Suppose that $\mu \in \mathcal{P}_{\text{inv}}(\varphi_{\text{per}})$ and let $A_R = \overline{B}_R(0) \times \mathbb{S}^1_{2\pi} \subset \mathbb{R}^3_{2\pi}$ for $R > 0$. We claim that

$$\forall R_1, R_2 > 0\ \exists t \in [0,\infty[: \quad A_{R_1} \subset \left(\varphi^t_{\text{per}}\right)^{-1}(A^{\text{c}}_{R_2})\,. \tag{3.74}$$

Indeed, from Theorem 3.21(b) it follows that for the parameter values under consideration the solutions of (1.2) are uniformly unbounded w.r.t initial values and phases, cf. Definition 3.20(b). Thus we find $\tau_* \geq 2\pi$ such that for $\tau \in [\tau_*,\infty[$

$$\inf_{|\xi_0| \leq R_1, \tau_0 \in [0,2\pi]} \mathcal{E}(\tau;\xi_0,\tau_0) \geq 2R_2^2\,. \tag{3.75}$$

If (3.74) were wrong, then $\varphi^t_{\text{per}}(\xi_0, e^{i\tau_0}) \in A_{R_2}$ for some $|\xi_0| \leq R_1$ and $\tau_0 \in [0,2\pi]$, and thus by definition of φ_{per}, cf. (2.29) and (3.1), and by (3.75) we obtain for $t \in [\tau_* - \tau_0, \infty[$

$$R_2^2 \geq x^2(\tau_0 + t;\xi_0,\tau_0) + \dot{x}^2(\tau_0 + t;\xi_0,\tau_0) = \mathcal{E}(\tau_0 + t;\xi_0,\tau_0) \geq 2R_2^2\,.$$

Hence (3.74) holds, and since $A_R \uparrow \mathbb{R}^3_{2\pi}$ as $R \uparrow \infty$, this gives a contradiction by invariance of μ. □

Thus it remains to investigate $\mathcal{P}_{\text{inv}}(\varphi_{\text{per}})$ for $\gamma \in]1,4/\pi[$ in the resonant case. Recall from Corollary 3.19(b2) that here we have a unique (2π)-periodic solution x_γ with uniformly asymptotically stable graph $\text{gr}(x_\gamma) \subset \mathbb{R}^3_{2\pi}$, in the sense of Lemma 3.26.

Theorem 3.38 *For $\gamma \in]1,4/\pi[$ and $\eta = 1$ we have $\mathcal{P}_{\text{inv}}(\varphi_{\text{per}}) = \mathcal{P}_{\text{erg}}(\varphi_{\text{per}}) = \{\mu_\gamma\}$, where for $A \in \mathcal{B}(\mathbb{R}^3_{2\pi})$*

$$\mu_\gamma(A) = \frac{1}{2\pi}\int_0^{2\pi} \delta_{(x_\gamma(\tau),\dot{x}_\gamma(\tau))}(A_\tau)\,d\tau = \frac{1}{2\pi}\lambda^1\left(\{\tau \in [0,2\pi] : \varphi^\tau_{\text{per}}(\xi_\gamma,(1,0)) \in A\}\right). \tag{3.76}$$

Here $A_\tau = \{\xi \in \mathbb{R}^2 : (\xi, e^{i\tau}) \in A\} \subset \mathbb{R}^2$ and $\xi_\gamma = (x_\gamma(0), \dot{x}_\gamma(0))$, i.e. $\varphi^\tau_{\text{per}}(\xi_\gamma,(1,0)) = (x_\gamma(\tau), \dot{x}_\gamma(\tau), e^{i\tau})$ for $\tau \in [0,\infty[$.

Proof: By Lemma 4.1 we only have to show that $\mathcal{P}_{\text{inv}}(\varphi_{\text{per}}) = \{\mu_\gamma\}$. For simplicity we write $\delta_{[\tau]} = \delta_{(x_\gamma(\tau),\dot{x}_\gamma(\tau))}$. "$\supseteq$": Let $t \in [0,\infty[$, $t = 2\pi k + t_{\text{mod}}$, with $k \in \mathbb{N}_0$ and $t_{\text{mod}} \in [0,2\pi[$. From the definitions, and since x_γ is (2π)-periodic, it follows that $\delta_{[\tau]}\Big(\big((\varphi_{\text{per}}^t)^{-1}A\big)_\tau\Big) = 1$ iff $\delta_{[\tau+t]}(A_{\tau+t}) = 1$ iff $\delta_{[\tau+t_{\text{mod}}]}(A_{\tau+t_{\text{mod}}}) = 1$, and hence

$$
\begin{aligned}
(\varphi_{\text{per}}(\mu_\gamma))(A) &= \frac{1}{2\pi}\int_0^{2\pi} \delta_{[\tau+t_{\text{mod}}]}(A_{\tau+t_{\text{mod}}})\,d\tau = \frac{1}{2\pi}\int_{t_{\text{mod}}}^{2\pi} \delta_{[\tau]}(A_\tau)\,d\tau + \frac{1}{2\pi}\int_{2\pi}^{2\pi+t_{\text{mod}}} \delta_{[\tau]}(A_\tau)\,d\tau \\
&= \mu_\gamma(A)\,,
\end{aligned}
$$

where the last equality follows again from the periodicity of x_γ. Therefore $\mu_\gamma \in \mathcal{P}_{\text{inv}}(\varphi_{\text{per}})$, and we also note that $\mu_\gamma(\text{gr}(x_\gamma)) = 1$, cf. the notation in Section 3.3.2. "$\subseteq$": Fix $\mu \in \mathcal{P}_{\text{inv}}(\varphi_{\text{per}})$. First we will show with an argument analogous to one in the proof of Lemma 3.34 that $\mu(\text{gr}(x_\gamma)) = 1$. For $\varepsilon > 0$ define $V_\varepsilon = \{(\xi, e^{i\tau}) \in \mathbb{R}^3_{2\pi} : \text{dist}_{\mathbb{R}^3_{2\pi}}((\xi, e^{i\tau}), \text{gr}(x_\gamma)) \le \varepsilon\}$. If $R > 0$ and $\varepsilon > 0$ are fixed, then by Lemma 3.26 there exists $t \in [0,\infty[$ such that $\varphi_{\text{per}}^t(\overline{B}_R(0) \times \mathbb{S}^1_{2\pi}) \subset V_\varepsilon$. Hence $\mu(\overline{B}_R(0) \times \mathbb{S}^1_{2\pi}) \le \mu(V_\varepsilon)$ by the invariance of μ, and with $R \to \infty$ and $\varepsilon \to 0^+$ this implies $\mu(\text{gr}(x_\gamma)) = 1$. Define the random variable $X : [0,\infty[\to \text{gr}(x_\gamma)$, $X(\tau) = (x_\gamma(\tau), \dot{x}_\gamma(\tau), e^{i\tau})$. Then $X(\tau + 2\pi k) = X(\tau)$ for $k \in \mathbb{N}_0$, and $X(\tau) = X(\sigma)$ iff $\tau = \sigma$, if $\tau, \sigma \in [0,2\pi[$. For $0 \le \tau_1 \le \tau_2$ let $|[\tau_1,\tau_2]| = X([\tau_1,\tau_2])$. The periodicity of x_γ implies $\text{gr}(x_\gamma) \cap (\varphi_{\text{per}}^t)^{-1}(|[\tau_1 + t, \tau_2 + t]|) = |[\tau_1,\tau_2]|$ for $t \in [0,\infty[$. Thus $\mu(|[\tau_1 + t, \tau_2 + t]|) = \mu(|[\tau_1,\tau_2]|)$ by the invariance of μ. Taking $\tau_1 = \tau_2$, this in particular yields $\mu(\{X(\tau)\}) = 0$ for every $\tau \in [0,2\pi[$, and hence $\mu(|[\tau_1,\tau_2]|) = \mu(X([\tau_1,\tau_2[))$ for $0 \le \tau_1 < \tau_2$. Fix $n \in \mathbb{N}$. Then because of $\text{gr}(x_\gamma) = X([0,2\pi[) = \dot{\bigcup}_{l=0}^{n-1} X([2\pi l/n, 2\pi(l+1)/n[)$ we obtain from the translation invariance that $\mu(X([0,2\pi/n[)) = 1/n$, and thus by translation invariance $\mu(X([\tau,\tau+2\pi/n])) = \mu(X([\tau,\tau+2\pi/n[)) = 1/n$ for $\tau \in [0,\infty[$ and $n \in \mathbb{N}$. Hence approximation clearly gives $\mu(X([\tau_1,\tau_2[)) = (\tau_2 - \tau_1)/(2\pi)$ for all $0 \le \tau_1 \le \tau_2 < 2\pi$. Note that by (3.76) for $0 \le \tau_1 \le \tau_2 < 2\pi$ also $\mu_\gamma(X([\tau_1,\tau_2[)) = (2\pi)^{-1}\lambda^1([\tau_1,\tau_2[) = (\tau_2 - \tau_1)/(2\pi)$. To exploit this, we remark that $X : [0,2\pi[\to \text{gr}(x_\gamma)$ is bijective with measurable inverse $X^{-1} : (\xi, e^{i\tau}) \mapsto \tau$. Hence $X(\mathcal{B}([0,2\pi[)) = \mathcal{B}(\text{gr}(x_\gamma))$. Let $\mathcal{E} = \{[\tau_1,\tau_2[: 0 \le \tau_1 \le \tau_2 < 2\pi\}$. Then $\mathcal{B}([0,2\pi[) = \sigma(\mathcal{E})$, the generated σ-algebra, and thus $\sigma(X(\mathcal{E})) = \sigma((X^{-1})^{-1}(\mathcal{E})) = (X^{-1})^{-1}(\sigma(\mathcal{E})) = X(\mathcal{B}([0,2\pi[)) = \mathcal{B}(\text{gr}(x_\gamma))$, i.e. $\mathcal{B}(\text{gr}(x_\gamma))$ is generated by sets of the form $X([\tau_1,\tau_2[)$ with $0 \le \tau_1 \le \tau_2 < 2\pi$. From this we conclude that $\mu = \mu_\gamma$ on $\mathcal{B}(\text{gr}(x_\gamma))$, and therefore finally $\mu = \mu_\gamma$, because both measures are supported on $\text{gr}(x_\gamma)$. $\square$

Hence for $\gamma \in]1, 4/\pi[$ we can apply the MET Theorem 4.3 only to the single ergodic measure μ_γ from Theorem 3.38. Before doing this, we have to extend the cocycle T_{per} to $\text{gr}(x_\gamma)$, or equivalently, T to

$$G_\gamma = \{\varphi^\tau(\xi_\gamma, 0) = (x_\gamma(\tau), \dot{x}_\gamma(\tau), \tau) : \tau \in [0,\infty[\} \subset \mathbb{R}^3\,.$$

To explain the reason for this, we first note that although in Section 3.2 we described the relevant sets M_k, E_k, and D_k only in case that $\gamma \in [0,1[$, it should be clear what are the analogs in the present situation of $\gamma > 1$. In particular, here $M_\infty = \emptyset$, and thus $G = G_{<\infty}$ and $t_i(y) < \infty$ for all $i \in \mathbb{N}_0$ and $y \in \mathbb{R}^3$, since trajectories cannot end in M_∞. Due to the very special form of x_γ, cf. Corollary 3.19(b2) we have

$$
\begin{aligned}
&y_0 = (\xi_\gamma, 0) = (b^-(\tau_1^\gamma), 0, 0) \in M_{\text{left}}\,, \quad t_1(y_0) = \tau_1^\gamma\,, \quad \xi_1(y_0) = (b^-(\tau_1^\gamma), 0, \tau_1^\gamma) \in M_+\,, \quad t_2(y_0) = \tau_2^\gamma\,, \\
&\xi_2(y_0) = (-b^-(\tau_1^\gamma), 0, \tau_2^\gamma) \in M_{\text{right}}\,, \quad t_3(y_0) = \pi + \tau_1^\gamma\,, \quad \xi_3(y_0) = (-b^-(\tau_1^\gamma), 0, \pi + \tau_1^\gamma) \in M_-\,, \\
&t_4(y_0) = \pi + \tau_2^\gamma\,, \quad \xi_4(y_0) = (b^-(\tau_1^\gamma), 0, \pi + \tau_2^\gamma) \in M_{\text{left}}\,, \quad t_5(y_0) = 2\pi + \tau_1^\gamma\,, \\
&\xi_5(y_0) = (b^-(\tau_1^\gamma), 0, 2\pi + \tau_1^\gamma) \in M_+, \ldots, \text{ etc.}
\end{aligned}
$$

Hence $\xi_1(y_0) \in E_+$, $\xi_3(y_0) \in E_-$, $\xi_5(y_0) \in E_+$, $\ldots$, etc. Consequently, $y_0 \notin G_{<\infty} = G$ here, cf. Lemma 2.3, and therefore by the periodicity of x_γ and the forward invariance of G even $G_\gamma \subset \mathbb{R}^3 \setminus G$. Now application of the MET will yield a set $\Gamma_\gamma \subset \mathbb{R}^3_{2\pi}$ with $\mu_\gamma(\Gamma_\gamma) = 1$ such that for every $(\xi, e^{i\tau}) \in \Gamma_\gamma$

the statements of the MET hold. Since μ_γ is supported on $\mathrm{gr}(x_\gamma)$, we thus can take advantage only for $(\xi,e^{i\tau})\in\mathrm{gr}(x_\gamma)$. But by definition in Theorem 2.14, $T(t,y)=0$ for the corresponding $y=(\xi,\tau)\in G_\gamma\subset\mathbb{R}^3\setminus G$, and we set $T(t,y)=0$ only to have T defined on the whole of $\mathbb{R}^3$. Thus we of course cannot expect that the corresponding Lyapunov exponents $-\infty,-\infty,-\infty$ are related to the dynamical behaviour.

Although this seems to be a problem, we can remedy in a simple way, since the trajectory of x_γ in $\mathbb{R}^3$, i.e. G_γ, is a forward invariant set. This allows us to extend the definition of T from G to $G\dot{\cup}G_\gamma$ and so to find the cocycle being appropriate for an application of the MET.

Lemma 3.39 *If $\gamma\in]1,4/\pi[$, $\eta=1$, we obtain a cocycle $T:[0,\infty[\times\mathbb{R}^3\to\mathbb{R}^3$ by letting $T(t,y)$ be as before in Theorem 2.14 if $y\in\mathbb{R}^3\setminus G_\gamma$, whereas for $t\in[0,\infty[$ and $y=\varphi^s(y_0)=\varphi^s(\xi_\gamma,0)\in G_\gamma$ we adopt the following definition, cf. the notation in Section 3.4.1. Choose $i\in\mathbb{N}_0$ such that $s\in[t_i(y_0),t_{i+1}(y_0)[$ and $k\in\mathbb{N}_0$ as well as $l\in\{0,1,2,3\}$ such that $i=4k+l$.*
<u>Case 1:</u> $l=0$ or $l=2$. Let

$$\begin{aligned}
t\in[0,t_1(y)[:&\quad T(t,y)=A(\alpha=0)\,,\\
t\in[t_1(y),t_2(y)[:&\quad T(t,y)=A(s+t,t_{i+1}(y_0),1)\circ A(\alpha=0)\,,\\
t\in[t_2(y),t_3(y)[:&\quad T(t,y)=A(\alpha=0)\circ A(t_{i+2}(y_0),t_{i+1}(y_0),1)\circ A(\alpha=0)\,,\\
t\in[t_3(y),t_4(y)[:&\quad T(t,y)=A(s+t,t_{i+3}(y_0),1)\circ A(\alpha=0)\circ A(t_{i+2}(y_0),t_{i+1}(y_0),1)\circ A(\alpha=0)\,,
\end{aligned}$$

..., etc. <u>Case 2:</u> $l=1$ or $l=3$. Let

$$\begin{aligned}
t\in[0,t_1(y)[:&\quad T(t,y)=A(s+t,s,1)\,,\\
t\in[t_1(y),t_2(y)[:&\quad T(t,y)=A(\alpha=0)\circ A(t_{i+1}(y_0),s,1)\,,\\
t\in[t_2(y),t_3(y)[:&\quad T(t,y)=A(s+t,t_{i+2}(y_0),1)\circ A(\alpha=0)\circ A(t_{i+1}(y_0),s,1)\,,\\
t\in[t_3(y),t_4(y)[:&\quad T(t,y)=A(\alpha=0)\circ A(t_{i+3}(y_0),t_{i+2}(y_0),1)\circ A(\alpha=0)\circ A(t_{i+1}(y_0),s,1)\,,
\end{aligned}$$

..., etc.

Proof: Since G_γ is forward invariant, we have to check $T(t+s,y)=T(t,\varphi^s(y))T(s,y)$ only for $y\in G_\gamma$. But this is straightforward from the definition, taking into account that $A(t,\bar{t},1)\circ A(\bar{t},s,1)=A(t,s,1)$, cf. Lemma 3.29. Note how T is constructed on G_γ: If e.g. $y=\varphi^s(y_0)=(x_\gamma(s),\dot{x}_\gamma(s),s)\in M_+$, then $l=1$ above by the form of the trajectory of x_γ, and hence we let $T(t,y)=\partial_y\Phi_{f_+}(t,y)=A(s+t,s,1)$, cf. (3.67), for $t\in[0,t_1(y)[=[0,t_{i+1}(y_0)-s[$. At time $t=t_1(y)$ we must have a switching from M_+ to M_{left}, and therefore we define in accordance with Lemma 3.31(c), $T(t,y)=A(\alpha=0)\circ(\lim_{\varrho\to t_1(y)^-}T(\varrho,y))=A(\alpha=0)\circ A(t_{i+1}(y_0),s,1)$ for $t\in[t_1(y),t_2(y)[=[t_{i+1}(y_0)-s,t_{i+2}(y_0)-s[$, and so on. □

Now we can apply the MET to the extended cocycle, or to be more precise, to its periodic analogue T_{per}, cf. Section 2.5. Of course in the present case of a uniformly stable periodic solution, the values of T on $\mathbb{R}^3\setminus G_\gamma$ are unimportant.

Theorem 3.40 *If $\gamma\in]1,4/\pi[$, $\eta=1$, the Lyapunov exponents equal $-\infty,\pi^{-1}\ln|\cos(\tau_2^\gamma-\tau_1^\gamma)|,0$, i.e. these values are obtained from the MET Theorem 4.3 applied with μ_γ from Theorem 3.38 and the extended cocycle from Lemma 3.39. Here τ_1^γ and τ_2^γ are from Corollary 3.19(b2).*

Proof: Again the integrability conditions are satisfied, cf. Corollary 3.32 and (2.30). Therefore we find $\Gamma_\gamma\subset\mathbb{R}^3_{2\pi}$ with $\mu_\gamma(\Gamma_\gamma)=1$ such that for every $(\xi_0,e^{i\tau_0})\in\Gamma_\gamma$ the results stated in the MET can be used. Since $\mu_\gamma(\mathrm{gr}(x_\gamma))=1$, in particular $\Gamma_\gamma\cap\mathrm{gr}(x_\gamma)\neq\emptyset$. Because Γ_γ is forward invariant w.r.t. φ_{per}, thus by periodicity $(\xi_\gamma,(1,0))=\varphi^0_{\mathrm{per}}(\xi_\gamma,(1,0))\in\Gamma_\gamma$. Hence even $\mathrm{gr}(x_\gamma)\subset\Gamma_\gamma$, and this corresponds to the fact that all the $y\in G_\gamma\subset\mathbb{R}^3$ are admissible to calculate $T(t,y)$ with. In particular, we may take $y=y_0=(\xi_\gamma,0)$. From Lemma 3.39 we explicitly know the form of $T(t,y_0)$, and with $t_i=t_i(y_0)$

we have $t_{2i}-t_{2i-1}=\tau_2^\gamma-\tau_1^\gamma$ for $i\in\mathbb{N}$, cf. the remarks before Lemma 3.39. We claim that for $k\in\mathbb{N}_0$ and $t\in[t_{4k},t_{4k+1}[$

$$T(t,y_0)=\begin{pmatrix}\cos(\tau_2^\gamma-\tau_1^\gamma)^{2k} & 0 & \alpha_k\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix},\quad \text{with}\quad \alpha_k=\sum_{j=1}^{2k}\cos(\tau_2^\gamma-\tau_1^\gamma)^{2k-j}P(t_{2j},t_{2j-1})\,. \tag{3.77}$$

Indeed, we have $\alpha_0:=0$ and $T(t,y_0)=A(\alpha=0)$ for $t\in[0,t_1[$. If (3.77) already holds for some $k\in\mathbb{N}_0$, then by definition for $t\in[t_{4(k+1)},t_{4(k+1)+1}[$, because of $\xi_{4k}(y_0)\in M_{\text{left}}$, $\xi_{4k+1}(y_0)\in M_+$, $\xi_{4k+2}(y_0)\in M_{\text{right}}$, $\xi_{4k+3}(y_0)\in M_-$, and $\xi_{4k+4}(y_0)\in M_{\text{left}}$,

$$T(t,y_0)=A(\alpha=0)\circ A(t_{4k+4},t_{4k+3},1)\circ A(\alpha=0)\circ A(t_{4k+2},t_{4k+1},1)\circ\Big(\lim_{s\to t_{4k+1}^-}T(s,y_0)\Big)\,.$$

By multiplication of the matrices and using $\cos(t_{4k+4}-t_{4k+3})\cos(t_{4k+2}-t_{4k+1})\cos(\tau_2^\gamma-\tau_1^\gamma)^{2k}=\cos(\tau_2^\gamma-\tau_1^\gamma)^{2(k+1)}$ as well as the fact that

$$\cos(t_{4k+4}-t_{4k+3})\cos(t_{4k+2}-t_{4k+1})\alpha_k+\cos(t_{4k+2}-t_{4k+1})P(t_{4k+2},t_{4k+1})+P(t_{4k+4},t_{4k+3})$$

equals α_{k+1}, this gives (3.77) for $k+1$. With e_j, $j=1,2,3$, being the unit vectors in $\mathbb{R}^3$, we conclude from the criterion (4.2) in the MET Theorem 4.3 and (2.30) in Section 2.5 that the Lyapunov exponents equal $\{\rho^{(1)},\rho^{(2)},\rho^{(3)}\}$, where

$$\rho^{(j)}=\lim_{k\to\infty}\left(\frac{1}{t_{4k}}\right)\ln|T(t_{4k},y_0)e_j|\,,$$

since $t_{4k}\to\infty$ as $k\to\infty$. Because of $t_{4k}=(2k-1)\pi+\tau_2^\gamma$ for $k\in\mathbb{N}$, we thus obtain from (3.77) that $\rho^{(2)}=-\infty$ and $\rho^{(1)}=\pi^{-1}\ln|\cos(\tau_2^\gamma-\tau_1^\gamma)|\in[-\infty,0[$, the latter, since $\tau_2^\gamma-\tau_1^\gamma\in]0,\pi[$ by Corollary 3.19(b2). Finally, it follows from Lemma 3.28 that

$$|\alpha_k|\le\sum_{j=1}^{2k}|P(t_{2j},t_{2j-1})|\le c_*\sum_{j=1}^{2k}(t_{2j}-t_{2j-1})\le c_*t_{4k}\,,$$

and therefore $|T(t_{4k},y_0)e_3|^2=1+\alpha_k^2\le 1+c_*^2t_{4k}^2$ yields

$$0\le\left(\frac{1}{t_{4k}}\right)\ln|T(t_{4k},y_0)e_3|\le\left(\frac{1}{2t_{4k}}\right)\ln(1+c_*^2t_{4k}^2)\to 0\quad\text{as}\quad k\to\infty\,,$$

hence $\rho^{(3)}=0$. □

To summarize Theorem 3.36, Theorem 3.40, and Lemma 3.37, we obtain the following bifurcation picture w.r.t. $\gamma\in[0,\infty[$ in the resonant case $\eta=1$: for $\gamma\in[0,1[$ the exponents are $-\infty,0,0$, whereas they equal $-\infty,\pi^{-1}\ln|\cos(\tau_2^\gamma-\tau_1^\gamma)|,0$ for $\gamma\in]1,4/\pi[$. If $\gamma\in]4/\pi,\infty[$, the MET does not apply.

Remark 3.41 The exponents are continuous at $\gamma=1$. To see this, let $\lambda_\gamma^{(2)}=\pi^{-1}\ln|\cos(\tau_2^\gamma-\tau_1^\gamma)|$ with $\tau_1^\gamma\in]0,\pi/2[$ and $\tau_2^\gamma\in]\pi/2,\pi[$, cf. Theorem 3.18(e) and Corollary 3.19(b2). These values were obtained in [10, Prop. 3] as the unique solutions of

$$\operatorname{cotan}\tau_2^\gamma=-\frac{\tau_2^\gamma-\tau_1^\gamma-\sin\tau_1^\gamma\cos\tau_1^\gamma}{\sin^2\tau_1^\gamma},\quad 2/\gamma=\frac{1}{2}\frac{\sin^2\tau_1^\gamma}{\sin\tau_2^\gamma}+\frac{1}{2}\sin\tau_2^\gamma+\sin\tau_1^\gamma\,,$$

or equivalently

$$\tau_2^\gamma-\tau_1^\gamma=\frac{\sin\tau_1^\gamma}{\sin\tau_2^\gamma}\sin(\tau_2^\gamma-\tau_1^\gamma)\,,\quad 4\sin\tau_2^\gamma=\gamma(\sin\tau_1^\gamma+\sin\tau_2^\gamma)^2\,.$$

This shows that $\tau_1^\gamma\uparrow\pi/2$ and $\tau_2^\gamma\downarrow\pi/2$ as $\gamma\to1^+$. Hence $\lambda_\gamma^{(2)}\to0$ as $\gamma\to1^+$. In addition, we note that $\tau_1^\gamma\downarrow0$ and $\tau_2^\gamma\uparrow\pi$ for $\gamma\to(4/\pi)^-$, and therefore also $\lambda_\gamma^{(2)}\to0$ as $\gamma\to(4/\pi)^-$.

4 Appendix

4.1 On ergodic theory

We introduce some standard notation. For a given measurable semiflow $\varphi:[0,\infty[\times M\to M$ on a complete separable metric manifold M we let $\mathcal{P}_{\text{inv}}=\mathcal{P}_{\text{inv}}(\varphi)=\{\mu:\forall t\in[0,\infty[:\varphi^t(\mu)=\mu\}$ denote the set of all $(\varphi^t)_{t\geq 0}$-invariant probability measures on $(M,\mathcal{B}(M))$, where $\mathcal{B}(M)$ is the Borel σ-algebra and $(\varphi^t(\mu))(A):=\mu((\varphi^t)^{-1}(A))$. In general, $\mathcal{P}_{\text{inv}}$ can be empty, cf. Lemma 3.37.

We call $(\varphi^t)_{t\geq 0}$-invariant a set $A\in\mathcal{B}(M)$, if $(\varphi^t)^{-1}(A)=A$ for all $t\in[0,\infty[$, and the set of ergodic invariant probability measures is

$$\mathcal{P}_{\text{erg}}=\mathcal{P}_{\text{erg}}(\varphi)=\{\mu\in\mathcal{P}_{\text{inv}}:A\in\mathcal{B}(M)\text{ is }(\varphi^t)_{t\geq 0}-\text{invariant}\Rightarrow\mu(A)\in\{0,1\}\}.$$

Since we did't find a suitable reference, we also include a proof of

Lemma 4.1 $\mathcal{P}_{\text{erg}}=\operatorname{ext}(\mathcal{P}_{\text{inv}})$ *is the set of extreme points of the convex set* $\mathcal{P}_{\text{inv}}$.

Proof: Let $\mathcal{I}=\{A\in\mathcal{B}(M):A\text{ is }(\varphi^t)_{t\geq 0}-\text{invariant}\}$ denote the σ-algebra of invariant sets. Then $\mu\in\mathcal{P}_{\text{erg}}$ iff $\mu\in\mathcal{P}_{\text{inv}}$ and μ is trivial on $\mathcal{I}$. "$\supseteq$": If $\mu(A)\in]0,1[$ for some $\mu\in\operatorname{ext}(\mathcal{P}_{\text{inv}})$ and $A\in\mathcal{I}$, then by the invariance of A and μ, $\nu(B)=\mu(B\cap A)/\mu(A)$ defines a probability measure $\nu\in\mathcal{P}_{\text{inv}}$, and the same is true for $\bar{\nu}(B)=\mu(B\cap A^c)/\mu(A^c)$. Because of $\mu=\mu(A)\,\nu+(1-\mu(A))\,\bar{\nu}$, μ cannot be extreme. "$\subseteq$": We intend to apply [15, Cor. 7.4]. To see that this is possible, define for every $t\in[0,\infty[$ a probability kernel π_t from $\mathcal{B}(M)$ to $\mathcal{B}(M)$ through $\pi_t(A,y)=1_A(\varphi^t(y))$. Then $\mu\pi_t=\varphi^t(\mu)$ for $\mu\in\mathcal{P}(M)$, and thus $\mathcal{P}_\Pi=\{\mu\in\mathcal{P}(M):\forall t\in[0,\infty[:\mu\pi_t=\mu\}=\mathcal{P}_{\text{inv}}$. Therefore, if we want to show that a fixed $\mu\in\mathcal{P}_{\text{erg}}\subset\mathcal{P}_{\text{inv}}$ is ergodic, we have to prove that μ is trivial on $\mathcal{I}_\Pi(\mu)=\{A\in\mathcal{B}(M):\forall t\in[0,\infty[:1_A(\varphi^t(y))=1_A\ \mu-\text{a.e.}\}$. For that, fix $A\in\mathcal{I}_\Pi(\mu)$. Then μ-a.e. for all $i\in\mathbb{N}_0$ we have $1_A(\varphi^i(\cdot))=1_A$ by assumption. Consequently, the Birkhoff Ergodic Theorem, cf. [27, Thm. 1.2], implies $1_A(\cdot)=\mu(A)$ μ-a.e., and thus $\mu(A)\in\{0,1\}$ as required. Hence [15, Cor. 7.4] applies to yield $\mu\in\operatorname{ext}(\mathcal{P}_{\text{inv}})$. □

Example 4.2 *We have* $\operatorname{ext}(\mathcal{P}([a,b]))=\{\delta_x:x\in[a,b]\}$.

Proof: Clearly, "$\supseteq$" holds. "$\subseteq$": Fix $\mu\in\operatorname{ext}(\mathcal{P}([a,b]))$. It is known that $\mathcal{P}=\mathcal{P}([a,b])$ is convex and a compact Hausdorff space in the weak topology. Moreover, $\Delta:[a,b]\to\mathcal{P}$, $x\mapsto\delta_x$, is $\mathcal{B}([0,a])-\mathcal{B}(\mathcal{P})$-measurable, and hence $\bar{\mu}=\Delta(\mu)$ is a probability measure on $(\mathcal{P},\mathcal{B}(\mathcal{P}))$, given through $\bar{\mu}(\mathcal{M})=\mu(\{x\in[a,b]:\delta_x\in\mathcal{M}\})$ for $\mathcal{M}\in\mathcal{B}(\mathcal{P})$. From the transformation rule we obtain

$$\int_{\mathcal{P}}\nu\,\bar{\mu}(d\nu)=\int_{\mathcal{P}}\nu\,\Delta(\mu)(d\nu)=\int_{[a,b]}\delta_x\,\mu(dx)=\mu,$$

i.e. $\bar{\mu}$ has barycenter μ. Because also $\int_{\mathcal{P}}\nu\,\delta_\mu(d\nu)=\mu$, and since μ is extreme, we conclude from [1, Cor. I.2.4] that $\bar{\mu}=\delta_\mu$. By applying this to $\mathcal{M}=\{\delta_x:x\in[a,b]\}$ we find $1=\delta_\mu(\mathcal{M})$, and consequently $\mu\in\mathcal{M}$. □

Define $\ln^+(s)=\max\{\ln(s),0\}$ for $s\in[0,\infty[$.

Theorem 4.3 (Oseledets' Multiplicative Ergodic Theorem) *Suppose that* $\varphi:[0,\infty[\times\mathbb{R}^d\to\mathbb{R}^d$ *is a measurable semiflow and* $\mu\in\mathcal{P}_{\text{inv}}$. *Let* $T:[0,\infty[\times\mathbb{R}^d\to\mathcal{L}(\mathbb{R}^d)$ *be a measurable cocycle for* $(\varphi^t)_{t\geq 0}$, *i.e.* T *is measurable and* $T(t+s,y)=T(t,\varphi^s(y))T(s,y)$ *for* $t,s\in[0,\infty[$ *and* $y\in\mathbb{R}^d$.

Then, if the integrability conditions

$$\sup_{\theta\in[0,1]}\ln^+|T(\theta,\cdot)|\in L^1(\mathbb{R}^d,\mu)\quad\textit{and}\quad\sup_{\theta\in[0,1]}\ln^+|T(1-\theta,\varphi^\theta(\cdot))|\in L^1(\mathbb{R}^d,\mu)\tag{4.1}$$

are satisfied, there exists $\Gamma\subset\mathbb{R}^d$ *with* $\mu(\Gamma)=1$ *such that* $\varphi^t(\Gamma)\subset\Gamma$ *for* $t\in[0,\infty[$, *and the following assertions hold for all* $y\in\Gamma$:

(a) $\lim_{t\to\infty}\Big(T(t,y)^*T(t,y)\Big)^{1/2t}=\Lambda_y$ *exists.*

(b) Let $\exp\lambda_y^{(1)}<\ldots<\exp\lambda_y^{(k)}$ *be the eigenvalues of* Λ_y *(where* $k=k(y)$*, the* $\lambda_y^{(i)}$ *are real and* $\lambda_y^{(1)}$ *may be* $-\infty$*), and* $U_y^{(1)},\ldots,U_y^{(k)}$ *the corresponding eigenspaces. Let* $m_y^{(i)}=\dim U_y^{(i)}$*. Then the functions* $y\mapsto\lambda_y^{(i)},m_y^{(i)}$ *are* $(\varphi^t)_{t\geq 0}$*-invariant. With* $V_y^{(0)}=\{0\}$ *and* $V_y^{(i)}=U_y^{(1)}+\ldots+U_y^{(i)}$ *one has for* $i=1,\ldots,k$

$$\lim_{t\to\infty}\frac{1}{t}\ln|T(t,y)\bar{y}|=\lambda_y^{(i)}\quad \textit{when}\quad \bar{y}\in V_y^{(i)}\setminus V_y^{(i-1)}.$$

If in addition $\mu\in\mathcal{P}_{\text{erg}}$*, then* $y\mapsto k(y)$ *resp.* $y\mapsto\lambda_y^{(i)}$ *are constant on* Γ*, and denoted by* k *resp.* $\lambda^{(i)}$*. In this case, if there are linearly independent* $\bar{y}_1,\ldots,\bar{y}_d\in\mathbb{R}^d$ *and sequences* $(t_n^j)_{n\in\mathbb{N}}$ *for* $j=1,\ldots,d$ *with* $t_n^j\to\infty$ *as* $n\to\infty$ *such that* $\lim_{n\to\infty}\left(\frac{1}{t_n^j}\right)\ln|T(t_n^j,y)\bar{y}_j|=\rho^{(j)}\in[-\infty,\infty[$ *for* $j=1,\ldots,d$*, then*

$$\{\lambda^{(i)}:1\leq i\leq k\}=\{\rho^{(j)}:1\leq j\leq d\}. \tag{4.2}$$

Proof: See [35, Thm. B3, p. 304]. The ergodic part follows like in the discrete case, cf. [22, Thm. 3.1], while the added criterion is straightforward. □

4.2 On boundedness of solutions to differential inclusions

We want to derive a criterion to prove the boundedness of all solutions for certain "dissipative" differential inclusions. We consider

$$\ddot{x}+x+F(\dot{x})\ni p(\tau)\quad\text{a.e.}, \tag{4.3}$$

first with a forcing $p\in C(\mathbb{R})$ being not necessarily periodic. For simplicity we assume the global unique solvability of every initial value problem corresponding to (4.3), i.e. for every $\xi_0=(x_0,v_0)\in\mathbb{R}^2$ and $\tau_0\in\mathbb{R}$ there exists a unique solution $x(\cdot;\xi_0,\tau_0)$ of (4.3) on $[\tau_0,\infty[$ with initial values $x(\tau_0;\xi_0,\tau_0)=x_0$ and $\dot{x}(\tau_0;\xi_0,\tau_0)=v_0$. We recall the notation $\mathcal{E}(\tau;\xi_0,\tau_0)=x^2(\tau;\xi_0,\tau_0)+\dot{x}^2(\tau;\xi_0,\tau_0)$ for $\tau\in[\tau_0,\infty[$ from (3.48) in Section 3.3.

Lemma 4.4 *Let* $F:\mathbb{R}\to 2^{\mathbb{R}}\setminus\{\emptyset\}$ *be an increasing multivalued map, i.e.*

$$(F(v)-F(\bar{v}))(v-\bar{v})\geq 0,\quad v,\bar{v}\in\mathbb{R}, \tag{4.4}$$

let $p\in C(\mathbb{R})$*, and fix* $\xi_0=(x_0,v_0)\in\mathbb{R}^2$*,* $\bar{\xi}_0=(\bar{x}_0,\bar{v}_0)\in\mathbb{R}^2$*, and* $\tau_0,\bar{\tau}_0\in\mathbb{R}$ *with* $\tau_0\leq\bar{\tau}_0$*. For* $\tau\in[\bar{\tau}_0,\infty[$ *let*

$$\Delta_{\mathcal{E}}(\tau;\xi_0,\tau_0,\bar{\xi}_0,\bar{\tau}_0)=\Big(x(\tau;\xi_0,\tau_0)-x(\tau;\bar{\xi}_0,\bar{\tau}_0)\Big)^2+\Big(\dot{x}(\tau;\xi_0,\tau_0)-\dot{x}(\tau;\bar{\xi}_0,\bar{\tau}_0)\Big)^2. \tag{4.5}$$

Then

(a) for $\tau\in[\tau_0,\infty[$,

$$\Delta_{\mathcal{E}}(\tau;\xi_0,\tau_0,\bar{\xi}_0,\tau_0)\leq\Delta_{\mathcal{E}}(\tau_0;\xi_0,\tau_0,\bar{\xi}_0,\tau_0)=(x_0-\bar{x}_0)^2+(v_0-\bar{v}_0)^2=|\xi_0-\bar{\xi}_0|^2. \tag{4.6}$$

(b) for $\tau\geq\sigma\geq\bar{\tau}_0$,

$$\Delta_{\mathcal{E}}(\tau;\xi_0,\tau_0,\xi_0,\bar{\tau}_0)\leq\Delta_{\mathcal{E}}(\sigma;\xi_0,\tau_0,\xi_0,\bar{\tau}_0). \tag{4.7}$$

Proof: Ad (a): Let $x(\tau)=x(\tau;\xi_0,\tau_0)$, $\bar{x}(\tau)=x(\tau;\bar{\xi}_0,\tau_0)$, and let $\Delta_{\mathcal{E}}(\tau)$ denote the left-hand side of (4.6). Choose functions w resp. $\bar{w}$ such that $\ddot{x}+x+w=p(\tau)$, $\ddot{\bar{x}}+\bar{x}+\bar{w}=p(\tau)$, $w(\tau)\in F(\dot{x}(\tau))$, and $\bar{w}(\tau)\in F(\dot{\bar{x}}(\tau))$, all a.e. in $[\tau_0,\infty[$. Differentiation gives $\frac{1}{2}\dot{\Delta}_{\mathcal{E}}(\tau)=-[w(\tau)-\bar{w}(\tau)][\dot{x}(\tau)-\dot{\bar{x}}(\tau)]\le 0$ a.e. in $[\tau_0,\infty[$ by assumption (4.4). Therefore $\Delta_{\mathcal{E}}(\tau)\le\Delta_{\mathcal{E}}(\tau_0)$ for $\tau\in[\tau_0,\infty[$. Ad (b): With $x(\tau)=x(\tau;\xi_0,\tau_0)$ and $\bar{x}(\tau)=x(\tau;\xi_0,\bar{\tau}_0)$, the calculation from (a) can be repeated. □

Example 4.5 The conclusion of Lemma 4.4 is valid for the inclusions $\ddot{x}+x+\alpha\dot{x}+\beta\,\mathrm{Sgn}(\dot{x})\ni p(\tau)$ a.e., with $\alpha,\beta\ge 0$, since in this case (4.4) holds. In particular, Lemma 4.4 applies to the pendulum with dry friction (1.2).

By means of Lemma 4.4 we obtain a criterion for the boundedness of solutions of (4.3).

Theorem 4.6 *Let $F:\mathbb{R}\to 2^{\mathbb{R}}\setminus\{\emptyset\}$ satisfy (4.4), and let $p\in C(\mathbb{R})$. Suppose that there exists an initial value $\xi_*\in\mathbb{R}^2$ and $c_1>0$ such that*

$$\sup_{\tau\in[0,\infty[}\mathcal{E}(\tau;\xi_*,0)\le c_1\,. \tag{4.8}$$

Then for every $R>0$ there exists a constant $c_2>0$ such that

$$\sup\Big\{\mathcal{E}(\tau;\xi_0,\tau_0):|\xi_0|\le R,\tau_0\in[0,\infty[,\tau\in[\tau_0,\infty[\Big\}\le c_2\,. \tag{4.9}$$

Proof: For $\xi_0=(x_0,v_0)\in\mathbb{R}^2$ with $|\xi_0|\le R$ and $\tau\in[0,\infty[$ it follows from Hölder's inequality and Lemma 4.4(a) that

$$\mathcal{E}^{1/2}(\tau;\xi_0,0)\le\Delta_{\mathcal{E}}^{1/2}(\tau;\xi_0,0,\xi_*,0)+\mathcal{E}^{1/2}(\tau;\xi_*,0)\le|\xi_0-\xi_*|+c_1^{1/2}\le R+|\xi_*|+c_1^{1/2}=:c_*\,.$$

Thus for $\tau_0\in[0,\infty[$ and $\tau\in[\tau_0,\infty[$ by application of Lemma 4.4(b) with $0,\tau_0$ in place of $\tau_0,\bar{\tau}_0$ there,

$$\begin{aligned}\mathcal{E}^{1/2}(\tau;\xi_0,\tau_0)&\le\Delta_{\mathcal{E}}^{1/2}(\tau;\xi_0,\tau_0,\xi_0,0)+\mathcal{E}^{1/2}(\tau;\xi_0,0)\le\Delta_{\mathcal{E}}^{1/2}(\tau_0;\xi_0,\tau_0,\xi_0,0)+c_*\\&\le|\xi_0|+\mathcal{E}^{1/2}(\tau_0;\xi_0,0)+c_*\le 3R+2|\xi_*|+2c_1^{1/2}\,.\end{aligned}$$

Hence $c_2=\left(3R+2|\xi_*|+2c_1^{1/2}\right)^2$ is a suitable choice. □

Example 4.7 In the setting of Theorem 4.6, let (4.3) have an ω-periodic solution $x:[0,\omega]\to\mathbb{R}$ for some $\omega>0$. Then (4.8) holds with $\xi_*=(x(0),\dot{x}(0))$ and $c_1=\max_{t\in[0,\omega]}x^2(t)+\max_{t\in[0,\omega]}\dot{x}^2(t)$. Hence Theorem 4.6 applies if F satisfies (4.4).

4.3 Some technical lemmas

Lemma 4.8 *Let $\gamma,\eta>0$, $\tau_*,x_*\in\mathbb{R}$, and $v_*>0$. Then for the solution of $\ddot{x}+x=b^-(\tau)=\gamma\sin(\eta\tau)-1$ in $[\tau_*,\infty[$, $x(\tau_*)=x_*$, $\dot{x}(\tau_*)=v_*$, there exists a time $\tau_{**}>\tau_*$ such that $\dot{x}(\tau_{**})<0$.*

Proof: We assume that $\dot{x}(\tau)\ge 0$ in $]\tau_*,\infty[$. If we suppose that $x(\tilde{\tau})+1=\delta>0$ for some $\tilde{\tau}\in[\tau_*,\infty[$, then for $\tau\in[\tilde{\tau},\infty[$ we have $\ddot{x}(\tau)+\delta\le\ddot{x}(\tau)+x(\tau)+1=\gamma\sin(\eta\tau)$, and thus $0\le\dot{x}(\tau)\le\dot{x}(\tilde{\tau})+2\gamma/\eta-\delta(\tau-\tilde{\tau})$ in $[\tilde{\tau},\infty[$, a contradiction. Consequently, $x(\tau)+1\le 0$ in $[\tau_*,\infty[$, and hence $\ddot{x}(\tau)\ge\ddot{x}(\tau)+x(\tau)+1=\gamma\sin(\eta\tau)$ in $[\tau_*,\infty[$. Choose $\delta\in]0,\pi/\eta[$ and the sequence $\tau_k\to\infty$ with $\tau_1=\tau_*$, $\tau_{k+1}=\tau_k+2\pi/\eta$, and $|\cos(\eta\tau_1)-\cos(\eta\tau)|\le\eta v_*/2\gamma$ for $\tau\in[\tau_k-\delta,\tau_k+\delta]$. Thus for $\tau\in[\tau_k-\delta,\tau_k+\delta]$ we obtain the estimate $\dot{x}(\tau)=\dot{x}(\tau_1)+\int_{\tau_1}^{\tau}\ddot{x}(\sigma)d\sigma\ge v_*+\gamma(\cos(\eta\tau_1)-\cos(\eta\tau))/\eta\ge v_*/2$. This implies $x(\tau_k+\delta)\ge x(\tau_k-\delta)+\delta v_*$ for $k\in\mathbb{N}$, and therefore $x_*=x(\tau_1)\le x(\tau_2-\delta)\le x(\tau_2+\delta)-\delta v_*\le x(\tau_3-\delta)-\delta v_*\le x(\tau_3+\delta)-2\delta v_*\le\dots$. Inductively we find $x_*+(k-1)\delta v_*\le x(\tau_k+\delta)\le -1$ for $k\in\mathbb{N}$, a contradiction. □

Lemma 4.9 *Let $d\geq 2$, $f\in C^1(\mathbb{R}^d,\mathbb{R}^d)$, and let Φ_f denote the global flow generated by f, i.e. $x(\cdot)=\Phi_f(\cdot,x_0)$ uniquely solves $\dot{x}=f(x)$ and $x(0)=x_0$ for $x_0\in\mathbb{R}^d$.*

(a) Suppose that $C\subset\mathbb{R}^d$ is a C^1-manifold of dimension $\leq d-2$ and global chart $\alpha:\mathbb{R}^{d-2}\to C$, $y\mapsto x=\alpha(y)$. Then $\lambda^d(\{x\in\mathbb{R}^d: \exists t\in[0,\infty[: \Phi_f(t,x)\in C\})=0$.

(b) If $[a,b]\subset\mathbb{R}$, and if $N\subset\mathbb{R}^d$ has $\lambda^d(N)=0$, then $\lambda^d(\Phi_f([a,b]\times N))=0$.

Proof: Ad (a): Let $\varrho:]-\infty,0[\times\mathbb{R}^{d-2}\to\mathbb{R}^d$, $\varrho(s,y)=\Phi_f(s,\alpha(y))$. Then ϱ is C^1 and

$$\varrho(]-\infty,0[\times\mathbb{R}^{d-2})=\{x\in\mathbb{R}^d: \exists t\in]0,\infty[: \Phi_f(t,x)\in C\}.$$

Since $\lambda^d(\varrho(]-\infty,0[\times\mathbb{R}^{d-2}))=0$, (a) follows. Ad (b): Because $\Phi_f\in C^1(\mathbb{R}\times\mathbb{R}^d)$, this is an immediate consequence of a result related to Sard's lemma, cf. [37, Thm. 3.1]. $\square$

Lemma 4.10 *Consider the matrices*

$$B(\theta,\alpha,P,P')=\begin{pmatrix}\cos\theta & \alpha\sin\theta & P\\ -\sin\theta & \alpha\cos\theta & P'\\ 0 & 0 & 1\end{pmatrix}\in\mathbb{R}^{3\times 3} \text{ and } \tilde{B}(\theta,\alpha)=\begin{pmatrix}\cos\theta & \alpha\sin\theta\\ -\sin\theta & \alpha\cos\theta\end{pmatrix}\in\mathbb{R}^{2\times 2} \qquad (4.10)$$

for $\theta,\alpha,P,P'\in\mathbb{R}$. Let $\theta_i,\alpha_i,P_i,P_i'\in\mathbb{R}$ for $i\in\mathbb{N}_0$ be given, and define $B_i=B(\theta_i,\alpha_i,P_i,P_i')$ as well as $\tilde{B}_i=\tilde{B}(\theta_i,\alpha_i)$. Let $\prod_{i=0}^{k} B_i=B_k\cdot\ldots\cdot B_0$. Then the following holds for every $k\in\mathbb{N}_0$.

(a) The first two lines of $\left(\prod_{i=0}^{k} B_i\right)(1,0,0)^{\mathrm{tr}}\in\mathbb{R}^3$ are $\left(\prod_{i=0}^{k}\tilde{B}_i\right)(1,0)^{\mathrm{tr}}\in\mathbb{R}^2$, and there is a zero in the third line.

(b) The first two lines of $\left(\prod_{i=0}^{k} B_i\right)(0,1,0)^{\mathrm{tr}}\in\mathbb{R}^3$ are $\left(\prod_{i=0}^{k}\tilde{B}_i\right)(0,1)^{\mathrm{tr}}\in\mathbb{R}^2$, and there is a zero in the third line.

(c) The first two lines of $\left(\prod_{i=0}^{k} B_i\right)(0,0,1)^{\mathrm{tr}}\in\mathbb{R}^3$ are

$$\sum_{i=0}^{k}\left(\prod_{j=k+2-i}^{k}\tilde{B}_j\right)\begin{pmatrix}P_{k-i+1}\\ P'_{k-i+1}\end{pmatrix}\in\mathbb{R}^2,$$

and there is a 1 in the third line. (Here we define empty products to be the unit matrix in $\mathbb{R}^2$.)

In addition, $\|\tilde{B}(\theta,\alpha)\|_2=1$ for all $\theta\in\mathbb{R}$ and $\alpha\in[0,1]$. So, assuming above that also $\alpha_i\in[0,1]$ for $i\in\mathbb{N}$, we obtain for every $k\in\mathbb{N}$

(d) $\left|\left(\prod_{i=0}^{k} B_i\right)(1,0,0)^{\mathrm{tr}}\right|_{\mathbb{R}^3}\leq 1$,

(e) $\left|\left(\prod_{i=0}^{k} B_i\right)(0,1,0)^{\mathrm{tr}}\right|_{\mathbb{R}^3}\leq 1$, *and*

(f) $\left|\left(\prod_{i=0}^{k} B_i\right)(0,0,1)^{\mathrm{tr}}\right|_{\mathbb{R}^3} \le 1+\sum_{i=0}^{k}\left|\begin{pmatrix} P_i \\ P_i' \end{pmatrix}\right|_{\mathbb{R}^2}.$

Proof: (a) and (b) are similar, so we consider (a). Since $B_0(1,0,0)^{\mathrm{tr}} = (\cos\theta_0, -\sin\theta_0, 0)^{\mathrm{tr}}$ and $\tilde{B}_0(1,0)^{\mathrm{tr}} = (\cos\theta_0, -\sin\theta_0)^{\mathrm{tr}}$, the case $k=0$ follows. If the claim already holds for $k\in\mathbb{N}_0$ and $(\prod_{i=0}^{k}\tilde{B}_i)(1,0)^{\mathrm{tr}} =: (u_k, v_k)^{\mathrm{tr}}$, then by induction hypotheses

$$\begin{aligned}(\prod_{i=0}^{k+1} B_i)(1,0,0)^{\mathrm{tr}} &= B_{k+1}(\prod_{i=0}^{k} B_i)(1,0,0)^{\mathrm{tr}} = B_{k+1}(u_k, v_k, 0)^{\mathrm{tr}} \\ &= (u_k\cos\theta_{k+1} + v_k\,\alpha_{k+1}\sin\theta_{k+1}, -u_k\sin\theta_{k+1} + v_k\,\alpha_{k+1}\cos\theta_{k+1}, 0)^{\mathrm{tr}},\end{aligned}$$

and also

$$\begin{aligned}(\prod_{i=0}^{k+1} \tilde{B}_i)(1,0)^{\mathrm{tr}} &= \tilde{B}_{k+1}(\prod_{i=0}^{k} \tilde{B}_i)(1,0)^{\mathrm{tr}} = \tilde{B}_{k+1}(u_k, v_k)^{\mathrm{tr}} \\ &= (u_k\cos\theta_{k+1} + v_k\,\alpha_{k+1}\sin\theta_{k+1}, -u_k\sin\theta_{k+1} + v_k\,\alpha_{k+1}\cos\theta_{k+1})^{\mathrm{tr}},\end{aligned}$$

and therefore we have shown (a).

Concerning (c), we first note that $B_0(0,0,1)^{\mathrm{tr}} = (P_0, P_0', 1)^{\mathrm{tr}}$, and thus the claim is true for $k=0$. Assuming it already holds for $k\in\mathbb{N}_0$, we obtain

$$\begin{aligned}(\prod_{i=0}^{k+1} B_i)(0,0,1)^{\mathrm{tr}} &= B_{k+1}(\prod_{i=0}^{k} B_i)(0,0,1)^{\mathrm{tr}} \\ &= \begin{pmatrix} \cos\theta_{k+1} & \alpha_{k+1}\sin\theta_{k+1} & P_{k+1} \\ -\sin\theta_{k+1} & \alpha_{k+1}\cos\theta_{k+1} & P_{k+1}' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sum_{i=0}^{k}\left(\prod_{j=k+2-i}^{k}\tilde{B}_j\right)\begin{pmatrix} P_{k-i+1} \\ P_{k-i+1}' \end{pmatrix} \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \tilde{B}_{k+1}\left[\sum_{i=0}^{k}\left(\prod_{j=k+2-i}^{k}\tilde{B}_j\right)\begin{pmatrix} P_{k-i+1} \\ P_{k-i+1}' \end{pmatrix}\right] + \begin{pmatrix} P_{k+1} \\ P_{k+1}' \end{pmatrix} \\ 1 \end{pmatrix},\end{aligned}$$

and hence (c) follows, because

$$\tilde{B}_{k+1}[\sum_{i=0}^{k}(\prod_{j=k+2-i}^{k}\tilde{B}_j)(P_{k-i+1}, P_{k-i+1}')^{\mathrm{tr}}] + (P_{k+1}, P_{k+1}')^{\mathrm{tr}} = \sum_{i=0}^{k+1}(\prod_{j=k+3-i}^{k+1}\tilde{B}_j)(P_{k-i+2}, P_{k-i+2}')^{\mathrm{tr}}.$$

Since $\tilde{B}(\theta,\alpha)^{\mathrm{tr}}\tilde{B}(\theta,\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & \alpha^2 \end{pmatrix}$ has spectral radius $1\vee\alpha^2 = 1$ for $\alpha\in[0,1]$, we conclude that $\|\tilde{B}(\theta,\alpha)\|_2 = 1$ for $\theta\in\mathbb{R}$ and $\alpha\in[0,1]$. This yields (d), (e), and (f), using $\sqrt{1+x^2}\le 1+x$, $x\in[0,\infty[$, for the latter. □

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