An Elliptic Quasi-Variational Inequality with Gradient Constraints and some of its Applications

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Abstract

We consider a class of quasi-variational inequalities for certain second order elliptic operators, where the set of admissible functions is required to satisfy an implicit gradient bound, which depends on the solutions itself. We give sufficient conditions for the existence of a solution, and we apply our results to stationary problems arising in superconductivity, in thermoplasticity, and in electrostatics with implicit ionization threshold.

1 Introduction

Variational inequalities with gradient constraints, in particular those arising in the theory of elastoplastic torsion of bars, have been studied deeply in the seventies (see [9] or [13] and the references therein). The intrinsic interest in explicit gradient bounds within the theory of elliptic partial differential equations can be seen in the delicate questions arising in the case of nonconstant gradient constraints for regularity of solutions, see, e.g., [16], or for the relation with unilateral problems, cf., e.g., [15] and its bibliography. For instance, it is well-known that the notion of unilateral problem and the notion of variational inequality in general are not equivalent if the gradient constraint is not a constant. In addition, their natural relations with the associated obstacle problems become a much more delicate problem in this case.

In the more general situation when the gradient constraint may depend on the solution itsself through a (local or nonlocal) functional relation, we have to investigate, instead of variational inequalities, the so-called quasi-variational inequalities. These quasi-variational problems (also called implicit variational inequalities) were introduced in [3] in connection with some stochastic impuls control problems, and thereafter extensively investigated (see [2] and its bibliography).

But as a matter of fact, the applications of quasi-variational inequalities so far mostly have been concerned with obstacle type problems, and it seems to us that no applications to problems with implicit gradient constraints have been considered up to now.

Motivated by Prigozhin's work [12], where an interesting critical-state model in superconductivity is formulated as an evolutionary quasi-variational inequality, we consider in this paper a simpler stationary situation, for which we are able to give sufficient conditions for the existence of a solution. We exploit the mathematical analogies of this magnetostatic problem in superconductivity theory with a variant of the elastoplastic torsion problem in which we allow the plasticity threshold to vary with the stress dependent temperature. Somehow dual to the superconductivity problem, we also analyze a model of ionizable media in electrostatics, extending the one of Duvaut and J.L. Lions from [8].

This paper consists of four more sections. In the first one we consider an abstract model problem in order to make clear the structure of proof for the more concrete examples to follow. The abstract Theorem 1 below gives sufficient conditions for the existence of solutions to a certain quasi-variational inequality, and its proof rests on the application of Schauder theory and a basic new lemma on the convergence of convex sets of gradient constraint type in the sense of Mosco. In the Section 3, after a brief physical motivation, we apply the existence theorem to the case of the stationary magnetization of a superconductor of type-II, with an extension of Bean's critical-state model. In Section 4 we develop the example of thermoplastic torsion of a bar subject to a lateral nonuniform heating or cooling through a stress-dependent law of Newton type. Finally, in Section 5, we solve a quasi-variational ionization problem in electrostatics, where the ionization threshold is allowed to vary at each point with the electrical energy.

2 An abstract model problem

In this section we consider a class of quasi-variational inequalities for second order elliptic operators of the type

$$\langle Au, v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\,dx\,,\quad u,v\in W_0^{1,p}(\Omega)\,,\quad 1<p<\infty\,, \tag{2.1}$$

where $\Omega\subset\mathrm{I\!R}^N$ with $N\geq 2$ is a bounded open domain. Recall that in this case we have the usual continuous embeddings

$$W_0^{1,p}(\Omega)\subset L^q(\Omega)\quad\text{for}\quad 1\leq p<N,\, 1\leq q\leq Np/(N-p)\quad\text{or}\quad p=N, 1\leq q<\infty\,, \tag{2.2}$$

and

$$W_0^{1,p}(\Omega)\subset C^0(\overline{\Omega})\quad\text{for}\quad p>N\,, \tag{2.3}$$

with the second embedding of (2.2) and (2.3) being compact, and the first embedding of (2.2) being compact for $1\leq p<N$, $1\leq q<Np/(N-p)$, cf. [1, Thm. 5.4, Thm. 6.2].

Given a measurable $\varphi:\Omega\to\mathrm{I\!R}$ with $\varphi(x)\geq 0$ a.e. we define the closed convex set with gradient constaint

$$K(\varphi)=\{v\in W_0^{1,p}(\Omega): |\nabla v(x)|\leq\varphi(x)\text{ a.e. in }\Omega\}\,. \tag{2.4}$$

Then in particular $0\in K(\varphi)\neq\emptyset$. Fix $\nu>0$ and define an (in general nonlinear) operator

$$\Phi: C\subset X\to L^\infty_\nu(\Omega)=\{\varphi\in L^\infty(\Omega):\varphi(x)\geq\nu\text{ a.e. in }\Omega\} \tag{2.5}$$

on some closed convex subset C of a Banach space X, where X will be choosen later to be $W_0^{1,p}(\Omega)$, $L^q(\Omega)$, or $C^0(\overline{\Omega})$.

For fixed $f\in W^{-1,p'}(\Omega)=[W_0^{1,p}(\Omega)]'$, where $p'=p/(p-1)$, we consider the following quasi-variational inequality:

$$\text{find}\quad u\in K(\Phi(u)):\quad \langle Au-f, v-u\rangle\geq 0\quad\forall v\in K(\Phi(u))\,. \tag{2.6}$$

Our main abstract result is

Theorem 1 *There exists at least one solution of (2.6), provided one of the following assumptions holds, with Φ from (2.5):*

(i) $X=W_0^{1,p}(\Omega)$, $1<p<\infty$, *and Φ is completely continuous (i.e. continuous and compact);*

(ii) $X=L^q(\Omega)$, $1<p<N$ *and* $1\leq q<Np/(N-p)$, *or* $p=N$ *and* $1\leq q<\infty$, *and Φ is continuous;*

(iii) $X=C^0(\overline{\Omega})$, $p>N$, *and Φ is continuous;*

(iv) $X=C^0(\overline{\Omega})$, *and Φ is continuous such that*

$$\|\Phi(w)\|_{L^r(\Omega)}\leq c_0+c_1\|w\|^\alpha_{L^{\sigma p}(\Omega)} \tag{2.7}$$

for some constants $c_0,c_1\geq 0$, $\alpha\geq 0$, *and* $r>N$, *and additionally* $1<p<N$ *and* $1/p\leq\sigma\leq N/(N-p)$, *or* $p=N$ *and* $1/p\leq\sigma$.

where in case (iv) we assume $C = X = C^0(\overline{\Omega})$, whereas in the cases (i), (ii), and (iii) we choose $C = \overline{B}_{R_f}(0) = \{u \in X : \|u\|_X \leq R_f\}$ with R_f being some constant determined by f. [Note that thus in particular it is enough that Φ satisfies the respective assumptions on every $\overline{B}_R(0)$ for large R.]

Proof: For a fixed $\varphi \in L^\infty_\nu(\Omega)$ we consider first the variational inequality

$$\text{find} \quad u \in K(\varphi): \quad \langle Au - f, v - u\rangle \geq 0 \quad \forall v \in K(\varphi). \tag{2.8}$$

By definition (2.1), $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is a strictly monotone operator, and hence by well-known results (cf. [2, p. 234], for instance) there exists a unique solution of (2.8), denoted by $u = S\varphi$. If $\varphi_n \to \varphi$ in $L^\infty(\Omega)$, then $K(\varphi_n) \to K(\varphi)$ in the sense of Mosco, by Lemma 1 below. Since A is also strongly coercive, Mosco's stability theorem (cf. [11] or [13, p. 99]) implies

$$u_n = S\varphi_n \to S\varphi = u \quad \text{in} \quad W_0^{1,p}(\Omega),$$

and this in turn yields continuity of the solution map $S : L^\infty_\nu(\Omega) \to W_0^{1,p}(\Omega)$.

On the other hand, Poincaré's inequality and $v = 0 \in K(\varphi)$ in (2.8) imply the a priori estimate

$$\|S\varphi\|_{W_0^{1,p}(\Omega)} = \|u\|_{W_0^{1,p}(\Omega)} \leq c\|f\|^{1/(p-1)}_{W^{-1,p'}(\Omega)} =: c_f \tag{2.9}$$

independently of $\varphi \in L^\infty_\nu(\Omega)$.

Next note that the fixed points of the application $T = S \circ \Phi : C \subset X \to L^\infty_\nu(\Omega) \to W_0^{1,p}(\Omega)$ are solutions of (2.6). We intend to apply the Schauder fixed point theorem in the cases (i), (ii), (iii) to find such fixed points. For that, in case (i) we can take $R_f = c_f$ and $C = \overline{B}_{R_f}(0) \subset W_0^{1,p}(\Omega)$, because then $T(C) \subset C$ by (2.9), and $T(C)$ is compact by assumption. Concerning case (ii), let c_q denote the constant of the compact embedding (2.2) and define $R_f = c_q c_f$, i.e. $C = \overline{B}_{R_f}(0) \subset L^q(\Omega)$. Then again T is a compact self-map of C. In case (iii) one can argue in the same way by compactness of (2.3). Finally, to treat case (iv), we are going use the the Leray-Schauder principle (cf. [17, p. 245]) in $C^0(\overline{\Omega})$ by showing that $T : C^0(\overline{\Omega}) \to W_0^{1,r}(\Omega)$ is a bounded map, and that the set $\mathcal{S} = \{w \in C^0(\overline{\Omega}) : w = \lambda Tw \text{ for some } \lambda \in [0,1]\} \subset C^0(\overline{\Omega})$ is bounded. Concerning the latter, fix $w \in \mathcal{S}$ and let $u = Tw$. Since u solves (2.8) with $\varphi = \Phi(w)$, we have in particular that $u \in K(\varphi)$, and thus $|\nabla u(x)| \leq \varphi(x)$ a.e. in Ω. Consequently we obtain from (2.7) and (2.9)

$$\begin{aligned}\||\nabla w|\|_{L^r(\Omega)} &= \lambda \||\nabla u|\|_{L^r(\Omega)} \leq \|\varphi\|_{L^r(\Omega)} = \|\Phi(w)\|_{L^r(\Omega)} \leq c_0 + c_1 \|w\|^\alpha_{L^{\sigma p}(\Omega)} \\ &\leq c_0 + c_1 \|u\|^\alpha_{L^{\sigma p}(\Omega)} \leq c_0 + c_1 c_q^\alpha \|u\|^\alpha_{W_0^{1,p}(\Omega)} \leq c_0 + c_1 c_q^\alpha c_f^\alpha,\end{aligned}$$

where the estimate next to the last follows from (2.2) and $1 \leq q = \sigma p \leq Np/(N-p)$ for $1 < p < N$. Thus Poincaré's inequality implies the boundedness of $\mathcal{S} \subset W^{1,r}(\Omega)$, and hence in particular the boundedness of $\mathcal{S} \subset C^0(\overline{\Omega})$, by (2.2). As the boundedness of $T : C^0(\overline{\Omega}) \to W_0^{1,r}(\Omega)$ may be seen analogously, the Leray-Schauder principle applies to yield a fixed point of T also in the case (iv). $\square$

Remark 1 Let $j : [0, \infty[\to \mathbb{R}$ be continuous such that $j(t) \geq \nu > 0$ for $t \geq 0$. Then the Nemytskii operator $(\Phi(u))(x) = j(u(x))$, $x \in \Omega$, is well-defined and satisfies the assumptions of Theorem 1(iv), if

$$j(t) \leq c_0 + c_1 t^\alpha, \quad t \geq 0, \tag{2.10}$$

with some constants $c_0, c_1 \geq 0$ and $0 \leq \alpha < p/(N-p)$ if $1<p<N$, or $\alpha \geq 0$ if $p=N$. Indeed, this can be easily seen, e.g., in the case $1<p<N$ and $0<\alpha<p/(N-p)$, by taking $r = \frac{Np}{\alpha(N-p)} > N$ and $\sigma = N/(N-p)$ in (iv) to obtain from (2.10) and $\alpha r = \sigma p$ that

$$\|\Phi(w)\|^r_{L^r(\Omega)} \leq c + c\int_\Omega |w(x)|^{\alpha r}\, dx = c + c\|w\|^{\alpha r}_{L^{\sigma p}(\Omega)},$$

and thus (2.7).

Remark 2 Theorem 1 may be extended to cover more general quasilinear elliptic operators A in $W^{1,p}(\Omega)$ of the type

$$\langle Au, v\rangle = \int_\Omega [a(x,u,\nabla u)\cdot \nabla v + b(x,u,\nabla u)v + c(x)uv]\, dx + \int_{\partial\Omega} d(x,u)v\, d\sigma$$

under appropriate structural conditions as in [6, 15, 7], assuring the unique solvability of the corresponding auxiliary problem (2.8).

In the proof of Theorem 1 we have used in a crucial way – but not yet shown – that for $\varphi_n, \varphi \in L^\infty_\nu(\Omega)$ with $\varphi_n \to \varphi$ in $L^\infty(\Omega)$ we have $K(\varphi_n) \to K(\varphi)$ in the sense of Mosco (cf. [11]), meaning that

(i) If $K(\varphi_{n'}) \ni v_{n'} \rightharpoonup v$ in $W^{1,p}_0(\Omega)$ for some subsequence $(n') \subset \mathbb{N}$, then $v \in K(\varphi)$ (2.11)

(ii) $\forall\, v \in K(\varphi)\ \exists\, v_n \in K(\varphi_n): \quad v_n \to v$ in $W^{1,p}_0(\Omega)$. (2.12)

This turns out to be a special case of the following lemma (which will be useful later also in the more general setting stated here).

Lemma 1 *Let $\xi \in L^\infty(\Omega, \mathbb{R}^n)$ be such that $|\xi(x)| \leq \nu_0$ a.e. in Ω for some $\nu_0 \geq 0$, and define*

$$K_\xi(\varphi) = \{v \in W^{1,p}_0(\Omega) : |\nabla v(x) + \xi(x)| \leq \varphi(x) \text{ a.e. in}\, \Omega\}.$$

Then, if $\nu > \nu_0$, $\varphi_n, \varphi \in L^\infty_\nu(\Omega)$, and $\varphi_n \to \varphi$ in $L^\infty(\Omega)$, we have $K_\xi(\varphi_n) \to K_\xi(\varphi)$ in the sense of Mosco.

Proof: To show (2.11), define for $\delta > 0$ the set $\mathcal{M}_\delta = \{w \in W^{1,p}_0(\Omega) : |\nabla w(x) + \xi(x)| \leq \varphi(x) + \delta \text{ a.e. in}\, \Omega\}$. Then $\mathcal{M}_\delta$ is closed and convex, hence weakly closed. Let $v_{n'} \in K(\varphi_{n'})$ and $v_{n'} \rightharpoonup v$ in $W^{1,p}_0(\Omega)$ for some subsequence $(n') \subset \mathbb{N}$. Since $v_{n'} \in \mathcal{M}_\delta$ for large n', also $v \in \mathcal{M}_\delta$ for all $\delta > 0$, and consequently $v \in K_\xi(\varphi)$. Concerning(2.12), we fix some $v \in K_\xi(\varphi)$ and choose $v_n \in K_\xi(\varphi_n)$ as the projection of v on $K_\xi(\varphi_n)$, i.e. the solution of

$$\|v_n - v\|_{W^{1,p}_0(\Omega)} \leq \|w - v\|_{W^{1,p}_0(\Omega)} \quad \forall w \in K_\xi(\varphi_n). \tag{2.13}$$

Since $L^\infty_\nu(\Omega) \ni \varphi_n \to \varphi$ in $L^\infty(\Omega)$ and $\nu > \nu_0$, we may define a sequence of reals $\alpha_n \to 1/2$ with $0 \leq \alpha_n \leq 1/2$ such that for a.e. $x \in \Omega$

$$\frac{\varphi(x) - \varphi_n(x)}{\varphi_n(x) - \nu_0} \leq \left|\frac{\varphi - \varphi_n}{\varphi_n - \nu_0}\right|_{L^\infty(\Omega)} =: 1 - 2\alpha_n.$$

This in turn implies that $w_n = v/2 + \alpha_n v_n \in K_\xi(\varphi_n)$, because

$$|\nabla w_n(x) + \xi(x)| = \Big|[\nabla v(x) + \xi(x)]/2 + \alpha_n[\nabla v_n(x) + \xi(x)] + [1/2 - \alpha_n]\xi(x)\Big|$$
$$\leq \varphi(x)/2 + \alpha_n \varphi_n(x) + [1/2 - \alpha_n]\nu_0 \leq \varphi_n(x) \quad \text{a.e. in } \Omega$$

by definition of α_n. Hence it follows from (2.13) with $w = w_n$ that

$$\|v_n - v\|_{W_0^{1,p}(\Omega)} \leq \|\alpha_n v_n - v/2\|_{W_0^{1,p}(\Omega)} \leq \alpha_n \|v_n - v\|_{W_0^{1,p}(\Omega)} + (1/2 - \alpha_n)\|v\|_{W_0^{1,p}(\Omega)},$$

and therefore

$$\|v_n - v\|_{W_0^{1,p}(\Omega)} \leq \frac{(1/2 - \alpha_n)}{1 - \alpha_n}\|v\|_{W_0^{1,p}(\Omega)} \to 0 \quad \text{for} \quad n \to \infty,$$

as was to be proved. □

Remark 3 It would be interesting to drop the restriction on the strict positivity of the φ_n in Lemma 1, but this seems to be a difficult problem. Of course that assumption is not necessary in the special case when $\varphi_n \geq \varphi$ a.e., since then $K_\xi(\varphi_n) \supset K_\xi(\varphi)$, and hence we can choose $v_n = v$ in (2.12), whereas to show (2.11), $\varphi_n \to \varphi$ in $L^1(\Omega)$ is sufficient.

3 The stationary magnetization of a superconductor

Recently some extensions of Bean's critical-state model for the magnetization of type-II superconductors have been proposed in the physical literature. In particular, Bossavit introduced in [4] a convex analysis approach to the relevant constitutive relations, which is valid in general three-dimensional situations, and Prigozhin in [12] has shown its equivalence to a quasi-variational inequality. Here we restrict ourselves to a simple stationary model in longitudinal geometry, where the main unknown is the magnetic field, assumed to be of the form $\mathbf{H} = (0, 0, u(x))$. Here $x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2$, i.e., we consider the case of transverse stationary currents.

Maxwell's equations reduce to

$$\operatorname{rot}\mathbf{E} = 0 \quad \text{and} \quad \operatorname{rot}\mathbf{H} = \mathbf{J}, \tag{3.1}$$

with $\mathbf{E}$ denoting the electric field, and $\mathbf{J}$, the unknown density of induced currents, is then given by $\mathbf{J} = (\partial u/\partial x_2, -\partial u/\partial x_1, 0) = \nabla^* u$. Following [4] and [5], we shall assume that the electric field inside the superconductor depends on the current density via a law $\mathbf{J} \in \Gamma(\mathbf{E})$. This can be regarded as an extension of Ohm's classical law $\mathbf{E} = \rho\mathbf{J}$, where the scalar resistivity $\rho = \rho(|\mathbf{J}|)$ can be taken as a highly nonlinear function depending on $j = |\mathbf{J}|$. In Bean's model, with a power-law graph as shown in Fig. 1 below, the current density cannot exceed some critical value j_c, hence enforcing $|\nabla^* u| = |\nabla u| \leq j_c$.

– Fig. 1 [see at the end of the paper] –

In general, the critical value j_c may additionally depend on the absolute value $|\mathbf{H}|=|u|$ of the magnetic field $\mathbf{H}=(0,0,u(x))$. This naturally yields a quasi-variational inequality (as observed by Prigozhin [12] for the special degenerate case $e_c=0$), since the constitutive relation for $\mathbf{E}$ is now $(1<p<\infty)$

$$\mathbf{E}=\begin{cases} \rho_0|\nabla u|^{p-2}\nabla^* u & \text{if} \quad |\nabla u|<j_c(|u|) \\ (\rho_0 j_c^{p-2}+\lambda)\nabla^* u & \text{if} \quad |\nabla u|=j_c(|u|), \end{cases} \tag{3.2}$$

where $\rho_0>0$ is a given constant, and $\lambda\geq 0$ can be regarded as an (unknown) Lagrange multiplier related to the inequality constraint (cf. also [12, Section 3]), acting only in the superconducting region $\{x\in\Omega : |\nabla u(x)|=j_c(|u(x)|)\}$.

In order to complete the mathematical formulation, we assume the boundary of the cross section $\Omega\subset\mathbb{R}^2$ to be Lipschitz, and we prescribe

$$u=u_1 \quad \text{on} \quad \partial\Omega, \tag{3.3}$$

with $u_1\in C^{0,1}(\overline{\Omega})$ being a given Lipschitz continuous function.

Recalling equation (3.1), the constitutive relation (3.2), and the boundary condition (3.3), we may formulate this stationary superconductivity problem as the following quasi-variational inequality for the magnetic field u:

$$\text{find} \quad u\in K_1(u): \quad \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla(v-u)\,dx\geq 0 \quad \forall v\in K_1(u), \tag{3.4}$$

where for a given non-negative continuous function j_c we define

$$K_1(u)=\{v\in W^{1,p}(\Omega) : v|_{\partial\Omega}=u_1,\, |\nabla v(x)|\leq j_c(|u(x)|) \text{ a.e. in } \Omega\}. \tag{3.5}$$

It is clear that (3.4) is a problem of the same type as was studied in Section 2, with the difference being that we now have a non-homogeneous Dirichlet boundary condition. Assuming

$$\|\nabla u_1\|_{L^\infty(\Omega)}<\nu\leq j_c \tag{3.6}$$

for some $\nu>0$, we remark that $u_1\in K_1(\varphi)$ for every $\varphi\in C^0(\overline{\Omega})$, and we may consider the corresponding translated problem for $\hat{u}=u-u_1$, where the respective family of nonempty convex sets is then

$$\hat{K}(w)=\{v\in W_0^{1,p}(\Omega) : |\nabla v(x)+\nabla u_1(x)|\leq j_c(|w(x)+u_1(x)|) \text{ a.e. in } \Omega\}.$$

By using Lemma 1 with $\xi=\nabla u_1$ and $\nu_0=\|\nabla u_1\|_{L^\infty(\Omega)}$, and by also taking into account Remark 1, we can argue exactly as in the proof of Theorem 1(iv) to obtain

Theorem 2 *Under the assumption (3.6) there exists at least one solution u of the quasi-variational inequality (3.4), (3.5) for $p>2$. If $1<p<2$ resp. $p=2$, then there exists at least one solution, if in addition j_c satisfies a growth condition (2.10) with some $0\leq\alpha<p/(2-p)$ resp. $0\leq\alpha<\infty$.* $\square$

4 The torsion problem with variable threshold of plasticity

In this section we consider the elastoplastic torsion of an isotropic and homogeneous cylinder with a simply connected cross section $\Omega\subset\mathbb{R}^2$ having Lipschitz boundary $\partial\Omega$, defining a lateral contour that is assumed to be stress-free, but subject to a non-homogeneous heating or cooling. It is then well-know (cf., e.g., [9] or [13] and the references therein), that the problem may be described by the stress function $u(x)$, $x=(x_1,x_2)\in\Omega$, the derivatives of which yield the only two non-zero components of the stress tensor σ_{ij}, i.e.,

$$\sigma_{13}=\frac{\partial u}{\partial x_2} \quad \text{and} \quad \sigma_{23}=-\frac{\partial u}{\partial x_1}.$$

By the von Mises criterion for an elastic perfectly plastic material, the usual yield limit (or threshold of plasticity) $|\nabla u|\leq\gamma$ (for some $\gamma>0$) divides the cross section in an elastic zone $\{x\in\Omega:|\nabla u(x)|<\gamma\}$ and a plastic zone $\{x\in\Omega:|\nabla u(x)|=\gamma\}$, the latter (when existing) being connected with the boundary $\partial\Omega$. In that case, u is the unique solution to the variational inequality (2.8), with A from (2.1), $p=2$, $\varphi\equiv\gamma$, and $f=-2\mu\theta$, where the positive constants μ and θ represent, respectively, the second Lamé coefficient and the angle of rotation.

In thermoplasticity, in general the yield limit is a function of the temperature, cf. [10, Paragraph 19]. Hence we have to ensure that $|\nabla u(x)|\leq\gamma(\vartheta(x))$, $x\in\Omega$, where $\vartheta(x)$ denotes the temperature at a point x, which in general is supposed to be non-homogeneous, and γ is a known function for which we assume

$$\gamma:\mathbb{R}\to\mathbb{R} \quad \text{is continuous and} \quad \gamma(t)\geq\nu>0, \quad t\in\mathbb{R}. \tag{4.1}$$

In equilibrium, and neglecting internal heat sources, the temperature field $\vartheta=\vartheta(x)$ is harmonic, i.e., we require

$$\Delta\vartheta=0 \quad \text{in} \quad \Omega, \tag{4.2}$$

whereas we shall assume a linear heat flux boundary condition

$$\frac{\partial\vartheta}{\partial n}=\alpha(x,[u])(\vartheta_1-\vartheta) \quad \text{on} \quad \partial\Omega. \tag{4.3}$$

Here $\vartheta_1=\vartheta_1(x)$ is a given extremal temperature such that

$$\vartheta_1\in H^1(\Omega)\cap L^\infty(\Omega), \quad |\vartheta_1(x)|\leq M \quad \text{a.e. on} \quad \partial\Omega, \tag{4.4}$$

with $[u](x)$ denoting the local elastoplastic state of the material near a point $x\in\partial\Omega$, and α is a coefficient function such that $\alpha:\partial\Omega\times\mathbb{R}\to\mathbb{R}$ is Carathéodory with

$$0<\alpha_*\leq\alpha(x,r)\leq\alpha^*, \quad \text{a.e. } x\in\partial\Omega,\, r\in\mathbb{R}. \tag{4.5}$$

Since $u=0$ on $\partial\Omega$, and in order to avoid delicate considerations on the general regularity properties of ∇u on $\partial\Omega$, we shall in fact use the strain energy $[u]_\beta(x)$ stored in a suitable neighbourhood $\Sigma_\beta(x)=\{y\in\Omega:|y-x|<\beta\}$ of each point $x\in\partial\Omega$ as a suitable approximation of the local elastoplastic state $[u](x)$, i.e., we replace $[u]$ in (4.3) through $[u]_\beta$, with

$$[u]_\beta(x)=\int_{\Sigma_\beta(x)}|\nabla u(y)|^2\,dy, \quad x\in\partial\Omega, \tag{4.6}$$

where $\beta>0$ is a fixed suitable parameter.

Lemma 2 *For each $u \in H_0^1(\Omega)$ consider the Neumann problem (4.2), (4.3) with $[u]_\beta$ instead of $[u]$, under the assumptions (4.4) and (4.5). Then this problem has a unique solution $\vartheta \in H^1(\Omega) \cap C^{0,\lambda}(\overline{\Omega})$, for some $\lambda \in (0,1)$, such that*

$$|\vartheta(x)| \leq M \quad on \quad \overline{\Omega}. \tag{4.7}$$

In addition, the solution map $\Theta : H_0^1(\Omega) \ni u \mapsto \vartheta \in C^0(\overline{\Omega})$ is completely continuous.

Proof: For fixed $u \in H_0^1(\Omega)$, the function $\alpha_u(x) = \alpha(x,[u]_\beta(x))$, $x \in \partial\Omega$, belongs to $L^\infty(\partial\Omega)$. Hence (4.2),(4.3) has the variational form

$$\text{find} \quad \vartheta \in H^1(\Omega): \quad \int_\Omega \nabla\vartheta \cdot \nabla\zeta \, dx + \int_{\partial\Omega} \alpha_u \vartheta \zeta \, d\sigma = \int_{\partial\Omega} \alpha_u \vartheta_1 \zeta \, d\sigma \quad \forall \zeta \in H^1(\Omega), \tag{4.8}$$

which, by assumption (4.5), admits a unique solution ϑ. By the standard weak maximum principle (take $\zeta = (\vartheta - M)^+$ and $\zeta = (\vartheta + M)^-$ in (4.8)), we obtain (4.7), and the Hölder continuity up to the boundary follows from a well-known result of De Giorgi-Stampacchia, cf., e.g., [13, Thm. 7.1], and the references cited in the remark following this theorem.

From definition (4.6), clearly the map $H_0^1(\Omega) \ni u \mapsto [u]_\beta \in L^\infty(\partial\Omega)$ is continuous, hence the same can be said for $H_0^1(\Omega) \ni u \mapsto \alpha_u \in L^\infty(\partial\Omega)$. Consequently, the results on the continuous dependence of solutions to (4.8) on the data yields the continuity of the solution map $u \mapsto \vartheta$ for the $H_0^1(\Omega)$-topologies. Since the estimate of ϑ in $C^{0,\lambda}(\overline{\Omega})$ depends only on the L^∞-norms of ϑ_1 and α_u, we may use (4.4) and (4.5) to find that this estimate in fact is independent of u. Hence the compactness of the embedding $C^{0,\lambda}(\overline{\Omega}) \subset C^0(\overline{\Omega})$ yields the conclusion. $\square$

With the help of Lemma 2 and assumption (4.1), we may define the completely continuous operator

$$\Phi : H_0^1(\Omega) \to L_\nu^\infty(\Omega), \quad \Phi(u) = \gamma(\Theta(u)),$$

and formulate the thermoelastic torsion problem in terms of the following quasi-variational inequality in $H_0^1(\Omega)$:

$$\text{find} \quad u \in K(\Phi(u)): \quad \int_\Omega \nabla u \cdot \nabla(v-u)\, dx \geq -2\mu\theta \int_\Omega (v-u)\, dx \quad \forall v \in K(\Phi(u)), \tag{4.9}$$

where K is defined by (2.4) with $p=2$. Hence, as an application of Theorem 1(i), we obtain

Theorem 3 *Under the preceeding assumptions, namely (4.1), (4.4), and (4.5), there exists at least one solution of the thermoplastic quasi-variational inequality (4.9).*

Remark 4 The local elastoplastic state $[u]$ in (4.3) can be replaced by any other continuous operator from $H_0^1(\Omega)$ to $L^\infty(\partial\Omega)$, as is $[u]_\beta$ from (4.6). Moreover, due to the a priori estimate (4.7), the upper boundedness condition $\alpha(x,r) \leq \alpha^*$ from (4.5) also may be weakened to be of the form $\alpha(x,r) \leq \alpha_1(x)\alpha_2(r)$, where α_1 and α_2 are positive functions, respectively in $L^{1+\varepsilon}(\partial\Omega)$ and $C^0(\mathbb{R})$, by assumption of the De Giorgi-Stampacchia $C^{0,\lambda}$-estimate. Analogously, Theorem 3 remains valid, if we replace (4.3) by a mixed boundary condition

$$\frac{\partial\vartheta}{\partial n} = \alpha(x,[u])(\vartheta_1 - \vartheta) \text{ on } \partial\Omega_1 \quad \text{and} \quad \vartheta = \vartheta_2 \text{ on } \partial\Omega_2,$$

if $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$, and if ϑ_2 also satisfies (4.4).

5 An ionization problem in electrostatics

In this section we consider a quasi-variational extension of an ionization model, proposed by Duvaut and J.L. Lions, cf. [8, p. 329 ff.] and also [14]. Let $\Omega \subset \mathbb{R}^N$, $N=2$ or $N=3$, be an open and bounded domain with Lipschitz boundary $\partial\Omega = \Gamma_0' \cup \Gamma_1 \cup \Gamma_0''$ such that $\overline{\Gamma_0'} \cap \overline{\Gamma_0''} = \emptyset$, and $\Gamma_0', \Gamma_0'' \subset \partial\Omega$ are relatively open with positive $(N-1)$-dimensional Lebesgue measure.

The conservation of the electrical charge, in a stationary field, relates the electrical current $\mathbf{J} = \mathbf{J}(x)$, $x \in \Omega$, to the density of charge sources $\rho = \rho(x)$ by

$$\operatorname{div} \mathbf{J} = \rho \quad \text{in} \quad \Omega .$$

We consider the electric field $\mathbf{E} = \mathbf{E}(x)$ being given by a potential u through

$$\mathbf{E} = -\nabla u ,$$

and we assume a fixed potential difference between Γ_0' and Γ_0'', while Γ_1 (when $\Gamma_1 \neq \emptyset$) is insulated. This leads to the boundary conditions

$$u = u_0 \text{ on } \Gamma_0 = \Gamma_0' \cup \Gamma_0'' \quad \text{and} \quad \mathbf{J} \cdot \mathbf{n} = 0 \text{ on } \Gamma_1 , \tag{5.1}$$

with a nonconstant given function u_0, and $\mathbf{n}$ denoting the outer unit normal to $\partial\Omega$.

A main point is to introduce a suitable relation between the electric current $\mathbf{J}$ and the electric field $\mathbf{E}$. For stable media, Ohm's law suggests $\mathbf{J} = \sigma \mathbf{E}$ as being the right choice, where $\sigma > 0$ is the constant electric conductivity. If the medium is assumed to be ionizable, then the electric conductivity can no longer be assumed to be constant, and the relation between $\mathbf{J}$ and $\mathbf{E}$ often is taken to be of the form

$$\mathbf{J} = \begin{cases} \sigma \mathbf{E} & : \quad |\mathbf{E}| < E_0 \\ (\sigma + \lambda) \mathbf{E} & : \quad |\mathbf{E}| = E_0 \end{cases} , \tag{5.2}$$

with $\sigma > 0$, a Lagrange multiplier $\lambda \geq 0$, and a positive ionization threshold value $E_0 > 0$, cf. [8, Fig. 1, p. 329].

However, physical experiments suggest that (5.2) only may be considered as an approximation of the true ionization law, which is observed in practice to be of the following form, cf. [8, Fig. 2, p. 329],

– Fig. 2 [see at the end of the paper] –

and which shows a delay to the ionization, corresponding to a certain physical instability. We suggest here to describe this phenomenon by letting E_0 vary locally with the electrical energy $|\mathbf{E}|^2$ in a suitable neighbourhood of each point $x \in \overline{\Omega}$:

$$E_0 = \gamma([u](x)) \quad \text{with} \quad [u](x) = \int_{\Omega_\delta(x)} |\nabla u(y)|^2 \, dy , \quad x \in \overline{\Omega} , \tag{5.3}$$

where $\Omega_\delta(x) = \{ y \in \Omega : |y - x| < \delta \}$, cf. (4.6), $\delta > 0$ is fixed, and γ satisfies (4.1).

Hence we can formulate the stationary ionization problem with the ionization law from Figure 2 through the following quasi-variational inequality:

$$\text{find} \quad u \in K(u): \quad \sigma \int_\Omega \nabla u \cdot \nabla(v-u)\,dx \geq \int_\Omega \rho(v-u)\,dx \quad \forall v \in K(u), \tag{5.4}$$

which is still of the form (2.6), with $p=2$, where we let for each $w \in H^1(\Omega)$

$$K(w) = \{v \in H^1(\Omega) : v|_{\Gamma_0} = u_0\,,\ |\nabla v(x)| \leq \gamma([w](x)) \text{ a.e. in } \Omega\}.$$

Define $\Phi : H^1(\Omega) \to L^\infty_\nu(\Omega) \cap C^0(\overline{\Omega})$ through $\Phi(w) = \gamma([w])$. Then from (5.3) and from assumption (4.1) on γ it follows that Φ is well-defined, and since moreover

$$|[w](x) - [w](\bar{x})| = \int_{D_\delta(x,\bar{x})} |\nabla w(y)|^2\,dy, \quad x, \bar{x} \in \overline{\Omega},$$

with $D_\delta(x,\bar{x}) = (\Omega_\delta(x) \setminus \Omega_\delta(\bar{x})) \cup (\Omega_\delta(\bar{x}) \setminus \Omega_\delta(x))$, we obtain the continuity of Φ. Moreover, the Arzelà-Ascoli theorem implies that $\Phi : H^1(\Omega) \to C^0(\overline{\Omega})$ maps bounded sets to relatively compact sets.

Theorem 4 *Assume $\rho \in L^2(\Omega)$, $\sigma > 0$, and that $u_0 \in W^{1,\infty}(\Omega)$ is such that $\|\nabla u_0\|_{L^\infty(\Omega)} < \nu \leq \inf_{\mathbb{R}} \gamma$ for $\gamma \in C^0(\overline{\Omega})$. Then there exists at least one solution of (5.4).*

Proof: As in Section 3 we may translate the problem via $\hat{u} = u - u_0$ and consider the correspondingly shifted K to $\hat{K}$, since by assumption $u_0 \in K(w)$ for every $w \in H^1(\Omega)$. The a priori estimate for (5.4) follows from Poincaré's inequality, cf. [13, Thm. 5.8, p. 75], in the subspace $\{\hat{u} \in H^1(\Omega) : \hat{u}|_{\Gamma_0} = 0\}$. Consequently, the arguments of Section 2 can easily be adapted in the present case, due to the compactness of Φ defined above. $\square$

Remark 5 Again Theorem 4 can be extended to cover more general boundary conditions, e.g. the resistance condition on Γ_0'' in (5.1) may be replaced by an unknown constant; cf. [14] for the formulation.

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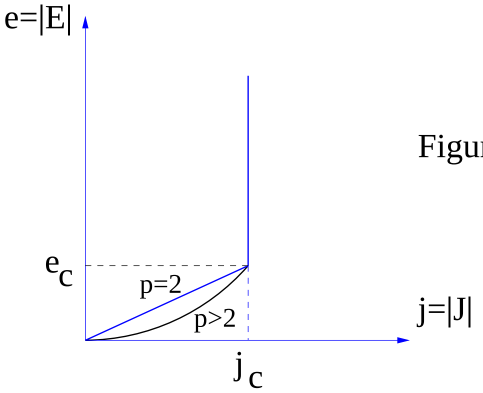

Figure 1

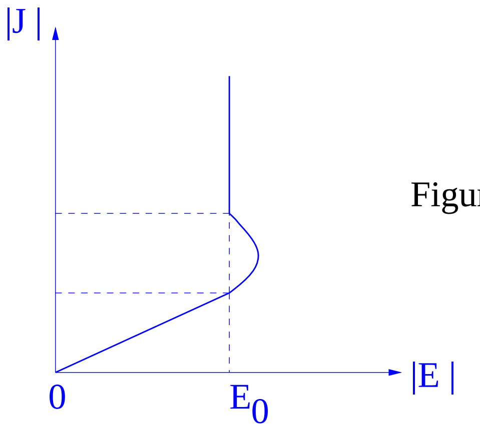

Figure 2