On Zeros of Semicondensing Maps

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1. Introduction

In [1] Schöneberg introduced a degree of mapping for so - called semicondensing vector fields which may be thought of to represent most kind of maps $F:\overline{\Omega}\to X$ where both condensing operators and operators of monotone type are involved in the definition of F, and $\Omega\subset X$ is a bounded open set. If we replace $\overline{\Omega}$ by a closed bounded convex $D\subset X$, and if a natural boundary condition at ∂D is satisfied (which is nothing but the weakly inward condition for $I-F$), one can still ask for conditions ensuring that F has a zero in D.
We start with some preliminaries in section 2. As a warm-up, in section 3 we prove the corresponding existence theorem in case $\mathring{D}\neq\emptyset$ by simply applying the Leray-Schauder-principle for the above mentioned degree. In general the domain D will suffer from $\mathring{D}=\emptyset$; in this case one tried (in similar situations) to use the relationship between results concerning differential equations in Banach spaces and results concerning the existence of zeros or fixed points (see e.g. [2], Example 24.12. or [3], Section 3). However, to prove the existence of solutions to the corresponding initial value problems, at present the uniform continuity of F is needed, a condition which seems to be unnatural and one wants to get rid of; see [4], p.155/156 for additional comments. Our Lemma 4.1. now shows that a different approach to the existence problem via a variational inequality (which is formulated in terms of the semi-inner products $(\cdot,\cdot)_\pm$) is possible. Due to the "bad" linearity and convexity properties of $(\cdot,\cdot)_\pm$ in the second argument we can solve this variational inequality only in the Hilbert space case where $(\cdot,\cdot)_\pm$ reduces to $\langle\cdot,\cdot\rangle$, the inner product in X. This will be the subject of section 4. Finally, in section 5 we discuss the existence of zeros of semicondensing maps in Banach spaces which have uniformly convex dual spaces X^* and admit a Galerkin projection scheme with some additional properties.
Our results apply, in particular, in case that $F=I-F_1-F_2$, where F_1 satisfies a certain compactness condition and F_2 satisfies a certain monotonicity condition; we need not to assume that F_1,F_2 or F is uniformly continuous.

2. Preliminaries

Throughout this paper X will denote an infinite dimensional Banach space with norm $|\cdot|$ and dual space X^*; $\langle\cdot,\cdot\rangle$ is the duality pairing between X and X^*. For a set $A\subset X$, we shall write $\operatorname{int}(A)=\mathring{A}$, $\operatorname{cl}(A)=\overline{A}$ and ∂A for the interior of A, the closure of A and the boundary of A, respectively. Furthermore, $\operatorname{dist}(x,A)=\inf_{y\in A}|x-y|$ denotes the distance of $x\in X$ to A, and we shall use

$$B_r(x_0)=\{x\in X: |x-x_0|<r\} \quad \text{and} \quad \overline{B}_r(x_0)=\overline{B_r(x_0)}$$

for $x_0\in X$ and $r>0$. We let $a\wedge b=\min\{a,b\}$ and $a\vee b=\max\{a,b\}$ denote the minimum resp. maximum of two reals a,b. For a bounded $B\subset X$,

$$\begin{aligned}\alpha(B) = \inf\{d>0: &\text{there is a finite system } B_1,\dots,B_n\subset X \\ &\text{with } B=\bigcup_{i=1}^{n} B_i \text{ and } \operatorname{diam} B_i\le d,\ 1\le i\le n\},\end{aligned}$$

is the Kuratowski measure of noncompactness. About α more than needed in the sequel may be found in Proposition 7.2. of [2].
We shall denote by $\mathcal{F}$ the (in general multivalued) duality mapping of X, where

$$\mathcal{F}x=\{x^*\in X^*: |x^*|=|x|,\ x^*(x)=|x|^2\}.$$

With the aid of $\mathcal{F}:X\to 2^{X^*}\setminus\{\emptyset\}$, the semi-inner products $(\cdot,\cdot)_\pm: X\times X\to\mathbb{R}$ are definied as

$$(x,y)_+=\max\{y^*(x): y^*\in\mathcal{F}y\} \quad \text{and} \quad (x,y)_-=\min\{y^*(x): y^*\in\mathcal{F}y\}$$

for $x,y\in X$. Some essential properties of $\mathcal{F}$ and $(\cdot,\cdot)_\pm$ are collected in Propositions 12.3. and 13.1. of [2].
Sometimes infinite dimensional problems can be reduced to finite-dimensional ones by means of Galerkin projection schemes $\{X_n,P_n\}_{n\in\mathbb{N}}$, where X_n is a finite dimensional subspace of X, $P_n: X\to X_n$ is a continuous linear projection onto X_n, and $\lim_{n\to\infty}P_n x=x$ for every $x\in X$. We will need

Lemma 2.1. *Let X be a Banach space with Galerkin projection scheme. Then*

(a) $B\subset X$ is relatively compact iff $\sup_{y\in B}|y-P_n y|\to 0$ as $n\to\infty$.

(b) If $\sup_{n\in\mathbb{N}}|P_n|=1$, then $(x,y)_-\le(P_n x,y)_-\le(x,y)_+$ for all $n\in\mathbb{N}$, $x\in X$ and $y\in X_n$.

(c) In case X^* is strictly convex $\mathcal{F}$ is single-valued. Therefore $(x,y)=\langle x,\mathcal{F}y\rangle=(x,y)_\pm$ implies $(x,y)=(P_n x,y)$ for all $n\in\mathbb{N}$, $x\in X$ and $y\in X_n$. Furthermore $(\cdot,\cdot)$ is linear in the first argument.

Proof. [2], Proposition 21.1. (a) and (c). □

To solve the equation $Fx=0$ in D, which is equivalent to $(I-F)x=x$, we shall impose a boundary condition at ∂D. For this purpose let us recall that $F:D\to X$ satisfies the subtangential condition (ST) at ∂D iff

$$\lim_{\lambda\to 0+} \lambda^{-1}\operatorname{dist}(x-\lambda Fx, D)=0 \text{ for all } x\in\partial D. \tag{ST}$$

In case D is closed convex, F satisfies (ST) iff $I-F$ is weakly inward; see, for example, Paragraph 18.3. in [2]. Now we turn to semicondensing vector fields. To reformulate Definition 1.1. from [1], let us define the class $\mathcal{M}=\{\omega:[0,\infty)\to[0,\infty):\omega$ is continuous and strictly increasing with $\omega(0)=0$ and $\omega(r)\to\infty$ as $r\to\infty\}$.

Definition 2.2. *Let $D\subset X$ be closed bounded convex. A continuous $F:D\to X$ is called semicondensing if there exist a continuous bounded map $G:D\times D\to X$ and an $\omega\in\mathcal{M}$ such that*

(a) $Fx=G(x,x)$ for $x\in D$;

(b) $\{G(\cdot,y):y\in D\}$ ist equicontinuous (pointwise);

(c) For all $A\subset D$ with $\alpha(A)>0$ there exist an ε with $0\le\varepsilon<\omega(\alpha(A))$ and a finite covering $A=\bigcup_{k=1}^{K} A_k$ such that

$$\omega(|y-\bar{y}|)|y-\bar{y}| \le (G(x,y)-G(\bar{x},\bar{y}),y-\bar{y})_+ + \varepsilon|y-\bar{y}|$$

holds for all $y,\bar{y}\in D$ and all $x,\bar{x}$ in the same A_k.

Let us remark that a lot about mappings having diagonal representations $Fx=G(x,x)$ of various types can be found in Paragraph 12 and 13 of [5]; see also [6].
The following example, which may be proved along the lines of Example 1.3. in [1] by choosing $G(x,y)=y-F_1x-F_2y$ and $\omega(r)=(1-k_2)r$, shows that the class of semicondensing maps covers, in particular, some better-known subclasses.

Example 2.3. *Let $D\subset X$ be closed bounded convex, $F_1,F_2:D\to X$ continuous and bounded. If there are $k_1\in[0,\infty)$ and $k_2\in\mathbb{R}$ such that $k_1+k_2<1$,*

(a) $\alpha(F_1B)\le k_1\alpha(B)$ for $B\subset D$, and

(b) $(F_2x-F_2y,x-y)_-\le k_2|x-y|^2$ for $x,y\in D$,

then $F=I-F_1-F_2$ is semicondensing.

Definition 2.2. implies in particular that in case $\overline{\Omega}=D$ for $\Omega=\mathring{D}\neq\emptyset$, a semicondensing map is semicondensing in the sense of Definition 1.1. in [1]. As a consequence, we can apply the degree theory for semicondensing vector fields. However, in case $\mathring{D}=\emptyset$ this degree theory cannot be used, and then we do not need condition (b). Furthermore, F need not have a diagonal representation $G:D\times D\to X$ as in (c). This leads to

Definition 2.4. *Let $D\subset X$ be closed bounded convex, and let $F:D\to X$ be continuous and bounded. F is called weakened semicondensing, if there is an $\omega\in\mathcal{M}$ such that the following holds: For all $A\subset D$ with $\alpha(A)>0$ there exist an ε with $0\le\varepsilon<\omega(\alpha(A))$ and a finite covering $A=\bigcup_{k=1}^{K} A_k$ such that*

$$\omega(|x-\bar{x}|)|x-\bar{x}| \le (Fx-F\bar{x},x-\bar{x})_+ + \varepsilon|x-\bar{x}| \tag{2.1}$$

for all $x,\bar{x}$ in the same A_k. In this case ω is called an estimate for F.

We want to close this section by proving the following result on (weakened) semicondensing maps which will be needed later on.

Lemma 2.5. *Let $D \subset X$ be closed bounded convex, $F : D \to X$ weakened semicondensing. Suppose that $(x_n)_{n\in\mathbb{N}} \subset D$ satisfies*

$$\forall\, \delta > 0\, \exists\, n_0 \in \mathbb{N}\, \forall\, n,m \in \mathbb{N} : n,m \geq n_0 \implies (Fx_n - Fx_m, x_n - x_m)_+ \leq \delta\,. \tag{2.2}$$

Then $\alpha(\{x_n : n \in \mathbb{N}\}) = 0$.

Proof. Suppose on the contrary that $\alpha(A) > 0$ for $A = \{x_n : n \in \mathbb{N}\} \subset D$. Choose an estimate $\omega \in \mathcal{M}$ for F, an ε with $0 \leq \varepsilon < \omega(\alpha(A))$ and a covering $A = \bigcup_{k=1}^{K} A_k$ of A in accordance with Definition 2.4. Fix $\delta > 0$ and choose the corresponding $n_0 \in \mathbb{N}$ from (2.2). We let $A_0 = \{x_n : n \geq n_0\} \subset D$. Notice that $\alpha(A_0) = \alpha(A) > 0$ and $A_0 = \bigcup_{k=1}^{K} A_k^0$ with $A_k^0 = A_k \cap A_0$. Furthermore, with $R = \operatorname{diam} D$, we find an $\eta \in (0, \delta \wedge \alpha(A_0)]$ such that

$$r,s \in [0,R],\, |\,r - s\,| \leq \eta \implies |\,r\,\omega(r) - s\,\omega(s)\,| \leq \delta\,. \tag{2.3}$$

By definition of $\alpha(A_0)$ we have $A_0 = \bigcup_{l=1}^{L} B_l$ with $B_l \subset A_0$ such that $\operatorname{diam} B_l \leq \alpha(A_0) + \eta$. Therefore $A_0 = \bigcup_{(k,l)\in I} (A_k^0 \cap B_l)$ where $I = \{1,\dots,K\} \times \{1,\dots,L\}$. Since I is finite, we find a $(k_0,l_0) \in I$ with $\operatorname{diam}(A_{k_0}^0 \cap B_{l_0}) \geq \alpha(A_0) > \alpha(A_0) - \eta$, hence $x, \bar{x} \in A_{k_0}^0 \cap B_{l_0}$ such that $|\,x - \bar{x}\,| \geq \alpha(A_0) - \eta$. This in turn implies that (2.1) can be applied to x and $\bar{x}$ because $x, \bar{x} \in A_{k_0}^0 \subset A_{k_0}$. Furthermore we get $|\,x - \bar{x}\,| \leq \operatorname{diam} B_{l_0} \leq \alpha(A_0) + \eta$, and therefore $|\,\alpha(A_0) - |\,x - \bar{x}\,|\,| \leq \eta$. So, using (2.3), (2.1) and (2.2), because of $x = x_n$ and $\bar{x} = x_m$ for some $n,m \geq n_0$, we finally arrive at

$$\begin{aligned}
&\alpha(A)\,\omega(\alpha(A)) = \alpha(A_0)\,\omega(\alpha(A_0)) \\
&\leq \Big|\alpha(A_0)\,\omega(\alpha(A_0)) - |\,x - \bar{x}\,|\,\omega(|\,x - \bar{x}\,|)\Big| + |\,x - \bar{x}\,|\,\omega(|\,x - \bar{x}\,|) \\
&\leq \delta + (Fx - F\bar{x}, x - \bar{x})_+ + \varepsilon\,|\,x - \bar{x}\,| \\
&\leq 2\delta + \varepsilon\,(\,\alpha(A_0) + \eta\,) \leq 2\delta + \varepsilon\,(\,\alpha(A) + \delta\,)\,.
\end{aligned}$$

Since δ has been arbitrary, $\alpha(A) > 0$ yields the contradiction $\omega(\alpha(A)) \leq \varepsilon$. $\square$

3. The Case $\operatorname{int}(D) \neq \emptyset$

Let us start with a simple but satisfactory theorem which implicitely can be found in [4], Theorem 2, or [1], Corollary 3.3.

Theorem 3.1. *Let $D \subset X$ be closed bounded convex with $\mathring{D} \neq \emptyset$, $F : D \to X$ semicondensing and satisfying (ST) at ∂D. Then F has a zero.*

Proof. Let $\Omega = \mathring{D} \neq \emptyset$ and notice that $\overline{\Omega} = D$. Because $F : \overline{\Omega} \to X$ is semicondensing in the sense of Definition 1.1. in [1] we only have to verify that F satisfies the Leray-Schauder boundary condition

$$\exists x_0 \in \Omega\, \forall x \in \partial\Omega\, \forall \lambda > 0 : Fx + \lambda(x - x_0) \neq 0\,. \tag{LS}$$

on $\partial\Omega = \partial D$. Since $F_0 = I - F$ is weakly inward, Exercise 18.4. in [2] implies

$$\exists x_0 \in \Omega \, \forall x \in \partial\Omega \, \forall \mu > 1 : F_0 x - x_0 \neq \mu(x - x_0)$$

where the $x_0 \in \Omega$ may be chosen arbitrary. This yields (LS) for every $x_0 \in \Omega$ and completes the proof, because Theorem 2.4. from [1] can be applied to $H : [0,1] \times \overline{\Omega} \to X, (t,x) \mapsto tFx + (1-t)(x - x_0)$. □

In case $D = K_r$ is a conical segment this yields a simple generalization of Theorem 20.4. in [2] which is very useful for the investigation of integral equations in $X = C([0,1])$; see [7].

Theorem 3.2. *Let $K \subset X$ be a cone (i.e. a closed convex set such that $\lambda K \subset K$ for all $\lambda \geq 0$ and $K \cap (-K) = \{0\}$) with $\mathring{K} \neq \emptyset$. Fix $r > 0$ and suppose that the semicondensing $F : K_r = K \cap \overline{B}_r(0) \to X$ satisfies the boundary conditions*

(a) $x \in K, |x| = r, Fx + \lambda x = 0 \Longrightarrow \lambda \leq 0$;

(b) $x \in \partial K, |x| \leq r, x^* \in K^*, x^*(x) = 0 \Longrightarrow x^*(Fx) \leq 0$.

Then F has a zero.

Proof. W.l.o.g. we can assume that "$[\ldots] \Rightarrow \lambda < 0$" holds in (a). The proof of Theorem 20.4. in [2] shows that in case $F : \overline{\mathring{K}_r} = K_r \to X$ does not satisfy (LS) we can find $(x_n)_{n \in \mathbb{N}} \subset K_r$ with $|x_n| = r$ for $n \in \mathbb{N}$ and some $\lambda_0 \in [0,\infty)$ such that $(F + \lambda_0 I)(x_n) \to 0$ holds. By Theorem 1.5. in [1], $(F + \lambda_0 I) : \overline{\mathring{K}_r} \to X$ is semicondensing too, and therefore proper. Hence we may assume $x_n \to x_0$ for some $x_0 \in K_r$ with $|x_0| = r$. But this implies $Fx_0 + \lambda_0 x_0 = 0$, contradicting assumption (a). □

Sometimes it is useful to find zeros $x \neq 0$ because $F(0) = 0$ is already known. Then one can try to apply the following result which generalizes Theorem 2 in [8].

Theorem 3.3. *Let $K \subset X$ be a cone with $\mathring{K} \neq \emptyset$. Fix $r > 0$ and let $F : K_r \to X$ be a semicondensing map satisfying (a) and (b) from Theorem 3.2.. Suppose there exist $e \in K \setminus \{0\}$ and $\varrho \in (0,r)$ such that*

(c) $|x| = \varrho, Fx + \lambda e = 0 \Longrightarrow \lambda \geq 0$.

Then F has a zero in $\{x \in K : \varrho \leq |x| \leq r\}$.

Proof. The proof is mainly the same as the one given for Theorem 2 in [8]. For we can define semicondensing maps by

$$F_n : \overline{\mathring{K}_r} = K_r \to X, \quad x \mapsto Fx - \varphi_n(|x|)e$$

with φ_n from the cited theorem. Then application of Theorem 3.2. to F_n yields the existence of a zero $x_n \in K_r$ of F_n with $|x_n| > \varrho - 1/n$ and w.l.o.g. $|x_n| < \varrho$. Furthermore, by passing to a subsequence if necessary, we find a

$$\lambda_0 = -\lim_{n \to \infty} \varphi_n(|x_n|) \in (-\infty, 0] \quad \text{with} \quad Fx_n + \lambda_0 e \to 0.$$

Using assumption (c) the claim follows. □

4. The Hilbert Space Case

We start with a general observation concerning zeros of maps satisfying (ST).

Lemma 4.1. *Let X be a Banach space, $D\subset X$ closed bounded convex, $F:D\to X$ such that (ST) holds. Then the following five statements are equivalent:*

(a) F has a zero

(b) $\exists x_0\in D\forall x\in D:(Fx_0,x_0-x)_+\le 0$; (c)$\exists x_0\in D\forall x\in D:(Fx_0,x_0-x)_-\le 0$;

(d) $\exists x_0\in D\forall x\in D:(x_0-x,Fx_0)_+\le 0$; (e)$\exists x_0\in D\forall x\in D:(x_0-x,Fx_0)_-\le 0$.

Proof. Clearly the implications $(a)\Rightarrow(b)$, $(b)\Rightarrow(c)$, $(a)\Rightarrow(d)$ and $(d)\Rightarrow(e)$ are valid. To prove $(c)\Rightarrow(a)$ we note that $\lim_{\lambda\to 0^+}\lambda^{-1}\varrho(x_0-\lambda Fx_0,D)=0$ because of (ST). Therefore Proposition 26.1. in [2] shows that there exist $\delta>0$ and a function $v:[0,\delta]\to X$ such that $w(t)=x_0-tFx_0+v(t)\in D$ for $t\in[0,\delta]$ as well as $v(t)/t\to 0$ as $t\to 0^+$. Hence (c) with $x=w(t)$ implies

$$0\ge(Fx_0,x_0-w(t))_-=t^{-1}(|x_0-w(t)|^2+(v(t),x_0-w(t))_-)\,.$$

We obtain

$$|x_0-w(t)|^2\le-(v(t),x_0-w(t))_-\le|v(t)|\,|x_0-w(t)|\,,$$

and therefore

$$|Fx_0|^2=t^{-2}|[x_0-w(t)]+v(t)|^2\le 4\left|\frac{v(t)}{t}\right|^2\to 0\quad\text{as}\quad t\to 0^+,$$

hence $(a)\iff(b)\iff(c)$. The proof of $(e)\Rightarrow(a)$ is similar. □

To solve such a variational inequality, at least when X is a Hilbert space, we shall need the following particular case of a theorem by F.E. Browder; see Theorem 32.C. in [9], for example.

Lemma 4.2. *Let X be a reflexive Banach space, $D\subset X$ closed bounded convex, $T:D\to X^*$ pseudomonotone demicontinuous and bounded. Then there is an $x_0\in D$ such that $(Tx_0)(x_0-x)\le 0$ for all $x\in D$.*

Here $T:D\to X^*$ is called pseudomonotone (see (H3) on p.867 of [9]), if $(x_n)_{n\in\mathbb{N}}\subset D$, $x_n\rightharpoonup x$ and $\limsup_{n\to\infty}(Tx_n)(x_n-x)\le 0$ imply $(Tx)(x-y)\le\liminf_{n\to\infty}(Tx_n)(x_n-y)$ for all $y\in D$. T is called demicontinous (see Definition 26.1. on p.554 of [9]), if $(x_n)_{n\in\mathbb{N}}\subset D$ and $x_n\to x$ imply $Tx_n\rightharpoonup^* Tx$. Because X is assumed to be reflexive, the weak-$*$-convergence may be replaced by the weak convergence in X^*.

After this preparation we can prove our main result.

Theorem 4.3. *Let X be a Hilbert space, $D\subset X$ closed bounded convex, $F:D\to X$ weakened semicondensing and satisfying (ST) at ∂D. Then F has a zero.*

Proof. At first, $(\cdot,\cdot)_+ = (\cdot,\cdot)_- = \langle\cdot,\cdot\rangle$, if the inner product in X is denoted by $\langle\cdot,\cdot\rangle$. Fix a sequence $(x_n)_{n\in\mathbb{N}} \subset D$ with $x_n \rightharpoonup x \in X$ and $\limsup_{n\to\infty}\langle Fx_n, x_n - x\rangle \leq 0$. To prove the theorem by using Lemma 4.1. and by applying Lemma 4.2. to $T: D \to X^*$, $x \mapsto \langle Fx, \cdot\rangle$, it is clearly enough to prove that even $x_n \to x$ is true. Now $x_n \rightharpoonup x \in D$ shows that we only have to make sure that every subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence of its own. Because the limsup-condition is satisfied for subsequences, we may use indices $n \in \mathbb{N}$ for simplification. Furthermore, since X is reflexive and D and F are bounded we may assume $Fx_n \rightharpoonup z$ for some $z \in X$. Let $(\varepsilon_n)_{n\in\mathbb{N}} \subset (0,1]$ be decreasing to zero. By passing to a further subsequence, we also can assume that

$$\forall\, n,m \in \mathbb{N} : n > m \Longrightarrow \langle z - Fx_n, x_m\rangle + \langle Fx_m, x - x_n\rangle \leq \varepsilon_m \,. \tag{4.4}$$

Now the assumptions of Lemma 2.5. are satisfied for $(x_n)_{n\in\mathbb{N}} \subset D$. Indeed, let $\delta > 0$ be given. Because of $\limsup_{n\to\infty}\langle Fx_n, x_n - x\rangle < \delta/4$, $\varepsilon_n \to 0$, $x_n \rightharpoonup x$ and $Fx_n \rightharpoonup z$, we find an $n_0 \in \mathbb{N}$ such that

$$\sup_{n\geq n_0}\langle Fx_n, x_n - x\rangle \leq \delta/4 \tag{4.5}$$

as well as

$$\varepsilon_n \leq \delta/4 \quad \text{and} \quad \langle z, x - x_m\rangle + \langle Fx_n - z, x\rangle \leq \delta/4 \quad \text{for} \quad n,m \geq n_0 \,. \tag{4.6}$$

Fix $n,m \geq n_0$ with (w.l.o.g.) $n > m$. Then (4.5), (4.6) and (4.4) yield

$$\begin{aligned}
&\langle Fx_n - Fx_m, x_n - x_m\rangle \\
&= \langle Fx_n, x_n - x\rangle + \langle Fx_n, x - x_m\rangle + \langle Fx_m, x_m - x\rangle + \langle Fx_m, x - x_n\rangle \\
&\leq \delta/2 + \langle z, x - x_m\rangle + \langle Fx_n - z, x\rangle + \langle z - Fx_n, x_m\rangle + \langle Fx_m, x - x_n\rangle \\
&\leq \delta/2 + \delta/4 + \varepsilon_m \leq \delta \,.
\end{aligned}$$

Therefore Lemma 2.5. shows that $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence. $\square$

Even for the (weakened) semicondensing maps considered in Example 2.3. this gives something new, namely the following partial answer to Problem 2 in [4].

Corollary 4.4. *Let X be a Hilbert space, $D \subset X$ closed bounded convex, $F_1, F_2 : D \to X$ continuous and bounded. Let $F_1 + F_2$ be weakly inward and suppose that there are $k_1 \in [0,\infty)$ and $k_2 \in \mathbb{R}$ with $k_1 + k_2 < 1$ such that (a), (b) from Example 2.3. are satisfied. Then $F_1 + F_2$ has a fixed point.*

5. Existence in Spaces with Special Galerkin Schemes

In this section we investigate the existence of zeros in Banach spaces which admit a Galerkin scheme with some additional properties. To some extent this approach is suggested by Example 1 of [4].

Theorem 5.1. *Let X be a Banach space with Galerkin projection scheme $\{X_n, P_n\}_{n\in\mathbb{N}}$, $D \subset X$ closed bounded convex with $0 \in D$, $F : D \to X$ weakened semicondensing and satisfying (ST) at ∂D. Furthermore, suppose that*

(1) X^* is uniformly convex and $x_n \rightharpoonup x$ implies $\mathcal{F}(x_n) \rightharpoonup^* \mathcal{F}(x)$;

(2) $X_n \subset X_{n+1}$, $P_n(D) \subset D$ for all $n \in \mathbb{N}$ and $\sup_{n\in\mathbb{N}} |P_n| = 1$.

Then F has a zero.

Proof. 1. Let $D_n = D \cap X_n$. Since $(P_n F)|_{D_n}$ is compact and satisfies (ST) at $\partial_{X_n} D_n$, the boundary of D_n in X_n, we may apply Lemma 18.4. in [2] to get $P_n F x_n = 0$ for some $x_n \in D_n$. Therefore, by means of these x_n we also find a zero of F if we are able to show that F is A-proper with respect to the Galerkin scheme $\{X_n, P_n\}$, as was introduced by Petryshin. This means that given any sequence $(x_n) \subset D$ with $x_n \in D_n$ for all $n \in \mathbb{N}$, every subsequence $(x_{n'})$ satisfying $P_{n'} F x_{n'} \to y$ for some $y \in X$ has a convergent subsequence with limit x satisfying $Fx = y$; see Definition 1.1 B on p.227 of [10].
2. To prove that F is A-proper the subsequence according to the above definition again will be indexed by $n \in \mathbb{N}$. Furthermore we can assume $x_n \rightharpoonup x$ for some $x \in D$. The estimate

$$|Fx - y| \le |Fx - Fx_n| + |Fx_n - P_n F x_n| + |P_n F x_n - y|$$

and Lemma 2.1.(a) show that it is again enough to find a convergent subsequence of (x_n), and using Lemma 2.5. this reduces to finding a subsequence satisfying (2.2). For fix a sequence $(\varepsilon_n)_{n\in\mathbb{N}} \subset (0,1]$ decreasing to zero. Let $C = (\sup_{z\in D} |Fz|) \vee |y| \vee 2(\sup_{z\in D} |z|)$. Because X^* is assumed to be uniformly convex, the semi-inner product $(\cdot,\cdot) : \overline{B}_C(0) \times \overline{B}_C(0) \to \mathbb{R}$ is uniformly continuous. Therefore the modulus of continuity ϱ of $(\cdot,\cdot)$, defined as $\varrho(d) = \sup\{ |(w,z) - (\bar{w},\bar{z})| : w,\bar{w},z,\bar{z} \in \overline{B}_C(0),\, |w-\bar{w}|, |z-\bar{z}| \le d\}$ for $d \ge 0$, is continuous and increasing in $\mathbb{R}_0^+$ with $\varrho(0) = 0$. Hence, because of $P_n x \to x$ and $P_n F x_n \to y$ there exists an $n_1 \in \mathbb{N}$ such that

$$n \ge n_1 \Longrightarrow \varrho(|x - P_n x|) \le \varepsilon_1/2 \quad \text{and} \quad \varrho(|y - P_n F x_n|) \le \varepsilon_1/4 . \tag{5.7}$$

Lemma 2.1.(c), $\mathcal{F}(-z) = -\mathcal{F}(z)$, the definition of ϱ, and (5.7) imply

$$\begin{aligned}
&(Fx_n - Fx_{n_1}, x_n - x_{n_1}) \\
&= [(y, x - x_{n_1}) - (y, P_{n_1} x - x_{n_1})] + [(y, x_n - x_{n_1}) - (y, x - x_{n_1})] \\
&\quad [(P_n F x_n, x_n - x_{n_1}) - (y, x_n - x_{n_1})] + [(y, x_{n_1} - P_{n_1} x_n) - (y, x_{n_1} - P_{n_1} x)] \\
&\quad + [(P_{n_1} F x_{n_1}, x_{n_1} - P_{n_1} x_n) - (y, x_{n_1} - P_{n_1} x_n)] \\
&\quad + [(F x_{n_1}, x_{n_1} - x_n) - (F x_{n_1}, x_{n_1} - x)] \\
&\quad + [(F x_{n_1}, x_{n_1} - x) - (F x_{n_1}, x_{n_1} - P_{n_1} x)] \\
&\quad + [(F x_{n_1}, x_{n_1} - P_{n_1} x) - (F x_{n_1}, x_{n_1} - P_{n_1} x_n)] \\
&\le 2\,\varrho(|P_{n_1} x_n - P_{n_1} x|) + |(y, x_n - x_{n_1}) - (y, x - x_{n_1})| \\
&\quad + |(F x_{n_1}, x_n - x_{n_1}) - (F x_{n_1}, x - x_{n_1})| + 3\varepsilon_1/2
\end{aligned}$$

for all $n \ge n_1$. Now P_{n_1} has finite dimensional range X_{n_1} and thus is a completely continuous operator. Hence $x_n \rightharpoonup x$ yields $\varrho(|P_{n_1} x_n - P_{n_1} x|) \to 0$, and we have $\mathcal{F}(x_n - x_{n_1}) \rightharpoonup^* \mathcal{F}(x - x_{n_1})$ as $n \to \infty$. Therefore we find an $n_2 > n_1$ such that

$$\begin{aligned}
n \ge n_2 \Longrightarrow (Fx_n - Fx_{n_1}, x_n - x_{n_1}) \le 3\varepsilon_1, \quad &\varrho(|x - P_n x|) \le \varepsilon_2/2 \\
\text{and} \quad &\varrho(|y - P_n F x_n|) \le \varepsilon_2/4 .
\end{aligned}$$

Iterating the process beginning with (5.7), we clearly obtain a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ satisfying the hypotheses of Lemma 2.5. Therefore the proof is complete. $\square$

Theorem 5.1. and Example 2.3. show that the answer to open problem 2 in [4] is yes in the cases described by the following corollary. In particular, this applies when $X = l_p$ with $p \in (1,\infty)$ and $D = l_p^+ \cap \overline{B}_r(0)$ for some $r > 0$; see Exercise 4 on p.122 of [2]. Here $l_p^+ = \{(x_n)_{n\in\mathbb{N}} \in l_p : x_n \geq 0 \text{ for } n \in \mathbb{N}\}$.

Corollary 5.2. *Let X be a Banach space with Galerkin projection scheme $\{X_n, P_n\}_{n\in\mathbb{N}}$, $D \subset X$ closed bounded convex with $0 \in D$, $F_1, F_2 : D \to X$ continuous and bounded. Let $F_1 + F_2$ be weakly inward and suppose that there are $k_1 \in [0,\infty)$ and $k_2 \in \mathbb{R}$ with $k_1 + k_2 < 1$ such that (a), (b) from Example 2.3. are satisfied. Finally, suppose that (1), (2) from Theorem 5.1. hold. Then $F_1 + F_2$ has a fixed point.*

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