Radiation Reaction and Center Manifolds

Markus Kunze

Mathematisches Institut der Universität Köln
Weyertal 86, D-50931 Köln, Germany
email: mkunze@mi.uni-koeln.de

Herbert Spohn

Zentrum Mathematik and Physik Department, TU München
D-80290 München, Germany
email: spohn@mathematik.tu-muenchen.de

Abstract

We study the effective dynamics of a mechanical particle coupled to a wave field and subject to the slowly varying potential $V(\varepsilon q)$ with ε small. To lowest order in ε the motion of the particle is governed by an effective Hamiltonian. In the next order one obtains "dissipative" terms which describe the radiation reaction. We establish that this dissipative dynamics has a center manifold which is repulsive in the normal direction and which is global, in the sense that for given data and sufficiently small ε the solution stays on the center manifold forever. We prove that the solution of the full system is well approximated by the dissipative effective dynamics on its center manifold.

1 Introduction

At the beginning of this century, in the context of the Maxwell-Lorentz equations, radiation reaction was one of the most outstanding problems in theoretical physics. It was left sort of unfinished when theoreticians turned to quantum electrodynamics. In this paper we study radiation reaction in the mathematically somewhat more accessible case of a scalar wave field. We believe that our results provide good indications on the effective dynamics for a charge coupled to the Maxwell field [12].

To explain in more detail the physical context we have to set up the model first. We consider a particle, position $q(t) \in \mathbb{R}^3$ and momentum $p(t) \in \mathbb{R}^3$, with "charge" distribution ρ of total charge

$$\mathsf{e} = \int d^3x \rho(x) \neq 0.$$

We require that ρ is smooth, radial, and supported in a ball of radius R_ρ,

$$\rho \in C_0^\infty(\mathbb{R}^3)\,, \quad \rho(x) = \rho_r(|x|)\,, \quad \rho(x) = 0 \quad \text{for} \quad |x| \geq R_\rho. \tag{C}$$

The particle is coupled to the scalar wave field $\phi(x,t)$ with the canonically conjugate momentum field $\pi(x,t)$, $x \in \mathbb{R}^3$. In addition the particle is subject to an external potential, V, whose properties will be listed below. We assume that the potential is *slowly varying* on the scale of the charge distribution, i.e., on the scale set by R_ρ. Formally we introduce the dimensionless parameter ε, $\varepsilon \ll 1$, and consider the scale of potentials $V(\varepsilon q)$, $\varepsilon \to 0$. The equations of motion for the coupled system are

$$\begin{aligned} &\dot\phi(x,t) = \pi(x,t), \qquad \dot\pi(x,t) = \Delta\phi(x,t) - \rho(x-q(t)), \\ &\dot q(t) = \frac{p(t)}{\sqrt{1+p(t)^2}}, \quad \dot p(t) = -\varepsilon\nabla V(\varepsilon q(t)) + \int d^3x\, \phi(x,t)\nabla\rho(x-q(t))\,. \end{aligned} \tag{1.1}$$

The dynamics governed by (1.1) has three distinct time scales, well-separated as $\varepsilon \to 0$. On the *microscopic* time scale, $t = \mathcal{O}(1)$, the particle moves along an essentially straight line and the field adjusts itself stationarily. On a time scale $\mathcal{O}(\varepsilon^{-1})$, that we call the *macroscopic* scale, the particle feels the potential and responds to it with an effective kinetic energy which incorporates the coupling to the field. This scale was studied in [5]. The particle looses energy through radiation at a rate roughly proportional to $\ddot q(t)^2$. Thus on the macroscopic time scale, friction through radiation is of order ε. To resolve such an effect we have to go to even longer times or to look with higher precision. The *friction* time scale is the subject of our paper.

The dynamics of (1.1) is of Hamiltonian form. We need a few facts in the case the external potential vanishes, $V = 0$. Then (1.1) has the energy

$$\mathcal{H}_0(\phi,\pi,q,p) = (1+p^2)^{1/2} + \frac{1}{2}\int d^3x\,(|\pi(x)|^2 + |\nabla\phi(x)|^2) + \int d^3x\,\phi(x)\rho(x-q)$$

and the conserved total momentum

$$\mathcal{P}(\phi,\pi,q,p) = p + \int d^3x\,\phi(x)\nabla\pi(x).$$

The minimum of $\mathcal{H}_0$, at fixed $\mathcal{P}$, is attained at

$$S_{q,v} = (\phi_v(x-q), \pi_v(x-q), q, p_v) \tag{1.2}$$

where $v \in \mathcal{V} = \{v : |v| < 1\}$, $p_v = v/\sqrt{1-v^2}$, $\pi_v = -v \cdot \nabla \phi_v$, and $\hat{\phi}_v(k) = -\hat{\rho}(k)/[k^2 - (v \cdot k)^2]$; the hat denotes Fourier transform. We call $S_{q,v}$ the soliton centered at q, v. It has the normalized energy

$$\mathcal{E}_s(v) = \mathcal{H}_0(S_{q,v}) - \mathcal{H}_0(S_{q,0}) = (1-v^2)^{-1/2} - 1 + 3m_e \left[\frac{2-v^2}{2(1-v^2)} - \frac{1}{2|v|} \log \frac{1+|v|}{1-|v|} \right]$$

and the total momentum

$$\mathcal{P}_s(v) = \mathcal{P}(S_{q,v}) = v(1-v^2)^{-1/2} + 3m_e v \left[\frac{1}{2v^2(1-v^2)} - \frac{1}{4|v|^3} \log \frac{1+|v|}{1-|v|} \right]. \tag{1.3}$$

Here $m_e = \frac{1}{3} \int d^3k \, |\hat{\rho}(k)|^2 k^{-2}$ is the mass of the particle due to the coupling to the field. We note that because of the Hamiltonian structure we have the identity $v(d\mathcal{P}_s/dv) = (d\mathcal{E}_s/dv)$. It is shown in [6] that the map $v \mapsto \mathcal{P}_s(v)$ is invertible from $\mathcal{V}$ to $\mathbb{R}^3$, with inverse $P \mapsto v(P)$, and thus $E(P) := \mathcal{E}_s(v(P))$ is well defined.

Taking $S_{q,v}$ as initial conditions for (1.1) with $V = 0$ we obtain a solution travelling at constant velocity v,

$$S_{q,v}(t) = (\phi_v(x - q - vt), \pi_v(x - q - vt), q + vt, p_v) , \quad v \in \mathcal{V}.$$

Let us call $\{S_{q,v} : q \in \mathbb{R}^3, v \in \mathcal{V}\}$ the six–dimensional soliton manifold, $\mathcal{S}$. Thus, for $V = 0$, if we start initially on $\mathcal{S}$ the solution remains on $\mathcal{S}$ and moves along the straight line $t \mapsto q^0 + v^0 t$. In fact, if we start close to $\mathcal{S}$, then $\mathcal{S}$ is approached asymptotically, [6]. When the particle is subject to a slowly varying external potential, then the rough picture is that the solution will remain close to $\mathcal{S}$ in the course of time. For simplicity we assume throughout that the initial datum for (1.1) lies exactly on $\mathcal{S}$, i.e.,

$$(\phi(0), \pi(0), q(0), p(0)) = S_{q^0,v^0} , \tag{1.4}$$

possible generalizations being discussed below.

At this point it is instructive to transform (1.1) to the macroscopic space-time scale in such a way that the field energy remains constant. Then the macroscopic variables, denoted by a $'$, are

$$t = \varepsilon^{-1} t' , \quad q = \varepsilon^{-1} q' , \quad x = \varepsilon^{-1} x' , \quad q(t) = \varepsilon^{-1} q'(t') , \quad \text{and} \quad \phi(x,t) = \sqrt{\varepsilon}\phi'(x',t').$$

We also set

$$\rho_\varepsilon(x) = \varepsilon^{-3} \rho(\varepsilon^{-1} x) .$$

In particular, $\rho_\varepsilon(x) = 0$ for $|x| \geq \varepsilon R_\rho$ and $\int d^3x \rho_\varepsilon(x) = \int d^3x \rho(x)$. With this convention, omitting the primes and indicating explicitly the ε-dependence of $q'(t')$, we arrive at

$$\begin{gathered} \ddot{\phi}(x,t) = \Delta\phi(x,t) - \sqrt{\varepsilon}\rho_\varepsilon(x - q^\varepsilon(t)), \qquad (1.5) \\ \dot{q}^\varepsilon(t) = v^\varepsilon(t) , \quad m_0(v^\varepsilon(t))\dot{v}^\varepsilon(t) = -\nabla V(q(t)) + \sqrt{\varepsilon} \int d^3x \, \phi(x,t) \nabla \rho_\varepsilon(x - q^\varepsilon(t)). \end{gathered}$$

Here $m_0(v)$ is the 3×3 matrix defined through $m_0(v)\dot{v} = \gamma\dot{v} + \gamma^3(v \cdot \dot{v})v$ with $\gamma(v) = 1/\sqrt{1-v^2}$. Rather than momenta as in (1.1), we use velocities which turns out to be more convenient in our context. The initial soliton (1.4) transforms to

$$S^\varepsilon_{q^0,v^0} = (\phi^\varepsilon_{v^0}(x - q^0), \pi^\varepsilon_{v^0}(x - q^0), q^0, v^0), \tag{1.6}$$

where $\hat{\phi}_v^\varepsilon(k) = -\sqrt{\varepsilon}\hat{\rho}(\varepsilon k)/[k^2 - (v\cdot k)^2]$ and $\pi_v^\varepsilon = -v\cdot\nabla\phi_v^\varepsilon$. Thus, on the macroscopic scale, the total charge is $\sqrt{\varepsilon}\int d^3x\rho(x)$, whereas $m_e = \frac{1}{3}\varepsilon\int d^3k|\hat{\rho}_\varepsilon(k)|^2k^{-2}$ is independent of ε. Eqs. (1.5) are again of Hamiltonian form. The energy

$$\mathcal{H}_{mac}(\phi,\pi,q,v) = \gamma(v) + V(q) + \frac{1}{2}\int d^3x\,(|\pi(x)|^2 + |\nabla\phi(x)|^2) + \sqrt{\varepsilon}\int d^3x\,\phi(x)\rho_\varepsilon(x-q) \quad (1.7)$$

is conserved under (1.5). It is bounded from below, as $\mathcal{H}_{mac}(\phi,\pi,q,v) \geq V(q) - 3m_e$ independently of ε.

There is another, very instructive way to think about the initial value problem (1.5), (1.6). We prescribe initial data at $t=-\tau$, $\tau>0$, which have finite energy and some smoothness. We refer to [6] for the precise conditions. We solve (1.5) for $V=0$ up to time $t=0$. Then in the limit $\tau\to\infty$ the data at $t=0$ are exactly of the form (1.6). For $t>0$ the external forces are acting. Clearly this causes some mismatch, which is reflected by a non–smoothness of the fields (ϕ,π) at the light cone $\{x:|x|=t, t>0\}$ in the limit $\varepsilon\to 0$.

Under suitable assumptions on V and for $|\rho|_{L^2}$ sufficiently small we proved in [5] that

$$|\dot{q}^\varepsilon(t)| \leq \bar{v} < 1\,, \quad |\ddot{q}^\varepsilon(t)| \leq C\,, \quad \text{and} \quad |\dddot{q}^\varepsilon(t)| \leq C \quad (1.8)$$

uniformly in ε and $t\in\mathbb{R}$, and that the limit

$$\lim_{\varepsilon\to 0^+} q^\varepsilon(t) = r(t) \quad (1.9)$$

exists. Here $r(t)$ is the solution of Hamilton's equations of motion with the effective Hamiltonian $E(p)+V(q)$, cf. the definition of $E(p)$ below (1.3), which in terms of velocities read

$$\dot{r} = u\,, \quad m(u)\dot{u} = -\nabla V(r), \quad (1.10)$$

with initial data $r(0)=q^0$, $u(0)=v^0$. Here $m(u)=m_0(u)+m_f(u)$, where $m_f(u)$ is the additional "mass" due to the coupling to the field defined by

$$m_f(u)\dot{u} = 3m_e\Big(\varphi(|u|)\dot{u} + |u|^{-1}\varphi'(|u|)(u\cdot\dot{u})u\Big) \quad (1.11)$$

as a 3×3 matrix, where $\varphi(|v|)$ is the function appearing in the square brackets of Eq. (1.3). Note that the energy

$$H(r,u) = \mathcal{E}_s(u) + V(r) \quad (1.12)$$

is conserved by the solutions to (1.10).

With this background information let us return to the radiation reaction as discussed by Abraham, Lorentz, Schott, and Dirac, cf. [14] for an excellent account. Of course, these theoretical physicists were interested in the electrodynamics of moving charges. We take here the liberty to transcribe their arguments to the case of a scalar wave equation. For the sake of discussion we reintroduce the bare mass m_0 and state the equations for small velocities only. In our proof below, however, we will handle all $v\in\mathcal{V}$.

At the beginning of this century the hope was to define a structureless elementary charge through a point charge limit. For this program, one had to model the charge distribution phenomenologically with the understanding that finer details should become irrelevant in the limit. In (1.5) we adopted the Abraham model of a rigid charge distribution. The point charge limit then corresponds to taking in (1.5) the charge distribution ρ_ε instead of $\sqrt{\varepsilon}\rho_\varepsilon$. With this choice

$\mathsf{e} = \int d^3x \rho_\varepsilon(x) = \int d^3x \rho(x)$, whereas the electromagnetic mass equals $\frac{1}{3}\int d^3k |\hat{\rho}_\varepsilon(k)|^2 k^{-2} = \varepsilon^{-1} m_e$. A formal Taylor expansion leads to the effective equation of motion

$$m_0 \ddot{r} = -\nabla V(r) - \varepsilon^{-1} m_e \ddot{r} + a \mathsf{e}^2 \dddot{r}, \tag{1.13}$$

valid for small velocities $\dot{r}$, with some constant $a > 0$. Eq. (1.13) is the nonrelativistic limit of the Lorentz-Dirac equation, [10]. The standard argument, reproduced in many textbooks, e.g. [3], (with the notable exception of Landau and Lifshitz [9]) is to lump m_0 and $\varepsilon^{-1} m_e$ together and to take the limits $\varepsilon \to 0$ and $m_0 \to -\infty$ at constant $m_0 + \varepsilon^{-1} m_e = m_{exp}$, the experimentally observed mass of the particle. Then (1.13) reads as

$$m_{exp} \ddot{r} = -\nabla V(r) + a \mathsf{e}^2 \dddot{r} \, . \tag{1.14}$$

Since this equation is of third order, one needs besides q^0, v^0 also $\dot{u}(0)$ as initial condition which has to be extracted somehow from the initial data of the full system. Even worse, (1.14) has solutions which are exponentially unbounded in time, the famous run-away solutions. Thus one needs an additional criterion to single out the solutions of physical relevance. Dirac [1], and later Haag [2], argued that physical solutions have to satisfy the asymptotic condition

$$\lim_{t \to \infty} \ddot{r}(t) = 0, \tag{1.15}$$

as a substitute for the missing initial condition $\ddot{r}(0)$. The validity of the asymptotic condition has been checked only in trivial cases; see [10]. For general V one should expect the solutions to (1.13) to be chaotic. Physical and unphysical solutions might be badly mixed up. On a more practical level, the physical solutions are unstable and therefore difficult to compute numerically. To put it in the words of W. Thirring [13]: "*...(1.14) has not only crazy solutions and there are attempts to separate sense from nonsense through special initial conditions. But one hopes that the true solution to the problem will look differently and that the nature of the equations of motion is not so highly unstable that the act of balance can be achieved only through a stroke of good fortune in the initial conditions.*"

This is indeed the case, as we are going to show in this paper, and our resolution requires just a little twist. If according to (1.5) we adopt the macroscopic time scale, then (1.13) reads

$$(m_0 + m_e) \ddot{r} = -\nabla V(r) + \varepsilon a \mathsf{e}^2 \dddot{r} \tag{1.16}$$

which just reflects that radiation reaction is a small correction to the Hamiltonian motion. The bare mass m_0 should be kept strictly positive. Otherwise, $\mathcal{H}_{mac}$ from (1.7) is not bounded from below and (1.5) has solutions increasing exponentially in time, a phenomenon completely unrelated to run-away solutions, however.

In (1.16) the highest derivative appears with a small prefactor. Such differential equations are studied in geometric singular perturbation theory. From there we know that (1.16) has a six-dimensional invariant center manifold $\mathcal{I}_\varepsilon$, which is only $\mathcal{O}(\varepsilon)$ away from the Hamiltonian manifold $\mathcal{I}_0 = \{(q, \dot{q}, \ddot{q}) : (m_0 + m_e)\ddot{q} = -\nabla V(q)\}$. For initial conditions slightly off $\mathcal{I}_\varepsilon$ the solution moves away from $\mathcal{I}_\varepsilon$ exponentially fast. On $\mathcal{I}_\varepsilon$, $\dot{q}$ is bounded away from 1, $\ddot{q}$ is bounded, and the motion is governed by an effective second order equation, cf. Eq. (4.9) below, which gives *precisely* the physical solutions. To establish such a result we have to prove that the solution to (1.5) stays indeed close to $\mathcal{I}_\varepsilon$.

In our paper we carry out this program, essentially under the same conditions as in [5], namely a sufficiently differentiable V and $|\rho|_{L^2}$ small. Our main additional estimate is

$$| \dddot{v}^\varepsilon (t)| \leq C \tag{1.17}$$

uniformly in ε and $t \in \mathbb{R}$. Thereby we can bound one further order in the rigorous Taylor expansion and obtain, setting $\dot{q}^\varepsilon = v^\varepsilon$,

$$m(v^\varepsilon)\dot{v}^\varepsilon = -\nabla V(q^\varepsilon) + \varepsilon a(v^\varepsilon)\ddot{v}^\varepsilon + \varepsilon b(v^\varepsilon, \dot{v}^\varepsilon) + \varepsilon^2 f^\varepsilon(t), \quad t \geq \varepsilon t_1, \tag{1.18}$$

with $|f^\varepsilon(t)| \leq C$ and coefficient functions a, b that will be defined below. Clearly (1.18) should be compared with

$$\dot{r} = u\,, \quad m(u)\dot{u} = -\nabla V(r) + \varepsilon a(u)\ddot{u} + \varepsilon b(u, \dot{u}). \tag{1.19}$$

Our crucial observation is that the condition $|u(t)| \leq$ const. < 1 for all t holds only on the center manifold $\mathcal{I}_\varepsilon$. Thus the a priori estimate $|\dot{q}^\varepsilon(t)| \leq \bar{v} < 1$, see (1.8), together with the initial conditions $r(0) = q^0$, $u(0) = v^0$, *uniquely* singles out that solution of (1.19) which is to be compared with the true solution.

Since on the error term $f^\varepsilon(t)$ in (1.18) we only know that it is uniformly bounded, the difference $|q^\varepsilon(t) - r(t)|$, with $r(t)$ having initial conditions on $\mathcal{I}_\varepsilon$, can be bounded at best as εe^{ct}. Thus on the time scale $t = \mathcal{O}(1)$ we seem to be back to the result (1.9) already proved in [5]. To distinguish, from this point of view, between (1.19) and (1.10) we would have to control the difference with a precision of order ε^2. At present we do not know whether this is possible, but nevertheless we can prove the weaker statement

$$|H(q^\varepsilon(t), v^\varepsilon(t)) - H(r(t), u(t))| \leq \text{const.}\, \varepsilon^2, \tag{1.20}$$

where H is the energy from (1.12). Thus on a surface of constant energy the difference $|q^\varepsilon(t) - r(t)|$ could be of order ε, whereas along ∇H it must be of order ε^2. In addition to (1.20) it may also be shown that in fact $|q^\varepsilon(t) - r(t)| \sim \varepsilon^3$ on the short time scale $t = \mathcal{O}(\varepsilon)$, a result that is quite natural from the viewpoint of singularly perturbed ODEs. On the original time scale of (1.1) this amounts at least to an estimate with precision ε^2 over time intervals of length $\mathcal{O}(1)$, a result that could not have been obtained from the bounds in [5].

2 Main results

We give some more details and state our main results precisely. First we have to establish the bound (1.17).

Lemma 2.1 *For $|\rho|_{L^2}$ sufficiently small we have*

$$\sup_{t \in \mathbb{R}} |\dddot{v}^\varepsilon(t)| \leq C$$

for every solution of (1.5) which starts on the soliton manifold $\mathcal{S}$. Both the constant C and the bound for $|\rho|_{L^2}$ depend only on the initial data.

The bound of Lemma 2.1 may be used to Taylor expand the self-force

$$F_s^\varepsilon(t) = \sqrt{\varepsilon} \int d^3x\, \phi(x,t) \nabla \rho_\varepsilon(x - q^\varepsilon(t)) \tag{2.1}$$

in (1.5) as

$$F_s^\varepsilon(t) = -m_f(v^\varepsilon(t))\dot{v}^\varepsilon(t) + \varepsilon a(v^\varepsilon(t))\ddot{v}^\varepsilon(t) + \varepsilon b(v^\varepsilon(t), \dot{v}^\varepsilon(t)) + \mathcal{O}(\varepsilon^2)\,, \quad t \geq \varepsilon t_1\,, \tag{2.2}$$

which together with the second equation in (1.5) yields (1.18). Here m_f is defined in (1.11), and $t_1 = 2R_\rho/(1-\bar{v})$ is the microscopic time the wave equation needs to forget its data because of the compact support of ρ and the velocity bound, cf. assumption (C) and (1.8). The coefficient functions are given by

$$\begin{aligned} a(v)\ddot{v} &= (\mathsf{e}^2/24\pi)(\ddot{v}\cdot\nabla_v)\nabla_v\gamma^2 = (\mathsf{e}^2/12\pi)[\gamma^4\ddot{v} + 4\gamma^6(v\cdot\ddot{v})v]\,, &(2.3)\\ b(v,\dot{v}) &= (\mathsf{e}^2/32\pi)(\dot{v}\cdot\nabla_v)^2\nabla_v\gamma^2 = (\mathsf{e}^2/4\pi)[2\gamma^6(v\cdot\dot{v})\dot{v} + \gamma^6\dot{v}^2 v + 6\gamma^8(v\cdot\dot{v})^2 v], &(2.4)\end{aligned}$$

$\dot{v},\ddot{v}\in\mathbb{R}^3$, with $\gamma = 1/\sqrt{1-v^2}$, $|v|<1$.

Next we explain the existence and the role of the center-like manifolds $\mathcal{I}_\varepsilon$ in greater detail. We refer to [11, 4] for further background on geometric singular perturbation theory. To rewrite (1.19) as a singular perturbation problem, let

$$x = (r,u)\in\mathbb{R}^3\times\mathcal{V}\,,\quad y=\dot{u}\in\mathbb{R}^3\,,\quad f(x,y) = (x_2,y)\in\mathcal{V}\times\mathbb{R}^3\,,\quad\text{and}$$
$$g(x,y,\varepsilon) = a(x_2)^{-1}[m(x_2)y + \nabla V(x_1) - \varepsilon b(x_2,y)]\,.$$

Then (1.19) reads as

$$\dot{x} = f(x,y)\,,\quad \varepsilon\dot{y} = g(x,y,\varepsilon)\,. \tag{2.5}$$

We intend to apply the results from [11] to (2.5) in order to find a center-like manifold for the perturbed problem near the corresponding manifold for the $(\varepsilon=0)$-problem. With $h(x) = -m(x_2)^{-1}\nabla V(x_1)$, let

$$\mathcal{I}_0 = \{(x,y): g(x,y,0)=0\} = \{(r,u,\dot{u}): m(u)\dot{u} = -\nabla V(r)\} = \{(x,h(x)): x\in\mathbb{R}^3\times\mathcal{V}\} \tag{2.6}$$

be this invariant manifold for (2.5) with $\varepsilon=0$. The flow on $\mathcal{I}_0$ is governed by the equation $\dot{x} = f(x,h(x))$, or stated differently, $m(\dot{r})\ddot{r} = -\nabla V(r)$, the familiar Hamiltonian flow.

To see that $\mathcal{I}_0$ is perturbed to some $\mathcal{I}_\varepsilon$ with ε small, we have to modify the functions $a(u)$, $m(u)$, and $b(u,\dot{u})$ for $|u|$ close to one due to the singularity at $|u|=1$. This will cause no problems later on, since we already have the a priori bound $|v^\varepsilon(t)|\le\bar{v}<1$ for the velocity of the true system. In (4.4) below, we will fix a small $\bar{\delta}=\bar{\delta}(\bar{v})>0$ satisfying some estimates; $\bar{\delta}$ depends only on bounds for the initial data, since $\bar{v}$ does so. Let

$$\mathcal{K}_{1-\bar{\delta}} = \mathbb{R}^3\times\{u\in\mathbb{R}^3: |u|\le 1-\bar{\delta}\},$$

We continue $a(u)$, $m(u)$, and $b(u,\dot{u})$ with their values at $|u| = 1-\bar{\delta}$ to the missing infinite strip $1-\bar{\delta}<|u|<1$. Then the basic assumptions (I), (II) from [11, p. 45] are satisfied, since $\mathcal{I}_0$ is also what is called normally hyperbolic, i.e. repulsive in the direction normal to $\mathcal{I}_0$ at an ε-independent rate, see Lemma 4.1 below. Hence we find $\varepsilon_0=\varepsilon_0(\bar{\delta})>0$ and a C^1-function $h(x,\varepsilon) = h_\varepsilon(x):\mathbb{R}^3\times\mathcal{V}\times]0,\varepsilon_0]\to\mathbb{R}^3$ such that for $\varepsilon\le\varepsilon_0$,

$$\mathcal{I}_\varepsilon = \{(x,h_\varepsilon(x)): x\in\mathbb{R}^3\times\mathcal{V}\}$$

is forward invariant for the flow (1.19) with the modified functions a,m,b. Since the modified equation agrees with (1.19) in the interior of $\mathcal{K}_{1-\bar{\delta}}$, we conclude that $\mathcal{I}_\varepsilon$ is locally invariant for the flow (1.19), i.e. the solution of the modified equation is the solution to the original equation as long as it does not reach the boundary set $\{(x,h_\varepsilon(x)) = (r,u,h_\varepsilon(r,u)): |u| = 1-\bar{\delta}\}$. The flow for $\varepsilon=0$ is then perturbed to $\dot{x} = f(x,h_\varepsilon(x))$ for $\varepsilon\le\varepsilon_0$.

We will show in Theorem 4.4 below that for $\varepsilon\in]0,\varepsilon_1]$, with $\varepsilon_1>0$ sufficiently small, all solutions of (1.19) starting at points $(r,u,h_\varepsilon(r,u))\in\mathcal{I}_\varepsilon$ with $|u|\le\bar{v}$, will indeed stay away from the boundary $\{(r,u,h_\varepsilon(r,u)) : |u| = 1-\bar{\delta}\}$ for all future times. In addition, $\nabla V(r(t)) \to 0$ and $\ddot{r}(t) \to 0$ as $t \to \infty$, which is just the asymptotic condition (1.15) postulated by Dirac and Haag. If the potential is sufficiently confining, then the solution trajectory on $\mathcal{I}_\varepsilon$ not only approaches the set of critical points for V in the long-time limit, but it converges to some definite critical point. Moreover, we will show that for all solutions on the center manifold, $\dot{u}(t)$ and $\ddot{u}(t)$ are bounded, and $u(t)$ is bounded away from 1, uniformly in ε and t. Conversely, every such solution to (1.19) has to lie on $\mathcal{I}_\varepsilon$. Thus $\mathcal{I}_\varepsilon$ indeed characterizes the physical solutions.

To summarize, we have established now the existence of a center manifold $\mathcal{I}_\varepsilon$ with a well-defined (semi-) flow on it that gives a unique solution to (1.19) for initial velocities bounded by $\bar{v}$.

For the potential $V \in C^3(\mathbb{R}^3)$ we assume that it is bounded in the sense

$$\inf_{q\in\mathbb{R}^3} V(q) > -\infty \quad \text{and} \quad \sup_{q\in\mathbb{R}^3} \Big(|V(q)| + |\nabla V(q)| + |\nabla\nabla V(q)| + |\nabla\nabla\nabla V(q)|\Big) < \infty\,. \qquad (U)$$

The method works equally well for $V \in C^3(\mathbb{R}^3)$ which is confining, i.e.,

$$V(q) \to \infty \quad \text{as} \quad |q| \to \infty\,, \qquad (U')$$

as will be made more precise in Section 4, cf. Theorem 4.8.

Our main result is the following

Theorem 2.2 *Assume (U) or (U') for the potential, and let the initial data $(\phi^0(x), \pi^0(x), q^0, v^0)$ for (1.5) be given by (1.6). Let $|\rho|_{L^2}$ and $\varepsilon \le \varepsilon_1$ be sufficiently small, and introduce the center manifolds $\mathcal{I}_\varepsilon$ for the comparison dynamics (1.19) as explained above. At time $\varepsilon t_1 = \varepsilon 2R_\rho/(1-\bar{v})$ we match the initial values, $r(\varepsilon t_1) = q^\varepsilon(\varepsilon t_1)$, $u(\varepsilon t_1) = v^\varepsilon(\varepsilon t_1)$, for the motion on the center manifold, i.e., the initial data for the comparison dynamics are*

$$(q^\varepsilon(\varepsilon t_1), v^\varepsilon(\varepsilon t_1), h_\varepsilon(q^\varepsilon(\varepsilon t_1), v^\varepsilon(\varepsilon t_1))) \in \mathcal{I}_\varepsilon\,.$$

Then for every $\tau > 0$ there exists $c(\tau) > 0$ such that for all $t \in [\varepsilon t_1, \varepsilon t_1 + \tau]$

$$|q^\varepsilon(t) - r(t)| \le c(\tau)\varepsilon\,, \quad |v^\varepsilon(t) - u(t)| \le c(\tau)\varepsilon\,, \quad \text{and} \quad |\dot{v}^\varepsilon(t) - \dot{u}(t)| \le c(\tau)\varepsilon. \qquad (2.7)$$

In addition we have the bound

$$|H(q^\varepsilon(t), v^\varepsilon(t)) - H(r(t), u(t))| \le c(\tau)\varepsilon^2. \qquad (2.8)$$

Remarks 2.3 (i) As already mentioned at the end of the introduction, we can also show

$$|q^\varepsilon(t) - r(t)| \le c(\tau)\varepsilon^3 \quad \text{and} \quad |v^\varepsilon(t) - u(t)| \le c(\tau)\varepsilon^2 \qquad (2.9)$$

for $t \in [\varepsilon t_1, \varepsilon t_1 + \varepsilon\tau]$, i.e., $t = \mathcal{O}(\varepsilon)$, cf. Proposition 5.1.

(ii) The construction of the center manifolds and the upper bound for $|\rho|_{L^2}$ rely only on bounds for the data, but not on properties of a particularly chosen solution. Our main technical assumption is a sufficiently small $|\rho|_{L^2}$ which is presumably not necessary.

(iii) In [5] we did not require the true solution to start on the soliton manifold, but instead to start close to it. We refer the criterion [5, Thm. 2.6] for an "adiabatic" family of solutions. The same generality could be achieved in the present context, using an appropriately modified version of [5, Thm. 2.6]. In Section 8 we derive the relevant estimates, in particular (8.9), in full generality containing a non-zero initial difference $Z(0)$. The corresponding generalization of Theorem 2.2 is then straightforward. However, since we did not want to obscure our main achievement through technicalities, we decided to elaborate here the more accessible case of a trajectory starting right on the soliton manifold. In the same spirit we do not consider arbitrary time intervals of length τ, but only the particular $[\varepsilon t_1, \varepsilon t_1 + \tau]$.

(iv) The existence of solutions to (1.1) is discussed in [5, Lemma 2.2]. For every initial value $Y^0 = (\phi^0(x), \pi^0(x), q^0, p^0) \in \mathcal{E}$ we find a unique (weak) solution $Y(\cdot) \in C(\mathbb{R}, \mathcal{E})$ such that $Y(0) = Y^0$. Here the state space is $\mathcal{E} = D^{1,2}(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$ [where $D^{1,2}(\mathbb{R}^3) = \{\phi \in L^6(\mathbb{R}^3) : |\nabla\phi| \in L^2(\mathbb{R}^3)\}$] with norm $|Y|_{\mathcal{E}} = |\nabla\phi|_{L^2} + |\pi|_{L^2} + |q| + |p|$.

Having such fairly precise information on the particle trajectory we can also determine the adiabatic limit $\varepsilon \to 0$ of the fields (ϕ, π) in (1.5) through the solution of the inhomogeneous wave equation. We generate the initial data as explained in the introduction. On the level of the comparison dynamics this means to extend $r(t)$ and $u(t)$ to negative times $t \leq 0$ by $r(t) = q^0 + tv^0$ resp. $u(t) = v^0$. Let the retarded time t_{ret}, depending on x and t, be the unique solution of $t_{\mathrm{ret}} = t - |x - r(t_{\mathrm{ret}})|$, and let $\hat{n}(x,t) = (x - r(t_{\mathrm{ret}}))/|x - r(t_{\mathrm{ret}})|$.

Theorem 2.4 *Under the conditions of Theorem 2.2 and for the fields (ϕ, π) from (1.5) we have for $x \neq r(t)$ the pointwise limits*

$$\lim_{\varepsilon\to 0} \frac{1}{\sqrt{\varepsilon}}\, \phi(x,t) = -\frac{\mathsf{e}}{4\pi|x - r(t_{\mathrm{ret}})|} \Big(1 - \hat{n}(x,t)\cdot u(t_{\mathrm{ret}})\Big)^{-1} \tag{2.10}$$

and, except for the light cone $\{x : |x| = t > 0\}$,

$$\begin{aligned} \lim_{\varepsilon\to 0} \frac{1}{\sqrt{\varepsilon}}\, \pi(x,t) \;&=\; -\frac{\mathsf{e}}{4\pi|x - r(t_{\mathrm{ret}})|} \Big(1 - \hat{n}(x,t)\cdot u(t_{\mathrm{ret}})\Big)^{-3} \hat{n}(x,t)\cdot \dot{u}(t_{\mathrm{ret}}) \\ &\quad -\frac{\mathsf{e}}{4\pi|x - r(t_{\mathrm{ret}})|^2} \Big(1 - \hat{n}(x,t)\cdot u(t_{\mathrm{ret}})\Big)^{-3} \big(\hat{n}(x,t)\cdot u(t_{\mathrm{ret}}) - u(t_{\mathrm{ret}})^2\big). \end{aligned} \tag{2.11}$$

The paper is organized as follows. Since the proof of Lemma 2.1 is rather technical, we moved it to an appendix, Section 8. The derivation of the representation (2.2) of the self-force term is the contents of Section 3. In Section 4 we give supplementary remarks on the behaviour of solutions on the center manifold, whereas in Section 5 we carry out the proofs of Theorem 2.2 and Proposition 5.1. Section 6 contains the proof of Theorem 2.4, and finally in Section 7 we determine the amount of energy radiated to infinity.

3 Representation of the self-force

In this section we show that the self-force $F_s^\varepsilon(t)$ from (2.1) can be written in the form (2.2). We carry out this computation on the original fast time scale corresponding to (1.1) since we will need some of the arguments from [5]. Thus we consider

$$F_s(t) = \int d^3x\, \phi(x,t)\nabla\rho(x - q(t)).$$

Since $\phi(x,t) = \phi_r(x,t) + \phi_0(x,t)$, where $\ddot{\phi}_0 = \Delta\phi_0$ with initial values $\phi_0(x,0) = \phi^0(x)$ and $\pi_0(x,0) = \pi^0(x)$, and since

$$\phi_r(x,t) = -\frac{1}{4\pi}\int_0^t \frac{ds}{t-s} \int_{|y-x|=t-s} d^2y\, \rho(y - q(s))$$

is the retarded potential, we can decompose accordingly,

$$F_s(t) = F_0(t) + F_r(t) = \langle \phi_0(\cdot,t), \nabla\rho(\cdot - q(t))\rangle + \langle \phi_r(\cdot,t), \nabla\rho(\cdot - q(t))\rangle \,.$$

Lemma 3.1 *The function $F_0(t)$ vanishes for $t \geq t_1 = 2R_\rho/(1-\bar v)$.*

Proof: Let $U(t)$ denote the group generated by the free wave equation in $D^{1,2}(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)$. Then (1.4) and Fourier transformation implies $(\phi^0(x), \pi^0(x)) = -\int_{-\infty}^0 ds\,[U(-s)\bar\rho(\cdot - q^0 - v^0 s)](x)$ with $\bar\rho(x) = (0, \rho(x))$. Thus Kirchhoff's formula yields, as a consequence of $|v^0| < 1$, that $\phi_0(x,t) = 0$ for $|x - q^0| \leq t - R_\rho$. Since $|q(t) - q^0| \leq \bar v t$, the claim follows. □

Hence to show (2.2) it is enough to prove

Lemma 3.2 *For $t \geq t_1$,*

$$F_r(t) = -m_f(v(t))\dot v(t) + a(v(t))\ddot v(t) + b(v(t), \dot v(t)) + \mathcal{O}(\varepsilon^3)\,,$$

cf. (1.11), (2.3), and (2.4).

Proof: We follow the proof of [5, Lemma 5.1], but expand

$$q(s) = q(t) - v(t)(t-s) + \frac{1}{2}\dot v(t)(t-s)^2 - \frac{1}{6}\ddot v(t)(t-s)^3 + \mathcal{O}(\varepsilon^3)$$

up to third order, which is allowed by Lemma 2.1. Through Fourier transformation we arrive at

$$F_r(t) = (-i)\int_0^t ds \int d^3k\,|\hat\rho(k)|^2 \frac{k}{|k|}\,\sin|k|(t-s)\, e^{-i(k\cdot v)(t-s)}\, e^{-i[-\frac{1}{2}(k\cdot\dot v)(t-s)^2 + \frac{1}{6}(k\cdot\ddot v)(t-s)^3]} + \mathcal{O}(\varepsilon^3)\,,$$

with $v = v(t)$, etc.. As in [5, Lemma 5.1], here and in the following $\int_0^t ds(\ldots)$ can be changed forth and back to $\int_{t-T}^t ds(\ldots)$ for all $t, T \geq t_1$. Because

$$e^{-i[-\frac{1}{2}(k\cdot\dot v)(t-s)^2 + \frac{1}{6}(k\cdot\ddot v)(t-s)^3]} = 1 + \frac{i}{2}(k\cdot\dot v)(t-s)^2 - \frac{i}{6}(k\cdot\ddot v)(t-s)^3 - \frac{1}{8}(k\cdot\dot v)^2(t-s)^4 + \mathcal{O}(\varepsilon^3)$$

for $t - s = \mathcal{O}(1)$ by (8.1) below, we obtain, for $t, T \geq t_1$,

$$F_r(t) = (-i)\int_0^T d\tau \int d^3k\,|\hat\rho(k)|^2 \frac{k}{|k|}\,\sin|k|\tau\, e^{-i(k\cdot v)\tau}\left[1 + \frac{i}{2}(k\cdot\dot v)\tau^2 - \frac{i}{6}(k\cdot\ddot v)\tau^3 - \frac{1}{8}(k\cdot\dot v)^2\tau^4\right] + \mathcal{O}(\varepsilon^3)\,.$$

Let

$$I_p = \int_0^T d\tau\,\frac{\sin|k|\tau}{|k|}\, e^{-i(k\cdot v)\tau}\tau^p\,, \quad p = 0, \ldots, 4\,.$$

Then

$$(\ddot v\cdot\nabla_v)\nabla_v I_1 = -k(k\cdot\ddot v)I_3 \quad\text{and}\quad (\dot v\cdot\nabla_v)^2\nabla_v I_1 = ik(k\cdot\dot v)^2 I_4\,.$$

Our claim now follows from Lemma 3.3 below, $\int d^3k\,|\hat\rho(k)|^2 k I_0 \to 0$, and $(1/2)\int d^3k\,|\hat\rho(k)|^2 k(k\cdot\dot v)I_2 \to -m_f(v(t))\dot v(t)$ for $T \to \infty$; see [5, Appendix A]. □

Lemma 3.3 *We have the identity*

$$\int_0^\infty dt\, t\int d^3k|\hat\rho(k)|^2 \frac{\sin|k|t}{|k|}\, e^{-i(k\cdot v)t} = (\mathsf{e}^2/4\pi)\gamma^2\,.$$

Proof: Since $\hat{\rho}(k) = \hat{\rho}_r(|k|)$ is radial, and by transformation to polar coordinates,

$$\int d^3k|\hat{\rho}(k)|^2 \frac{\sin|k|t}{|k|}\, e^{-i(k\cdot v)t} = \frac{4\pi}{t|v|}\int_0^\infty dR\,|\hat{\rho}_r(R)|^2 \sin(Rt)\sin(Rt|v|)\,.$$

Thus for fixed $T>0$,

$$\int_0^T dt\, t\int d^3k|\hat{\rho}(k)|^2 \frac{\sin|k|t}{|k|}\, e^{-i(k\cdot v)t} = \frac{2\pi}{|v|}\int_0^\infty \frac{dR}{R}|\hat{\rho}_r(R)|^2 \left(\frac{\sin(R(1-|v|)T)}{1-|v|} - \frac{\sin(R(1+|v|)T)}{1+|v|}\right).$$

To complete the proof we only need to verify that $\int d^3k|\hat{\rho}(k)|^2\, |k|^{-3}\sin|k|T \to \mathsf{e}^2/4\pi$ as $T\to\infty$. To see this, let $\hat{\psi}(k) = |k|^{-3}\sin(|k|T)$. Then $\int d^3k|\hat{\rho}(k)|^2\,\hat{\psi}(k) = (2\pi)^{-3/2}\int d^3x\rho(x)\int d^3y\,\rho(y)\psi(x-y)$ and we are going to show $\psi(x)\to\sqrt{\pi/2}$ as $T\to\infty$. We have, by transformation to polar coordinates,

$$(2\pi)^{3/2}\psi(x) = \int d^3k\,\hat{\psi}(k)e^{-ik\cdot x} = 4\pi\int_0^\infty ds\,\frac{\sin(s)}{s}\,\frac{\sin(s|x|/T)}{s|x|/T} \to 2\pi^2 \ \text{ for } T\to\infty\,.$$

This completes the proof. □

4 More about the center manifold

In this section we explain the behaviour of solutions on the center manifold. First we show that the unperturbed manifold $\mathcal{I}_0$ from (2.6) is hyperbolic in normal direction.

Lemma 4.1 *The eigenvalues of $D_yg(x,y,0) = a(x_2)^{-1}m(x_2)$ are bounded below by a positive constant, uniformly in $x=(r,u)$ with $r\in\mathbb{R}^3$ and $|u|\le 1-\delta$, for all prescribed $\delta\in]0,1]$.*

Proof: By [8, Thm. 2, p. 185], $a(u)$ and $m(u)$ can be simultaneously transformed to diagonal form through a single non-singular matrix B. In addition, denoting by $b_j\neq 0$ the jth column of B and by λ_j the jth eigenvalue of $a(u)^{-1}m(u)$, one has $\lambda_j a(u)b_j = m(u)b_j$, $j=1,2,3$. Multiplication by b_j leads to $\lambda_j(\mathsf{e}^2/12\pi)\gamma^3[\gamma b_j^2 + 4\gamma^3(v\cdot b_j)^2] \ge \gamma b_j^2 + \gamma^3(v\cdot b_j)^2$, and thus $\lambda_j \ge (3\pi/\mathsf{e}^2)\gamma^{-3}$. □

Since $a(u)$, $m(u)$ are modified to be constant outside $|u|\le 1-\bar{\delta}$, their corresponding eigenvalues are uniformly bounded below for $|u|<1$. As a consequence of Lemma 4.1 the manifolds $\mathcal{I}_\varepsilon$ are unstable at some exponential rate $e^{\mu t}$ for solutions in the normal direction.

We note that, by [11, Thm. 2.1],

$$\sup\{|h_\varepsilon(r,u)| : (r,u)\in\mathbb{R}^3\times\mathcal{V}, \varepsilon\in]0,\varepsilon_0]\} \le c = c(\bar{\delta}). \tag{4.1}$$

Our next aim is to prove global existence of solutions to (1.19) forward in time which start over $\mathcal{K}_{\bar{v}} = \mathbb{R}^3\times\{u\in\mathbb{R}^3 : |u|\le\bar{v}\}$ on the center manifold, provided $\varepsilon\le\varepsilon_1$ with $\varepsilon_1>0$ sufficiently small. For this purpose we introduce a suitable Lyapunov function.

Lemma 4.2 *Let*

$$G_\varepsilon(r,u,\dot{u}) = H(r,u) - \varepsilon(a(u)\dot{u})\cdot u = \mathcal{E}_s(u) + V(r) - \varepsilon(a(u)\dot{u})\cdot u\,.$$

Then along solutions $(r(t),u(t),\dot{u}(t))$ of (1.19) we have

$$\frac{d}{dt}G_\varepsilon(r,u,\dot{u}) = -\varepsilon(\mathsf{e}^2/12\pi)\,[6\gamma^8(u\cdot\dot{u})^2 + \gamma^6\dot{u}^2]. \tag{4.2}$$

Proof: Straightforward calculation, observing that $(a(u)\dot{u})\cdot u = (\mathsf{e}^2/12\pi)\gamma^6\,(1+3u^2)(u\cdot\dot{u})$. $\square$

Through the Lyapunov function G_ε we can control the long time behaviour.

Theorem 4.3 *Let (U) or (U') hold and let any global solution $(r(t),u(t))$ of (1.19) be given such that $\sup_{t\geq 0}|u(t)|\leq \bar{u}(\varepsilon)<1$ and $\sup_{t\geq 0}|\dot{u}(t)|\leq c(\varepsilon)$, for possibly ε-dependent constants $\bar{u}(\varepsilon)$ and $c(\varepsilon)$. Then*

$$\dot{u}(t)\to 0,\quad \ddot{u}(t)\to 0,\quad \textit{and}\quad \nabla V(r(t))\to 0\quad \textit{as}\quad t\to\infty.$$

Proof: Denoting by $c(\varepsilon)$ or $C(\varepsilon)$ general ε-dependent constants, by Lemma 4.2 we have along a trajectory

$$\begin{aligned}
c(\varepsilon)\int_0^T \dot{u}^2\,dt &\leq -\int_0^T \frac{d}{dt}G_\varepsilon\,dt\\
&= -\mathcal{E}_s(u(T)) - V(r(T)) + \varepsilon(a(u(T))\dot{u}(T))\cdot u(T)\\
&\quad +\mathcal{E}_s(u^0) + V(r^0) - \varepsilon(a(u^0)\dot{u}^0)\cdot u^0\\
&\leq C(\varepsilon,\text{data}).
\end{aligned}$$

For the last estimate observe $\inf_{r\in\mathbb{R}^3}V(r)>-\infty$ in both cases (U) and (U'). Thus $\int_0^\infty \dot{u}^2\,dt\leq C(\varepsilon,\text{data})$ and, by (1.19), also $\sup_{t\geq 0}|\ddot{u}(t)|\leq C(\varepsilon,\text{data})$. Hence we conclude $\dot{u}(t)\to 0$ as $t\to\infty$. Next, differentiation of (1.19) yields $\sup_{t\geq 0}|\dddot{u}(t)|\leq C(\varepsilon,\text{data})$, and thus from $\dot{u}(t)\to 0$ we find $\ddot{u}(t)\to 0$. Therefore $\nabla V(r(t))\to 0$ follows from the equation (1.19). $\square$

In the demonstration of the following theorem we use the sublevel sets $\{G_\varepsilon\leq c\}=\{(r,u,\dot{u}): G_\varepsilon(r,u,\dot{u})\leq c\}$ and $\{H\leq c\}=\{(r,u): H(r,u)\leq c\}$ for $c\in\mathbb{R}$. However, before proceeding, we first have to introduce an appropriate $\bar{\delta}=\bar{\delta}(\bar{v})>0$ small to modify the functions $a(u)$, $m(u)$, and $b(u,\dot{u})$ outside $|u|\leq 1-\bar{\delta}$, cf. Section 2. To do this, we assume (U) from now on. The case (U') is discussed in the remarks below. Since V is bounded and $\bar{v}<1$, we can find $c_0\in\mathbb{R}$ such that $\mathcal{K}_{\bar{v}}\subset\{H\leq c_0\}$. Then as a consequence of $\mathcal{E}_s(u)\to\infty$ for $|u|\to 1$, we have

$$s_0=\sup\Big\{|u|:(r,u)\in\{H\leq c_0+1\}\text{ for some }r\in\mathbb{R}^3\Big\}<1\,. \tag{4.3}$$

Let us define

$$\bar{\delta}=\min\{(1-\bar{v})/2,(1-s_0)/2\}>0. \tag{4.4}$$

Theorem 4.4 *Assume the potential V to satisfy the condition (U). Then there exists $\varepsilon_1>0$ depending only upon $\bar{v}$ such that for $\varepsilon\in]0,\varepsilon_1]$ all solutions of (1.19) starting at points $(r,u,h_\varepsilon(r,u))\in\mathcal{I}_\varepsilon$, $|u|\leq\bar{v}$, stay away from the boundary $\{(r,u,h_\varepsilon(r,u)):|u|=1-\bar{\delta}\}$ for all future times. In particular, solutions exist globally.*

Proof: Let us denote the bound $c(\bar{\delta})$ from (4.1) by c_1 and let us fix $c_a>0$ such that $|a(u)|\leq c_a$ for all $|u|<1$. We recall that $a(u)$ was modified to be constant outside $|u|\leq 1-\bar{\delta}$. We define $\varepsilon_1=\min\{\varepsilon_0,(2c_ac_1)^{-1}\}>0$.

Let $(r,u)\in\mathcal{K}_{\bar{v}}$. Then $G_\varepsilon(r,u,h_\varepsilon(r,u))=H(r,u)-\varepsilon(a(u)h_\varepsilon(r,u))\cdot u\leq c_0+c_ac_1\varepsilon$. Because of Lemma 4.2 the set $\{G_\varepsilon\leq c_0+c_ac_1\varepsilon\}$ is forward invariant and the solution remains in this set for all future times. On the other hand, since $\bar{v}\leq 1-2\bar{\delta}<1-\bar{\delta}$, the solution of the modified problem is a solution to (1.19) and stays on $\mathcal{I}_\varepsilon$, at least for a short times. For the fixed time span where this holds the solution is of the form $(r_1,u_1,h_\varepsilon(r_1,u_1))$ and we have $H(r_1,u_1)=$

$G_\varepsilon(r_1, u_1, h_\varepsilon(r_1, u_1)) + \varepsilon(a(u_1)h_\varepsilon(r_1, u_1)) \cdot u_1 \leq c_0 + c_a c_1 \varepsilon + c_a c_1 \varepsilon = c_0 + 2c_a c_1 \varepsilon \leq c_0 + 1$ for $\varepsilon \leq \varepsilon_1$. Therefore by (4.3), $|u_1| \leq s_0 \leq 1 - 2\bar{\delta} < 1 - \bar{\delta}$. This argument shows that in fact the solution is confined to $\{(r, u, \dot{u}) : |u| \leq 1 - 2\bar{\delta}\}$. Hence the solution of the modified problem exists, is a solution to (1.19), and stays on $\mathcal{I}_\varepsilon$ for all future times. □

Corollary 4.5 *In the setting of Theorem 4.4, for solutions of (1.19) starting on $\mathcal{I}_\varepsilon$,*

$$\sup\{|u(t)| : t \in \mathbb{R}, \varepsilon \in]0, \varepsilon_1]\} \leq 1 - 2\bar{\delta} < 1\,, \quad \textit{and}$$
$$\sup\{|\dot{u}(t)| : t \in \mathbb{R}, \varepsilon \in]0, \varepsilon_1]\} + \sup\{|\ddot{u}(t)| : t \in \mathbb{R}, \varepsilon \in]0, \varepsilon_1]\} \leq c(\bar{\delta}). \tag{4.5}$$

In particular by Theorem 4.3

$$\dot{u}(t) \to 0, \quad \ddot{u}(t) \to 0, \quad \textit{and} \quad \nabla V(r(t)) \to 0 \quad \textit{as} \quad t \to \infty.$$

Proof: The first estimate was mentioned already in the preceding proof. For the second we note that (4.1) applies, since the trajectory stays on the center manifold, $\dot{u}(t) = h_\varepsilon(r(t), u(t))$. Concerning the last bound, we may write

$$h_\varepsilon(r, u) = -m(u)^{-1}\nabla V(r) + h_{1,\varepsilon}(r, u) \quad \text{with} \quad |h_{1,\varepsilon}(r, u)| \leq c(\bar{\delta})\varepsilon\,, \quad (r, u) \in \mathbb{R}^3 \times \mathcal{V}, \tag{4.6}$$

see [11, Thm. 2.9]. Thus by (1.19), $|\varepsilon \ddot{u}| \leq |a(u)^{-1}||m(u)h_{1,\varepsilon}(r, u) - \varepsilon b(u, \dot{u})| \leq c(\bar{\delta})\varepsilon$. □

Solutions on $\mathcal{I}_\varepsilon$ are uniformly bounded, in the sense of the corollary; in general a bound on $r(t)$ cannot be expected, e.g. in a scattering situation. Conversely, as to be shown next, solutions with uniformly bounded $u(t)$, $\dot{u}(t)$, and $\ddot{u}(t)$ are confined to the center manifolds.

Proposition 4.6 *Suppose we have a family $(r^\varepsilon(t), u^\varepsilon(t))$, $\varepsilon \in]0, \varepsilon_2]$, of solutions to (1.19) such that*

$$\sup\{|u^\varepsilon(t)| : t \in \mathbb{R}, \varepsilon \in]0, \varepsilon_2]\} \leq \bar{u} < 1\,, \quad \textit{and}$$
$$\sup\{|\dot{u}^\varepsilon(t)| : t \in \mathbb{R}, \varepsilon \in]0, \varepsilon_2]\} + \sup\{|\ddot{u}^\varepsilon(t)| : t \in \mathbb{R}, \varepsilon \in]0, \varepsilon_2]\} \leq c_2.$$

Then for sufficiently small ε the solutions have to lie on $\mathcal{I}_\varepsilon$.

Proof: Note that we can construct $\mathcal{I}_\varepsilon$ here by modifying $a(u), m(u)$, and $b(u, \dot{u})$ to be constant outside, say, $\{u : |u| \leq (1 + \bar{u})/2\}$. According to [11, Thm. 2.1 (ii)] there exists $\delta > 0$ such that for all ε small and solutions $(x(t), y(t))$ to (2.5) the condition $\sup_{t\in\mathbb{R}} |y(t) - h(x(t))| \leq \delta$ implies that the solution has to lie on $\mathcal{I}_\varepsilon$. With $x(t) = (r^\varepsilon(t), u^\varepsilon(t))$ and $y(t) = \dot{u}^\varepsilon(t)$, this condition is verified since we obtain from (1.19) and the assumed bounds $|\dot{u}^\varepsilon(t) + m(u^\varepsilon(t))^{-1}\nabla V(r^\varepsilon(t))| \leq c\varepsilon \leq \delta$, the latter for ε small. □

The asymptotic condition, $\dddot{r}(t) \to 0$, of Dirac and Haag is also sufficient for a solution to lie on $\mathcal{I}_\varepsilon$, in the following sense.

Proposition 4.7 *Suppose a family $(r^\varepsilon(t), u^\varepsilon(t))$, $\varepsilon \in]0, \varepsilon_2]$, of solutions to (1.19) is given such that*

$$\sup\{|u^\varepsilon(t)| : t \in \mathbb{R}, \varepsilon \in]0, \varepsilon_2]\} \leq \bar{u} < 1$$

and $\dddot{r}^\varepsilon(t) = \dot{u}^\varepsilon(t) \to 0$ as $t \to \infty$ for each $\varepsilon \in]0, \varepsilon_2]$. Then for sufficient small ε the solutions have to lie on $\mathcal{I}_\varepsilon$.

Proof: Fix $\delta > 0$. Since Theorem 4.3 applies, we find in the notation of Proposition 4.6

$$|y(t) - h(x(t))| = |\dot{u}^\varepsilon(t) + m(u^\varepsilon(t))^{-1}\nabla V(r^\varepsilon(t))| \le \delta/2$$

for $t \ge t(\varepsilon)$, with some $t(\varepsilon)$. Thus the solution remains $(\delta/2)$-close to $\mathcal{I}_0$ after time $t(\varepsilon)$, and hence by (4.6) also δ-close to $\mathcal{I}_\varepsilon$ for ε small. Since $\mathcal{I}_\varepsilon$ is normally hyperbolic (repulsive) at an ε-independent rate and since $\delta > 0$ was arbitrary, this can only happen if the solution was already contained in $\mathcal{I}_\varepsilon$. □

Corollary 4.5 provides a partial information on the long time behavior of the solutions to (2.5) on the center manifold. Roughly one can distinguish two classes. (i) (scattering): The particle enters a domain where $-\nabla V = 0$ at r_1 with velocity u_∞. If the straight line trajectory $r_1 + u_\infty t$, $t \ge 0$, is contained in this domain, then the particle travels freely to infinity. Physically this is a scattering trajectory. In this case $\lim_{t\to\infty} u(t) = u_\infty \ne 0$, whereas the position has no limit. (ii) (bounded motion): We assume that $|r(t)| \le const.$ and that within this ball the critical points of V form a discrete set. Then by Corollary 4.5 and by continuity we have $\lim_{t\to\infty} u(t) = 0$ and $\lim_{t\to\infty} r(t) = r_\infty$, where r_∞ is one of the critical points of V. If r_∞ is a stable critical point, then the relaxation is exponentially fast, as can be seen from linearization around the fixed point. Clearly (i) and (ii) do not exhaust all possibilities. The critical points of V could lie on a sphere. If V is confining, one would still expect convergence to a definite r_∞. Moreover, V could vanish inside a ball. If $-\nabla V$ is pointing towards the ball, then close to each turning point the particle looses energy. Thus $\lim_{t\to\infty} u(t) = 0$, whereas the position has no limit. The potential could decrease so slowly at infinity that no definite velocity is approached. All these cases have to be studied separately.

Up to now we discussed bounded potentials satisfying (U). In the introduction we claimed that our results remain valid also for confining potentials satisfying (U'). In this case, since V is unbounded, we have no longer $\mathcal{K}_{\bar{v}} \subset \{H \le c_0\}$ for some $c_0 \in \mathbb{R}$ as in Theorem 4.4 above. However, by energy conservation, one can derive the a priori bound $\sup_{t\in\mathbb{R}} |q^\varepsilon(t)| \le \bar{M}$ for solutions to the true system (1.5) on the macroscopic time scale. Thus the motion is bounded also in the q-direction and it suffices to build the center manifold for the effective equation (1.19) over the bounded domain

$$\mathcal{K}_{\bar{M},\bar{v}} = \{r \in \mathbb{R}^3 : |r| \le \bar{M}\} \times \{u \in \mathbb{R}^3 : |u| \le \bar{v}\},$$

enlarged to a suitable $\mathcal{K}_{\bar{M}+1,1-\bar{\delta}}$ such that solutions starting over $\mathcal{K}_{\bar{M},\bar{v}}$ stay away from the boundary of $\mathcal{K}_{\bar{M}+1,1-\bar{\delta}}$ for $\varepsilon > 0$ sufficiently small. In this manner we obtain

Theorem 4.8 *Assume (U') holds for the potential, and let $\mathcal{K}_{\bar{M},\bar{\delta}}$ be defined as above. Then there exists $\varepsilon_1 > 0$ depending only on the initial data such that for $\varepsilon \in]0, \varepsilon_1]$ all solutions of (1.19) starting at points $(r, u, h_\varepsilon(r, u)) \in \mathcal{I}_\varepsilon$, $(r, u) \in \mathcal{K}_{\bar{M},\bar{\delta}}$, exist globally. Moreover, these solutions are uniformly bounded,*

$$\begin{gathered}\sup\{|r(t)| : t \in \mathbb{R}, \varepsilon \in]0, \varepsilon_1]\} \le c(\bar{\delta}),\\ \sup\{|u(t)| : t \in \mathbb{R}, \varepsilon \in]0, \varepsilon_1]\} \le 1 - 2\bar{\delta} < 1, \quad \text{and}\\ \sup\{|\dot{u}(t)| : t \in \mathbb{R}, \varepsilon \in]0, \varepsilon_1]\} + \sup\{|\ddot{u}(t)| : t \in \mathbb{R}, \varepsilon \in]0, \varepsilon_1]\} \le c(\bar{\delta}).\end{gathered} \tag{4.7}$$

In addition,

$$\dot{u}(t) \to 0, \quad \ddot{u}(t) \to 0, \quad \text{and} \quad \nabla V(r(t)) \to 0 \quad \text{as} \quad t \to \infty.$$

Proof: The proof is similar to the one of Theorem 4.4. Concerning the boundedness, note that again for some $c_0 \in \mathbb{R}$ and $\varepsilon > 0$ small,

$$\mathcal{K}_{\bar{M},\bar{\delta}} \subset \{H \le c_0\} \subset \{(r,u) : G_\varepsilon(r,u,h_\varepsilon(r,u)) \le c_0 + c_a c_1 \varepsilon\} \subset \{H \le c_0 + 1\}.$$

Thus all solutions starting over $\mathcal{K}_{\bar{M},\bar{\delta}}$ will remain on the manifolds over $\{H \le c_0 + 1\}$. Since this set is independent of ε and compact by (U'), the solutions must be uniformly bounded, because h_ε is uniformly bounded. □

On the center manifold the motion is governed by the (second order) equation

$$\dot{x} = f(x, h_\varepsilon(x)). \tag{4.8}$$

Since the existence of h_ε is established only abstractly, Eq. (4.8) is somewhat implicit. From [11, (2.9-1) & Thm. 2.9] we know that h_ε depends smoothly on ε. Thus (4.8) can be expanded in ε. Including the first Taylor term we pick up an error of order ε^2, which is of the same order as the error between the true and the comparison dynamics on the center manifold. For consistency we should stop then at this order. We make the ansatz

$$h_\varepsilon(r,u) = h_0(r,u) + \varepsilon h_1(r,u) + h_{2,\varepsilon}(r,u)\,, \quad |h_{2,\varepsilon}(r,u)| \le c(\bar{\delta})\varepsilon^2, \quad (r,u) \in \mathbb{R}^3 \times \mathcal{V}.$$

Then

$$m(u) h_0(r,u) = -\nabla V(r),$$

and $h_1(r,u)$ is determined through

$$D_x h_0(r,u) f(r,u,h_0(r,u)) = D_y g(r,u,h_0(r,u),0) h_1(r,u) + D_\varepsilon g(r,u,h(r,u),0),$$

see [11, (2.9-1) & Thm. 2.9]. Computing the respective derivatives one arrives at

$$\begin{aligned} m(u)h_1(r,u) \;&=\; a(u)\Big[-m(u)^{-1}\nabla^2 V(r)u + \Big(\frac{d}{du}m(u)\Big)^{-1}\big(\nabla V(r), m(u)^{-1}\nabla V(r)\big) \\ &\qquad + b(u, m(u)^{-1}\nabla V(r))\Big] \end{aligned}$$

and the effective second order equation

$$\dot{r} = u\,, \quad m(u)\dot{u} = -\nabla V(r) + \varepsilon m(u) h_1(r,u) \tag{4.9}$$

of the particle motion on the center manifold.

5 Comparison of the true and the effective system

In this section we prove Theorem 2.2. Since we have $|u(t_1)| = |v^\varepsilon(t_1)| \le \bar{v}$ by (1.8), Theorem 4.4, resp. Theorem 4.8, implies that the solution trajectory of the system with the modified functions $a(u)$, $m(u)$, and $b(u,\dot{u})$ is indeed a solution trajectory to (1.19). Recall that, by (1.18) and (1.19),

$$m(v^\varepsilon)\dot{v}^\varepsilon = -\nabla V(q^\varepsilon) + \varepsilon a(v^\varepsilon)\ddot{v}^\varepsilon + \varepsilon b(v^\varepsilon, \dot{v}^\varepsilon) + \varepsilon^2 f^\varepsilon(t), \quad t \ge \varepsilon t_1, \tag{5.1}$$

$$m(u)\dot{u} = -\nabla V(r) + \varepsilon a(u)\ddot{u} + \varepsilon b(u,\dot{u}), \tag{5.2}$$

with $|f^\varepsilon(t)| \le C$. Using the bounds (1.8) and (4.5), resp. (4.7), we infer the weaker estimate

$$\begin{aligned} m(v^\varepsilon)\dot v^\varepsilon &= -\nabla V(q^\varepsilon) + \mathcal{O}(\varepsilon), \quad t \ge t_1, \\ m(u)\dot u &= -\nabla V(r) + \mathcal{O}(\varepsilon), \end{aligned}$$

which has been proved already in [5]. Hence (2.7) follows by the argument there.

To show (2.8) we compute as in Lemma 4.2, using (5.1),

$$\begin{aligned} \frac{d}{dt} G_\varepsilon(q^\varepsilon(t), v^\varepsilon(t), \dot v^\varepsilon(t)) &= \varepsilon^2 f^\varepsilon(t) v^\varepsilon(t) \\ &\quad -\varepsilon(\mathsf{e}^2/12\pi)\Big[\gamma(\dot v^\varepsilon(t))^6 \dot v^\varepsilon(t)^2 + 6\gamma(\dot v^\varepsilon(t))^8 (v^\varepsilon(t)\cdot \dot v^\varepsilon(t))^2\Big], \quad t \ge \varepsilon t_1. \end{aligned}$$

Since $r(\varepsilon t_1) = q^\varepsilon(\varepsilon t_1)$ and $u(\varepsilon t_1) = v^\varepsilon(\varepsilon t_1)$, using the uniform bounds, we have for $t \ge \varepsilon t_1$

$$\begin{aligned} &|H(q^\varepsilon(t), v^\varepsilon(t)) - H(r(t), u(t))| \\ &\le \varepsilon \Big|(a(v^\varepsilon(t))\dot v^\varepsilon(t)) \cdot v^\varepsilon(t) - (a(u(t))\dot u(t)) \cdot u(t)\Big| \\ &\quad + \int_{\varepsilon t_1}^t ds \, \Big[\varepsilon^2 |f^\varepsilon(s) v^\varepsilon(s)| + \varepsilon(\mathsf{e}^2/12\pi)\Big(\Big|\gamma(\dot v^\varepsilon(s))^6 \dot v^\varepsilon(s)^2 - \gamma(\dot u(s))^6 \dot u^\varepsilon(s)^2\Big| \\ &\qquad + 6\Big|\gamma(\dot v^\varepsilon(s))^8 (v^\varepsilon(s)\cdot \dot v^\varepsilon(s))^2 - \gamma(\dot u(s))^8 (u(s)\cdot \dot u(s))^2\Big|\Big)\Big] \\ &\le C\varepsilon\Big[|v^\varepsilon(t) - u(t)| + |\dot v^\varepsilon(t) - \dot u(t)|\Big] + C\varepsilon^2 t + C\varepsilon \int_{\varepsilon t_1}^t ds \, \Big[|v^\varepsilon(s) - u(s)| + |\dot v^\varepsilon(s) - \dot u(s)|\Big] \\ &\le C\varepsilon^2(1+t) \le C\varepsilon^2, \end{aligned}$$

by (2.7) for $t = \mathcal{O}(1)$. This concludes the proof of Theorem 2.2. □

Finally we show that on a microscopic time scale our results track the true trajectory with a higher precision, cf. (2.9), Remark 2.3(i).

Proposition 5.1 *We have*

$$|q^\varepsilon(t) - r(t)| \le c\varepsilon^3 \quad \textit{and} \quad |v^\varepsilon(t) - u(t)| \le c\varepsilon^2, \quad t = \mathcal{O}(\varepsilon),$$

i.e., (2.9) holds.

Proof: Define $\Psi(s) = \Big(\varepsilon^{-1} q^\varepsilon(\varepsilon s) - \varepsilon^{-1} r(\varepsilon s), v^\varepsilon(\varepsilon s) - u(\varepsilon s), \varepsilon \dot v^\varepsilon(\varepsilon s) - \varepsilon \dot u(\varepsilon s)\Big)$ for $s \ge t_1$. Then $\dot\Psi(s) = A\Psi(s) + \theta(s)$, where $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $\theta(s) = \varepsilon^2(0, 0, \ddot v^\varepsilon(\varepsilon s) - \ddot u(\varepsilon s))$, whence

$$\begin{aligned} |\theta(s)| &\le c\varepsilon\Big(|q^\varepsilon(\varepsilon s) - r(\varepsilon s)| + |v^\varepsilon(\varepsilon s) - u(\varepsilon s)| + |\dot v^\varepsilon(\varepsilon s) - \dot u(\varepsilon s)| + \varepsilon^2\Big) \\ &\le c(|\Psi(s)| + \varepsilon^3), \quad s \ge t_1, \end{aligned}$$

by (5.1), (5.2), and the uniform bounds. Therefore by the variation of constants formula and Gronwall's inequality for $s \in [t_1, t_1 + \tau]$, $|\Psi(s)| \le c(\tau)(|\Psi(s_1)| + \varepsilon^3)$. Consequently, $q^\varepsilon(\varepsilon t_1) = r(\varepsilon t_1)$ and $v^\varepsilon(\varepsilon t_1) = u(\varepsilon t_1)$ yields

$$\varepsilon^{-1}|q^\varepsilon(t) - r(t)| + |v^\varepsilon(t) - u(t)| \le c(\tau)\Big(\varepsilon|\dot v^\varepsilon(\varepsilon t_1) - \dot u(\varepsilon t_1)| + \varepsilon^3\Big)$$

for $t \in [\varepsilon t_1, \varepsilon t_1 + \varepsilon\tau]$. By (5.1) and (5.2), $|\dot v^\varepsilon(\varepsilon t_1) - \dot u(\varepsilon t_1)| \le c\varepsilon$, so that (2.9) follows. □

6 Adiabatic limit of the fields

We prove Theorem 2.4. Let $U(t)$ again denote the fundamental solution of the wave equation in $D^{1,2}(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$. We set $Z(x,t) = (\phi(x,t), \pi(x,t))$ as well as $\bar{\rho}_\varepsilon = (0, \rho_\varepsilon)$. Then the inhomogeneous wave equation in (1.5) is solved as

$$Z(x,t) = [U(t)Z(\cdot,0)](x) - \sqrt{\varepsilon} \int_0^t ds\, [U(t-s)\bar{\rho}_\varepsilon(\cdot - q^\varepsilon(s))](x).$$

Since

$$Z(x,0) = -\sqrt{\varepsilon} \int_{-\infty}^0 ds\, [U(s)\bar{\rho}_\varepsilon(\cdot - q^0 - v^0 s)](x),$$

cf. Lemma 3.1, we have for $t > 0$

$$Z(x,t) = -\sqrt{\varepsilon} \int_{-\infty}^t ds\, [U(t-s)\bar{\rho}_\varepsilon(\cdot - q^\varepsilon(s))](x),$$

where we extended the position to negative times $t \le 0$ by $q^\varepsilon(t) = q^0 + v^0 t$. Thus by the solution formula for the wave equation

$$\frac{1}{\sqrt{\varepsilon}}\, \phi(x,t) = -\int \frac{d^3y}{4\pi|x-y|}\, \rho_\varepsilon(y - q^\varepsilon(t - |x-y|)) \tag{6.1}$$

and $\pi(x,t) = \dot{\phi}(x,t)$. For $\varepsilon \to 0$, $q^\varepsilon(t) \to r(t)$, cf. (1.9), with $r(t)$ extended to negative times by $r(t) = q^0 + v^0 t$. Moreover, $\rho_\varepsilon(x) = \varepsilon^{-3}\rho(\varepsilon^{-1}x) \to \mathsf{e}\delta_0$ in the sense of distributions. Hence the transformation $z = y - q^\varepsilon(t - |x-y|)$, $\det(dy/dz) = \left[1 - v^\varepsilon(t - |x-y|)\cdot(x-y)/|x-y|\right]^{-1}$, in (6.1) yields the pointwise convergence (2.10), except on the worldline of the particle, since the integrand in (6.1) is singular at $y = x$, i.e. for $x = r(t)$ which corresponds to $t_{\text{ret}} = t$.

The analogous argument works for $\pi(x,t)$. In the limit $\varepsilon \to 0$, π is discontinuous at the light cone $\{x : |x| = t\}$, which we avoided by assumption. □

7 Radiated Energy

Let $E_{R,q^\varepsilon(t)}(t+R)$ be the energy, particle plus field, at time $t+R$ in a ball of radius R centered at $q^\varepsilon(t)$. For $R > \varepsilon R_\rho$ this energy changes as

$$\begin{aligned}
&\frac{d}{dt}\Big(E_{R,q^\varepsilon(t)}(t+R)\Big) \\
&= \frac{d}{dt}\Big(\mathcal{H}_{mac}(t=0) - \frac{1}{2}\int_{\{|x-q^\varepsilon(t)|>R\}} d^3x \Big[|\pi(x,t+R)|^2 + |\nabla\phi(x,t+R)|^2\Big]\Big) \\
&= R^2 \int_{|\omega|=1} d^2\omega\, \pi(q^\varepsilon(t)+R\omega, t+R)\, \omega\cdot\nabla\phi(q^\varepsilon(t)+R\omega, t+R) \\
&\quad + \frac{R^2}{2}\int_{|\omega|=1} d\omega\, (\omega\cdot v^\varepsilon(t))\Big[|\pi(q^\varepsilon(t)+R\omega, t+R)|^2 + |\nabla\phi(q^\varepsilon(t)+R\omega, t+R)|^2\Big],
\end{aligned} \tag{7.1}$$

where we used that the total energy is conserved.

E_R changes because there is energy flowing back and forth between particle and field, and because energy is lost irreversibly to infinity. To separate both contributions we take the limit

$R \to \infty$. Using (6.1) and the relation $t + R - |q^\varepsilon(t) + R\omega - y| = t + \omega \cdot (y - q^\varepsilon(t)) + \mathcal{O}(1/R)$ for bounded $|y|$, we arrive at

$$\begin{aligned} I^\varepsilon(t) &= \lim_{R\to\infty} \frac{d}{dt}\Big(E_{R,q^\varepsilon(t)}(t+R)\Big) \\ &= -\varepsilon(4\pi)^{-2}\int_{|\omega|=1} d^2\omega\,(1-\omega\cdot v^\varepsilon(t))\,\Big[\int d^3y\,\rho_\varepsilon(y - q^\varepsilon(t+\omega\cdot[y-q^\varepsilon(t)])) \\ &\qquad\qquad \times\frac{\omega\cdot\dot v^\varepsilon(t+\omega\cdot[y-q^\varepsilon(t)])}{(1-\omega\cdot v^\varepsilon(t+\omega\cdot[y-q^\varepsilon(t)]))^2}\Big]^2 \end{aligned}$$

cf. [7, Sec. 3] for details on a similar calculation. In fact, there the ball of radius R was centered at the origin and the second summand in (7.1) is absent. To let $\varepsilon \to 0$, we again transform to $z = y - q^\varepsilon(t + \omega \cdot [y - q^\varepsilon(t)])$, $\det(dy/dz) = [1 - \omega \cdot v^\varepsilon(t + \omega \cdot [y - q^\varepsilon(t)])]^{-1}$, use $\rho_\varepsilon(x) \to \mathsf{e}\delta_0$ in the sense of distributions, and insert the identity $y = q^\varepsilon(t)$ for $z = 0$ to obtain

$$\begin{aligned} \lim_{\varepsilon\to 0}\varepsilon^{-1}\, I^\varepsilon(t) &= -\mathsf{e}^2(4\pi)^{-2}\int_{|\omega|=1} d^2\omega\,(1-\omega\cdot u(t))^{-5}\,(\omega\cdot\dot u(t))^2 \\ &= -(\mathsf{e}^2/12\pi)\,[6\gamma^8(u(t)\cdot\dot u(t))^2 + \gamma^6\dot u(t)^2], \end{aligned}$$

in agreement with (4.2).

Alternatively, we could first take the limit $\varepsilon \to 0$ in (7.1). Using Theorem 2.4 we find, with $(\overline{\phi}, \overline{\pi})$ denoting the limit fields from (2.10), (2.11),

$$\begin{aligned} I_R(t) &= \lim_{\varepsilon\to 0}\varepsilon^{-1}\,\frac{d}{dt}\Big(E_{R,q^\varepsilon(t)}(t+R)\Big) \\ &= R^2\int_{|\omega|=1} d^2\omega\,\overline{\pi}(r(t)+R\omega,t+R)\,\omega\cdot\nabla\overline{\phi}(r(t)+R\omega,t+R) \\ &\quad +\frac{R^2}{2}\int_{|\omega|=1} d^2\omega\,(\omega\cdot u(t))\,\Big[|\overline{\pi}(r(t)+R\omega,t+R)|^2 + |\nabla\overline{\phi}(r(t)+R\omega,t+R)|^2\Big]. \end{aligned}$$

Since both $\overline{\pi}$ and $\nabla\overline{\phi}$ have one term proportional to R^{-1} and other contributions of order R^{-2}, in the limit $R \to \infty$ only the product of the two leading terms survives and it follows that

$$\lim_{R\to\infty} I_R(t) = -\mathsf{e}^2(4\pi)^{-2}\int_{|\omega|=1} d^2\omega\,(1-\omega\cdot u(t))^{-5}\,(\omega\cdot\dot u(t))^2,$$

as before. We note that the radiated energy is of order ε and it therefore suffices to use the effective dynamics to order one, i.e., ignoring the radiation reaction.

8 Appendix: Proof of Lemma 2.1

In this appendix we prove Lemma 2.1. Since we need to use some identities from [5], we switch back to the original time scale of (1.1). Hence we have to show

Lemma 2.1 *For solutions of (1.1) with initial values satisfying (1.4), i.e., starting on the soliton manifold, and for $|\rho|_{L^2}$ sufficiently small we have*

$$\sup_{t\in\mathbb{R}} |\stackrel{\cdot\cdot}{v}(t)| \le C\varepsilon^3\,.$$

The constant C and the bound on $|\rho|_{L^2}$ depend only on the data.

Proof: From [5, Lemma 2.2 and Prop. 4.1] we already know the bounds

$$\sup_{t\in\mathbb{R}} |v(t)| \le \bar{v} < 1\,, \quad \sup_{t\in\mathbb{R}} |\dot{v}(t)| + \sup_{t\in\mathbb{R}} |\dot{p}(t)| \le C\varepsilon \quad \text{and} \quad \sup_{t\in\mathbb{R}} |\ddot{v}(t)| + \sup_{t\in\mathbb{R}} |\ddot{p}(t)| \le C\varepsilon^2\,, \tag{8.1}$$

for $|\rho|_{L^2}$ sufficiently small. The constants $\bar{v}$ and C appearing in (8.1) do not depend on the particular solution, but only on bounds for the initial values.

Denote

$$Z(x,t) = \left(\begin{array}{c} \varphi(x,t) \\ \psi(x,t) \end{array} \right) = \left(\begin{array}{c} \phi(x,t) - \phi_{v(t)}(x-q(t)) \\ \pi(x,t) - \pi_{v(t)}(x-q(t)) \end{array} \right).$$

Then, cf. [5], with $L(t)\phi = \nabla\phi\cdot v(t) + \dot{\phi}$,

$$\ddot{p}(t) = -\varepsilon^2\nabla^2 V(\varepsilon q(t))\cdot v(t) + \int d^3x\, (L(t)\varphi)(x+q(t),t)\nabla\rho(x) =: -\varepsilon^2\nabla^2 V(\varepsilon q(t))\cdot v(t) + M(t)\,.$$

Therefore

$$\dddot{p}\,(t) = -\varepsilon^2\nabla^2 V(\varepsilon q(t))\cdot \dot{v}(t) - \varepsilon^3\nabla^3 V(\varepsilon q(t))(v(t),v(t)) + \dot{M}(t)\,. \tag{8.2}$$

Below we will show

Lemma 8.1 *The estimate*

$$|\dot{M}(t)| \le C\left(\varepsilon^3 + |\rho|_{L^2}\int_0^t \frac{|\dddot{v}\,(s)|}{1+(t-s)^2}\,ds\right)$$

holds.

Then according to (8.2), (8.1), and assumption (U) on the potential,

$$|\dddot{p}\,(t)| \le C\left(\varepsilon^3 + |\rho|_{L^2}\int_0^t \frac{|\dddot{v}\,(s)|}{1+(t-s)^2}\,ds\right). \tag{8.3}$$

Since

$$|\dddot{v}\,| = \left|\frac{d}{dp}\Big(\frac{p}{\sqrt{1+p^2}}\Big)\dddot{p} + 3\frac{d^2}{dp^2}\Big(\frac{p}{\sqrt{1+p^2}}\Big)(\dot{p},\ddot{p}) + \frac{d^3}{dp^3}\Big(\frac{p}{\sqrt{1+p^2}}\Big)(\dot{p},\dot{p},\dot{p})\right| \le C(|\dddot{p}\,| + \varepsilon^3)\,,$$

the claim of Lemma 2.1 obtains from (8.3) by taking $|\rho|_{L^2}$ small enough. □

Thus it remains to give the

Proof of Lemma 8.1: First note

$$\dot{M}(t) = \int d^3x\, \Big\langle (\mathcal{L}(t)Z(\cdot,t))(x),\, \nabla\rho_*(x-q(t))\Big\rangle_{\mathbb{R}^2}\,, \quad \rho_*(x) = (\rho(x),0),$$

where $\mathcal{L}(t)Z = \nabla Z\cdot\dot{v}(t) + (\nabla^2 Z)(v(t),v(t)) + 2\nabla\dot{Z}\cdot v(t) + \ddot{Z}$. Because $\dot{Z} = AZ - B$, with

$$A(\phi,\pi) = (\pi,\Delta\phi) \quad \text{and} \quad B(x,t) = \left(\begin{array}{c} \nabla_v\phi_{v(t)}(x-q(t))\cdot\dot{v}(t) \\ \nabla_v\pi_{v(t)}(x-q(t))\cdot\dot{v}(t) \end{array} \right), \tag{8.4}$$

we obtain

$$\frac{d}{dt}(\mathcal{L}(t)Z) = A(\mathcal{L}(t)Z) - \mathcal{L}(t)B + 2\left[(\nabla^2 Z)(v,\dot{v}) + \nabla\dot{Z}\cdot\dot{v}\right] + \nabla Z\cdot\ddot{v}\,.$$

Let $U(t)$ again denote the group generated by the free wave equation on $D^{1,2}(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)$. Then

$$
\begin{aligned}
\dot M(t) &= \Big\langle U(t)[\mathcal{L}(0)Z(\cdot,0)],\, \nabla\rho_*(\cdot - q(t))\Big\rangle_{L^2(\mathbb{R}^2)} \\
&\quad + \int_0^t ds \Big[-\Big\langle U(t-s)[\mathcal{L}(s)B(\cdot,s)],\, \nabla\rho_*(\cdot - q(t))\Big\rangle_{L^2(\mathbb{R}^2)} \\
&\qquad + 2\Big\langle U(t-s)[(\nabla^2 Z(\cdot,s))(v(s),\dot v(s)) + \nabla\dot Z(\cdot,s)\cdot \dot v(s)],\, \nabla\rho_*(\cdot - q(t))\Big\rangle_{L^2(\mathbb{R}^2)} \\
&\qquad + \Big\langle U(t-s)[\nabla Z(\cdot,s)\cdot \ddot v(s)],\, \nabla\rho_*(\cdot - q(t))\Big\rangle_{L^2(\mathbb{R}^2)}\Big] \\
&=: T_0 + T_1 + T_2 + T_3\,.
\end{aligned}
$$

We estimate each term T_j separately, keeping all parts which do contain only initial values. Note that here according to (1.4) we have $Z(x,0) = 0$, so all these terms vanish. Nevertheless, we wanted to derive the general form of the estimate; see Remark 2.3(iii).

Estimate of T_3: Since $\dot Z = AZ - B$, we find

$$
Z(t) = U(t)Z(0) - \int_0^t ds\, U(t-s)B(\cdot,s), \tag{8.5}
$$

and hence

$$
\begin{aligned}
T_3 &= \int_0^t ds \Big\langle U(t)[\nabla Z(\cdot,0)\cdot \ddot v(s)],\, \nabla\rho_*(\cdot - q(t))\Big\rangle_{L^2(\mathbb{R}^2)} \\
&\quad - \int_0^t ds \int_0^s d\tau \Big\langle U(t-\tau)[\nabla B(\cdot,\tau)\cdot \ddot v(s)],\, \nabla\rho_*(\cdot - q(t))\Big\rangle_{L^2(\mathbb{R}^2)} \\
&=: T_{3,0} + T_{3,1}\,.
\end{aligned}
$$

Then Lemma 8.2 below and (8.1) imply through integration by parts in the d^3x-integral,

$$
T_{3,1} \le C\varepsilon^2 \int_0^t ds \int_0^s d\tau \,\frac{\varepsilon}{1+(t-\tau)^3} \le C\varepsilon^3\,.
$$

Estimate of T_0: This term is determined solely through the data.

Estimate of T_1: If we calculate the form of $\mathcal{L}(t)B = \nabla B\cdot\dot v + (\nabla^2 B)(v,v) + 2\nabla\dot B\cdot v + \ddot B$ explicitly from (8.4), fortunately many terms cancel, and we find with $\Phi_v = \begin{pmatrix}\phi_v\\ \pi_v\end{pmatrix}$,

$$
\mathcal{L}(t)B = \nabla_v\Phi_v(x-q)\cdot\, \ddot v\, + 3\nabla_v^2\Phi_v(x-q)(\dot v,\ddot v) + \nabla_v^3\Phi_v(x-q)(\dot v,\dot v,\dot v)\,.
$$

Now we may argue analogously to the estimate of T_3 and Lemma 8.2 to obtain with

$$
\begin{pmatrix}\tilde\phi(x)\\ \tilde\pi(x)\end{pmatrix} = [U(t-s)\mathcal{L}(s)B(\cdot,s)](x)\,,
$$

the estimate

$$
|\nabla\tilde\phi(x+q(t))| \le \frac{C}{1+(t-s)^2}\,(\varepsilon^3 + |\,\ddot v\,(s)|)\,,\quad |x|\le R_\rho\,,\quad t\ge s\,. \tag{8.6}
$$

Here we have used (8.1) and some of the estimates

$$
\begin{aligned}
|\nabla\nabla_v\phi_v(x)| + |\nabla\nabla_v^2\phi_v(x)| + |\nabla\nabla_v^3\phi_v(x)| &\le C(1+|x|)^{-2}, \\
|\nabla^2\nabla_v\phi_v(x)| + |\nabla^2\nabla_v^2\phi_v(x)| + |\nabla^2\nabla_v^3\phi_v(x)| &\le C(1+|x|)^{-3}, \\
|\nabla^3\nabla_v\phi_v(x)| + |\nabla^3\nabla_v^2\phi_v(x)| + |\nabla^3\nabla_v^3\phi_v(x)| &\le C(1+|x|)^{-4}, \\
|\nabla^4\nabla_v\phi_v(x)| + |\nabla^4\nabla_v^2\phi_v(x)| + |\nabla^4\nabla_v^3\phi_v(x)| &\le C(1+|x|)^{-5}, \\
|\nabla\nabla_v\pi_v(x)| + |\nabla\nabla_v^2\pi_v(x)| + |\nabla\nabla_v^3\pi_v(x)| &\le C(1+|x|)^{-3}, \\
|\nabla^2\nabla_v\pi_v(x)| + |\nabla^2\nabla_v^2\pi_v(x)| + |\nabla^2\nabla_v^3\pi_v(x)| &\le C(1+|x|)^{-4}, \\
|\nabla^3\nabla_v\pi_v(x)| + |\nabla^3\nabla_v^2\pi_v(x)| + |\nabla^3\nabla_v^3\pi_v(x)| &\le C(1+|x|)^{-5}, \qquad (8.7)
\end{aligned}
$$
$$(8.8)$$

for $x \in \mathbb{R}^3$ and $|v| \le \bar{v}$. From (8.6) we conclude

$$T_1 = -\int_0^t ds \int_{|x|\le R_\rho} d^3x\, \nabla\tilde{\phi}(x+q(t))\rho(x) \le C\left(\varepsilon^3 + |\rho|_{L^2}\int_0^t \frac{|\ddot{v}(s)|}{1+(t-s)^2}\,ds\right).$$

Estimate of T_2: Let $P(t)Z = \nabla^2 Z(\cdot,t)v(t) + \nabla\dot{Z}(\cdot,t)$. Then

$$\frac{d}{dt}(P(t)Z) = P(t)\dot{Z} + (\nabla^2 Z)\dot{v} = A(P(t)Z) - P(t)B + (\nabla^2 Z)\dot{v}\,.$$

Therefore by definition of T_2,

$$
\begin{aligned}
T_2 &= 2\int_0^t ds\,\big\langle U(t)[(P(0)Z(\cdot,0))\cdot\dot{v}(s)],\, \nabla\rho_*(\cdot-q(t))\big\rangle_{L^2(\mathbb{R}^2)} \\
&\quad +2\int_0^t ds\int_0^s d\tau\,\big\langle U(t-\tau)\big[-P(\tau)B(\cdot,\tau) + (\nabla^2 Z(\cdot,\tau))\dot{v}(\tau)\big]\cdot\dot{v}(s),\, \nabla\rho_*(\cdot-q(t))\big\rangle_{L^2(\mathbb{R}^2)} \\
&=: T_{2,0} + T_{2,1} + T_{2,2}\,.
\end{aligned}
$$

To estimate $T_{2,1}$, observe

$$P(t)B = \nabla\nabla_v\Phi_v(x-q)\cdot\ddot{v} + \nabla\nabla_v^2\Phi_v(x-q)(\dot{v},\dot{v})\,.$$

Hence we may argue as before to find $|T_{2,1}| \le C\varepsilon^3$. In order to bound $T_{2,2}$, similarly to the estimate of T_3 we again use (8.5) to get

$$
\begin{aligned}
T_{2,2} &= 2\int_0^t ds\int_0^s d\tau\,\big\langle U(t)[\nabla^2 Z(0)(\dot{v}(\tau),\dot{v}(s))],\, \nabla\rho_*(\cdot-q(t))\big\rangle_{L^2(\mathbb{R}^2)} \\
&\quad -2\int_0^t ds\int_0^s d\tau\int_0^\tau d\sigma\,\big\langle U(t-\sigma)[\nabla^2 B(\cdot,\sigma)(\dot{v}(\tau),\dot{v}(s))],\, \nabla\rho_*(\cdot-q(t))\big\rangle_{L^2(\mathbb{R}^2)} \\
&=: T_{2,2,0} + T_{2,2,1}\,.
\end{aligned}
$$

By (8.7) and the argument of Lemma 8.2 then

$$T_{2,2,1} \le \int_0^t ds\int_0^s d\tau\int_0^\tau d\sigma\,\frac{C\varepsilon^3}{1+(t-\sigma)^4} \le C\varepsilon^3\,.$$

Summarizing all above estimates for T_0–T_3, we hence arrive at

$$
\begin{aligned}
|\dot M(t)| \;\le\;& C\left(\varepsilon^3+|\rho|_{L^2}\int_0^t \frac{|\ddot v(s)|}{1+(t-s)^2}\,ds\right)\\
&+\left\langle U(t)[\mathcal{L}(0)Z(\cdot,0)],\,\nabla\rho_*(\cdot-q(t))\right\rangle_{L^2(\mathbb{R}^2)}\\
&+2\int_0^t ds\left\langle U(t)[(P(0)Z(\cdot,0))\cdot\dot v(s)],\,\nabla\rho_*(\cdot-q(t))\right\rangle_{L^2(\mathbb{R}^2)}\\
&+2\int_0^t ds\int_0^s d\tau\left\langle U(t)[\nabla^2 Z(0)(\dot v(\tau),\dot v(s))],\,\nabla\rho_*(\cdot-q(t))\right\rangle_{L^2(\mathbb{R}^2)}\\
&+\int_0^t ds\left\langle U(t)[\nabla Z(\cdot,0)\cdot\ddot v(s)],\,\nabla\rho_*(\cdot-q(t))\right\rangle_{L^2(\mathbb{R}^2)}. \qquad (8.9)
\end{aligned}
$$

Concerning the terms that contain data, these vanish here since $Z(x,0)=0$ as a consequence of (1.4). This completes the proof of Lemma 8.1. □

In case of solutions starting not on, but close, to the soliton manifold as discussed in Remark 2.3(iii), conditions on the data have to be imposed to ensure the last four terms in (8.9) can also be estimated by $C\varepsilon^3$. In [5, Thm. 2.6] and Section 4 of that paper details are carried out for derivatives of one order less.

Above we used the following lemma.

Lemma 8.2 *The estimate*

$$\|\nabla[U(t-\tau)\nabla B(\cdot,\tau)](\cdot+q(t))\|_{R_\rho}\le C\,\frac{\varepsilon}{1+(t-\tau)^3}\,,\qquad t\ge\tau\,, \qquad (8.10)$$

holds.

Proof: Such estimates have already been used in [5], but we nevertheless include some details of the argument. Let

$$\begin{pmatrix}\tilde\phi(x)\\ \tilde\pi(x)\end{pmatrix}=[U(t-\tau)\nabla B(\cdot,\tau)](x)$$

for fixed t,τ. By Kirchhoff's formula for the solution to the wave equation and by (8.4),

$$
\begin{aligned}
\nabla\tilde\phi(x+q(t)) \;=\;& \frac{1}{4\pi(t-\tau)^2}\int_{|y-x-q(t)|=(t-\tau)} d^2y\,\Big[(t-\tau)\nabla^2\nabla_v\pi_{v(\tau)}(y-q(\tau))\cdot\dot v(\tau)\\
&+\nabla^2\nabla_v\phi_{v(\tau)}(y-q(\tau))\cdot\dot v(\tau)\\
&+\nabla^3\nabla_v\phi_{v(\tau)}(y-q(\tau))(\dot v(\tau),y-x-q(t))\Big]\,. \qquad (8.11)
\end{aligned}
$$

Now $|x|\le R_\rho$ and $|y-x-q(t)|=(t-\tau)$ yields $|y-q(\tau)|\ge(t-\tau)-\bar v(t-\tau)-R_\rho=(1-\bar v)(t-\tau)-R_\rho$ by (8.1). As a consequence of (8.7), hence (8.10) follows from (8.11). □

Acknowledgement: We thank A. Komech for useful discussions.

References

[1] P.A.M. Dirac: Classical theory of radiating electrons. *Proc. Royal Soc. London* **A 167** (1938), 148-169.

[2] R. Haag: Die Selbstwechselwirkung des Elektrons. *Z. Naturforsch.* **10a**, 752-761 (1955).

[3] J.D. Jackson: *Classical electrodynamics.* 2nd edition, John Wiley & Sons, New York-London 1975.

[4] Ch. Jones: Geometric singular perturbation theory. In: *Dynamical Systems, Proceedings, Montecatini Terme 1994*, Ed. Johnson R., LNM 1609, Springer, Berlin-New York 1995, pp. 44-118.

[5] A. Komech, M. Kunze & H. Spohn: Effective dynamics for a mechanical particle coupled to a wave field. To appear in *Comm. Math. Phys.*

[6] A. Komech & H. Spohn: Soliton-like asymptotics for a classical particle interacting with a scalar wave field. *Nonlinear Anal.* **33** (1998), 13-24.

[7] A. Komech, H. Spohn & M. Kunze: Long-time asymptotics for a classical particle interacting with a scalar wave field. *Comm. Partial Differential Equations* **22** (1997), 307-335.

[8] P. Lancaster & M. Tismenetsky: *The theory of matrices.* 2nd edition, Academic Press, Orlando-New York 1985.

[9] L.D. Landau & E.M. Lifshitz: *Course of theoretical physics, vol. 2: the classical theory of fields.* 4th edition, Pergamon Press, Oxford-New York 1975.

[10] F. Rohrlich: *Classical charged particles.* 2nd edition, Addison-Wesley, Reading, MA, 1990.

[11] K. Sakamoto: Invariant manifolds in singular perturbation problems for ordinary differential equations. *Proc. Roy. Soc. Edinburgh Sect. A* **116**, (1990) 45-78.

[12] H. Spohn: Runaway charged particles and center manifolds. Preprint 1998.

[13] W. Thirring: *A course in mathematical physics, vol. 2: classical field theory.* Springer, New York-Wien 1978.

[14] A.D. Yaghjian: *Relativistic dynamics of a charged sphere.* Lecture Notes in Physics **m 11**, Springer, Berlin-New York 1992.