# An Introduction to Moreau's Sweeping Process

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**Abstract.** Starting from an elementary level, this rewiew paper summarizes some results and applications concerning first and second order sweeping processes.

## 1 Introduction

These lecture notes concern the so-called "sweeping process" that plays an important role in elastoplasticity, quasistatics, and dynamics. A sweeping process consists of two main ingredients: one that sweeps and one that is swept. As an example, consider a large ring that contains a smaller ball inside, and the ring will start to move at time $t = 0$. Depending on the motion of the ring, the ball will just stay where it is (in case it is not hit by the ring), or otherwise it is swept towards the interior of the ring. In this latter case the velocity of the ball has to point inwards to the ring in order not to leave; see Fig. 1.

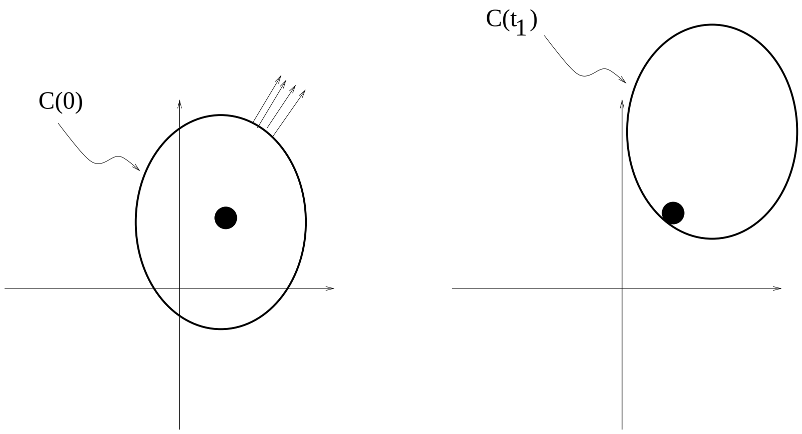


**Fig. 1.** A sweeping process

In more mathematical terms, this becomes

$$-u'(t) \in N_{C(t)}(u(t)) \quad \text{a.e. in} \quad [0,T], \quad u(0) = u_0 \in C(0), \tag{1}$$

where $u(t)$ is the position of the ball at time $t$, whereas $C(t)$ is the (set comprising the) ring at time $t$. The expression $N_{C(t)}(u(t))$ denotes the outward normal cone to the set $C(t)$ at position $u(t)$, so (1) just says that the velocity $u'(t)$ of the ball has to point inwards to the ring at almost every (a.e.) time $t \in [0, T]$. The restriction to "almost every" is due to the fact that usually we will find no smooth solution $t \mapsto u(t)$ of (1), but only solutions that are differentiable everywhere besides on some subset of $[0, T]$ of measure zero. The initial condition $u(0) \in C(0)$ states that the ball initially is contained in the ring.

Equ. (1) is the simplest instance of the sweeping process, introduced by Moreau in the seventies. In general, the time-dependent moving set $t \mapsto C(t)$ is given, and we intend to prove the existence of a (unique) solution $t \mapsto u(t)$ that will take values in some Hilbert space $H$ (above $H = \mathbb{R}^2$). Note also that $C(t)$ is allowed to change its shape while moving, contrary to the example of the ring which just moved by translation and kept its initial shape.

The paper is organized as follows. In Section 2 we start by introducing some concepts from convex analysis and functional analysis, like normal cones and functions of bounded variation. Then in the first main part of these notes, Section 3, we summarize some important results for the sweeping process (1) concerning existence and uniqueness of solutions. As expected, the better behaved the set will move (e.g. without jumps), the better will be the solution. So we will prove in Section 3.1 quite in some detail the existence of a unique Lipschitz continuous solution to (1) in case the set $t \mapsto C(t)$ moves in a Lipschitz continuous way (in a suitable sense, measured by the Hausdorff distance of sets). Afterwards in Section 3.2 we carefully analyze an example from elastoplasticity and show how the general theory from Section 3.1 applies. Then in Section 3.3 we discuss a variant of the sweeping process which arises naturally in the modeling of more complex problems, like quasi-variational inequalities. Contrary to (1) the moving set here may depend also on the current state $u(t)$, i.e. $C = C(t, u(t))$ instead of $C = C(t)$, and we will see in particular that not all such state-dependent sweeping processes will admit a solution. Finally in Section 3.4 we add some supplementary remarks.

In the second main part (Sections 4 and 5), we turn to problems which are of second order in time. In Section 4 we deal with

$$-q''(t) \in N_{V(q(t))}(q'(t)). \tag{2}$$

Note that this formulation does make sense only if the velocity $u(t) = q'(t)$ were an absolutely continuous function (thus admitting a derivative a.e.). However, under the usual assumptions that the multifunction $q \mapsto V(q)$ is continuous in the sense of Hausdorff distance and has closed convex values with nonempty interior, it turns out that $u$ in general is only a continuous function of bounded variation; therefore the differential inclusion (2) has to be rewritten accordingly. We will prove the existence of a solution under somewhat stronger technical assumptions on $V$, since this allows to highlight some additional technical difficulties that arise when using a discretization procedure (similar to the one in [38]) which is of great importance for the subsequent applications.

In Section 5, we consider such an application to the dynamics of unilateral contact. In Section 5.1 we present the corresponding mathematical formulation, introduced by Moreau, to model the dynamics of a particle or a system under frictionless constraints. The appropriate mathematical description is given by a differential inclusion which is a second order sweeping process of the above type, with an extra term to account for forces acting on the system. Then we outline the recent proof by Mabrouk [38] of the existence of a solution which is a motion $q$ with bounded variation velocity $u$. It should be stressed that purely inelastic as well as purely or partially elastic collisions can be handled with this technique. Finally, in Section 5.2, we give a very brief account on how the previous ideas apply to unilateral contact with friction and on some promising recent advances in the field.

## 2 Some preliminaries from convex analysis and functional analysis

In these notes we will be concerned with a Hilbert space $H$ which may have finite or infinite dimension. The inner product and norm, respectively, are denoted by $\langle \cdot, \cdot \rangle$ and $|\cdot|$, and $\overline{B}_r(x)$ denotes the closed ball of radius $r$ centered at $x \in H$.

For a closed convex $C \subset H$ and $x \in C$ the set

$$N_C(x) = \{\xi \in H : \langle \xi, c - x \rangle \leq 0 \ \ \forall c \in C\}$$

denotes the outward normal cone to $C$ at $x$. See Fig. 2 for some examples.

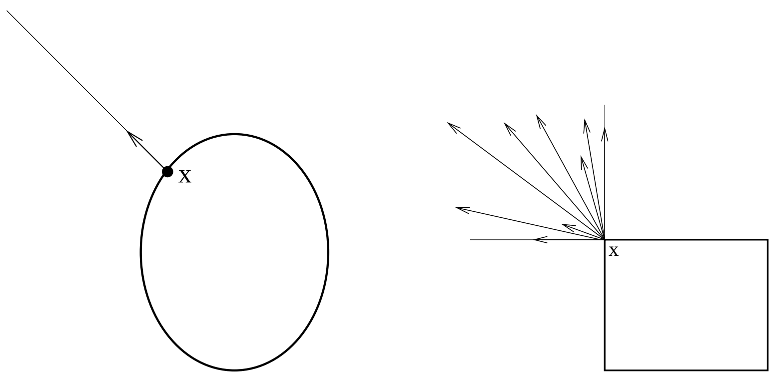


**Fig. 2.** Examples of normal cones

Observe that in fact we have drawn the sets $x + N_C(x)$ rather than $N_C(x)$.

*Remark 1.* We always have that $0 \in N_C(x)$, $N_{\{x\}}(x) = H$, and $N_C(x) = \{0\}$ for $x \in \mathrm{int}C$, the interior of $C$, if $\mathrm{int}C \neq \emptyset$. The latter relation shows the outward normal cone is only interesting for $x \in \partial C$, the boundary of $C$. Note that $\mathrm{int}C = \emptyset$ might well be the case, think e.g. of $C = \{(x,0) : 0 \leq x \leq 1\} \subset \mathbb{R}^2$. In this example we obtain $N_C((0,0)) = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1 \leq 0, \xi_2 \in \mathbb{R}\}$, $N_C((1,0)) = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1 \geq 0, \xi_2 \in \mathbb{R}\}$, and $N_C((x,0)) = \{(0, \xi_2) \in \mathbb{R}^2 : \xi_2 \in \mathbb{R}\}$ for $x \in ]0,1[$ as normal cones. $\diamondsuit$

If $x \in H$ and $C \subset H$ is closed convex, then there exists a unique $y \in C$ that minimizes the distance of $x$ to $C$. This $y$ is called the projection of $x$ onto $C$ and written as $y = \mathrm{proj}\,(x, C)$. Hence the characterizing properties are

$$y = \mathrm{proj}\,(x,C) \quad \text{if and only if} \quad y \in C, \quad |x - y| = \mathrm{dist}\,(x, C) \tag{3}$$

with $\mathrm{dist}\,(x,C) = \inf\{|x - c| : c \in C\}$. Equivalently,

$$y = \mathrm{proj}\,(x,C) \quad \text{if and only if} \quad y \in C, \quad \langle y - x, y - c\rangle \leq 0 \;\; \forall c \in C, \tag{4}$$

and hence

$$y = \mathrm{proj}\,(x,C) \quad \text{if and only if} \quad y \in C, \quad x - y \in N_C(y). \tag{5}$$

Moreover,

$$\mathrm{d_H}(C_1, C_2) = \max\Big\{ \sup_{x \in C_2} \mathrm{dist}\,(x, C_1),\ \sup_{x \in C_1} \mathrm{dist}\,(x, C_2)\Big\} \tag{6}$$

is the Hausdorff distance between the sets $C_1$ and $C_2$.

It will also be convenient to introduce the support function of a convex $C \subset H$ defined as

$$\delta^*(x, C) = \sup\{\langle x, c\rangle : c \in C\} \quad \text{for} \quad x \in H. \tag{7}$$

*Remark 2.* Note that $x \mapsto \delta^*(x,C)$ is a convex map on $C$, that is, $\delta^*(\lambda x + (1-\lambda)\bar{x}, C) \leq \lambda\delta^*(x,C) + (1-\lambda)\delta^*(\bar{x}, C)$ for $\lambda \in [0,1]$ and $x, \bar{x} \in H$. If $C$ is unbounded, it might be the case that $\delta^*(x,C) = +\infty$ for some $x \in H$. $\diamondsuit$

We will later need some further elementary geometrical facts.

**Lemma 1.** *The following assertions hold:*

*(a) If $U \subset H$ is a closed subspace and $V = U^{\perp}$ is the orthogonal complement of $U$ in $H$, then $N_V(x) = U$ for $x \in V$.*
*(b) If $C, D \subset H$ are nonempty, closed, and convex with $\mathrm{int}C \cap D \neq \emptyset$, then*

$$N_{C \cap D}(x) = N_C(x) + N_D(x) \quad \textit{for} \quad x \in C \cap D.$$

*(c) If $C \subset H$ is nonempty, closed, and convex and $x \in H$ is such that* $\operatorname{dist}(x, H \setminus C) \geq \rho > 0$, *then* $\rho|y| + \langle y, x\rangle \leq \delta^*(y, C)$ *for all* $y \in H$.

*(d) If $C \subset H$ is nonempty, closed, and convex and $\rho > 0$, then the set* $C_\rho = \{x \in H : \operatorname{dist}(x, H \setminus C) \geq \rho\}$ *is convex.*

**Proof:** (a) If $u \in U$, then $\langle u, v - x\rangle = 0$ for all $v \in V$, since also $x \in V$. Conversely, if $u \in N_V(x)$, then

$$\langle u, v\rangle \leq \langle u, x\rangle \quad \text{for all} \quad v \in V. \tag{8}$$

Assume $\langle u, v_0\rangle \neq 0$ for some $v_0 \in V$. Then we may suppose $\langle u, v_0\rangle > 0$, since otherwise we could replace $v_0$ by $(-v_0) \in V$. But then we can use $v = \lambda v_0 \in V$ in (8) and let $\lambda \to \infty$ to obtain a contradiction. Thus $\langle u, v\rangle = 0$ for all $v \in V$, i.e., $u \in V^\perp = \overline{U} = U$. (b) Firstly, for $\xi \in N_C(x)$, $\eta \in N_D(x)$, and $c \in C \cap D$ we find $\langle \xi + \eta, c - x\rangle = \langle \xi, c - x\rangle + \langle \eta, c - x\rangle \leq 0$, whence $\xi + \eta \in N_{C\cap D}(x)$. On the other hand, let $\xi \in N_{C\cap D}(x)$ for $x \in C \cap D$. Since generally $N_C(x) = N_{C-x}(0)$ for $x \in C$ we can assume without loss of generality that $x = 0 \in C \cap D$. Define

$$\mathcal{D} = D \times [0, \infty[ \quad \text{and} \quad \mathcal{C} = \{(c, t) \in C \times \mathbb{R} : t \leq \langle \xi, c\rangle\}.$$

We claim that $\mathcal{D} \cap \operatorname{int} \mathcal{C} = \emptyset$. Indeed, otherwise we would find $c_0 \in C \cap D$, $t_0 \in [0, \infty[$ and $\varepsilon > 0$ such that $|c - c_0| \leq \varepsilon$ and $|t - t_0| \leq \varepsilon$ for $c \in H$ and $t \in \mathbb{R}$ implies $t \leq \langle \xi, c\rangle$. As $\xi \in N_{C\cap D}(0)$, we have $\langle \xi, c_0\rangle \leq 0$, and hence $t_0 \in [0, \infty[$ and also $t \leq \langle \xi, c_0\rangle \leq 0$ for all $t$ with $|t - t_0| \leq \varepsilon$, a contradiction. On the other hand, $\operatorname{int} \mathcal{C} \neq \emptyset$, since e.g. $(\hat{c}, \langle \xi, \hat{c}\rangle - 1) \in \operatorname{int} \mathcal{C}$ for $\hat{c} \in \operatorname{int} C$. Since both $\mathcal{D}$ and $\mathcal{C}$ are convex subsets of $H \times \mathbb{R}$ that contain $(0, 0) \in H \times \mathbb{R}$, by Mazur's separation theorem [26, Thm. 7.12 (b)] we find $\eta \in H$ and $r \in \mathbb{R}$ such that

$$\begin{aligned} \mathcal{D} &\subset \{(u, t) \in H \times \mathbb{R} : \langle \eta, u\rangle + rt \geq 0\}, \\ \operatorname{int} \mathcal{C} &\subset \{(u, t) \in H \times \mathbb{R} : \langle \eta, u\rangle + rt < 0\}. \end{aligned} \tag{9}$$

Since by assumption $\hat{c} \in \operatorname{int} C \cap D$ for some $\hat{c} \in H$, we can use $(\hat{c}, 0) \in \mathcal{D}$ and $(\hat{c}, \langle \xi, \hat{c}\rangle - 1) \in \operatorname{int} \mathcal{C}$ to conclude from relations (9) that $r \neq 0$. As $0 \in D$ we find $(0, t) \in \mathcal{D}$ for all $t \in [0, \infty[$, and thus $rt \geq 0$ for all $t \in [0, \infty[$. This shows that in fact $r > 0$. Define $\xi_1 = -r^{-1}\eta$ and $\xi_2 = \xi + r^{-1}\eta$. Then $\xi = \xi_1 + \xi_2$. Moreover, if $d \in D$ then $(d, 0) \in \mathcal{D}$, and hence $\langle \xi_1, d\rangle = -r^{-1}\langle \eta, d\rangle \leq 0$ by (9), i.e., $\xi_1 \in N_D(0)$. Next, for $c \in C$ we take $(c, \langle \xi, c\rangle) \in \mathcal{C}$ to obtain $\langle \xi_2, c\rangle = r^{-1}(r\langle \xi, c\rangle + \langle \eta, c\rangle) \leq 0$, and therefore $\xi_2 \in N_C(0)$, i.e., $\xi \in N_D(0) + N_C(0)$. (c) Fix $y \neq 0$ and $\varepsilon > 0$ small. Then we obtain $x + (1 - \varepsilon)\rho\frac{y}{|y|} \in C$ by assumption, and hence $\delta^*(y, C) \geq \langle y, x + (1 - \varepsilon)\rho\frac{y}{|y|}\rangle = \langle y, x\rangle + (1 - \varepsilon)\rho|y|$. Thus $\varepsilon \to 0$ yields the claimed estimate. (d) Assume on the contrary that $|\lambda x + (1 - \lambda)\bar{x} - z| \leq \rho - \varepsilon$ for some $\lambda \in ]0, 1[$, $x, \bar{x} \in C_\rho$, $z \in H \setminus C$, and $\varepsilon > 0$. Let $x_1 = z + (1 - \lambda)(x - \bar{x})$ and $\bar{x}_1 = z + \lambda(\bar{x} - x)$. Then $|x - x_1| \leq \rho - \varepsilon$ and $|\bar{x} - \bar{x}_1| \leq \rho - \varepsilon$. As $x, \bar{x} \in C_\rho$ it follows that $x_1, \bar{x}_1 \in C$. But since $C$ is convex, then also $z = \lambda x_1 + (1 - \lambda)\bar{x}_1 \in C$, a contradiction. □

The variation of a function $u : [0,T] \to H$ is defined as

$$\begin{aligned} \mathrm{var}\,(u) &= \mathrm{var}\,(u,[0,T]) \\ &= \sup\Bigg\{ \sum_{i=0}^{N-1} |u(t_{i+1}) - u(t_i)| : \ 0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T \\ &\qquad\qquad\qquad\qquad\qquad \text{is a partition of } [0,T] \Bigg\}, \end{aligned} \tag{10}$$

and $u$ is called of bounded variation, if $\mathrm{var}\,(u) < \infty$.

*Example 1.* (a) If $u : [0,T] \to H$ is Lipschitz continuous, i.e., for some constant $L > 0$

$$|u(t) - u(s)| \le L|t-s|, \quad t, s \in [0,T],$$

then $u$ is of bounded variation, and $\mathrm{var}\,(u) \le LT$. To see this, consider any partition of $[0,T]$ as in the definition (10). Then

$$\sum_{i=0}^{N-1} |u(t_{i+1}) - u(t_i)| \le L \sum_{i=0}^{N-1} (t_{i+1} - t_i) = L(t_N - t_0) = LT.$$

(b) If $u : [0,T] \to \mathbb{R}$ is increasing, then $u$ is of bounded variation, with $\mathrm{var}\,(u) = u(T) - u(0)$. This follows from

$$\sum_{i=0}^{N-1} |u(t_{i+1}) - u(t_i)| = \sum_{i=0}^{N-1} (u(t_{i+1}) - u(t_i)) = u(T) - u(0).$$

Observe that $u$ is not necessarily continuous here.

(c) Let $u : [0,T] \to H$ be piecewise affine, i.e., there exist $0 = t_0^* < t_1^* < \ldots < t_{N-1}^* < t_N^* = T$ such that

$$u(t) = u_i + \left( \frac{t - t_i^*}{t_{i+1}^* - t_i^*} \right) (u_{i+1} - u_i), \quad t \in [t_i^*, t_{i+1}^*],$$

with $u_i = u(t_i^*)$. Then

$$\mathrm{var}\,(u) = \sum_{i=0}^{N-1} |u_{i+1} - u_i|.$$

Indeed, if we are given any partition of $[0,T]$, then we may add the points $0 = t_0^* < t_1^* < \ldots < t_{N-1}^* < t_N^* = T$ to this partition as this only increases the sum. However, if $t_i^* = s_0 < s_1 < \ldots < s_{M-1} < s_M = t_{i+1}^*$, then

$$\sum_{j=0}^{M-1} |u(s_{j+1}) - u(s_j)| = \sum_{j=0}^{M-1} \left( \frac{s_{j+1} - s_j}{t_{i+1}^* - t_i^*} \right) |u_{i+1} - u_i| = |u_{i+1} - u_i|,$$

and hence no additional contributions appear.

(d) To give an example of a function which is continuous but not of bounded variation, let $u(t) = t\cos(\pi/2t)$ for $t \in ]0,1]$ and $u(0) = 0$; see [1, Ch. 8]. Then $u$ is continuous. For the partition

$$t_0 = 0, \quad t_1 = \frac{1}{2n}, \quad t_2 = \frac{1}{2n-1}, \quad \ldots, \quad t_{2n} = 1$$

one calculates

$$\sum_{i=0}^{2n-1} |u(t_{i+1}) - u(t_i)| = t_1 + t_1 + t_3 + t_3 + \ldots + t_{2n-1} + t_{2n-1} = \sum_{k=1}^{n} \frac{1}{k},$$

and hence the supremum over all possible partitions equals $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$.

(e) A well-known further class of examples of functions that are continuous but of unbounded variation even on every subinterval are (almost every) paths of a Brownian motion; see [5, Cor. 12.27]. $\diamondsuit$

In order to prove the existence of solutions to certain variants of the sweeping process, we mostly will construct a sequence of approximating solutions whose variations are uniformly bounded. The following compactness theorem then allows to select a subsequence which converges towards some limit function, and the task is afterwards to show the limit function is indeed the desired solution.

**Theorem 1.** *Let $H$ be a Hilbert space and $(u_n)_{n\in\mathbb{N}}$ a sequence of functions $u_n : [0,T] \to H$ that is bounded uniformly in norm and variation, i.e.,*

$$|u_n(t)| \le M_1, \quad n \in \mathbb{N},\ t \in [0,T], \quad \textit{and} \quad \operatorname{var}(u_n) \le M_2, \quad n \in \mathbb{N}, \tag{11}$$

*for some constants $M_1, M_2 > 0$ independently of $n \in \mathbb{N}$ and $t \in [0,T]$. Then there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and a function $u : [0,T] \to H$ such that $\operatorname{var}(u) \le M_2$ and $u_{n_k}(t) \to u(t)$ weakly in $H$ for all $t \in [0,T]$, i.e.,*

$$\langle u_{n_k}(t), z\rangle \to \langle u(t), z\rangle \quad \textit{for all} \quad z \in H \quad \textit{as} \quad k \to \infty.$$

**Proof:** See [42, p. 10]. $\square$

The next lemma summarizes some facts related to weak convergence.

**Lemma 2.** *Let $u_n \to u$ weakly in $H$.*

*(a) $|u| \le \liminf_{n\to\infty} |u_n|$ holds.*
*(b) If $u_n \in C + \overline{B}_{\varepsilon_n}(0)$ for some closed convex $C \subset H$ and some sequence $\varepsilon_n \to 0$, then $u \in C$.*

**Proof:** (a) See [71, Thm. V.1.1 ii)]. (b) If $u \notin C$, then by the separating hyperplane theorem, cf. [64, Thm. 3.4], there exists $z \in H$ and $\gamma \in \mathbb{R}$ such that $\langle z, u\rangle < \gamma < \langle z, c\rangle$ for all $c \in C$. For every $n \in \mathbb{N}$ choose $r_n \in H$ and $c_n \in C$ with $|r_n| \le \varepsilon_n$ and $u_n = c_n + r_n$. Then according to the weak convergence and since $|\langle z, r_n\rangle| \le |z||r_n| \to 0$, $\langle z, c_n\rangle = \langle z, u_n\rangle - \langle z, r_n\rangle \to \langle z, u\rangle$. Thus from $\gamma < \langle z, c_n\rangle$ we obtain the contradiction $\langle z, u\rangle < \gamma \le \langle z, u\rangle$. □

Later we will also need a result on the behaviour of certain integral functionals w.r.t. to weak convergence of functions taking values in $H$.

**Lemma 3.** *Let $(v_n)_{n\in\mathbb{N}}$ be a sequence of functions $v_n : [0, T] \to H$ such that $v_n \to v_*$ in the weak-star topology of $L^\infty([0, T]; H)$, i.e.,*

$$\int_0^T \langle v_n(t), \varphi(t)\rangle\, dt \to \int_0^T \langle v_*(t), \varphi(t)\rangle\, dt \quad \textit{as} \quad n \to \infty$$

*for all $\varphi \in L^1([0, T]; H)$. [The latter means that $\varphi : [0, T] \to H$ is (strongly) measurable and $\int_0^T |\varphi(t)|dt < \infty$.] Assume that for each $t \in [0, T]$ the set $C(t) \subset H$ is nonempty, closed, and convex such that $\mathrm{d_H}(C(t), C(s)) \le L|t-s|$ for $t, s \in [0, T]$ with some constant $L > 0$. Let*

$$\Phi(v) = \int_0^T \delta^*(v(t), C(t))\, dt \quad \textit{for} \quad v \in L^\infty([0, T]; H).$$

*Then $\Phi$ is lower semicontinuous, i.e.,*

$$\Phi(v_*) \le \liminf_{n\to\infty} \Phi(v_n).$$

**Proof:** See [63, Corollary, p. 227]. The proof rests on the fact that $f(v, t) = \delta^*(v, C(t))$ is measurable and $f(\cdot, t)$ is convex for all $t \in [0, T]$. One may also consult [23] for similar results in more general settings. □

**Lemma 4.** *Let $u : [0, T] \to H$ be an absolutely continuous function. Then*

$$\int_0^T \langle u'(t), u(t)\rangle dt = \frac{1}{2}|u(T)|^2 - \frac{1}{2}|u(0)|^2.$$

**Proof:** Under the assumption made it can be shown by consideration of the difference quotient that

$$\frac{1}{2}\frac{d}{dt}\Big(|u(t)|^2\Big) = \langle u'(t), u(t)\rangle \tag{12}$$

at every $t \in ]0, T[$ where $u$ has a derivative. Integration yields the claim. □

We remark that absolute continuity of a function is a somewhat technical mathematical notion which for our purpose can be stated as "$u$ is (up to a constant) the indefinite integral of its derivative".

# 3 First-order problems

## 3.1 The Lipschitz continuous sweeping process

In this section we assume that $t \mapsto C(t)$ is Lipschitz continuous, i.e.,

$$\mathrm{d_H}(C(t), C(s)) \leq L|t-s|, \quad t, s \in [0,T], \tag{13}$$

for some constant $L > 0$. Let us first consider an example where the set $C(t)$ is driven by a point $\xi(t)$.

*Example 2.* Let $C \subset H$ be a fixed nonempty, closed, and convex set and $\xi : [0,T] \to H$ a function which is Lipschitz continuous with constant $L > 0$. Let $C(t) = C + \xi(t)$ for $t \in [0,T]$. Then $t \mapsto C(t)$ satisfies (13). To see this, fix $x \in C(s)$. Then $y = [x - \xi(s)] + \xi(t) \in C + \xi(t) = C(t)$, and thus $\mathrm{dist}\,(x, C(t)) \leq |x-y| = |\xi(t) - \xi(s)| \leq L|t-s|$. Hence also $\sup_{x \in C(s)} \mathrm{dist}\,(x, C(t)) \leq L|t-s|$, and similarly $\sup_{y \in C(t)} \mathrm{dist}\,(y, C(s)) \leq L|t-s|$. Therefore by the definition of $\mathrm{d_H}$ in (6) we obtain the claim. ◊

Our aim is to show that for a Lipschitz continuous moving set there exists a unique solution to (1). Let us first fix what will be called a solution.

**Definition 1.** A function $u : [0,T] \to H$ is a solution of (1), if

(a) $u(0) = u_0$;
(b) $u(t) \in C(t)$ for all $t \in [0,T]$;
(c) $u$ is differentiable at almost every point $t \in ]0,T[$;
(d) $-u'(t) \in N_{C(t)}(u(t))$ for almost every $t \in ]0,T[$.

The following theorem is a very basic existence result.

**Theorem 2 (Existence).** *Assume the map $t \mapsto C(t)$ satisfies (13) and $C(t) \subset H$ is nonempty, closed, and convex for every $t \in [0,T]$. Let $u_0 \in C(0)$. Then there exists a solution $u : [0,T] \to H$ of (1) which is Lipschitz continuous of constant $L$. In particular, $|u'(t)| \leq L$ for almost every $t \in ]0,T[$.*

**Proof:** The following discretization scheme lies at the heart of many proofs for sweeping processes. We fix $n \in \mathbb{N}$ and choose a time discretization

$$0 = t_0^n < t_1^n < \ldots < t_{N-1}^n < t_N^n = T, \text{ with } (t_{i+1}^n - t_i^n) \leq \frac{1}{n}, \; 0 \leq i \leq N-1. \tag{14}$$

E.g. we can set $t_i^n = i/n$, but we need not fix the discretization explicitly. The value of $N \in \mathbb{N}$ will depend on $n$, and $N \to \infty$ for $n \to \infty$. We define the step approximations $u^n : [0,T] \to H$ as follows. Let

$$u_0^n = u_0, \quad u_{i+1}^n = \mathrm{proj}\,(u_i^n, C(t_{i+1}^n)) \in C(t_{i+1}^n), \quad 0 \leq i \leq N-1. \tag{15}$$

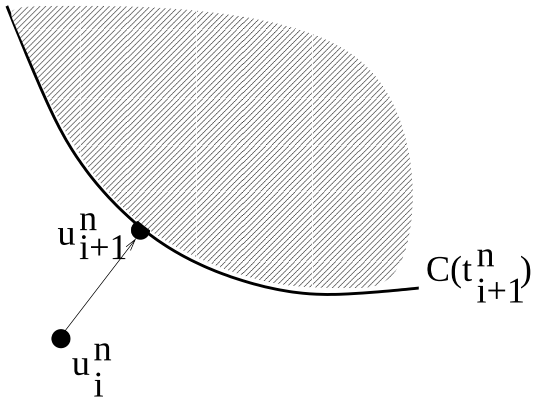


**Fig. 3.** The catching up algorithm

This is the "catching up" algorithm, since the approximation $u_{i+1}^n$ is made to catch up with the set $C(t_{i+1}^n)$ through projection; see Fig. 3. Recall that we have to achieve $u(t) \in C(t)$ for the solution.
The $u_n$ are defined via linear interpolation,

$$u_n(t) = u_i^n + \left(\frac{t - t_i^n}{t_{i+1}^n - t_i^n}\right)(u_{i+1}^n - u_i^n), \quad t \in [t_i^n, t_{i+1}^n]; \tag{16}$$

see Fig. 4 below.

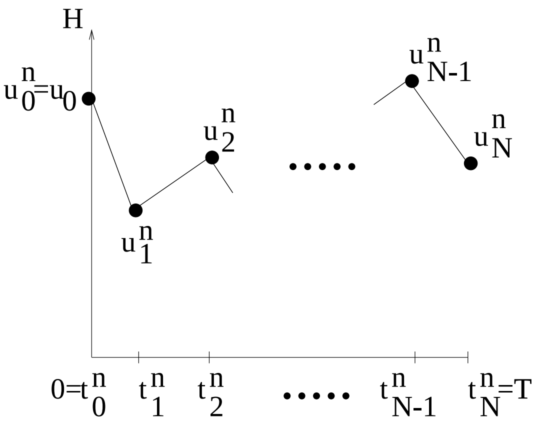


**Fig. 4.** The approximate solution $u_n$

We intend to find a subsequence of $(u_n)_{n\in\mathbb{N}}$ that converges to a solution of (1). To this end we wish to apply Theorem 1, and we have to derive the uniform bounds in norm and variation in (11). First note that by (15) and since $u_i^n \in C(t_i^n)$,

$$\begin{aligned}|u_{i+1}^n - u_i^n| &= |\mathrm{proj}\,(u_i^n, C(t_{i+1}^n)) - u_i^n| = \mathrm{dist}\,(u_i^n, C(t_{i+1}^n)) \\ &\le \mathrm{d_H}(C(t_i^n), C(t_{i+1}^n)) \le L|t_i^n - t_{i+1}^n|, \qquad (17)\end{aligned}$$

where we also have used (3) and (13). Together with Example 1(c) this leads to

$$\begin{aligned}\mathrm{var}\,(u_n) &= \sum_{i=0}^{N-1} |u_n(t_{i+1}^n) - u_n(t_i^n)| = \sum_{i=0}^{N-1} |u_{i+1}^n - u_i^n| \le L \sum_{i=0}^{N-1} (t_{i+1}^n - t_i^n) \\ &= LT =: M_2. \qquad (18)\end{aligned}$$

From (17) we get $|u_{i+1}^n| \le |u_n^i| + L(t_{i+1}^n - t_i^n)$, and thus by induction $|u_i^n| \le |u_0| + Lt_i^n$. According to (16) and (17) this yields

$$\begin{aligned}|u_n(t)| &\le |u_i^n| + L|u_{i+1}^n - u_i^n| \le |u_0| + Lt_i^n + L(t_{i+1}^n - t_i^n) = |u_0| + Lt_{i+1}^n \\ &\le |u_0| + LT =: M_1 \qquad (19)\end{aligned}$$

for $t \in [t_i^n, t_{i+1}^n]$, whence for all $n \in \mathbb{N}$ and $t \in [0,T]$, since the bound is independent of $n$ and $t$. Summarizing (18) and (19), we have shown that (11) in Theorem 1 holds. Thus we may extract a subsequence of $(u_n)_{n\in\mathbb{N}}$ (still indexed by $n \in \mathbb{N}$ for simplicity) and we find $u$ with $\mathrm{var}\,(u) \le LT = M_2$ such that $u_n(t) \to u(t)$ as $n \to \infty$ weakly in $H$ for all $t \in [0,T]$.

To show $u$ is a solution of (1), note first that $u_n(0) = u_0$ implies $u(0) = \mathrm{w} - \lim u_n(0) = u_0$. Thus (a) of Definition 1 is satisfied; $\mathrm{w} - \lim$ denotes the weak limit. To exploit (17) further, fix $t, s \in [0,T]$. Then $t \in [t_j^n, t_{j+1}^n]$ and $s \in [t_i^n, t_{i+1}^n]$ for some $0 \le j, i \le N-1$. Without loss of generality we can assume $i \le j$, i.e., $t_i^n \le t_j^n$. From (16) and (17) we obtain

$$\begin{aligned}&|u_n(t) - u_n(s)| \\ &\le |u_n(t) - u_n(t_j^n)| + \sum_{k=i+1}^{j-1} |u_n(t_{k+1}^n) - u_n(t_k^n)| + |u_n(t_{i+1}^n) - u_n(s)| \\ &\le \left(\frac{t - t_j^n}{t_{j+1}^n - t_j^n}\right) |u_{j+1}^n - u_j^n| + \sum_{k=i+1}^{j-1} |u_{k+1}^n - u_k^n| + \left(\frac{t_{i+1}^n - s}{t_{i+1}^n - t_i^n}\right) |u_{i+1}^n - u_i^n| \\ &\le \sum_{k=i}^{j} |u_{k+1}^n - u_k^n| \le L \sum_{k=i}^{j} (t_{k+1}^n - t_k^n) = L(t_{j+1}^n - t_i^n) \\ &= L([t-s] + [s - t_i^n] + [t_{j+1}^n - t]) \le L\left(|t-s| + \frac{2}{n}\right), \qquad (20)\end{aligned}$$

the latter by (14). Since $\mathrm{w} - \lim_{n\to\infty}(u_n(t) - u_n(s)) = u(t) - u(s)$, Lemma 2(a) and (20) yield

$$|u(t) - u(s)| \le \liminf_{n\to\infty} |u_n(t) - u_n(s)| \le L|t-s|, \quad t, s \in [0,T].$$

Consequently, $u$ is Lipschitz continuous with constant $L$, and hence in particular differentiable almost everywhere in $]0,T[$; cf. [6, Appendix, Sect. A.2]. Thus we have verified (c) of Definition 1. To see (b), we infer from (17) and (14) that

$$\left(\frac{t-t_i^n}{t_{i+1}^n-t_i^n}\right)|u_{i+1}^n-u_i^n| \le L(t-t_i^n) \le L(t_{i+1}^n-t_i^n) \le \frac{L}{n}$$

for $t\in[t_i^n,t_{i+1}^n]$. Therefore by (16), (15), and (13) for $t\in[t_i^n,t_{i+1}^n]$

$$\begin{aligned} u_n(t) &\in C(t_i^n)+\overline{B}_{L/n}(0)\subset C(t)+\overline{B}_{L(t-t_i^n)}(0)+\overline{B}_{L/n}(0) \\ &\subset C(t)+\overline{B}_{2L/n}(0), \end{aligned} \tag{21}$$

where in the last step we have used (14). Hence (21) in fact holds for all $n\in\mathbb{N}$ and $t\in[0,T]$, and thus Lemma 2(b) yields $u(t)\in C(t)$ for all $t\in[0,T]$. Finally we need to verify (d) of Definition 1, the most difficult step of the proof. First note that

$$\langle u_{i+1}^n-u_i^n, u_{i+1}^n-\hat{c}\rangle \le 0,\quad \hat{c}\in C(t_{i+1}^n), \tag{22}$$

by (15) and (4). From (16), (17), and (14) we obtain

$$|u_n(t)-u_{i+1}^n| = |u_n(t)-u_n(t_{i+1}^n)| \le L(t_{i+1}^n-t) \le \frac{L}{n},\quad t\in[t_i^n,t_{i+1}^n].$$

Since by (13), $C(t)\subset C(t_{i+1}^n)+\overline{B}_{L(t_{i+1}^n-t)}(0)\subset C(t_{i+1}^n)+\overline{B}_{L/n}(0)$ for $t\in[t_i^n,t_{i+1}^n]$, we find from (22) and (17) that

$$\begin{aligned} &\langle u_{i+1}^n-u_i^n, u_n(t)-c\rangle \\ &= \langle u_{i+1}^n-u_i^n, u_{i+1}^n-\hat{c}\rangle + \langle u_{i+1}^n-u_i^n, [u_n(t)-u_{i+1}^n]+[\hat{c}-c]\rangle \\ &\le |u_{i+1}^n-u_i^n|\left(\frac{L}{n}+\frac{L}{n}\right) \le \frac{2L}{n}(t_{i+1}^n-t_i^n) \end{aligned} \tag{23}$$

for $t\in[t_i^n,t_{i+1}^n]$ and $c\in C(t)$. In the interior $]t_i^n,t_{i+1}^n[$, $u_n$ is differentiable with derivative $u_n'(t)=(t_{i+1}^n-t_i^n)^{-1}(u_{i+1}^n-u_i^n)$, and hence by (23),

$$\langle u_n'(t), u_n(t)-c\rangle \le \frac{2L}{n},\quad t\in]t_i^n,t_{i+1}^n[,\quad c\in C(t). \tag{24}$$

The estimate (17) also shows

$$|u_n'(t)|\le L,\quad t\ne t_i^n,\quad \text{hence}\quad |u_n'|_{L^\infty([0,T];H)}\le L,\quad n\in\mathbb{N}.$$

Since $L^\infty([0,T];H)$ is the dual space of $L^1([0,T];H)$, it is a consequence of the Banach-Alaoglu theorem [64, Thm. 3.15] that we may extract a further subsequence (again indexed by $n\in\mathbb{N}$) such that $u_n'\to v_*$ for some

$v_* \in L^\infty([0,T];H)$, the convergence being in the weak-star topology on $L^\infty([0,T];H)$. This means that for all $\varphi \in L^1([0,T];H)$

$$\int_0^T \langle u_n'(t), \varphi(t)\rangle\,dt \to \int_0^T \langle v_*(t), \varphi(t)\rangle\,dt \quad \text{as} \quad n \to \infty.$$

Since $u_n$ is absolutely continuous,

$$u_n(t) = u_0 + \int_0^t u_n'(s)\,ds, \quad t \in [0,T]. \tag{25}$$

For fixed $t_0 \in [0,T]$ and $z \in H$ define

$$\varphi(t) = \begin{cases} z & : \quad 0 \le t \le t_0 \\ 0 & : \quad t_0 < t \le 1 \end{cases}.$$

Then $\varphi \in L^1([0,T];H)$ since $\int_0^T |\varphi(t)|dt = t_0|z|$. Taking the inner product of (25) with $z$ and using the weak convergence $u_n(t) \to u(t)$ it follows as $n \to \infty$ that

$$\begin{aligned}
\langle u(t_0), z\rangle \longleftarrow \langle u_n(t_0), z\rangle &= \langle u_0, z\rangle + \left\langle \int_0^{t_0} u_n'(t)\,dt, z \right\rangle \\
&= \langle u_0, z\rangle + \int_0^{t_0} \langle u_n'(t), z\rangle\,dt \\
&= \langle u_0, z\rangle + \int_0^{T} \langle u_n'(t), \varphi(t)\rangle\,dt \\
&\longrightarrow \langle u_0, z\rangle + \int_0^{T} \langle v_*(t), \varphi(t)\rangle\,dt = \left\langle u_0 + \int_0^{t_0} v_*(t)\,dt, z \right\rangle.
\end{aligned}$$

As this holds for arbitrary $z \in H$ we can conclude that

$$u(t) = u_0 + \int_0^t v_*(s)\,ds, \quad t \in [0,T].$$

This again shows that $u : [0,T] \to H$ is differentiable at almost every point $t \in ]0,T[$, and moreover $u'(t) = v_*(t)$ for almost every $t \in ]0,T[$. In particular, $(-u_n') \to (-u')$ in the weak-star topology on $L^\infty([0,T];H)$. According to Lemma 3 this yields

$$\int_0^T \delta^*(-u'(t), C(t))\,dt \le \liminf_{n\to\infty} \int_0^T \delta^*(-u_n'(t), C(t))\,dt; \tag{26}$$

recall the definition of $\delta^*$ from (7). Note also that by Lemma 4 and Lemma 2, and as a consequence of the weak convergence $u_n(T) \to u(T)$,

$$\int_0^T \langle u'(t), u(t)\rangle\,dt = \frac{1}{2}\Big(|u(T)|^2 - |u_0|^2\Big) \le \liminf_{n\to\infty} \frac{1}{2}\Big(|u_n(T)|^2 - |u_0|^2\Big)$$

$$= \liminf_{n\to\infty} \frac{1}{2}\Big(|u_n(T)|^2 - |u_n(0)|^2\Big)$$
$$= \liminf_{n\to\infty} \int_0^T \langle u_n'(t), u_n(t)\rangle\, dt. \tag{27}$$

Next, taking the supremum w.r.t. to $c \in C(t)$ in (24) and integrating over $[0,T]$ we find that

$$\int_0^T \Big[\delta^*(-u_n'(t), C(t)) + \langle u_n'(t), u_n(t)\rangle\Big]\, dt \le \frac{2LT}{n} \tag{28}$$

for $n \in \mathbb{N}$. Since $\big(\liminf_{n\to\infty} a_n\big) + \big(\liminf_{n\to\infty} b_n\big) \le \liminf_{n\to\infty}(a_n + b_n)$ for general sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, (28) together with (26) and (27) leads to

$$\int_0^T \Big[\delta^*(-u'(t), C(t)) + \langle u'(t), u(t)\rangle\Big]\, dt \le 0 \tag{29}$$

by taking $\liminf_{n\to\infty}$ on both sides of (28). We have already shown that $u(t) \in C(t)$ for $t \in [0,T]$. Therefore by definition of $\delta^*$, $\delta^*(-u'(t), C(t)) \ge \langle -u'(t), u(t)\rangle$, or stated differently,

$$\delta^*(-u'(t), C(t)) + \langle u'(t), u(t)\rangle \ge 0.$$

Whence (29) shows

$$\delta^*(-u'(t), C(t)) + \langle u'(t), u(t)\rangle = 0 \quad \text{for almost every} \quad t \in ]0,T[.$$

Thus for any $c \in C(t)$, $\langle -u'(t), u(t)\rangle = \delta^*(-u'(t), C(t)) \ge \langle -u'(t), c\rangle$, i.e., $\langle -u'(t), c - u(t)\rangle \le 0$. By definition of the normal cone we infer $-u'(t) \in N_{C(t)}(u(t))$ for almost every $t \in ]0,T[$. This completes the proof of the theorem. □

*Remark 3.* (a) In the context of Theorem 2, the solution to (1) cannot be expected to be differentiable at really every point $t \in ]0,T[$. Consider $H = \mathbb{R}$,

$$C(t) = \begin{cases} [t,1] & : \ 0 \le t \le \frac{1}{2} \\ [1-t,1] & : \ \frac{1}{2} \le t \le 1 \end{cases},$$

and $u_0 = 0 \in C(0) = [0,1]$. Then the solution is

$$u(t) = \begin{cases} t & : \ 0 \le t \le \frac{1}{2} \\ \frac{1}{2} & : \ \frac{1}{2} \le t \le 1 \end{cases}, \quad \text{with} \quad -u'(t) = \begin{cases} -1 & : \ 0 \le t < \frac{1}{2} \\ 0 & : \ \frac{1}{2} < t \le 1 \end{cases},$$

and hence belongs to $N_{C(t)}(u(t))$, as $N_{[t,1]}(t) = ]-\infty, 0]$ for $t \in [0,\frac{1}{2}]$ and $N_{[1-t,1]}(\frac{1}{2}) = \{0\}$ for $t \in ]\frac{1}{2},1]$, recall Remark 1. We obtain $\mathrm{d_H}(C(t),C(s)) = |t-s|$ for $t,s \in [0,\frac{1}{2}]$ and $t,s \in [\frac{1}{2},1]$. Moreover,

$$\mathrm{d_H}(C(t),C(s)) = |t-(1-s)| \le |t-s|, \quad t \in [0,\frac{1}{2}], \quad s \in [\frac{1}{2},1].$$

Hence (13) is satisfied with $L = 1$. On the other hand, $u$ is not differentiable at $t = \frac{1}{2}$, but Lipschitz with constant 1. In the sweeping process interpretation, $u_0$ is swept by $C(t)$ to $u = \frac{1}{2}$ and then remains resting at $\frac{1}{2}$, since the boundary of $C(t)$ starts to move in the opposite direction at time $t = \frac{1}{2}$.

(b) Note that the proof of Theorem 2 is constructive. The catching-up algorithm can be easily implemented numerically. See [52] for more on numerical methods for the sweeping process.

(c) In the proof of Theorem 2 it could be even shown that the sequence of approximations converges to a solution uniformly. However, this would not have led to great simplification, and the proof actually given explains a general technique useful also for other purposes. $\diamondsuit$

**Theorem 3 (Uniqueness).** *The solution of (13) is unique in the class of absolutely continuous functions.*

**Proof:** Let $u$ be a solution of $-u'(t) \in N_{C(t)}(u(t))$, $u(0) = u_0$, and let $v$ be a solution of $-v'(t) \in N_{C(t)}(v(t))$, $v(0) = v_0$. Since $u(t) \in C(t)$ and $v(t) \in C(t)$ for $t \in [0,T]$, it follows that $\langle -u'(t), v(t) - u(t)\rangle \leq 0$ and also $\langle -v'(t), u(t) - v(t)\rangle \leq 0$ for almost every $t \in ]0,T[$. Due to Equ. (12) in Lemma 4 we find

$$\frac{1}{2}\frac{d}{dt}\left(|u(t) - v(t)|^2\right) = \langle u'(t) - v'(t), u(t) - v(t)\rangle \leq 0$$

almost everywhere in $]0,T[$. Integration yields

$$|u(t) - v(t)|^2 \leq |u(0) - v(0)|^2 = |u_0 - v_0|^2, \quad t \in [0,T]. \tag{30}$$

In particular, if $u_0 = v_0$, we get the claimed uniqueness. $\square$

From (30) we also get

**Corollary 1 (Dependence on data).** *Under the above assumptions, if $u$ and $v$ are two solutions with $u(0) = u_0$ and $v(0) = v_0$, then*

$$|u(t) - v(t)| \leq |u_0 - v_0|, \quad t \in [0,T].$$

**Theorem 4 (Dependence on the moving set).** *Let $t \mapsto C(t)$ and $t \mapsto D(t)$ be two moving sets which satisfy (13) with Lipschitz constants $L_C$ and $L_D$, respectively. Assume that both $C(t)$ and $D(t)$ are nonempty, closed, and convex for every $t \in [0,T]$. Then, if $u$ denotes the solution to the sweeping process with $t \mapsto C(t)$ and initial value $u(0) = u_0$, and if $v$ denotes the solution to the sweeping process with $t \mapsto D(t)$ and initial value $v(0) = v_0$, the estimate*

$$|u(t) - v(t)|^2 \leq |u_0 - v_0|^2 + 2(L_C + L_D)\int_0^t \Delta(s)\,ds\,, \quad t \in [0,T], \tag{31}$$

*holds, where*

$$\Delta(t) = \mathrm{d_H}(C(t), D(t)), \quad t \in [0,T].$$

**Proof:** For fixed $t \in [0,T]$ we have $u(t) \in C(t) \subset D(t) + \overline{B}_{\Delta(t)}(0)$. Hence there exist vectors $d(t) \in D(t)$ and $r(t) \in H$ such that $u(t) = d(t) + r(t)$ and $|r(t)| \leq \Delta(t)$. It can be shown that it is possible to choose the maps $t \mapsto d(t)$ and $t \mapsto r(t)$ as being measurable. Similarly, we find $v(t) = c(t) + s(t)$ with $c(t) \in C(t)$ and $|s(t)| \leq \Delta(t)$. Arguing as in the proof of Theorem 3 we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(|u(t) - v(t)|^2\Big) \\
&= \langle u'(t) - v'(t), u(t) - v(t)\rangle \\
&= \langle u'(t), u(t) - c(t)\rangle + \langle v'(t), v(t) - d(t)\rangle - \langle u'(t), s(t)\rangle - \langle v'(t), r(t)\rangle \\
&\leq -\langle u'(t), s(t)\rangle - \langle v'(t), r(t)\rangle \leq (|u'(t)| + |v'(t)|)\Delta(t).
\end{aligned}$$

According to Theorem 2, $|u'(t)| \leq L_C$ and $|v'(t)| \leq L_D$ almost everywhere, thus integration yields (31). □

### 3.2 An application: Evolution of a quasi-static mechanical system

This section concerns a problem from elastoplasticity, for which we want to show how the sweeping process arises and how the theory developed in Section 3.1 can be successfully applied to yield a solution of the underlying evolution problem. Most of the material is taken from [43, Sect. 6].

The system we will deal with consists of two main ingredients: (displacements of) states and forces. The force $f$ acting on the velocity $u'$ of a state $u$ gives rise to $\langle u', f\rangle$, the power of the force $f$. Then

$$\int_{t_0}^{t_1} \langle u'(t), f\rangle\, dt = \int_{t_1}^{t_2} \frac{d}{dt}\langle u(t), f\rangle\, dt = \langle u(t_2), f\rangle - \langle u(t_1), f\rangle \cong \langle \delta u, f\rangle$$

is the work of the force $f$ corresponding to the displacement $\delta u$. The classical approach to plasticity requires introducing, besides the "visible" elements describing the configuration, additional internal variables (also called hidden parameters). Thus a state $u$ in fact consists of $u = (x, p)$, denoting the "visible" component by $x$ and the hidden (or "plastic") component by $p$. Usually $x$ is the part that experiences external forces or that may be submitted to constraints. Let us consider an example, see Fig. 5.
The system is comprised by two particles $x$ and $p$ which move in the plane $H = \mathbb{R}^2$. Let $U = \mathbb{R} \times \{0\}$ be the horizontal axis through the origin. This axis is driven by some function $g(t)$ to move parallel to $U$, and $x$ is constrained to remain in this set, i.e., we require

$$x \in U + g. \tag{32}$$

The point $p$ influences the motion of $x$, since it is assumed to be coupled to $x$ by means of a spring. In addition, an external force $c(t)$ is applied to $x$,

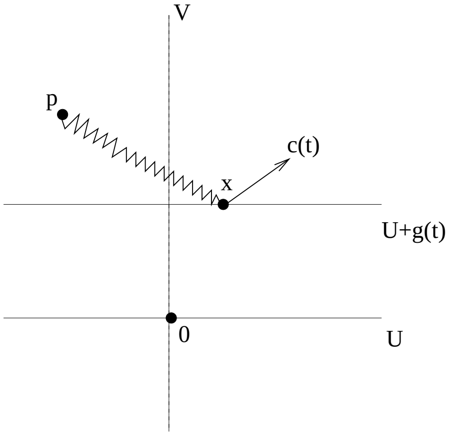


**Fig. 5.** An example system

and moreover, $x$ moves without experiencing dry friction. We suppose that at each instant $t$ the various forces equilibrate each other. In particular, the force applied to $p$ by the spring has to counter-balance the dry friction resistance of the plane to $p$. Let us now consider in more detail the forces acting on $p$ and $x$.

**($p$–1)** As just mentioned, there is a force $f \in H$ describing "plastic resistance" against the velocity $p'$ of $p$. Usually one imposes the law

$$p' \in N_C(-f), \tag{33}$$

with $C \subset H$ being a fixed nonempty closed convex subset of $H$ consisting of the negatives $(-\hat{f})$ of forces $\hat{f}$; the minus sign is a little confusing but conventionally used here. The law (33) is motivated as follows: By definition of the normal cone this means $\langle p', c+f\rangle \leq 0$ for all $c \in C$. Taking $c = -\hat{f}$ with another force $\hat{f}$, we then ensure by (33) that

$$\langle p', \hat{f}\rangle \geq \langle p', f\rangle \quad \text{for all forces} \quad \hat{f}.$$

Thus $f$ is to be taken as one of those forces acting on $p'$ that minimize the power $\langle p', \hat{f}\rangle$ over all forces $\hat{f}$. Since the quantity $-\langle p', \hat{f}\rangle$ has the interpretation of the dissipated power of $\hat{f}$ acting on $p'$, (33) often is called the "principle of maximal dissipation".

**($p$–2)** The second force that $p$ experiences is the force $s$ of "elastic restoring towards $x$", described by a law

$$s = A(e), \quad e = x - p. \tag{34}$$

The difference $e$ is called the deviation. Here $A : H \to H$ is chosen an invertible linear matrix, generalizations being possible.

**($x$–1)** The first force that acts on $x$ is the "reaction" $r$ of the constraint $U+g(t)$: In order for $x$ not to leave this set, cf. (32), we require that

$$-r \in N_{U+g(t)}(x). \tag{35}$$

**($x$–2)** There is also a "load" $c$ on $x$.

**($x$–3)** Finally $x$ experiences the force $-s$ of "elastic restoring towards $p$" according to ($p$–2).

Since we analyzed the forces acting at some fixed instant of time $t$, the given functions $g$ and $c$ will depend on time. We intend to understand the evolution of the state variables $x(t)$ and $p(t)$ under the basic assumption that the motion is sufficiently slow for no inertia to being accounted for, i.e., the acting forces just equilibrate each other. For that reason the system is called quasi-static. Due to ($p$–1)–($p$–2) and ($x$–1)-($x$–3) we therefore assume

$$s+f=0, \quad r+c-s=0. \tag{36}$$

Now we are going to show how the whole problem can be conveniently recast and solved as a sweeping process. For simplicity we let $A=I$, the identity matrix, in (34). Thus $s=e=x-p$. The more general case $A\neq I$ could be treated by taking as inner product in $H$ the expression $\langle u,v\rangle = (Au,v)$, with $(\cdot,\cdot)$ the usual Euclidean inner product; the quantity $\frac{1}{2}(Au,u)$ is the elastic energy of $u$. Then (32)-(36) can be summarized to

$$\left.\begin{array}{r} x(t)\in U+g(t), \quad p'(t)\in N_C(s(t)), \\ x(t)=p(t)+s(t), \quad c(t)-s(t)\in N_{U+g(t)}(x(t)) \end{array}\right\} \tag{37}$$

for (almost every) $t\in[0,T]$, with some fixed $T>0$. Introducing $V=U^{\perp}$, the orthogonal complement of $U$ in the Hilbert space $H$, we find the equivalent form

$$x(t)\in U+g(t), \quad s(t)\in V+c(t), \quad x(t)=p(t)+s(t), \quad p'(t)\in N_C(s(t)) \tag{38}$$

for (almost every) $t\in[0,T]$. Indeed, omitting the variable $t$ for the moment, assume first that (37) holds. Then $x-g=u_0\in U$ and $0\geq\langle c-s,u+g-x\rangle = \langle c-s,u-u_0\rangle$ for all $u\in U$, whence

$$\langle u_0,c-s\rangle \geq \langle u,c-s\rangle, \quad u\in U. \tag{39}$$

Suppose there exists $u_1\in U$ such that $\langle u_1,c-s\rangle\neq 0$. Without loss of generality we may assume $\langle u_1,c-s\rangle>0$, since otherwise we may replace $u_1$ by $-u_1\in U$; recall $U$ is a subspace of $H$. But then also $u=\lambda u_1\in U$ for $\lambda>0$, and this gives a contradiction in (39) as $\lambda\to\infty$. Thus $\langle u,s-c\rangle=0$ for all $u\in U$, i.e., $s-c\in V$. Conversely, if $s\in V+c$ and $u\in U$, then $\langle c-s,u\rangle=0$, in particular $\langle c-s,x-g\rangle=0$. Therefore if $u\in U$, then

$\langle c - s, u + g - x \rangle = 0$, and consequently $c - s \in N_{U+g}(x)$. Hence we have shown that in fact (38) is equivalent to (37).

To solve (38) for the unknown functions $x(t)$ and $p(t)$, we make the following general hypotheses.

**(H1)** $H$ is a separable Hilbert space, $U \subset H$ a subspace, and $V = U^\perp$;

**(H2)** $C \subset H$ is nonempty, closed, convex, and bounded;

**(H3)** both $g : [0,T] \to V$ and $c : [0,T] \to U$ are given Lipschitz continuous functions;

**(H4)** $\operatorname{int} C \cap (V + c(t)) \neq \emptyset$ for $t \in [0,T]$.

The assumption in $(H3)$ that $g(t) \in V$ and $c(t) \in U$ is no real restriction, since e.g. for a general $g(t)$ we may decompose $g(t) = g_U(t) + g_V(t)$ with $g_U(t) \in U$ and $g_V(t) \in V$, and then deal with $x(t) \in [U + g_U(t)] + g_V(t)$. This amounts to allowing $U(t) := U + g_U(t)$ to move with time in a Lipschitz continuous manner, but this would introduce no serious additional difficulties. Concerning $(H4)$, note that a necessary condition for (38) to have a solution is $C \cap (V + c(t)) \neq \emptyset$ for $t \in [0,T]$, because $s(t) \in C \cap (V + c(t))$. Thus $(H4)$ may be interpreted as a "safe load assumption", since the condition has to hold with a certain "safety margin".

The next step in rewriting (38) consists in a kind of Lyapunov-Schmidt reduction by projection of the equations on $U$ and $U^\perp = V$. For this, we introduce new variables $z$ and $y$ by

$$z = \operatorname{proj}_U(p) \quad \text{and} \quad y = -\operatorname{proj}_V(p). \tag{40}$$

Then (38) is equivalent to

$$z'(t) - y'(t) \in N_C(y(t) + c(t) + g(t)), \quad z(t) \in U, \quad y(t) \in V \tag{41}$$

for (almost every) $t \in [0,T]$. To see this, first suppose (38) is satisfied. Then $(x - g) \in U$, $(s - c) \in V$, and $p + c - g = (x - g) + (c - s) \in U \oplus V$. Therefore $x - g = \operatorname{proj}_U(p + c - g)$ and also $c - s = \operatorname{proj}_V(p + c - g)$. Since the projection is linear on subspaces, and due to $g \in V$ and $c \in U$, we find $x - g = \operatorname{proj}_U(p + c - g) = \operatorname{proj}_U(p) + c$, whence $z = \operatorname{proj}_U(p) = x - c - g$, and similarly $y = -\operatorname{proj}_V(p) = s - c - g$. We get $z' - y' = p' \in N_C(s) = N_C(y + c + g)$. On the other hand, let (41) hold and define

$$p = z - y, \quad x = z + c + g, \quad \text{and} \quad s = y + c + g. \tag{42}$$

Then $x - g = z + c \in U$, $s - c = y + g \in V$, $x = p + s$, and in addition $p' = z' - y' \in N_C(y + c + g) = N_C(s)$. Accordingly, we have shown (41) and (38) are equivalent. Through the transformations (40) and (42) we established a one-to-one correspondence between (41) and (38) resp. (37).

Now the connection of (41) to the sweeping process will become obvious.

**Lemma 5.** *If $z : [0,T] \to U$ and $y : [0,T] \to V$ are a solution to (41), then $y$ solves the sweeping process $-y'(t) \in N_{C(t)}(y(t))$, with*

$$C(t) = (C - c(t) - g(t)) \cap V. \tag{43}$$

**Proof:** Firstly, by (41) we have $y(t) \in V$ and $y(t) + c(t) + g(t) \in C$, i.e., $y(t) \in C(t)$. Next, fix $\hat{c} \in C$ such that $\hat{c} - c(t) - g(t) \in V$. Since the image of $z$ is contained in $U$ and $U$ is a vector space, also $z'(t) \in U$ for almost every $t \in [0, T]$. Hence $\langle z'(t), \hat{c} - c(t) - g(t) - y(t)\rangle = 0$ as also $y(t) \in V$, and therefore by (41)

$$\langle -y'(t), [\hat{c} - c(t) - g(t)] - y(t)\rangle = \langle z'(t) - y'(t), \hat{c} - c(t) - g(t) - y(t)\rangle \le 0.$$

Consequently, we have shown $-y'(t) \in N_{C(t)}(y(t))$ at times $t$ where $y'(t)$ does exist. $\square$

According to the orthogonal decomposition above, it turns out that by means of the sweeping process for $y$ in Lemma 5 we already can construct a solution $(z, y)$ to the full problem (41). Once this has been done, we have also proved that the original problem (37) has a solution $(x, p)$. In particular this applies to the example model from Fig. 5 above.

**Theorem 5.** *Assume that the hypotheses $(H1)$–$(H4)$ are satisfied and that the compatibility conditions*

$$z_0 \in U, \quad y_0 \in V, \quad y_0 + c(0) + g(0) \in C \tag{44}$$

*hold for the initial values $z_0$ and $y_0$. Then there exists a Lipschitz continuous solution $z : [0, T] \to U$ and $y : [0, T] \to V$ of (41) with $z(0) = z_0$ and $y(0) = y_0$.*

**Proof:** First we intend to solve the sweeping process

$$-y'(t) \in N_{C(t)}(y(t)) \quad \text{a.e. in} \quad [0, T], \quad y(0) = y_0 \tag{45}$$

with $C(t)$ from (43), by application of Theorem 2. According to (44) we have $y_0 \in (C - c(0) - g(0)) \cap V = C(0)$. Moreover, since $V$ is a vector space and $g(t) \in V$, $(H4)$ implies $C(t) \neq \emptyset$ for $t \in [0, T]$, and by $(H2)$ each $C(t)$ is closed and convex. By Lemma 6 below, $t \mapsto C(t)$ is Lipschitz continuous with some constant $L > 0$. Thus by Theorem 2, (45) has a Lipschitz continuous solution $y : [0, T] \to H$, and $y(t) \in V$, since $y(t) \in C(t)$ for $t \in [0, T]$. Let

$$F(t) = \{w \in H : \delta^*(w, C) = \langle w, y(t) + c(t) + g(t)\rangle\}, \quad t \in [0, T],$$

with $\delta^*$ from (7). We are going to show that

$$\mathcal{F}(t) = F(t) \cap (U - y'(t)) \neq \emptyset \quad \text{a.e. in} \quad [0, T]. \tag{46}$$

Indeed, let $t$ be a point where (45) holds. Observe that

$$\mathrm{int}(C - c(t) - g(t)) \cap V = [\mathrm{int}(C) - c(t) - g(t)] \cap V \neq \emptyset,$$

as $g(t) \in V$ and $\operatorname{int}(C) \cap (V + c(t)) \neq \emptyset$ due to $(H4)$. Moreover, $y(t) \in C(t) \subset V$. Hence it follows from Lemma 1(b) and (a) that

$$\begin{aligned} N_{C(t)}(y(t)) &= N_{[C-c(t)-g(t)]\cap V}(y(t)) = N_{C-c(t)-g(t)}(y(t)) + N_V(y(t)) \\ &= N_{C-c(t)-g(t)}(y(t)) + U. \end{aligned}$$

Accordingly we may decompose $-y'(t) = w - u$. Then $w = u - y'(t) \in U - y'(t)$, and additionally $w \in F(t)$, since for $\hat{c} \in C$ we obtain $\langle w, c(t) + g(t) + y(t)\rangle \geq \langle w, \hat{c}\rangle$, i.e., $\langle w, c(t) + g(t) + y(t)\rangle \geq \delta^*(w, C)$. As $y(t) \in C(t) \subset C - c(t) - g(t)$ or $c(t) + g(t) + y(t) \in C$, we conclude $w \in F(t)$. This ends the proof of (46). By modification of $\mathcal{F}(t)$ on a set of measure zero in $[0, T]$, we may assume that in fact $\mathcal{F}(t) \neq \emptyset$ for all $t \in [0, T]$. Since $F(t)$ is closed for $t \in [0, T]$, we see that in addition $\mathcal{F}(t)$ is closed. Moreover, with some technical efforts it can be shown that $\mathcal{F} : [0, T] \to 2^H$, the power set of $H$, is measurable, i.e., for any open $B \subset H$ the set $\{t \in [0, T] : \mathcal{F}(t) \cap B \neq \emptyset\}$ is measurable. As $H$ is assumed to be separable, a result from the theory of set-valued maps, cf. [27, Prop. 3.2], shows that there exists a measurable selection $\zeta : [0, T] \to H$ of $\mathcal{F}$. This means that $\zeta(t) \in \mathcal{F}(t)$ for $t \in [0, T]$, or according to the definition of $\mathcal{F}(t)$,

$$\delta^*(\zeta(t), C) = \langle \zeta(t), y(t) + c(t) + g(t)\rangle \quad \text{and} \quad \zeta(t) + y'(t) \in U \tag{47}$$

for almost all $t \in [0, T]$. We intend to set

$$z(t) = z_0 + \int_0^t [\zeta(s) + y'(s)]\, ds, \quad t \in [0, T], \tag{48}$$

if the integral exists. In that case, as $z_0 \in U$ and $\zeta(s) + y'(s) \in U$ for almost every $s$, we would have $z(t) \in U$ for $t \in [0, T]$, since $U$ is a subspace. Moreover, $z'(t) = \zeta(t) + y'(t)$ almost everywhere in conjunction with the first relation in (47) would prove $z'(t) - y'(t) = \zeta(t) \in N_C(y(t) + c(t) + g(t))$ almost everywhere in $[0, T]$, i.e., $(z, y)$ is the desired solution of (41).

Thus it suffices to derive a bound on $\zeta(s)$, since this also will show that the integral in (48) does exist. For $\rho > 0$ let

$$C_\rho = \{w \in H : \operatorname{dist}(w, H \setminus C) \geq \rho\} \subset C.$$

We claim that there exists $\rho > 0$ such that

$$C_\rho \cap (V + c(t)) \neq \emptyset \quad \text{for all} \quad t \in [0, T]. \tag{49}$$

In fact, otherwise we could choose a sequence $\rho_j \to 0$ and find times $t_j \in [0, T]$ such that $C_{\rho_j} \cap (V + c(t_j)) = \emptyset$ for all $j$. By definition of $C_{\rho_j}$ hence $\operatorname{dist}(v + c(t_j), H \setminus C) \leq \rho_j$ for $v \in V$. If we assume without loss of generality that $t_j \to t_0 \in [0, T]$ as $j \to \infty$, we infer from $x \mapsto \operatorname{dist}(x, A)$ being even Lipschitz continuous for a general $A \subset H$ that $\operatorname{dist}(v + c(t_0), H \setminus C) = 0$ for all $v \in V$. Therefore $v + c(t_0) \in \overline{H \setminus C} = H \setminus \operatorname{int} C$, i.e., $V + c(t_0) \subset H \setminus \operatorname{int} C$, in contradiction to $(H4)$. Consequently, (49) is satisfied for some small $\rho > 0$. Clearly $C_\rho$ is closed, and it is also convex by Lemma 1(d). Moreover, the argument of Lemma 6 below implies that the moving set $t \mapsto H(t) = C_\rho \cap (V + c(t))$ is Lipschitz continuous as $\operatorname{int} C_\rho \neq \emptyset$ for $\rho$ small. By solving the corresponding sweeping process with an arbitrary initial value chosen in $H(0)$, it follows that there exists a (Lipschitz) continuous function $h : [0,T] \to H$ such that

$$h(t) \in C_\rho \cap (V + c(t)) \quad \text{for} \quad t \in [0,T]. \tag{50}$$

This function $h$ will finally help us to estimate $\zeta$ from above. We first note that

$$\rho|w| + \langle w, h(t)\rangle \leq \delta^*(w, C), \quad t \in [0,T], \quad w \in H, \tag{51}$$

according to the definition of $C_\rho$ and by Lemma 1(c). Taking $w = \zeta(t)$ in (51) we obtain from (47) that for almost every $t \in [0,T]$

$$\begin{aligned}\rho|\zeta(t)| &\leq \delta^*(\zeta(t), C) - \langle \zeta(t), h(t)\rangle = \langle \zeta(t), y(t) + c(t) + g(t)\rangle - \langle \zeta(t), h(t)\rangle \\ &= \langle \zeta(t) + y'(t), y(t) + c(t) + g(t) - h(t)\rangle \\ &\quad - \langle y'(t), y(t) + c(t) + g(t) - h(t)\rangle. \end{aligned} \tag{52}$$

Recall from (47) that $\zeta(t) + y'(t) \in U$. In addition, $c(t) - h(t) \in V$ by (50), and we have $g(t) \in V$ by $(H3)$ and also $y(t) \in C(t) \subset V$. Since $U$ and $V$ are orthogonal the first term in (52) therefore vanishes and we get

$$\begin{aligned}\rho|\zeta(t)| &\leq -\langle y'(t), y(t) + c(t) + g(t) - h(t)\rangle \\ &\leq |y'(t)|\Big(|y(t)| + |c(t)| + |g(t)| + |h(t)|\Big) \\ &\leq L\Big(|y|_{L^\infty([0,T];H)} + |c|_{L^\infty([0,T];H)} + |g|_{L^\infty([0,T];H)} + |h|_{L^\infty([0,T];H)}\Big).\end{aligned}$$

The latter is a finite constant, whence it follows that even $\zeta \in L^\infty([0,T];H)$. In particular, the integral in (48) is well-defined, and $z$ is Lipschitz continuous with constant $|\zeta|_{L^\infty([0,T];H)} + L$. This completes the proof of Theorem 5. □

The following lemma was needed in the proof of Theorem 5.

**Lemma 6.** *Under the assumptions of Theorem 5, the moving set $t \mapsto C(t)$ from (43) is Lipschitz continuous.*

**Proof:** Since both $g$ and $c$ are Lipschitz continuous, it is sufficient to prove that $\tilde{C}(t) = C(t) + c(t) + g(t) = C \cap (V + c(t))$ is Lipschitz continuous; recall that $g(t) \in V$. Fix $w \in \tilde{C}(s)$ and choose $a$ (depending on $t$) such that $a \in V + c(t)$ and $\overline{B}_\rho(a) \subset C$; this is possible by (49). According to a geometrical inequality of Moreau, see [42, p. 24], we have $\operatorname{dist}(w, \tilde{C}(t)) \leq (1 + \rho^{-1}|w - a|)(\operatorname{dist}(w, C) + \operatorname{dist}(w, V + c(t)))$. As $w \in C$, the first distance is zero. Moreover, $[w - c(s)] + c(t) \in V + c(t)$, so we get $\operatorname{dist}(w, \tilde{C}(t)) \leq (1 + \rho^{-1}|w - a|)\operatorname{lip}(c)|t - s|$, with $\operatorname{lip}(c)$ the Lipschitz constant of $c$. As $w, a \in C$

and $C$ is bounded by $(H2)$, it follows that $\sup_{w\in\tilde{C}(s)} \operatorname{dist}(w,\tilde{C}(t)) \le K|t-s|$ for some constant $K > 0$. Since the role of $s$ and $t$ can be interchanged, $t \mapsto \tilde{C}(t)$ is Lipschitz continuous with Lipschitz constant $K$. $\square$

*Remark 4.* (a) Since the component $y$ of a solution $(z,y)$ to (41) is a solution to the sweeping process (45) according to Lemma 5, and since this solution to the sweeping process is unique as a consequence of Theorem 3, it follows that the $y$-component of a solution to (41) is uniquely determined. Utilizing the transformation (42), we conclude that $s = x - p$ in (37) is uniquely determined, too.

(b) Quite often models from elastoplasticity can be formulated as a sweeping process which then is coupled to one or more other equations. This e.g. was the case for the variable $y$ above, see Lemma 5, where a full solution additionally required the determination of $z$ in (41). However, this turned out to be not too difficult if we first managed to solve the corresponding sweeping process for $y$. One may consult [44] for a similar approach to a model for an elastoplastic rectilinear rod with small longitudinal displacements, and cf. also [9,11–13,17,31]. $\diamondsuit$

### 3.3 The state-dependent sweeping process

The classical sweeping process (1) allows the moving set $C(t)$ to depend only on the time $t \in [0,T]$, and as we have seen in Section 3.1, a well-posed problem arises in that each such Lipschitz continuous sweeping process has a unique solution. Here, following [35,36], we study a generalization of the classical sweeping process in that we allow $C(t,u)$ to depend also on the current state $u = u(t)$. Thus we consider

$$-u'(t) \in N_{C(t,u(t))}(u(t)) \quad \text{a.e. in} \quad [0,T], \quad u(0) = u_0 \in C(0,u_0), \tag{53}$$

and instead of (13) we suppose

$$\mathrm{d_H}(C(t,u),C(s,v)) \le L_1\,|t-s| + L_2\,|u-v|, \quad t,s \in [0,T],\ u,v \in H. \tag{54}$$

Note that a solution of (53) in particular has to satisfy $u(t) \in C(t,u(t))$ for $t \in [0,T]$.

There is a special case of (53) which also deserves separate attention, namely parabolic quasi-variational inequalities of the form

$$\text{find} \quad v(t) \in \Gamma(v(t)) : \quad \langle v'(t) + f(t),\, w - v(t)\rangle \ge 0 \quad \forall w \in \Gamma(v(t)), \tag{55}$$

with some initial value $v(0) = v_0 \in \Gamma(v_0)$, where $v = v(t) : [0,T] \to H$, and $f : [0,T] \to H$ is some inhomogeneity, and $\Gamma(v) \subset H$ is a set of constraints. Written somewhat differently, (55) means that

$$-v'(t) \in N_{\Gamma(v(t))}(v(t)) + f(t) \quad \text{a.e. in} \quad [0,T], \quad v(0) = v_0 \in \Gamma(v_0). \tag{56}$$

Then, if $v$ is a solution of (56), and if we define

$$u(t) = v(t) + \int_0^t f(s)\,ds \quad \text{and} \quad C(t,u) = \Gamma\Big(u - \int_0^t f(s)\,ds\Big) + \int_0^t f(s)\,ds, \tag{57}$$

it follows that $u$ is a solution of (53), with initial value $u_0 = v_0 \in C(0, u_0)$. Thus indeed the quasi-variational inequalities (56) are particular cases of (53).

When dealing with (56), we will always suppose that $v \mapsto \Gamma(v)$ is Lipschitz with constant $L \geq 0$, i.e.,

$$\mathrm{d_H}(\Gamma(v), \Gamma(w)) \leq L\,|v - w|, \quad v, w \in H. \tag{58}$$

In case that $f \in L^\infty([0,T]; H)$ in (56) (which we will assume for simplicity, but $f \in L^1([0,T]; H)$ is sufficient), (58) implies that (54) holds for $C$ defined by (57), with $L_1 = (L+1)|f|_{L^\infty([0,T];H)}$ and $L_2 = L$. This might be a bad estimate for $L_1$, but it will turn out that it is only the size of $L_2$ in (54) which determines the existence of a solution to (53).

*Example 3.* Choose $H = L^2(0,1)$, the usual space of square-integrable real-valued functions on $(0,1)$ with norm defined by $|\phi|^2_{L^2(0,1)} = \int |\phi(x)|^2 dx$, and set

$$\Gamma(v) = \Big\{\phi \in H_0^1(0,1) : |\phi'(x)| \leq \Phi(x, v(x)) \text{ for a.e. } x \in (0,1)\Big\}, \tag{59}$$

for $v \in L^2(0,1)$, with $H_0^1(0,1)$ being the Sobolev space of functions $\phi \in L^2(0,1)$ that possess a (distributional) derivative $\phi' \in L^2(0,1)$. In addition, belonging to $H_0^1(0,1)$ somehow requires $\phi$ to "vanish at the boundary of $(0,1)$", and $|\phi|^2_{H_0^1(0,1)} = \int |\phi(x)|^2 dx + \int |\phi'(x)|^2 dx$ defines the norm in $H_0^1(0,1)$. The function $\Phi : (0,1) \times \mathbb{R} \to [0, \infty[$ in the definition of $\Gamma(v)$ is prescribed.

The study of problems of type (55), (59), is motivated through quasi-variational inequalities arising e.g. in the evolution of sandpiles; see [62]. In the sandpile example (where one has $x \in \Omega \subset \mathbb{R}^2$ rather than $x \in (0,1)$ as we are going to treat here), without simplifying assumptions the function $\Phi$ will be discontinuous, posing serious technical problems to showing the existence of solutions. We will return to this example again later. $\diamondsuit$

There are several other instances where state-dependent sweeping processes as (53) or parabolic quasi-variational inequalities of type (56) yield the correct mathematical description of the underlying practical problem. State-dependent sweeping processes as (53) occur, for example, in the treatment of 2-D or 3-D quasistatic evolution problems with friction, as treated in [25, Ch. II, III] (see also the account given in [42, p. 155-161]); the models investigated there are generalizations of the one introduced in Section 3.2. In a different context, the state-dependent sweeping process is used in micromechanical damage models (the so-called Gurson-models) for iron materials with memory to describe the evolution of the plastic strain in presence of small damages; cf. [70,7]. Examples of evolutionary quasi-variational inequalities may also be found in [2] and the references therein, cf. in particular p. 242f.

Returning now to the general problem of proving existence of solutions to (56) and (53) respectively, we start with a negative result.

*Example 4.* If $L > 1$ in (58), then in general (56) will not have a solution. To see this, let $H = \mathbb{R}$, $\Gamma(v) = [Lv, 1]$ for $v \leq 1/L$, and $\Gamma(v) = [1, Lv]$ for $v > 1/L$, $v \in \mathbb{R}$. Then $v \mapsto \Gamma(v)$ is Lipschitz continuous with constant $L$. In addition, $v_0 := 0 \in \Gamma(0) = [0, 1]$, but (56) with $f(t) = -1$, $t \in [0, 1]$, has no solution. Indeed, assume $v : [0, 1] \to H$ to be a solution. By continuity, $Lv(t) \leq 1$ for $t \in [0, \delta]$ with a suitable $\delta > 0$, hence $v(t) \in \Gamma(v(t)) = [Lv(t), 1]$ and $L > 1$ imply $v(t) \leq 0$ in $[0, \delta]$. On the other hand, for $x \leq 0$,

$$N_{\Gamma(x)}(x) = \begin{cases} ]-\infty, 0] & : \quad x = 0 \\ \{0\} & : \quad x < 0 \end{cases} \subset ]-\infty, 0]\,,$$

so that (56) yields $v'(t) \geq -f(t) = 1$ in $[0, \delta]$. But this gives the contradiction $v(t) \geq t > 0 \geq v(t)$ in $]0, \delta]$. ♢

As (56) is a special case of (53), this shows that in general we cannot expect to have a solution of either of the two systems for $L_2 > 1$ in (54) or $L > 1$ in (58), respectively. Moreover, the next more advanced example shows that usually we will not have solutions even for $L_2 = 1$ or $L = 1$.

*Example 5.* Let $H = \mathbb{R}^2$ and $\Gamma(v) = \{|v|\} \times [-1, 1]$ for $v \in \mathbb{R}^2$, and $f(t) = (0, -1)$. Take $v_0 = (0, 0) \in C(0)$, and notice that $\Gamma$ is Lipschitz continuous with constant $L = 1$. Suppose that (56) has a solution $u$ on some interval $[0, T]$. Then $v(t) = (v_1(t), v_2(t)) \in \Gamma(v(t))$ implies $v_1(t) = |v(t)|$, and therefore $v_2(t) = 0$ in $[0, T]$. For every $x = (x_1, x_2) \in \Gamma(v(t))$ we have

$$v_1'(t)(x_1 - v_1(t)) + (v_2'(t) + 1)(x_2 - v_2(t)) \geq 0,$$

and since necessarily $x_1 = v_1(t)$, this reduces to $x_2 \geq 0$ for all $x_2 \in [-1, 1]$, a contradiction. ♢

This already shows that (53) can be much more complex than the classical sweeping process (1), as the latter always has a solution. However, in case that $L_2 < 1$ the existence of a solution to (53) can be shown, and by the above transformation (57) this carries over to (56) for $L < 1$.

**Theorem 6.** *Let (54) hold for $C$ with some $0 \leq L_2 < 1$, and assume that $C(t, u) \subset H$ is nonempty, closed, and convex for $t \in [0, T]$, and $u \in H$. Suppose that*

$$C(t, A) \cap \overline{B}_R(0) = \left( \bigcup_{u \in A} C(t, u) \right) \cap \overline{B}_R(0) \subset H$$

*is relatively compact for all bounded* $A \subset H$ *and all* $R > 0$*. If* $u_0 \in C(0, u_0)$*, then (53) has a solution on* $[0, T]$.

The compactness assumption is for technical reasons and always satisfied in the finite dimensional case $H = \mathbb{R}^d$. We will not give the complete proof of the theorem, but rather explain what is the difference to the proof of Theorem 2. To avoid problems with compactness (recall that in an infinite dimensional space closed bounded sets need not be compact), we will restrict ourselves to $H = \mathbb{R}^d$. We start again by fixing a time discretization as in (14). Defining $u_0^n = u_0$, we thus have $u_0^n \in C(t_0^n, u_0^n)$. To mimic the catching-up algorithm, we then should define $u_1^n = \mathrm{proj}\,(u_0^n, C(t_1^n, *))$, and the problem is what to replace the "$*$" with. If we let $u_1^n = \mathrm{proj}\,(u_0^n, C(t_1^n, u_0^n))$, then we have an explicit discretization scheme, but it will turn out that there is no easy way to obtain the necessary bound for convergence of the approximations to a solution. Thus we should rather use the implicit scheme $u_1^n = \mathrm{proj}\,(u_0^n, C(t_1^n, u_1^n))$, but then the first problem is to solve this equation for $u_1^n$. The next lemma contains the key observation in this respect.

**Lemma 7.** *If* $t \in [0, T]$ *and* $u \in C(s, u)$ *for some* $s \in [0, T]$*, then there exists* $v \in H$ *such that* $v = \mathrm{proj}\,(u, C(t, v))$ *and* $|v - u| \le L_1 |t - s| / (1 - L_2)$.

**Proof:** Let $r = L_1 |s - t| / (1 - L_2)$, $D = \overline{B}_r(u)$, and $Fv = \mathrm{proj}\,(u, C(t, v))$ for $v \in D$. Then by (54) for $v \in D$

$$\begin{aligned} |Fv - u| &= |\mathrm{proj}\,(u, C(t, v)) - u| = \mathrm{dist}\,(u, C(t, v)) \le \mathrm{d_H}(C(s, u), C(t, v)) \\ &\le L_1\,|s - t| + L_2\,|u - v| \le L_1\,|s - t| + L_2\,r = r, \end{aligned}$$

and hence $F(D) \subset D$. Moreover, $F$ is continuous as may be seen from (54) and a geometrical inequality of Moreau for projections, cf. [42, Prop. 4.7, p. 26]. Therefore $F$ is a self-map of the ball $D \subset \mathbb{R}^d$, and consequently has a fixed point $v \in D$ by Browder's fixed point theorem; see [26, Thm. 3.2]. $\square$

Hence we may continue above and choose $u_1^n \in H$ such that $u_1^n = \mathrm{proj}\,(u_0^n, C(t_1^n, u_1^n))$, and moreover $|u_1^n - u_0^n| \le L_1 t_1^n / (1 - L_2)$. Thus in particular, $u_1^n \in C(t_1^n, u_1^n)$, and hence Lemma 7 applies again to yield $u_2^n$ such that $u_2^n = \mathrm{proj}\,(u_1^n, C(t_2^n, u_2^n))$ and also $|u_2^n - u_1^n| \le L_1 (t_2^n - t_1^n)/(1 - L_2)$. Iterating this procedure, for $i = 1, \dots, N$ we find $u_i^n$ with

$$u_i^n = \mathrm{proj}\,(u_{i-1}^n, C(t_i^n, u_i^n)) \quad \text{and} \quad |u_i^n - u_{i-1}^n| \le L_1 (t_i^n - t_{i-1}^n)/(1 - L_2). \tag{60}$$

It turns out that in fact (60) is all that we need to derive the necessary bounds in norm and variation for the sequence $(u_n)_{n \in \mathbb{N}}$; this is shown as in the proof of Theorem 2, where (17) was one of the main ingredients. The passage to the limit $n \to \infty$ in order to prove that the limit of the approximations indeed yields a solution then is a little more technical than before, and we omit the details.

Finally we are going to indicate an application of Theorem 6 in the setting of Example 3; recall that we have also proved the existence of a solution to (55) in case that $L < 1$ in (58), at least under an appropriate compactness condition, see the remarks below. Note that such condition now will be needed, as $H = L^2(0,1)$ is an infinite dimensional space. For the function $\Phi$ in (59) we assume that

$$0 < m \leq \Phi(x,v) \leq M, \quad |\Phi(x,v) - \Phi(x,w)| \leq K|v-w|, \tag{61}$$

for a.e. $x \in (0,1)$ and $v, w \in \mathbb{R}$. Under these hypotheses, we will obtain the existence of a solution to (55) for $K$ small. In addition, the solution will be quite regular.

**Theorem 7.** *Let (61) be satisfied for $\Phi$ and assume that $K < \frac{\sqrt{2}m}{2M+m}$. Then for every $v_0 \in L^2(0,1)$ such that $v_0 \in \Gamma(v_0)$, there exists a Lipschitz continuous solution $v : [0,T] \to L^2(0,1)$ of (55) with initial value $v(0) = v_0$.*

Again we will not give the complete proof, but indicate some steps. Firstly, it is known that the embedding of $H_0^1(0,1)$ into $L^2(0,1)$ is relatively compact. Since $\Phi$ is bounded above by $M$, all sets $\Gamma(v)$ are contained in some fixed ball in $H_0^1(0,1)$, which, by the preceding remark, will be relatively compact in $L^2(0,1)$. Without going into further particularities, this is enough to guarantee that the necessary compactness condition is satisfied. Therefore the main task is to verify (58). For this, first observe that $|\phi|_{L^2(0,1)} \leq \frac{1}{\sqrt{2}}|\phi'|_{L^2(0,1)}$ for $\phi \in H_0^1(0,1)$ due to Hölder's inequality; note $|\phi(x)| = |\int_0^x \phi'(y)\,dy| \leq \sqrt{x}\,|\phi'|_{L^2(0,1)}$ for $x \in (0,1)$. Let

$$\Gamma'(v) = \Big\{\varphi \in L^2(0,1) : \int_0^1 \varphi\,dx = 0,\ |\varphi(x)| \leq \Phi(x, v(x)) \ \text{a.e.}\Big\},$$

for $v \in L^2(0,1)$. Then $\{\phi' : \phi \in \Gamma(v)\} = \Gamma'(v)$ holds, since a function $\phi \in H_0^1(0,1)$ has $\phi(0) = \phi(1) = 0$. Hence the aforementioned observation yields

$$\mathrm{d_H}(\Gamma(v), \Gamma(w)) \leq \frac{1}{\sqrt{2}}\,\mathrm{d_H}(\Gamma'(v), \Gamma'(w)), \quad v, w \in L^2(0,1). \tag{62}$$

Suppose for the moment that we can show the following key lemma.

**Lemma 8.** *Let $f, g : (0,1) \to \mathbb{R}$ be measurable such that*

$$0 < m \leq f(x), g(x) \leq M \ \text{ a.e. } x \in (0,1), \tag{63}$$

*and define*

$$\Gamma'(f) = \Big\{\varphi \in L^2(0,1) : \int_0^1 \varphi\,dx = 0,\ |\varphi(x)| \leq f(x) \ \text{a.e.}\Big\}. \tag{64}$$

*Let* $v \in \Gamma'(f)$ *and* $w = \mathrm{proj}\,(v, \Gamma'(g))$. *Then*

$$|v - w|_{L^2(0,1)} \le \left(\frac{2M+m}{m}\right) |f - g|_{L^2(0,1)}. \tag{65}$$

Then (62) and (61) give the claim of Theorem 7, since for $v, w \in L^2(0,1)$

$$\begin{aligned} \mathrm{d_H}(\Gamma(v), \Gamma(w)) &\le \frac{1}{\sqrt{2}}\, \mathrm{d_H}(\Gamma'(v), \Gamma'(w)) \\ &\le \left(\frac{2M+m}{\sqrt{2}m}\right) |\Phi(\cdot, v(\cdot)) - \Phi(\cdot, w(\cdot))|_{L^2(0,1)} \\ &\le \left(\frac{2M+m}{\sqrt{2}m}\right) K\, |u - v|_{L^2(0,1)}, \end{aligned}$$

so that we can choose $L = \left(\frac{2M+m}{\sqrt{2}m}\right) K < 1$ to find (58) verified.

Hence one has to see why Lemma 8 holds. Since

$$|v - w|_{L^2(0,1)} \le |v - \varphi|_{L^2(0,1)}$$

for all $\varphi \in \Gamma'(g)$, it is necessary to construct a particular function $\varphi_* \in \Gamma'(g)$ such that

$$|v - \varphi_*|_{L^2(0,1)} \le \left(\frac{2M+m}{m}\right) |f - g|_{L^2(0,1)}.$$

With some efforts such $\varphi_*$ can be found through an appropriate modification of the given $v \in \Gamma'(f)$; see [36] for the details. Those are somewhat tedious and hence omitted here.

*Remark 5.* (a) Note that Theorem 6 does not make an assertion about the uniqueness of solutions to (53) and (56), respectively. First consider the following simple

*Example 6.* Let $H = \mathbb{R}$, $\Gamma(v) = [v, 1]$ for $v \le 1$ resp. $\Gamma(v) = [1, v]$ for $v > 1$, $v \in \mathbb{R}$. Define $v_0 = 0$ and $f(t) = 0$, $t \in [0,1]$. Since $N_{[x,1]}(x) = ]-\infty, 0]$ for $x < 1$ and $N_{\{x\}}(x) = \mathbb{R}$, all sufficiently regular functions $v : [0,1] \to \mathbb{R}$ with $v(0) = 0$, $v(t) \le 1$, and $v'(t) \ge 0$ for $t \in [0,1]$ are solutions to (56). $\diamondsuit$

However, this is a counter-example to uniqueness only for $L = 1$ in (58) corresponding to $L_2 = 1$ in (54), and we already know from Example 5 that there might be no solutions at all in that case. Recently examples have been found showing that in general solutions to (53) are non-unique in case that $L_2 < 1$, see [4,10], and these counter-examples even allow for an arbitrary small $L_2$.

(b) We have seen that (56) may have no solutions for $L = 1$ in (58) for dimensions $d \ge 2$; cf. Example 5. However, it can be shown that (56) with

$L = 1$ has a solution if $H = \mathbb{R}$. This is mainly due to the fact that "there are only two directions in $\mathbb{R}$"; see [35, Thm. 3.8]. This also enlightens a noteworthy difference between (53) and (56): The former in general will not have a solution if $L_2 = 1$ in (54), even for $H = \mathbb{R}$, as may be seen from the example $C(t, u) = [t + u, +\infty[$. Then $u_0 = 0 \in C(0, u_0)$, and (54) holds with $L_1 = L_2 = 1$. Nevertheless, the existence of a solution to (53) would imply $u(t) \in C(t, u(t))$ for $t > 0$, i.e., $u(t) \geq t + u(t)$, and this is impossible. $\diamondsuit$

### 3.4 Additional remarks

$\boxed{1.)}$ In Section 3.1 we proved the existence of a solution to the sweeping process under the basic hypothesis that the dependence $t \mapsto C(t)$ be Lipschitz continuous in the sense of (13). There are also a lot of results if this assumption is weakened as we are going to summarize in this remark.

The first variant is the rcbv sweeping process, where rcbv means "right-continuous and of bounded variation." A moving set $t \mapsto C(t)$ will be called rcbv, if

$$\mathrm{d_H}(C(t), C(s)) \leq r(t) - r(s), \quad 0 \leq s \leq t \leq T, \tag{66}$$

for some right-continuous non-decreasing function $r : [0, T] \to \mathbb{R}$. Since we take the difference on the right-hand side of (66), we may assume without loss of generality that $r(0) = 0$. As for any partition $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T$ of $[0, T]$ the estimate (66) yields

$$\sum_{i=0}^{N-1} \mathrm{d_H}(C(t_{i+1}), C(t_i)) \leq \sum_{i=0}^{N-1} [r(t_{i+1}) - r(t_i)] = r(T),$$

condition (66) can be interpreted as requiring that $t \mapsto C(t)$ be of bounded variation; recall (10) and also Example 1. Because the function $r(t) = Lt$ is rcbv, we see that in particular a Lipschitz continuous moving set $t \mapsto C(t)$ is rcbv. The converse is false, as is shown by the following example from [42, p. 29].

*Example 7.* In $H = \mathbb{R}$ define $C(t)$ for $t \in [0, 2]$ as $C(t) = [0, 2]$ for $t \in [0, 1[$ and $C(t) = [1, 2]$ for $t \in [1, 2]$. Then

$$\mathrm{d_H}(C(t), C(s)) = \begin{cases} 0 & : \quad s, t \in [0, 1[ \text{ or } s, t \in [1, 2] \\ 1 & : \quad s \in [0, 1[, t \in [1, 2] \end{cases},$$

and hence (13) is violated. Nevertheless, (66) is satisfied with $r(t) = 0$ for $t \in [0, 1[$ and $r(t) = 1$ for $t \in [1, 2]$, and $r$ is a rcbv function. $\diamondsuit$

In vague analogy to the results for the Lipschitz continuous sweeping process, one might expect that (1) with a rcbv moving set has a unique rcbv solution. However, this raises a problem, since continuous bv functions

$u$ in general are differentiable almost everywhere in $]0,T[$, but $\int_{t_0}^{t_1} u'(t)dt = u(t_1) - u(t_0)$ usually is verified only at a few points; see [54, Ch. 8] for an example of a Cantor-like function. Hence the formulation (1) of the sweeping process needs to be replaced by a weaker formulation. To do so, we first recall that a function $u : [0,T] \to H$ of bounded variation defines a $H$-valued measure $du$ on $[0,T]$ (called the differential measure or Stieltjes measure) as follows. Let $\phi : [0,T] \to H$ be a continuous function with compact support in $[0,T]$.

**Lemma 9.** *There is a unique $I \in H$, written $I = \int_0^T \phi\, du$, such that for every $\varepsilon > 0$ there exists a partition $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T$ of $[0,T]$ with the following property: If $0 = s_0 < s_1 < \ldots < s_{M-1} < s_M = T$ is any partition finer than the given one (i.e., with additional partition points) and if $\theta_k$ is chosen arbitrarily in $[s_k, s_{k+1}]$, $k = 0, \ldots, M-1$, then*

$$\left| \sum_{k=0}^{M-1} \phi(\theta_k)[u(s_{k+1}) - u(s_k)] - I \right| \le \varepsilon.$$

**Proof:** See [51, Prop. 6.1]. □

So we have well-defined integrals w.r.t. to $du$ of continuous functions with compact support. By a standard procedure in measure theory this can be used to define an $H$-valued measure $du$ on $[0,T]$. It turns out that for $du$ the following relations hold, with $[a,b] \subset [0,T]$.

$$\begin{aligned} \int_{[a,b]} du &= u^+(b) - u^-(a), \quad & \int_{]a,b]} du &= u^+(b) - u^+(a), \\ \int_{[a,b[} du &= u^-(b) - u^-(a), \quad & \int_{]a,b[} du &= u^-(b) - u^+(a). \end{aligned} \tag{67}$$

Here

$$u^+(a) = \lim_{t \searrow a} u(t) \quad \text{and} \quad u^-(a) = \lim_{t \nearrow a} u(t)$$

are the right resp. left limits of $u$ which can be shown for all $a \in [0,T[$ resp. $a \in ]0,T]$ to exist for functions of bounded variation $u$; see [51,42]. In particular, $u^+(a) - u^-(a) = \int_{\{a\}} du$ is the "jump" in $H$ at a point of discontinuity $a$. For a continuous function $u$ we have of course that $u^+ = u^- = u$.

Now the weak formulation of the sweeping process (1) reads as follows.

**Definition 2.** A function $u : [0,T] \to H$ of bounded variation is a solution to

$$-du \in N_{C(t)}(u(t)), \quad u_0 \in C(0), \tag{68}$$

if

(a) $u(0) = u_0$;

(b) $u(t) \in C(t)$ for all $t \in [0,T]$;
(c) there exists a positive measure $d\mu$ on $[0,T]$ relative to which $du$ has a density $u' \in L^1([0,T];H,d\mu)$ w.r.t. to $d\mu$ [i.e., $\int \phi\, du = \int \phi(t)u'(t)\, d\mu(t)$ for all reasonable $\phi$] such that

$$-u'(t) \in N_{C(t)}(u(t)) \quad \text{for} \quad d\mu - \text{almost every} \quad t \in [0,T].$$

Although this definition appears to be quite technical, it turns out to be natural since solutions to sweeping processes with merely rcbv moving sets $t \mapsto C(t)$ can behave quite irregularly and have "many jumps". The following theorem is a typical result about rbcv moving sets.

**Theorem 8.** *Let $t \mapsto C(t)$ be rcbv such that every $C(t) \subset H$ is nonempty, closed, and convex, and let $u_0 \in C(0)$. Then (68) has a unique rcbv solution.*

**Proof:** By the discretization approach (the catching-up algorithm), as in [45, Prop. 3b]. □

Under the additional assumption that

$$C(t) \cap \overline{B}_R(0) \subset H$$

is compact for every $t \in [0,T]$ and $R > 0$, an alternative proof using the so-called Yosida-Moreau approximations can be found in [40] and [42, p. 29].

We now have generalized the Lipschitz continuous sweeping process to the rcbv sweeping process, but there is a further generalization of Lipschitz continuous sweeping processes that deserves attention. A natural question is what can be said if $t \mapsto C(t)$ is merely continuous, i.e.,

$$\mathrm{d_H}(C(t), C(s)) \to 0, \quad t \to s. \tag{69}$$

On first sight, the situation is quite bad, as we can take e.g. $C(t) = \{v(t)\}$ with some continuous but nowhere differentiable function $v : [0,T] \to H$ which is not of bounded variation. Such functions do exist, examples are provided by e.g. (almost every) path of a Brownian motion, see [5, Thm. 12.25 & Cor. 12.27]. Since a prospective solution $u$ is to have $u(t) \in C(t)$, it is clear that $u(t)$ can be a very bad function for which no obvious kind of derivative could be defined. To indicate a way out, recall that it would be already sufficient to have $u$ of bounded variation. Moreover, in a general sweeping process the point to be swept will not move (and hence not increase its variation) as long as it stays in the interior of the moving set. This was exactly the problem with $C(t) = \{v(t)\}$, as this set has empty interior.

With some technical efforts one can then prove

**Theorem 9.** *Let $t \mapsto C(t)$ be continuous in the sense of (69) such that every $C(t) \subset H$ is nonempty, closed, and convex, and let $u_0 \in C(0)$. Assume that*

$$\operatorname{int} C(t) \neq \emptyset, \quad t \in [0,T].$$

*Then (68) has a unique solution which is continuous and of bounded variation. Here the measure $d\mu$ in Definition 2 can be chosen as $d\mu = |du|$, the "modulus" of $du$.*

**Proof:** See [42, p. 46]. □

We also note that there is a good approximation result for solutions to this problem by means of the Yosida-Moreau approximations; see [32]. Concrete examples for such rcbv or continuous sweeping processes can be obtained from the example in Section 3.2 by corresponding relaxation of the continuity assumptions on $g$ and $c$; see hypothesis $(H3)$.

$\boxed{2.)}$ This second remark concerns what has been called "degenerate sweeping processes", given by

$$-u'(t) \in N_{C(t)}(Au(t)) \quad \text{a.e. in} \quad [0,T], \quad u(0) = u_0 \in D(A), \quad Au_0 \in C(0). \tag{70}$$

Here $A$ is a "maximal monotone" operator with domain of definition $D(A)$. This concept of a maximal monotone operator can somehow be thought of being a nonlinear generalization of a selfadjoint linear operator; one may consult [6] for further information. For simplicity we only indicate some results in this linear case, see [33,34,37] for full details on the nonlinear problems. Sweeping processes of type (70) are quite general and may fail to have a solution, even for linear $A$; therefore they may be considered degenerate.

*Example 8.* Let $H = \mathbb{R}^2$, $T = 1$, $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $C(t) = [0,1] \times [t,1]$ for $t \in [0,1]$. Then $A$ is linear, bounded, selfadjoint, and even $\langle Au, u\rangle \geq 0$ for $u \in H$. Also $C(\cdot)$ is $\mathrm{d_H}$-Lipschitz continuous. However, there can be no solution $u$ of (70) with initial value $u_0 = (0,0) \in D(A)$, since this would imply $Au(t) = (u_1(t), 0) \in C(t)$ a.e. in $[0,1]$, a contradiction. ◊

If the monotonicity assumption on $A$ is sharpened to

$$\langle Au, u\rangle \geq \beta\, |u|^2 \quad \text{for} \quad u \in H \tag{71}$$

with some $\beta > 0$, then the situation is better, as (70) will have a unique solution.

**Theorem 10.** *Let $A : H \to H$ be linear, bounded, and selfadjoint such that (71) holds. If (13) is satisfied for $t \mapsto C(t)$ and if $Au_0 \in C(0)$, then (70) has a unique solution, and this solution is Lipschitz continuous.*

**Proof:** See [34, Thm. 2]. □

$\boxed{3.)}$ There are also some papers that deal with periodic solutions of sweeping processes or related models, under various assumptions. In this case one has to suppose that also $t \mapsto C(t)$ is $T$-periodic, i.e., $C(0) = C(T)$. See [19–21,31].

## 4 Second-order problems

In this section, we will analyze a general sweeping process problem of the form

$$-du \in N_{V(q(t))}(u(t)), \quad u(t) \in V(q(t)), \tag{72}$$

where $V$ is a closed convex valued multifunction with a suitable dependence on the state variable $q$, and $u$ is the first derivative of $q$. Hence $du$ somehow represents a second derivative of $q$, thus explaining why such sweeping processes are of second order. Our main concern is to describe a discretization procedure which is similar to those considered in the applications to dynamical problems. For (72), two main situations have been considered so far.

(a) If $q \mapsto V(q)$ is Lipschitz continuous in the sense of Hausdorff distance, then one obtains a Lipschitz continuous solution $u$.
(b) If $q \mapsto V(q)$ is continuous with nonempty interior, then there exists a continuous solution $u$ with bounded variation.

Below we will investigate a subclass of (b). It should also be noted that in addition Castaing studied (72) either in a finite-dimensional setting or assuming that $V$ is antimonotone (i.e., $-V$ is monotone); see [18], [42, Sect. 5.1], or [22, Thm. 6.1] for three different techniques.

Although the computations to follow look a little awkward, the main ingredients are Euclidean geometry and telescopic sums, so that the reader should not feel discouraged. However, on first reading this technical part can be omitted.

The discretization procedure that we will consider concerns more precisely the problem

$$-du \in N_{V(q(t))}(u(t)), \quad u(t) = \mathrm{Av}(u^-(t), u^+(t)) \in V(q(t)), \tag{73}$$

where $\mathrm{Av}(u^-(t), u^+(t))$ is a kind of linear interpolation, or weighted average, of $u^-(t)$ and $u^+(t)$, the left-limit and right-limit of $u$ at $t$, respectively; recall Remark 1.) in Section 3.4. For the operator Av our main choice will be

$$\mathrm{Av}(u^-, u^+) = u^+ + k(u^- - u^+), \quad 0 \le k \le 1/2, \tag{74}$$

which has the following properties:

$$\begin{aligned} u := \mathrm{Av}(u^-, u^+) &\in [(u^- + u^+)/2, u^+] \subset [u^-, u^+], \\ |u^+ - u| &\le k|u^+ - u^-|, \\ |u^+ - u| &\le k(1-k)^{-1}|u^- - u|, \\ |u^+ - u^-| &\le 2|u - u^-|; \end{aligned} \tag{75}$$

note that here we used the notation $[v, w] = \{\lambda w + (1-\lambda)v : \lambda \in [0,1]\} \subset \mathbb{R}^d$ for the line joining $v$ and $w$.

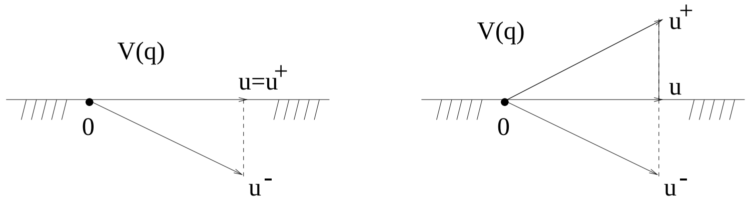


**Fig. 6.** (a) $k = 0$: inelastic (b) $k = 1/2$: elastic

In (74), the case $k = 0$, i.e., $u = u^+$, corresponds to purely inelastic collisions, while $k = 1/2$ corresponds to purely elastic collisions; see Fig. 6 (a) and (b).
The relation of $k$ to the usual restitution coefficient $e$ and to Moreau's dissipation index $\delta$ will be explained in the next section. The partial "inverse" operator $(u^-, u) \mapsto u^+$ will be denoted Iav, thus $\mathrm{Iav}(u^-, u) = (1-k)^{-1}(u - ku^-)$ for (75). Hence $\mathrm{Iav}(u^-, u) = u$ in the inelastic case and $\mathrm{Iav}(u^-, u) = u^- + 2(u - u^-)$ in the elastic case.

Assume now that $q \mapsto V(q)$ is Hausdorff continuous, with each set $V(q) \subset \mathbb{R}^d$ being closed, convex, and with nonempty interior. The initial data are $q_0 \in \mathbb{R}^d$ and $u_0 \in V(q_0)$. As in the proof of Theorem 2, for each $n \in \mathbb{N}$ we fix a partition

$$0 = t_0^n < t_1^n < ... < t_n^n = T$$

of the interval $[0, T]$, say, with equal stepsize $h = T/n$. By induction on $n$ we construct finite sequences $(q_i^n)$, $(u_i^n)$, and $(w_i^n)$, for $i = 0, \dots, n$, starting with $q_0^n = q_0$ and $u_0^n = w_0^n = u_0$, as follows. By integrating the constant velocity $u_i^n$ over the time interval $[t_i^n, t_{i+1}^n]$, we define

$$q_{i+1}^n = q_i^n + (t_{i+1}^n - t_i^n)u_i^n. \tag{76}$$

The differential inclusion in (73) will be discretized as

$$-(u_{i+1}^n - u_i^n) \in N_{V(q_{i+1}^n)}(\mathrm{Av}(u_i^n, u_{i+1}^n)). \tag{77}$$

This means that in (77) we update $V$ at $q_{i+1}^n$, and for the computation of the average we take into account that as $n \to \infty$ the "velocities" $u_i^n$ and $u_{i+1}^n$ should play the role of the left limit $u^-$ and the right limit $u^+$, respectively. Writing

$$w_{i+1}^n = \mathrm{Av}(u_i^n, u_{i+1}^n), \tag{78}$$

(hence $w_{i+1}^n = u_{i+1}^n$ in the "inelastic" case), we infer from (75) that

$$\begin{aligned} w_{i+1}^n &\in [(u_i^n + u_{i+1}^n)/2, u_{i+1}^n] \subset [u_i^n, u_{i+1}^n], \\ |u_{i+1}^n - w_{i+1}^n| &\le k|u_{i+1}^n - u_i^n|, \\ |u_{i+1}^n - w_{i+1}^n| &\le k(1-k)^{-1}|u_i^n - w_{i+1}^n|, \\ |u_{i+1}^n - u_i^n| &\le 2|w_{i+1}^n - u_i^n|. \end{aligned} \tag{79}$$

By (78), $u_{i+1}^n - u_i^n = (1-k)^{-1}(w_{i+1}^n - u_i^n)$, so that (77) is equivalent to

$$-(w_{i+1}^n - u_i^n) \in N_{V(q_{i+1}^n)}(w_{i+1}^n);$$

recall that $N_{V(q_{i+1}^n)}(w_{i+1}^n)$ is a cone and hence not affected by multiplication with a positive number. Thus by the characterization of projections (5)

$$w_{i+1}^n = \mathrm{proj}\,(u_i^n, V(q_{i+1}^n)) \in V(q_{i+1}^n) \tag{80}$$

is necessary. We summarize the algorithm as follows. If the values from step $i$ have already been determined, then we

(1.) update the state to $q_{i+1}^n$ by means of (76),

(2.) obtain $w_{i+1}^n$ by (80),

(3.) and introduce the new velocity $u_{i+1}^n$ by

$$u_{i+1}^n = \mathrm{Iav}(u_i^n, w_{i+1}^n) = (1-k)^{-1}(w_{i+1}^n - k u_i^n). \tag{81}$$

In particular, the estimates from (79) are valid. Notice that when $k \neq 0$ it may happen that $u_{i+1}^n \notin V(q_{i+1}^n)$; see Fig. 7 below. However, if $k = 0$, then $u_{i+1}^n = w_{i+1}^n \in V(q_{i+1}^n)$.

Finally we define the functions $u_n : [0,T] \to \mathbb{R}^d$ and $q_n : [0,T] \to \mathbb{R}^d$ by $u_n(t) = w_i^n$ for $t = t_i^n$,

$$u_n(t) = u_i^n \quad \text{for} \quad t \in ]t_i^n, t_{i+1}^n[,$$

and

$$q_n(t) = q_0 + \int_0^t u_n(s)\,ds. \tag{82}$$

Thus $q_n(t_i^n) = q_i^n$ by (76), and $\dot{q}_n(t) = u_n(t)$ Lebesgue almost everywhere in $[0,T]$.

Next we are going to investigate the convergence of this approximation scheme. Here the assumption that $V(q)$ has nonempty interior will be essential, similarly to what has briefly been mentioned in front of Theorem 9 in Section 3.4. In fact, this hypothesis will allow us to obtain estimates on the total variation of the derivatives (velocities) $u_n$. To explain this, note first that as $V(q_0)$ has nonempty interior it must contain some ball $\overline{B}_{2r}(a)$. As $q \mapsto V(q)$ is Hausdorff continuous, it then can be shown that for $q$ in a neighbourhood of $q_0$, $V(q)$ contains at least a ball with the same center $a \in \mathbb{R}^d$ but of a smaller radius, say

$$\overline{B}_r(a) \subset V(q); \tag{83}$$

see e.g. [42, Lemma 2.4.2] for a proof. With some a priori estimates and working if necessary in a smaller time interval $[0,T_0] \subset [0,T]$, we may suppose that (83) holds for all $q = q_i^n$; hence we may assume that already $T_0 = T$. The following geometrical result, due to Moreau, about projections into convex sets with nonempty interior is crucial for the subsequent estimates.

**Lemma 10.** *If $V$ is a closed convex set in a Hilbert space $H$ and $V$ contains a ball $\overline{B}_r(a)$, then*

$$|x - \mathrm{proj}\,(x, V)| \le \frac{1}{2r}(|x-a|^2 - |\mathrm{proj}\,(x, V) - a|^2) \quad \textit{for all} \quad x \in H.$$

**Proof:** See [46] and also [42, Lemma 0.4.3] □

Taking $x = u_i^n$ and $V = V(q_{i+1}^n)$, we thus obtain from (83) and (80)

$$|u_i^n - w_{i+1}^n| \le \frac{1}{2r}(|u_i^n - a|^2 - |w_{i+1}^n - a|^2). \tag{84}$$

The total variation of $u_n$, recall (10), is given by

$$\mathrm{var}\,(u_n) = \sum_{i=0}^{n-1}\Big(|u_{i+1}^n - w_{i+1}^n| + |w_{i+1}^n - u_i^n|\Big) = \sum_{i=0}^{n-1}|u_{i+1}^n - u_i^n|,$$

the latter because the points $u_i^n$, $w_{i+1}^n$, and $u_{i+1}^n$ are on a line.

In the simplest case $k = 0$, we have $u_{i+1}^n = w_{i+1}^n$, and an estimate on $\mathrm{var}\,(u_n)$ is easily obtained by (84) as we get a telescopic sum (Mengoli sum)

$$\begin{aligned}\mathrm{var}\,(u_n) &\le \sum_{i=0}^{n-1}\frac{1}{2r}(|u_i^n - a|^2 - |u_{i+1}^n - a|^2) \\ &= \frac{1}{2r}(|u_0^n - a|^2 - |u_n^n - a|^2) \le \frac{1}{2r}|u_0 - a|^2.\end{aligned} \tag{85}$$

The general "averaged" case $0 < k \le 1/2$ is much harder. From (79) and (84) we find

$$\mathrm{var}\,(u_n) \le \sum_{i=0}^{n-1} 2|w_{i+1}^n - u_i^n| \le \frac{1}{r}\sum_{i=0}^{n-1}(|u_i^n - a|^2 - |w_{i+1}^n - a|^2). \tag{86}$$

At this point we can invoke an observation of Mabrouk (see [38, Lemma 4]) to rearrange the above expression in order to obtain a telescopic sum.

**Lemma 11.** *With the above notation, the inequality*

$$|u_{i+1}^n - a|^2 \le |w_{i+1}^n - a|^2 + |u_{i+1}^n - w_{i+1}^n|^2$$

*holds.*

**Proof:** Since

$$|u_{i+1}^n - a|^2 = |w_{i+1}^n - a|^2 + |u_{i+1}^n - w_{i+1}^n|^2 + 2\langle u_{i+1}^n - w_{i+1}^n, w_{i+1}^n - a\rangle,$$

it remains to show that the inner product on the right-hand side is nonpositive. For all $z \in V(q_{i+1}^n)$ we have

$$\langle u_i^n - w_{i+1}^n, z - w_{i+1}^n\rangle \le 0, \tag{87}$$

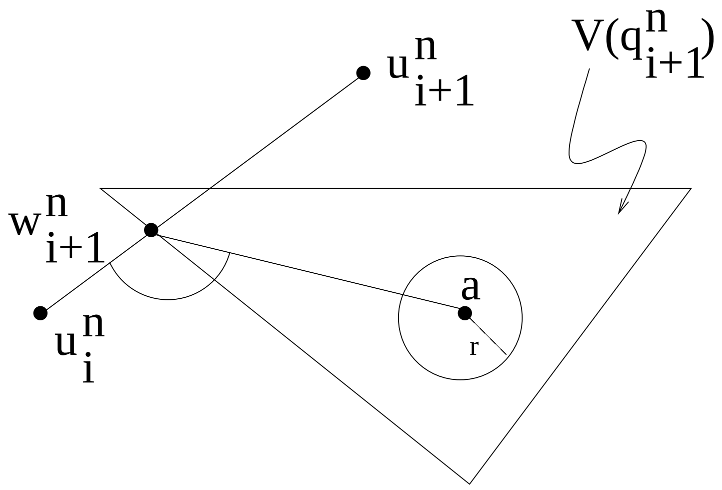


**Fig. 7.** Geometrical meaning of (88)

recall (80) and (4). Choosing $z = a \in V(q^n_{i+1})$ yields

$$\langle u^n_i - w^n_{i+1}, a - w^n_{i+1} \rangle \le 0, \tag{88}$$

see the Fig. 7: the angle is $\ge \pi/2$.
Furthermore, from (81) it follows that

$$w^n_{i+1} - u^n_{i+1} = k'(u^n_i - w^n_{i+1}), \quad k' = k(1-k)^{-1} \ge 0. \tag{89}$$

Hence we obtain

$$\langle w^n_{i+1} - u^n_{i+1}, a - w^n_{i+1} \rangle = k' \langle u^n_i - w^n_{i+1}, a - w^n_{i+1} \rangle \le 0$$

as desired. □

Thus $-|w^n_{i+1} - a|^2 \le -|u^n_{i+1} - a|^2 + |u^n_{i+1} - w^n_{i+1}|^2$, and so (86) implies

$$\begin{aligned} \operatorname{var}(u_n) &\le \frac{1}{r} \sum_{i=0}^{n-1} (|u^n_i - a|^2 - |u^n_{i+1} - a|^2 + |u^n_{i+1} - w^n_{i+1}|^2) \\ &\le \frac{1}{r}|u_0 - a|^2 + \frac{1}{r} \sum_{i=1}^{n} |u^n_i - w^n_i|^2, \end{aligned} \tag{90}$$

since part of the sum again "telescoped". In order to bound the extra terms, as compared with (85) in the case $k = 0$, we have to take into account the variation of $q \mapsto V(q)$. Assume that we already know some a priori estimate on the velocities, say

$$|u_n(t)| \le M_1, \quad n \in \mathbb{N}, \ t \in [0,T]. \tag{91}$$

Then by integration

$$|q_n(t)| \le M_2 := |q_0| + TM_1, \quad n \in \mathbb{N},\ t \in [0,T].$$

In the compact set $\overline{B}_{M_2}(0) \subset \mathbb{R}^d$ the continuous multifunction $V(\cdot)$ is uniformly continuous. In other words, if we introduce the function

$$\delta(\zeta) := \sup\Big\{\mathrm{d_H}(V(q), V(q')) : \, q, q' \in \overline{B}_{M_2}(0),\, |q - q'| \le \zeta\Big\},$$

then $\delta(\zeta) \to 0$ as $\zeta \to 0^+$. Since $|q_{i+1}^n - q_i^n| \le M_1(t_{i+1}^n - t_i^n) = M_1 T/n$, we then find

$$\mathrm{d_H}(V(q_i^n), V(q_{i+1}^n)) \le \delta_n := \delta(M_1 T/n) \to 0, \quad n \to \infty.$$

It turns out that when $k > 0$ the continuity assumption on $V$ has to be reinforced if one wants to obtain from (90) a finite upper bound on the total variations.

**Definition 3.** The multifunction $q \mapsto V(q)$ is called Hölder continuous of exponent $\alpha \in ]0,1]$, if there is $L > 0$ such that

$$\mathrm{d_H}(V(q), V(q')) \le L|q - q'|^\alpha, \quad q, q' \in H. \tag{92}$$

If $\alpha = 1$, this simply means that $V$ is a Lipschitz continuous multifunction. By (92), $\delta(\zeta) \le L\zeta^\alpha$, and thus in particular

$$\delta_n \le L M_1^\alpha T^\alpha n^{-\alpha}. \tag{93}$$

Note that although now the function $\delta(\cdot)$ is independent of $M_2$, the quantities $\delta_n$ do depend on $M_1$, the bound on $u_n$ from (91). To use (93) to estimate the last term in (90), we write $\eta_i^n := |w_i^n - u_i^n|$ and estimate it recursively. With $k' = k(1-k)^{-1}$ we find

$$\eta_{i+1}^n = k'|w_{i+1}^n - u_i^n| = k'\mathrm{dist}\,(u_i^n, V(q_{i+1}^n))$$

from (89). Thus for $i = 0$ we have

$$\eta_1^n = k'\mathrm{dist}\,(u_0, V(q_1^n)) \le k'\mathrm{d_H}(V(q_0), V(q_1^n)) \le k'\delta_n,$$

since $u_0 \in V(q_0)$. Similarly, as $w_i^n \in V(q_i^n)$,

$$\begin{aligned} \eta_{i+1}^n &= k'\mathrm{dist}\,(u_i^n, V(q_{i+1}^n)) \le k'\Big(|u_i^n - w_i^n| + \mathrm{dist}\,(w_i^n, V(q_{i+1}^n))\Big) \\ &\le k'(\eta_i^n + \delta_n), \end{aligned} \tag{94}$$

Notice that if $0 \le k < 1/2$ (this excludes the "elastic" case $k = 1/2$), then $0 \le k' < 1$, and by induction on (94)

$$\eta_i^n \le \delta_n(k' + k'^2 + \ldots + k'^i) \le \left(\frac{k'}{1-k'}\right)\delta_n = \left(\frac{k}{1-2k}\right)\delta_n. \tag{95}$$

Hence by (93)

$$\sum_{i=1}^{n} |u_i^n - w_i^n|^2 = \sum_{i=1}^{n} (\eta_i^n)^2 \leq k^2(1-2k)^{-2} n \delta_n^2 \leq \mathrm{const}\, n^{1-2\alpha}, \tag{96}$$

the constant depending on $M_1$ and $T$. Thus if the Hölder exponent satisfies $\alpha \in [1/2, 1]$, then for some $M_3 > 0$ and uniformly in $n \in \mathbb{N}$

$$\sum_{i=0}^{n-1} |u_{i+1}^n - w_{i+1}^n|^2 \leq M_3. \tag{97}$$

Combining this inequality with (90), we finally arrive at the desired uniform bound on the total variations

$$\mathrm{var}\,(u_n) \leq M_4, \quad n \in \mathbb{N},$$

for some constant $M_4 > 0$. By Theorem 1 we find subsequences – still denoted by $(u_n)$ and $(q_n)$ – and a function $u$ of bounded variation such that as $n \to \infty$

$$u_n(t) \to u(t) \quad \text{for every} \quad t \in [0, T], \tag{98}$$

and also

$$q_n(t) \to q(t) \quad \text{uniformly in} \quad t \in [0, T];$$

the uniform convergence is ensured by the Ascoli-Arzelà theorem. Taking limits in (82) we obtain

$$q(t) = q_0 + \int_0^t u(s)\, ds, \quad t \in [0, T]. \tag{99}$$

These observations can be used to prove the following theorem on the existence of a solution to (72).

**Theorem 11.** *Let $V$ be a multifunction with closed convex values having nonempty interior in $\mathbb{R}^d$, and assume $V$ is Hölder continuous with exponent $\alpha \in [1/2, 1]$, cf. (92). Define the "averaging" operator $\mathrm{Av}(\cdot,\cdot)$ in (74) with some $0 \leq k < 1/2$, and fix initial values $q_0 \in \mathbb{R}^d$ and $u_0 \in V(q_0)$. If the above sequence $(u_n)$ is uniformly bounded, cf. (91), then any pair $(u, q)$ of limit functions on a subsequence of such approximations of the "averaged" problem (73) is a solution of problem (72). More precisely, $u : [0, T] \to \mathbb{R}^d$ is a continuous function of bounded variation, $u(0) = u_0$, (99) holds, we have $u(t) \in V(q(t))$ for all $t \in [0, T]$, and*

$$-\frac{du}{|du|}(t) \in N_{V(q(t))}(u(t)), \quad |du| - a.e., \tag{100}$$

*where $du$ is the Stieltjes measure of $u$ and $|du|$ its measure of total variation.*

*Remark 6.* (a) If $k = 0$, the "bad" terms in the previous estimates vanish and so existence for (100) holds if $q \mapsto V(q)$ is merely continuous. This is the result obtained by Castaing in [18], already mentioned above. What is new here is the study of more general discretization procedures which will be of interest later.

(b) In the applications to dynamics given in the following Section 5, the necessary uniform bound on the velocities can be verified, as a result of some dissipation property inherent to problems describing inelastic collisions or friction. The case of purely elastic collisions, which corresponds to $k = 1/2$, may also be solved, cf. the references below.

(c) For the second order state-dependent problem (72) the Hausdorff continuity of the multifunction is sufficient, as opposed to the first order problems, where Lipschitz continuity was required, see Section 3.3. This is due to the usual smoothing effect of second order problems. ♢

The proof of the existence theorem will be divided into five lemmas, all more or less interesting in their own right. The reader may want to skip the remainder of this section and return to it only later.

From now on we assume $(u, q)$ are limit functions of the sequence $(u_n, q_n)$, which exist by the assumptions and the above arguments.

**Lemma 12.** *For all $t \in [0, T]$ we have $u(t) \in V(q(t))$.*

**Proof:** Let $t \in [0, T]$ and $n \in \mathbb{N}$. Then by construction we known $u_n(t)$ equals either $u_i^n$ or $w_i^n$ for some $i = i_n \in \{0, \dots, n\}$. Hence

$$\begin{aligned}
\operatorname{dist}(u_n(t), V(q(t))) &\le \operatorname{dist}(u_i^n, V(q(t))) + |u_i^n - w_i^n| \\
&\le \operatorname{dist}(u_i^n, V(q_i^n)) + \mathrm{d_H}(V(q_i^n), V(q(t))) + \eta_i^n \\
&\le \mathrm{d_H}(V(q_i^n), V(q(t))) + 2\eta_i^n,
\end{aligned}$$

since $w_i^n \in V(q_i^n)$ by (80). But $\eta_i^n \le \operatorname{const} \delta_n \to 0$ as $n \to \infty$ according to (95), uniformly in $i \in \{0, \dots, n\}$. Moreover, due to $q_i^n = q_n(t_i^n)$, and by (82) and (91),

$$\begin{aligned}
\mathrm{d_H}(V(q_i^n), V(q(t))) &\le L|q_i^n - q(t)|^\alpha \le L\Big(|q_n(t_i^n) - q_n(t)| + |q_n(t) - q(t)|\Big)^\alpha \\
&\le L\Big(\operatorname{const} n^{-1} + |q_n - q|_{L^\infty([0,T];\mathbb{R}^d)}\Big)^\alpha \to 0, \quad n \to \infty.
\end{aligned}$$

Therefore $\operatorname{dist}(u_n(t), V(q(t))) \to 0$ as $n \to \infty$, even uniformly in $[0, T]$. By (98), $u(t) = \lim_{n\to\infty} u_n(t)$, and thus taking the limit as $n \to \infty$ we obtain $\operatorname{dist}(u(t), V(q(t))) = 0$, i.e., $u(t)$ belongs to the closure of $V(q(t))$ which equals $V(q(t))$. □

**Lemma 13.** *The limit function $u$ is right-continuous.*

**Proof:** Fix $t_0 \in [0,T[$ and $\varepsilon > 0$. Since $u(t_0) \in V(q(t_0))$, we find $a_0$ and $r_0 > 0$ such that $B_{2r_0}(a_0) \subset V(q(t_0))$ and $|u(t_0) - a|^2/r_0 \le \varepsilon$; this is easy, cf. [42, Lemma 2.4.4]. Hence $B_{r_0}(a_0) \subset V(q_n(t))$ for all large $n \in \mathbb{N}$ and all $t$ close to $t_0$. By using the bound on the total variation of $u_n$ in a small interval $[t_0, t_0 + \sigma]$ instead of $[0,T]$, we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} \operatorname{var}(u_n; [t_0, t_0+\sigma]) \\
&\le \limsup_{n\to\infty} \left( \frac{1}{r_0} |u_n(t_0) - a_0|^2 + \frac{1}{r_0} \sum_{i=p_n}^{s_n} |u_i^n - w_i^n|^2 \right) \\
&\le \varepsilon + \frac{1}{r_0} \left( \frac{k}{1-2k} \right)^2 \limsup_{n\to\infty} \delta_n^2 \left( \sum_{i=p_n}^{s_n} 1 \right),
\end{aligned}$$

where the number of terms in the sum $\cong (s_n - p_n)$ is of the order of $\sigma/h = \sigma n/T$. Thus, as before in (96),

$$\begin{aligned}
\limsup_{n\to\infty} \operatorname{var}(u_n; [t_0, t_0+\sigma]) &\le \varepsilon + \limsup_{n\to\infty} \delta_n^2 \left( \frac{k}{1-2k} \right)^2 \frac{\sigma n}{r_0 T} \\
&\le \varepsilon + \operatorname{const} \sigma \limsup_{n\to\infty} n^{1-2\alpha} \le \varepsilon + \operatorname{const} \sigma \le 2\varepsilon,
\end{aligned}$$

if $\sigma$ is chosen small enough. For $t \in [t_0, t_0 + \sigma]$, we then have

$$|u(t) - u(t_0)| = \lim_{n\to\infty} |u_n(t) - u_n(t_0)| \le \limsup_{n\to\infty} \operatorname{var}(u_n; [t_0, t_0+\sigma]) \le 2\varepsilon.$$

Thus the lemma is proved. □

Since $u$ is of bounded variation, recall from Remark 1.) in Section 3.4 that the left limits $u^-(t_0) = \lim_{t \nearrow t_0} u(t)$ do exist.

**Lemma 14.** *We have $u^-(t_0) \in V(q(t_0))$ for $t_0 \in [0,T[$, in particular $u$ is continuous.*

**Proof:** 1.) Let $t_n \nearrow t_0$. By Lemma 12, $u(t_n) \in V(q(t_n))$, and we know that $q(t_n) \to q(t_0)$. By continuity of $V(\cdot)$ it then follows that $u^-(t_0) \in V(q(t_0))$.

2.) To verify the second claim, we only have to show $u$ is left-continuous. Fix $\varepsilon > 0$. As in Lemma 13 there is a ball $B_{2r_0}(a_0) \subset V(q(t_0))$ such that $|u^-(t_0) - a|^2/r_0 \le \varepsilon$. Then for sufficiently small $\sigma > 0$ we have analogously to the proof of Lemma 13 that $|u(t_0 - \sigma) - a_0|^2/r_0 \le 2\varepsilon$ and, say, that $\operatorname{var}(u; [t_0 - \sigma, t_0]) \le 3\varepsilon$. □

By the previous lemmas, to complete the proof of Theorem 11 all that remains is to verify (100). We start with a preliminary observation.

**Lemma 15.** *If $0 \le s \le t \le T$ and $z \in V(q')$ for all $q' \in \mathbb{R}^d$ in a neighbourhood of $q([s,t]) \subset \mathbb{R}^d$, then*

$$\langle z, u(t) - u(s)\rangle \ge \frac{1}{2}|u(t)|^2 - \frac{1}{2}|u(s)|^2. \tag{101}$$

**Proof:** Since $q_n \to q$ uniformly on $[0,T]$ and $q$ is Lipschitz continuous, we find $\varepsilon > 0$ such that $z \in V(q_n(\tau))$ for all $n \in \mathbb{N}$ sufficiently large (w.l.o.g. $n \ge 1$) and all $\tau \in [s-\varepsilon, t+\varepsilon]$. Fix $n \ge 1$. Then by definition of $u_n$ we obtain

$$\begin{aligned}\langle z, u_n(t) - u_n(s)\rangle &\ge \langle z, u_l^n - u_j^n\rangle - |z|\Big(|u_l^n - w_l^n| + |u_j^n - w_j^n|\Big)\\ &\ge \langle z, u_l^n - u_j^n\rangle - \mathrm{const}\,|z|\delta_n,\end{aligned} \tag{102}$$

as $\eta_i^n \le \mathrm{const}\,\delta_n$ for all $i \in \{0,\dots,n\}$; here $j$ and $l$, both depending on $n$, are chosen such that $s \in [t_j^n, t_{j+1}^n[$ and $t \in [t_l^n, t_{l+1}^n[$. We may as well assume that $s - \varepsilon \le t_j^n$ and $t_{l+1}^n \le t + \varepsilon$. Hence for $j \le i \le l$ we find $z \in V(q_n(t_{i+1}^n)) = V(q_{i+1}^n)$. Thus (87) implies $\langle z - w_{i+1}^n, w_{i+1}^n - u_i^n\rangle \ge 0$, and therefore

$$\langle z, u_l^n - u_j^n\rangle = \sum_{i=j}^{l-1}\langle z, u_{i+1}^n - u_i^n\rangle \ge \sum_{i=j}^{l-1}\langle w_{i+1}^n, u_{i+1}^n - u_i^n\rangle.$$

To estimate the right-hand side further, we can write

$$\begin{aligned}&\langle w_{i+1}^n, u_{i+1}^n - u_i^n\rangle\\ &= \langle u_{i+1}^n, u_{i+1}^n - u_i^n\rangle + \langle w_{i+1}^n - u_{i+1}^n, u_{i+1}^n - u_i^n\rangle\\ &= \frac{1}{2}|u_{i+1}^n|^2 - \frac{1}{2}|u_i^n|^2 + \frac{1}{2}|u_{i+1}^n - u_i^n|^2 + \langle w_{i+1}^n - u_{i+1}^n, u_{i+1}^n - u_i^n\rangle\\ &= \frac{1}{2}|u_{i+1}^n|^2 - \frac{1}{2}|u_i^n|^2 + \langle w_{i+1}^n - \frac{u_i^n + u_{i+1}^n}{2}, u_{i+1}^n - u_i^n\rangle.\end{aligned}$$

Now note the last inner product is nonnegative, according to $2w_{i+1}^n - u_i^n - u_{i+1}^n = (1-2k)(1-k)^{-1}(w_{i+1}^n - u_i^n)$ and $u_{i+1}^n - u_i^n = (1-k)^{-1}(w_{i+1}^n - u_i^n)$. It follows that

$$\langle z, u_l^n - u_j^n\rangle \ge \sum_{i=j}^{l-1}\left(\frac{1}{2}|u_{i+1}^n|^2 - \frac{1}{2}|u_i^n|^2\right) = \frac{1}{2}|u_l^n|^2 - \frac{1}{2}|u_j^n|^2,$$

whence (102) shows

$$\langle z, u_n(t) - u_n(s)\rangle \ge \frac{1}{2}|u_l^n|^2 - \frac{1}{2}|u_j^n|^2 - \mathrm{const}\,|z|\delta_n.$$

Finally, $|u_j^n - u_n(s)| \le \eta_j^n \le \mathrm{const}\,\delta_n$ and $|u_l^n - u_n(t)| \le \eta_l^n \le \mathrm{const}\,\delta_n$ together with (98) imply in the limit $n \to \infty$ that (101) holds. □

With the next lemma we finish the proof of Theorem 11. Notice that only this last lemma requires a more specific measure theoretic background. Some material on the subject may be found in [51].

**Lemma 16.** *For $|du|$-almost every $t$, one has*

$$-\frac{du}{|du|}(t) \in N_{V(q(t))}(u(t)). \tag{103}$$

**Proof:** By a theorem of Jeffery, cf. [42, p. 9], for $|du|$-a.e. $t$ the density $\frac{du}{|du|}(t)$ of the measure $du$ with respect to $|du|$ at $t$ can be calculated as

$$\frac{du}{|du|}(t) = \lim_{\varepsilon\to 0+} \frac{du([t, t+\varepsilon])}{|du|([t, t+\varepsilon])} = \lim_{\varepsilon\to 0+} \frac{u(t+\varepsilon) - u(t)}{|du|([t, t+\varepsilon])}.$$

The analogous result holds for the density $\frac{d(|u|^2/2)}{|du|}(t) = \langle u(t), \frac{du}{|du|}(t)\rangle$, where we have used the fact that $u$ is continuous. If $z$ is in the interior of $V(q(t))$, we find $\sigma > 0$ and a neighborhood of $q([t-\sigma, t+\sigma])$ such that $z \in V(q')$ for all $q'$ in this neighborhood, according to the continuity properties of $V(\cdot)$ and $q(\cdot)$. Hence Lemma 15 can be applied to yield

$$\langle z, u(t_0+\varepsilon) - u(t)\rangle \geq \frac{1}{2}|u(t+\varepsilon)|^2 - \frac{1}{2}|u(t)|^2$$

for $\varepsilon > 0$ small. Dividing by $|du|([t, t+\varepsilon])$ preserves the inequality, and letting $\varepsilon$ tend to zero we obtain

$$\left\langle z, \frac{du}{|du|}(t)\right\rangle \geq \left\langle u(t), \frac{du}{|du|}(t)\right\rangle.$$

By density, this inequality transfers to all $z \in V(q(t))$, and (103) follows. □

## 5 Applications to the dynamics of unilateral contact

### 5.1 Frictionless unilateral contact

In the early eighties, Moreau introduced a comprehensive mathematical formulation of the dynamics of a particle or a system under frictionless constraints, [47]. A detailed exposition of what he called "standard inelastic shocks" can be found in [48]. Moreau's formulation leads to quite natural and successful numerical procedures (mainly developed later for problems with friction). The first mathematical proof of existence of a solution to the system was given by Monteiro Marques in [39]; see also [42, Sects. 3.1 & 3.2]. The mathematical formulation and the existence proof recently have been extended by Mabrouk [38] to elastic or partially elastic cases. In this section, we will give an outline of this theory and of the proof of Mabrouk's result, and we also mention other related and/or recent developments. For more details, one may refer to [42] and especially to [8], where nonsmooth mechanical problems are very well introduced to the non-expert reader.

A mechanical system with a finite number of degrees of freedom can be represented by a point in the manifold of configurations, or alternatively using local coordinates, by a point $q$ in some Euclidean space $\mathbb{R}^d$. For brevity, we will write $E = \mathbb{R}^d$. The existence results mentioned above will thus be concerned with motions $t \mapsto q(t) \in E$. These motions are subject to a single smooth constraint, i.e., $q(t)$ has to remain in a fixed set

$$L = \{q \in E : f(q) \leq 0\},$$

with $f : E \to \mathbb{R}$ a given smooth function. In other words,

$$f(q(t)) \leq 0, \quad t \in I, \tag{104}$$

is required for the solution, where $I \subset \mathbb{R}$ is an interval, say $[0, T]$.

Contact with the boundary $S = \{q \in E : f(q) = 0\}$ may lead to a discontinuity of the velocity. Moreau thus assumed that the velocity $v = \dot{q}$ is a function of bounded variation (in short, a bv function), which is quite natural and advantageous. In fact, recall from Remark 1.) in Section 3.4 that such functions do have left and right limits, and therefore we will be able to consider left and right velocities and contact laws relating them. Moreover, the Stieltjes measure $dv$ of such a (possibly discontinuous) bv velocity will substitute the classical acceleration $\dot{v} = \ddot{q}$ of a smooth motion $q$. One reason for the usefulness of this approach is that by integration of the "generalized acceleration" $dv$ one obtains right and left limits of the velocity $v$, see the formulas (67) in Section 3.4. Since any bv function is continuous except at an at most countable set of points, the integration of either $v$, $v^+$, or $v^-$ with respect to Lebesgue measure all yields the same result, namely the motion $q$. Let us explain these points a little further. Classically, one writes

$$q(t) = q_0 + \int_0^t v(s)\, ds, \tag{105}$$

where $v$ is the usual velocity $v(t) = \dot{q}(t) = \lim_{h \to 0}(q(t+h) - q(t))/h$. In smooth dynamics, $v$ would be of class $C^1$, or at least absolutely continuous, and the acceleration $\dot{v} = \ddot{q}$ would satisfy

$$v(t) - v(s) = \int_s^t \dot{v}(\tau)\, d\tau.$$

Under the constraint (104), this no longer can be expected to hold, since contact with the boundary $S$ usually leads to discontinuities of the velocity. Therefore, we will instead work with velocities of bounded variation, i.e., $v \in \mathrm{bv}(I, E)$. More generally, in (105) we replace the "real" velocity $v$ (meaning the time-derivative of $q$ at each $t$) by any bv function $u$ which satisfies

$$q(t) = q_0 + \int_0^t u(s)\, ds, \quad t \in I. \tag{106}$$

Since $u \in \mathrm{bv}(I, E)$, $u$ possesses right and left limits $u^+$ and $u^-$, respectively, and $u$ has at most a countable number of discontinuity points. Hence $u = u^+ = u^-$ Lebesgue almost everywhere, and (106) implies that also

$$q(t) = q_0 + \int_0^t u^+(s)\,ds = q_0 + \int_0^t u^-(s)\,ds. \tag{107}$$

Moreover, by a simple computation,

$$\dot{q}^+(t) = \lim_{h\to 0+} \frac{q(t+h) - q(t)}{h} = \lim_{h\to 0+} \frac{1}{h}\int_t^{t+h} u(s)\,ds = u^+(t),$$

i.e., the right velocity $\dot{q}^+(t)$ equals the right limit $u^+(t)$; of course the analogous result holds for the left velocity. Thus, (107) means that the motion $q$ can be obtained by integration of either the right or the left velocity. At its continuity points, $u$ coincides with the actual velocity. From all the above, it is thus reasonable to call $u$ a (generalized) velocity of $q$.

Now we turn to the description of the dynamics, usually governed by

$$M\ddot{q} = p(t, q, \dot{q}) + R.$$

Here $M \in \mathbb{R}^{d\times d}$ is the mass (or inertia) matrix, $p$ is a given force field, say, of gravity, elastic, and viscous forces, and $R$ is the reaction force exerted by the boundary $S$, i.e., by the constraint. In the sequel we will assume for simplicity that $M = I$, hence we consider

$$R = \ddot{q} - p(t, q, \dot{q}) = \dot{u} - p(t, q, u). \tag{108}$$

Since we have already admitted that the acceleration $\ddot{q}$ is the "derivative" of a bv function, hence a measure, we thus have to take the next logical step: (108) implies that the reaction must be a measure as well,

$$dR = d\dot{q} - p(t, q, \dot{q})dt = du - p(t, q, u)dt, \tag{109}$$

where $dt$ is the Lebesgue measure on the interval $I$.

Away from contact, i.e., as long as $f(q(t)) < 0$, the reaction is zero, and hence $dR = 0$. Without going into more detail, this just means that $d\dot{q} = p(t, q, \dot{q})dt$, or the usual

$$\ddot{q} = p(t, q, \dot{q}), \quad \text{or} \quad \frac{du}{dt} = p(t, q, u).$$

In this case $u^+ = u^-$, and both functions are absolutely continuous (under reasonable assumptions on the forces $p$). Absolute continuity may also hold for periods of motion with persistent contact, $f(q(t)) \equiv 0$ during some time, usually with nonzero reaction.

When the velocity is discontinuous at an instant $t_s$, that is, $u^+(t_s) \neq u^-(t_s)$, the situation is different. "Restricted to $t_s$", $du$ coincides with a Dirac measure, since $du|_{\{t_s\}} = (u^+(t_s) - u^-(t_s))\delta_{t_s}$ because of

$$du(\{t_s\}) = \int_{\{t_s\}} du = u^+(t_s) - u^-(t_s).$$

As the Lebesgue measure of $\{t_s\}$ is zero, we have $\int_{\{t_s\}} p(t, q(t), u(t))dt = 0$, and therefore (109) implies that

$$dR(\{t_s\}) = \int_{\{t_s\}} dR = \int_{\{t_s\}} du = u^+(t_s) - u^-(t_s). \tag{110}$$

Consequently, the impact or contact laws relating a pre-impact velocity $u^-$ to a post-impact velocity $u^+$ can be expressed in terms of the reaction $dR$. To determine what are the conditions the left and right velocities have to satisfy, let $f(q(t_s)) = 0$. Since $f(q(t)) \leq 0$ for all $t$, $t = t_s$ is a maximum of that function. Taking right and left derivatives yields

$$\langle u^+(t_s), \nabla f(q(t_s))\rangle \leq 0 \quad \text{and} \quad \langle u^-(t_s), \nabla f(q(t_s))\rangle \geq 0. \tag{111}$$

Thus for all $t \in I$ we find

$$u^+(t) \in V(q(t)) \quad \text{and} \quad -u^-(t) \in V(q(t)),$$

where

$$V(q) = \begin{cases} \{w \in E : \quad \langle w, \nabla f(q)\rangle \leq 0\} & : \quad f(q) \geq 0 \\ \qquad\qquad E & : \quad f(q) < 0 \end{cases} \tag{112}$$

is called the "tangent cone to $L$ at the point $q$", although in fact $V(q)$ is defined for all $q \in E$ which is more convenient, especially in view of numerical applications. Therefore $V(q)$ is either the whole space or a halfspace, so that for all $q \in E$

$$\operatorname{int} V(q) \neq \emptyset. \tag{113}$$

If $f(q(t_s)) = 0$ and $\langle u^-(t_s), \nabla f(q(t_s))\rangle > 0$, it is clear from (111) that $u^+(t_s) \neq u^-(t_s)$: a collision or impact occurs. For perfect (frictionless) unilateral contacts, the reaction force exerted by the boundary $S = \partial L$ is normal and pointing inwards (no adhesion). Hence there exists $\lambda \geq 0$ such that

$$R = -\lambda \nabla f(q).$$

If we "integrate for the infinitesimally small duration of the impact", and assuming that the reaction does not change direction, we obtain by (110)

$$u^+(t_s) - u^-(t_s) = -\alpha \nabla f(q(t_s)) \tag{114}$$

for some $\alpha > 0$, the so-called "liaison percussion"; see Fig. 8.

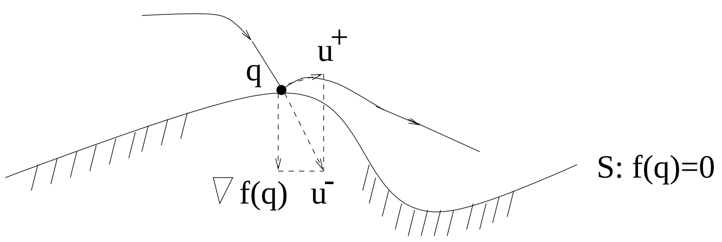


**Fig. 8.** A partially elastic collision at $q$

When $L = \{q : f(q) \leq 0\}$ is convex, this can be reformulated as $dR$ being inward normal to $L$ at $q$, or

$$-dR \in N_L(q), \tag{115}$$

since the outward normal cone is given by $N_L(q) = \{0\}$ if $q \in \operatorname{int} L$, and $N_L(q) = \{\alpha \nabla f(q) : \alpha \geq 0\}$ if $f(q) = 0$. (This might be extended to more general sets $L$, by using the appropriate notion of normal cone). Notice that (114) or (115) are not enough to determine $u^+ = u^+(t_s)$ given $u^- = u^-(t_s)$, as there is a range of possible choices for $u^+$ due to the fact that $\alpha$ is unknown.

To describe a **purely inelastic impact**, we just require that the post-impact velocity $u^+$ be tangential, i.e.,

$$u^+ \in T(q) = \{w \in E : \langle w, \nabla f(q) \rangle = 0\},$$

where $f(q) = 0$. Then the outward normal cone to $V(q)$ at $u = u^+$ is again

$$N_{V(q)}(u) = \{\alpha \nabla f(q) : \alpha \geq 0\},$$

so that, with $u = u^+$

$$-dR \in N_{V(q)}(u). \tag{116}$$

A **purely elastic impact** corresponds to conservation of the kinetic energy, $\mathcal{E}(\dot{q}^+) = \mathcal{E}(\dot{q}^-)$ with $\mathcal{E}(\dot{q}) = \frac{1}{2}|\dot{q}|^2$. Thus a purely elastic impact is characterized by

$$|u^+| = |u^-|. \tag{117}$$

This can simply be expressed as $\frac{1}{2}(u^+ + u^-) \in T(q) \subset V(q)$, and then

$$-dR \in N_{V(q)}\left(\frac{u^+ + u^-}{2}\right).$$

Besides these two classical impact laws, there are more general choices. In order to consider partially elastic, partially inelastic collisions, it suffices to assume $u = u(t_s)$ is a convex combination of $u^- = u^-(t_s)$ and of $u^+ = u^+(t_s)$ and then require that (116) holds. In fact, (116) forces $u$ to belong to $V(q)$. Moreover, $u$ cannot lie in the interior of $V(q)$, since otherwise the normal cone at $u$ would reduce to zero and the reaction would be zero, which is false. Even away from contact (116) has to hold, because then the reaction must be zero and in fact $V(q) = E$ implies that the corresponding outward normal cone is zero.

The general approach described above was adopted by Mabrouk [38], while Schatzman and Paoli [58] studied the problem in the form (115) and (117) (the purely elastic case was treated much earlier by Schatzman [66]). Here the convex combination is taken of the form

$$u = \frac{1}{1+e}\,u^+ + \frac{e}{1+e}\,u^-, \tag{118}$$

where $e \in [0,1]$ is the so-called restitution coefficient. Notice that $e = 0$ leads to $u = u^+$, i.e., purely inelastic impacts, while $e = 1$ leads to $u = \frac{1}{2}(u^+ + u^-)$, that is, the purely elastic case; cf. Fig. 9, and also Fig. 6 above.

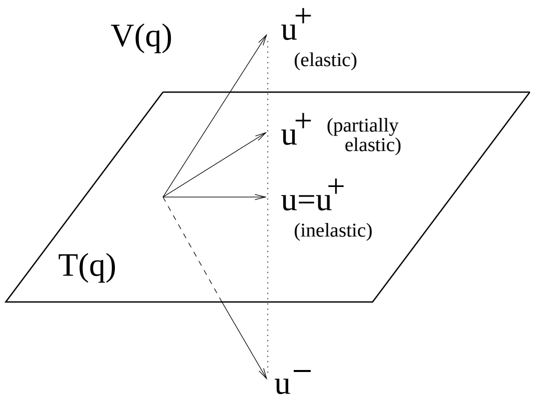


**Fig. 9.** Elastic, partially elastic, and inelastic impact

In the notation from (74) in Section 4, (118) reads as

$$u = \mathrm{Av}(u^-, u^+) \quad \text{with} \quad k = \frac{e}{1+e}\,.$$

Let us note in passing that therefore $k' = k(1-k)^{-1} = e$. Moreau [50, Remark 10.2] also proposed, at least informally, an extension of his model of standard inelastic impacts for the purpose of numerical computations. The extension consisted in the introduction of a "dissipation index" $\delta$ and the requirement that

$$u = \frac{1+\delta}{2}\,u^+ + \frac{1-\delta}{2}\,u^-;$$

hence with

$$e = \frac{1-\delta}{1+\delta}, \quad \delta = \frac{1-e}{1+e}, \quad k = \frac{1-\delta}{2}$$

we are back to (118).

Turning now to the more mathematical part of the problem, we say a function $u : I \to E$ of bounded variation is $e$-averaged, if it satisfies (118), and we write $u \in \mathrm{bvem}(I, E)$. This "normalization" only concerns the at most countable discontinuity points of $u$. Hence an $e$-averaged function $u$ obtained from $u^+$ and $u^-$ satisfying (107) will also satisfy (106), and its right and left limits are again $u^+$ and $u^-$, respectively. Moreover, since $u \in [u^-, u^+]$, the Stieltjes measure is not affected, i.e., $du = du^+ = du^-$; see [53, Section 6] for some information on such "aligned jumps". Notice that for $e = 0$, $\mathrm{bvem}(I, E)$ reduces to the standard space of right-continuous bv functions.

Summarizing the above discussion, we want to solve the following problem.

**Problem P:** *Given $q_0 \in L = \{q : f(q) \leq 0\}$, $u_0 \in V(q_0)$ (cf. (112)), and a force field $p : I \times E \times E \to E$, try to find $u \in bvem(I, E)$ with $u(0) = u_0$ such that, putting $q(t) = q_0 + \int_0^t u(s)ds$, we have $u(t) \in V(q(t))$ for $t \in I$, and moreover*

$$-dR = p(t, q(t), u(t))dt - du \in N_{V(q(t))}(u(t)), \tag{119}$$

*in the following sense: There exists a nonnegative measure $d\mu$ on the interval $I$ relative to which both the Lebesgue measure $dt$ and the Stieltjes measure $du$ possess densities $t'_\mu \in L^1(I, d\mu; \mathrm{I\!R})$ and $u'_\mu \in L^1(I, d\mu; E)$, respectively, such that the differential inclusion*

$$p(t, q(t), u(t))t'_\mu(t) - u'_\mu(t) \in N_{V(q(t))}(u(t)) \tag{120}$$

*holds for $d\mu$–almost every $t \in I$.*

*Remark 7.* (a) We can also request that (120) holds for every $t \in I$, by appropriately modifying the (representatives) of the densities in a set of measure zero.

(b) This formulation is essentially also valid for a more general set of constraints

$$L = \{q \in E : f_1(q) \leq 0, \dots, f_m(q) \leq 0\} \tag{121}$$

for $q \in \partial L$, where the tangent cone now is

$$V(q) = \{v \in E : \langle v, \nabla f_i(q) \rangle \leq 0 \text{ if } f_i(q) = 0\}. \tag{122}$$

(c) Mabrouk has shown in [38, Prop. 1] that any solution of Problem P has several expected properties, such as the generalized versions of the theorem of kinetic energy ($\langle u, dR \rangle = \langle u, pt'_\mu - u'_\mu \rangle = 0$) or of dissipation ($|u^+| \leq |u^-|$). More precisely, by [38, (16)] the dissipation is given by

$$\frac{1}{2}|u^+|^2 - \frac{1}{2}|u^-|^2 = -\frac{1}{2}\left(\frac{1-e}{1+e}\right)|u^+ - u^-|^2 = -\frac{\delta}{2}|u^+ - u^-|^2,$$

or $\mathcal{E}(u^+) - \mathcal{E}(u^-) = -\delta\mathcal{E}(u^+ - u^-) \leq 0$. This justifies that $\delta$ is called the dissipation index. ◊

According to the similarity of Problem P with the "averaged" second order sweeping process (73) from the previous section, it is natural to use a discretization procedure of the same type. This yields the following existence result, [38, Theorem 2]; recall $E = \mathbb{R}^d$.

**Theorem 12.** *Let the constraint function $f$ belong to $C^{1,\alpha}(E)$, with $\frac{1}{2} < \alpha \leq 1$, and let $\nabla f(q) \neq 0$ for all $q \in E$. Assume that the force field $p = p(t,q)$ is continuous and bounded. Then:*

*(i) The sequence $(u_n, q_n)$ to be constructed below has a subsequence which converges pointwise$\times$uniformly to a limit $(u,q)$, where $u \in \mathrm{bv}(I,E)$ and (106) holds.*
*(ii) $\mathrm{Av}(u^-, u^+)$ is a solution of Problem P.*

Before giving a sketch of the proof, let us make several comments.

*Remark 8.* (a) When the given forces $p$ are allowed to depend also on the velocity $u$, then it is necessary to suppose that $p$ be Lipschitz continuous with respect to the $u$-variable; see [38, Section 4.7]. If the global boundedness assumption on $p$ is weakened, the procedure still yields local solutions of Problem P.

(b) Uniqueness of solution has been shown when the data $(f,p)$ are real analytic, by Schatzman [67] in the one-dimensional case with one constraint. For elastic collisions inside a convex set, there is a specific result on uniqueness in [66]. Other results have been obtained by Percivale [60,61] and by Carriero and Pascali [16]. More recently, Ballard [3] made a significant extension of the uniqueness result to the situation mentioned in the next Remark (c). Nonuniqueness examples have been given e.g. by Schatzman [65] in the elastic case and by Ballard [3] in the inelastic case with a $C^\infty$ force.

(c) The existence of a solution in the general case of several constraints, cf. (121) and (122), with real analytic data and a non-constant mass matrix has recently been shown by Ballard [3].

(d) The existence of a solution in the situation of several constraints and more general (nonanalytic) data remains open. In the case of orthogonal constraints, there are strong indications (the last one being Ballard [3, Sect. 6 & 7]) that there is a solution. Notice that the use of a fixed point argument for some sweeping processes could only establish the existence of solution for a related simpler problem, which may be called "externally induced inelastic collisions"; see [42, Ch. 4].

(e) The elastic problem, say (115), (117), has attracted a lot of attention on its own; a part of the relevant references are mentioned in [42, Ch. 3]. Let us add here just Buttazzo and Percivale [14,15] and some of the work of Schatzman and Paoli. Existence for elastic collisions inside a convex set $L$ has been shown by Schatzman [65,66]. Paoli [57] proceeded to study dissipative collisions, still inside a convex set $L$. Recently, these results have been generalized to nonconvex $L$ with $C^3$ boundary, the mass matrix being allowed to vary, [59]. Some further insight might be gained by comparing this to Mabrouk's result, where $\partial L \in C^{1,\alpha}$ with $\frac{1}{2} < \alpha \leq 1$. $\diamondsuit$

Let us now start with a sketch of the proof of Theorem 12 by describing the discretization procedure. As in the previous section, the interval $I = [0, T]$ is partitioned by points $t_i^n$, and the approximating functions $u_n$ and $q_n$ are defined with the help of the specific values $q_i^n = q_n(t_i^n)$ and $u_i^n = u_n^+(t_i^n) = u_n^-(t_{i+1}^n)$. We discretize (118), (119) in the form of the following difference inclusion, cf. with (77):

$$(t_{i+1}^n - t_i^n)p(t_{i+1}^n, q_{i+1}^n) - (u_{i+1}^n - u_i^n) \in N_{V(q_{i+1}^n)}(\mathrm{Av}(u_i^n, u_{i+1}^n)).$$

Writing $p_{i+1}^n = p(t_{i+1}^n, q_{i+1}^n)$, $h = t_{i+1}^n - t_i^n$, and $w_{i+1}^n = \mathrm{Av}(u_i^n, u_{i+1}^n) = ku_i^n + (1-k)u_{i+1}^n$, we also have

$$(1-k)hp_{i+1}^n - (1-k)(u_{i+1}^n - u_i^n) \in N_{V(q_{i+1}^n)}(w_{i+1}^n),$$

where $k = e(1+e)^{-1}$. By (5), this is equivalent to

$$w_{i+1}^n = \mathrm{proj}\left(u_i^n + (1-k)hp_{i+1}^n, V(q_{i+1}^n)\right), \tag{123}$$

while by definition of $w_{i+1}^n$ and of the "inverse averaging function" Iav we have

$$u_{i+1}^n = \mathrm{Iav}(u_i^n, w_{i+1}^n).$$

The discretization procedure thus is quite similar to the one used in the previous section. The additional terms are of the order of the stepsize $h$, since

$$|(1-k)hp_{i+1}^n| \leq Mh,$$

where $M$ is a bound for the norm of the force $p$. Hence the estimates for the present problem, although being more complicated, resemble those we have been dealing with before. To further explain the analogy, note the multifunction $q \mapsto V(q)$ (the tangent cone from (112)) satisfies (113), but it is not Hausdorff continuous in the whole space as it has "jumps" at the boundary points of $L$. Nevertheless, $V$ is what is called lower semicontinuous, and this still ensures that $V(q)$ contains a fixed ball $B_r(a)$ for all $q$ in a neighbourhood of the initial position $q_0$. This fact is a crucial ingredient for estimating the total variations of the approximate velocities $u_n$. But first we have to obtain the required a priori estimate on the norm of the $u_n$, cf. (91). Using the projections on the $V(q)$'s and on their normal cones as well as Moreau's lemma of the two cones, one obtains, see [38, Prop. 2], that

$$|u_i^n| \leq |u_0| + ihM \leq |u_0| + TM, \quad \forall n, i.$$

Then the total variations can be bounded as in (90), but with some extra terms due to the contributions $(1-k)hp_i^n$ of the force. More precisely, recalling that $|u_{i+1}^n - w_{i+1}^n| \le k|u_{i+1}^n - u_i^n|$ by (79), we obtain

$$\operatorname{var}(u_n) \le c_1|u_0 - a|^2 + c_2 \sum_{i=0}^{n-1} |u_{i+1}^n - u_i^n|^2 + c_3 \sum_{i=0}^{n-1} h + c_4 \sum_{i=0}^{n-1} h^2,$$

whence

$$\operatorname{var}(u_n) \le c_5 + c_2 \sum_{i=0}^{n-1} |u_{i+1}^n - u_i^n|^2. \tag{124}$$

To deal with the sum, it could be shown that

$$|u_{i+1}^n - u_i^n| \le c_6 h + c_7 |\nu_{i+1}^n - \nu_i^n|, \tag{125}$$

see [38, p.829], where $\nu_i^n$ denotes the unit outward normal vector to the set $\{q : f(q) \le f(q_i^n)\}$ at $q_i^n$. Moreover, if $f \in C^{1,1/2}$ (i.e., if $\nabla f$ is Hölder continuous with exponent $\alpha = 1/2$), then there is $c_8 \ge 0$ such that

$$\sum_{i=0}^{n-1} |\nu_{i+1}^n - \nu_i^n|^2 \le c_8. \tag{126}$$

This is not surprising since $\nu_i^n$ is related to $\nabla f(q_i^n)$. One could also observe that if one intersects the halfspaces $V(q)$ with a fixed ball $B$, then the resulting multifunction $q \mapsto V(q) \cap B$ is 1/2-Hölder continuous. As shown in Section 4, cf. e.g. (97), this yields an adequate uniform estimate for the sum of the squared norms. To summarize, inequalities (124)-(126) imply an upper bound for the total variations of the velocities. Hence, there exist limit functions $u$ and $q$ such that, for subsequences still denoted $(u_n)$ and $(q_n)$, one has $u_n(t) \to u(t)$ pointwise and $q_n(t) \to q(t)$ uniformly for $t \in I$ as $n \to \infty$. Define $w = \mathrm{Av}(u^-, u^+)$, or more explicitly

$$w(t) = \frac{1}{1+e}\, u^+(t) + \frac{e}{1+e}\, u^-(t) = (1-k)u^+(t) + ku^-(t),$$

so that $w = u$ at continuity points and $q(t) = q_0 + \int_0^t w(s)\, ds$.

Then Mabrouk [38] proceeded to prove that $w$ is a solution of Problem P in the following steps.

Step 1.) It is shown that $f(q_n(t)) \le c_9 h$ for some constant $c_9$, so that in the limit $f(q(t)) \le 0$, i.e., $q(t) \in L$.

Step 2.) This immediately implies that

$$\dot{q}^+(t) = w^+(t) = u^+(t) \in V(q(t)),$$

i.e., the right velocity is kinetically admissible.

Step 3.) Similarly to Lemma 15, one can show next, see [38, Lemma 6], that for $f \in C^{1,1/2}$ and if $z \in V(q')$ for all $q'$ in a neighbourhood of the range $q([s,t])$, then

$$\langle z, u(t) - u(s)\rangle \geq \frac{1}{2}|u(t)|^2 - \frac{1}{2}|u(s)|^2 + \int_s^t \langle z - u(\tau), p(\tau, q(\tau))\rangle \, d\tau.$$

Step 4.) Now let $d\mu = |du| + dt$. Then the required densities do exist, and the previous inequality implies that for $d\mu$-almost every continuity point $t$ of $w$ (or equivalently, for $d\mu$-almost every continuity point of $u$),

$$\langle z, w'_\mu(t)\rangle \geq \langle w(t), w'_\mu(t)\rangle + \langle z - w(t), p(t, q(t))\rangle,$$

from which it follows that

$$p(t, q(t))t'_\mu(t) - u'_\mu(t) \in N_{V(q(t))}(u(t)).$$

To obtain this result, one uses again the theorem of Jeffery, cf. the proof of Lemma 16 or [53], and the fact that $w$ and $u$ differ only at discontinuity points.

Step 5.) To deal with the discontinuities, the assumption $f \in C^{1,\alpha}(E)$ for $\frac{1}{2} < \alpha \leq 1$ is used in its full strength for the first time; observe that up to now $f \in C^{1,1/2}(E)$ was sufficient. The arguments refine those in [39]. First, one checks that any discontinuity instant $t_s$ must correspond to a left velocity that is not admissible as a right velocity, i.e., $w^-(t_s) \notin V(q(t_s))$; this can be obtained similarly to Lemmas 13 and 14. Therefore one has to consider $q_s = q(t_s)$ with $f(q_s) = 0$ and $w^- = w^-(t_s) \notin V(q_s)$, i.e., $\langle w^-, \nabla f(q_s)\rangle > 0$; notice that $w^- = u^-(t_s)$. It is then not difficult to see that the condition to be satisfied at $t_s$ is

$$w(t_s) = \mathrm{proj}\,(w^-(t_s), V(q(t_s))) = \mathrm{proj}\,(w^-, V(q_s));$$

this may be compared with (123) with step $h = 0$ and $u_i^n$ standing for the left-limit $w^-$.

Step 6.) Since $w^-(t_s) \neq w^+(t_s)$, i.e., $u^-(t_s) \neq u^+(t_s)$, we find $\eta > 0$ with $|u(t_s+\eta) - u(t_s-\eta)| \geq 2\delta$, whence from the pointwise convergence it follows that $|u_n(t_s+\eta) - u_n(t_s-\eta)| \geq \delta > 0$ for all large $n \in \mathbb{N}$ and some $\delta > 0$. If all the $t_i^n \in ]t_s - \eta, t_s + \eta[$ were such that $f(q_n(t_i^n)) = f(q_i^n) < 0$, then $V(q_i^n) = E$ and by (123), $w_i^n = u_i^n + (1-k)hp_i^n$. This would imply that the variation of $u_n$ or of $w_n$ (with $w_n$ being defined through the $w_i^n$) in the above subinterval is bounded by its length $2\eta$ multiplied by the bound $M$ on the force $p$. With an appropriate choice of $\eta$ and $\delta$, this would lead to $|u_n(t_s+\eta) - u_n(t_s-\eta)| < \delta$, a contradiction. We conclude that, for large $n$, all the approximations $q_n$ must have $f(q_n(t_i^n)) \geq 0$ for some $t_i^n \in ]t_s - \eta, t_s + \eta[$.

Step 7.) Given the first such $t_i^n$, the above argument implies that the approximate velocities $u_n$ (hence also $w_n$) do not vary much before that instant. At

$t_i^n$, the definition of $w_i^n$ from (123) means that the required law is satisfied by the approximate functions. The harder part of the proof consists in showing that in the remaining subinterval $]t_i^n, t_s + \eta[$ the variations of $u_n$ and $w_n$ are small enough. This is done in [38, Lemma 9], where the stronger assumption on $f$ is used. By taking limits as $n \to \infty$ and $\eta \to 0$, the desired impact law follows. □

The extension of this existence result to the case of several constraints is certainly difficult, because a discontinuous dependence on the data can occur. Consider e.g. only two constraints $f_1(q) \leq 0$ and $f_2(q) \leq 0$. In the situation described in the last Step 7.) above, one can show similarly that for some $t_i^n$ either $f_1(q_i^n) \geq 0$ or $f_2(q_i^n) \geq 0$. However, limit functions of $(q_n)$ may depend on which of the two constraints is satisfied or violated first. A simple example of this behaviour is that of a ball in a purely inelastic billiard, moving towards a corner with an opening angle larger than $\pi/2$: it can go either way or be stopped at the corner. Nevertheless, this theoretical handicap does not preclude the successful application of discretization procedures based on generalizations of the above algorithm, often with appropriate modifications for better convergence and practicality; see for instance [49, Sect. 15].

### 5.2 Unilateral contact with friction

If friction is added to the dynamical model problem as described in the last section, then there is a vast literature (see e.g. [8]). In this final subsection we will give a very brief account on how the previous ideas apply to isotropic Coulomb friction. The formulation given here follows [42, p. 78-80]. It concerns only the simplest case considered already in the pioneering works of Moreau [49,50]; see also [30].

Let us assume that the classical isotropic Coulomb's law of friction holds when there is contact with the boundary $S = \{q \in E : f(q) = 0\}$. The friction coefficient $\nu$ may depend on the boundary point $q \in S$, i.e., $\nu = \nu(q)$. To each $q \in S$ we associate the friction cone $C(q)$, a (revolution) cone about the inward normal vector $n = n(q) = -|\nabla f(q)|^{-1}\nabla f(q)$, with opening angle $\alpha = \alpha(q)$ such that $\nu(q) = \tan\alpha(q)$. To be precise,

$$C = C(q) = \{v \in E : \langle v, n(q)\rangle \geq |v| \cos\alpha(q)\}.$$

In [42] it is assumed that $\alpha(q) \in ]0, \pi/2[$, i.e., that friction is finite in every direction, see Fig. 10.

The reaction force $r$ has to obey several restrictions: it belongs to the friction cone,

$$r \in C;$$

it vanishes when there is no contact,

$$f(q) < 0 \Rightarrow r = 0,$$

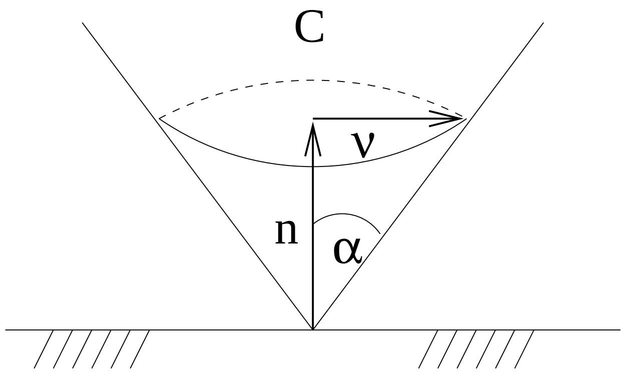


**Fig. 10.** The friction cone

or when there is contact and the right velocity $u$ points inward, that is

$$[f(q) = 0,\ \langle u, n\rangle > 0] \Rightarrow r = 0.$$

Finally, in the nontrivial case where the right velocity $u = \dot{q}^+$ is tangential, that is, when $u$ belongs to the tangent hyperplane $T = T(q)$, the classical form of the friction law can be equivalently expressed as

$$[f(q) = 0,\ \langle u, n\rangle = 0] \Rightarrow -u \in \operatorname{proj}_T\Big(N_C(r)\Big),$$

cf. [49, (11.15), p. 75-76]. That is to say $-u$ belongs to the orthogonal projection on $T$ of the outward normal cone to the friction cone $C$ at $r$. In the spirit of the previous subsection we introduce the reaction measure $dR = du - p(t, q(t))dt$, and we formulate the mathematical problem by replacing $r$ with the density

$$r'_\mu = \frac{dR}{d\mu} \in L^1(I, d\mu; E),$$

where $d\mu$ is any positive measure with respect to which $dR$ is absolutely continuous. Then we have the following result.

**Theorem 13.** *Let the constraint function $f \in C^{1,1}(E)$ satisfy $\nabla f(q) \neq 0$ for all $q \in E$. Assume that the force field $p = p(t, q) : I \times E \to E$ is continuous and bounded and that the angle of friction $\alpha(q)$ is a continuous function of $q \in E$ with values in $]0, \pi/2[$. Then, given $q_0 \in L = \{q : f(q) \leq 0\}$ and $u_0 \in V(q_0)$, there exists a right continuous bv function $u : I \to E$ (the right-velocity) defining the motion $q(t) = q_0 + \int_0^t u(s)\, ds$ for $t \in I$ such that $q(0) = q_0$, $u(0) = u_0$, $q(t) \in L$, and $u(t) \in V(q(t))$ for all $t \in I$. Moreover, the following implications hold $d\mu$-almost everywhere, where $d\mu$ and $r'_\mu$ are* *as above:*

$$
\begin{gathered}
f(q(t)) < 0 \Rightarrow r'_\mu(t) = 0; \\
[f(q(t)) = 0\,, \langle u(t), \nabla f(q(t))\rangle < 0] \Rightarrow r'_\mu(t) = 0; \\
[f(q(t)) = 0,\ \langle u(t), \nabla f(q(t))\rangle = 0] \Rightarrow -u(t) \in \operatorname{proj}_{T(q(t))}\Big(N_{C(q(t))}(r'_\mu(t))\Big).
\end{gathered}
\tag{127}
$$

This existence theorem was first given in [41], cf. also [42, Ch. 3]. It is based on a discretization technique similar to the one used above for frictionless contact problems. The algorithm is suggested by an equivalent form of (127) at a discontinuity point $t_s$: If $u^-(t_s) \neq u(t_s) = u^+(t_s)$, then $t_s$ is an atom of the measure $dR$, with $dR(\{t_s\}) = u(t_s) - u^-(t_s)$ and, a fortiori, an atom of $d\mu$. It follows that $r'_\mu(t_s) = \lambda(u(t_s) - u^-(t_s))$ for some $\lambda > 0$ and thus the inclusion in (127) is equivalent to

$$-u(t_s) \in \operatorname{proj}_{T(q(t_s))}\Big(N_{C(q(t_s))}(u(t_s) - u^-(t_s))\Big).$$

The latter was shown by Moreau [49, p.78-79] to be equivalent to

$$u(t_s) = \operatorname{proj}\Big(0, [u^-(t_s) + C(q(t_s))] \cap T(q(t_s))\Big); \tag{128}$$

see Fig. 11 below.

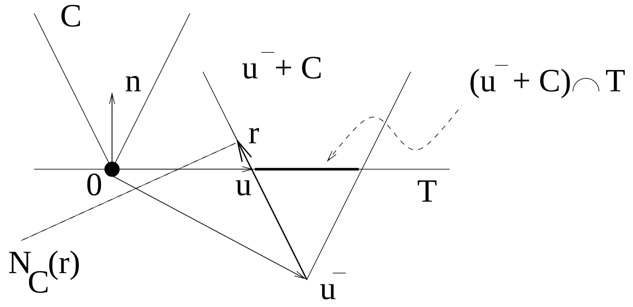


**Fig. 11.** Illustration of (128)

In the algorithm, this is translated roughly as follows; see [42, (3.3.8)]:

$$u^n_{i+1} = \operatorname{proj}\Big(0, [u^n_i + hp^n_{i+1} + C(q^n_{i+1})] \cap T(q^n_{i+1})\Big),$$

using the previous notation. The proof of existence, although more involved, follows the general scheme as in the frictionless case.

It should finally be mentioned that there is currently much interest in the study of nonsmooth dynamics with friction within the mathematical framework of measure differential inclusions (that is, differential inclusions involving bv functions and their Stieltjes measures). A full description of all the recent developments exceeds the scope of these notes, hence we will only mention a few of them.

Glocker [29] also used a measure equality to define the reaction. He considered so-called hemivariational inequalities, developed by Panagiotopoulos [55], which allow to include nonconvex potentials.

Frémond [28] invented a theory of rigid body collisions which may be formulated by means of a measure differential inclusion. The existence of solutions with bv velocities was shown by Cholet in [24].

Stewart [68] extended the existence theorem above, by combining the approach of measure differential inclusions with that of complementarity problems (for the latter see e.g. [56] or [69]). Stewart's remarkable existence result for the one contact case allows for more general friction laws, for a varying mass matrix, and a relaxed restriction on the friction cone. This restriction is satisfied in the celebrated Painlevé problem, which is thus shown to have a solution exhibiting shocks (impulsive contact forces without collisions); a numerical hint for that phenomenon can be found in [50, Sect. 15]. Stewart's approach is also numerically efficient in the case of multiple contacts. However, the corresponding existence theory is naturally not so complete. In that situation, even the formulation of the friction law is a matter of continuing research.

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