Effective Dynamics for the Maxwell–Lorentz System

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Abstract

We consider the Abraham model of a rigid charge distribution coupled to the electromagnetic field and subject to slowly varying external potentials. It is rigorously shown that the true dynamics is well-approximated by an effective dynamics, induced by the Lorentz-Dirac equation. We characterize all "physically relevant" solutions of the latter as lying on a center-like manifold of a singular perturbation problem. In particular, we characterize the famous "run-away"-solutions.

1 Introduction

The Maxwell–Lorentz equations describe the motion of a charged relativistic particle coupled to its self-generated electromagnetic field. Adopting the Abraham model of a rigid, velocity independent charge distribution [1, 13] and choosing units of space–time such that the speed of light $c = 1$ and unit of charge such that $\varepsilon_0 = 1 = \mu_0$, Maxwell's equations for the electric field E and the magnetic field B, respectively, read

$$\frac{\partial}{\partial t}B(x,t) = -\nabla \wedge E(x,t), \quad \frac{\partial}{\partial t}E(x,t) = \nabla \wedge B(x,t) - e\rho(x-q(t))v(t) \tag{1}$$

together with the constraints $\nabla \cdot E(x,t) = e\rho(x-q(t))$ and $\nabla \cdot B(x,t) = 0$. Here e denotes the charge of the particle, and $q(t) \in \mathbb{R}^3$ resp. $v(t) = \dot{q}(t)$ are position resp. velocity. The particle experiences a Lorentz force and hence moves according to

$$\dot{q}(t) = \frac{p(t)}{\sqrt{1+p(t)^2}}, \quad \dot{p}(t) = e\int d^3x\, \rho(x-q(t))\Big[(E(x,t) + E_{\mathrm{ex}}(x) + v(t) \wedge (B(x,t) + B_{\mathrm{ex}}(x))\Big], \tag{2}$$

where $E_{\mathrm{ex}} = -\nabla\phi_{\mathrm{ex}}$ and $B_{\mathrm{ex}} = \nabla \wedge A_{\mathrm{ex}}$ denote prescribed external fields given by potentials, and $p(t) = (1-v^2(t))^{-1/2}v(t)$ is the momentum. Finally, ρ is the charge distribution normalized to $\int d^3x\rho(x) = 1$, and we require $\rho(x)$ to be smooth, radial, and compactly supported, i.e., $\rho \in C_0^\infty(\mathbb{R}^3)$, $\rho(x) = \rho_r(|x|)$, and $\rho(x) = 0$ for $|x| \geq R_\rho$. Taking (formally) $\rho = \delta_0$ in (1), (2) corresponds to the particle being a point particle rather than a charged rigid sphere.

Since the particle is very small on the scale determined by $R_\rho = \mathcal{O}(1)$, it will take large times until it is to experience the effect of the external potentials. Mathematically this can be incorporated into the model by assuming the potentials ϕ_{ex} and A_{ex} to be slowly varying, and this amounts to replacing $E_{\mathrm{ex}}(x)$ and $B_{\mathrm{ex}}(x)$ in the Lorentz equation (2) by $\varepsilon E_{\mathrm{ex}}(\varepsilon x)$ and $\varepsilon B_{\mathrm{ex}}(\varepsilon x)$. We are now interested in the behaviour of the resulting system $(1)_\varepsilon$, $(2)_\varepsilon$ for $\varepsilon > 0$ small, that is to say, in the adiabatic limit. Mainly we would like to derive an ODE such that solutions $r(t)$ of this effective equation stay close to solutions $q(t)$ of the full Abraham-Lorentz system over all, or at least reasonably long times. The structure of this effective equation was first guessed by Abraham [1; p. 123, Eq. (85)] in 1905 and derived more systematically, but still on a formal level, in a famous paper by Dirac [3]. In our notation it reads

$$\begin{aligned}(m_0(u)+m_{\mathrm{f}}(u))\dot{u} &= e\,(E_{\mathrm{ex}}(r)+u\wedge B_{\mathrm{ex}}(r)) \\ &\quad +\varepsilon(e^2/6\pi)\left[\gamma^4(u\cdot\ddot{u})u+3\gamma^6(u\cdot\dot{u})^2u+3\gamma^4(u\cdot\dot{u})\dot{u}+\gamma^2\ddot{u}\right], \qquad (3)\end{aligned}$$

with $u=\dot{r}$. Here $m_0(u)\dot{u}=\gamma\dot{u}+\gamma^3(u\cdot\dot{u})u$ defines the velocity-dependent (bare) mass 3×3-matrix $m_0(u)$, with $\gamma=\gamma(u)=(1-u^2)^{-1/2}$ the relativistic factor. Moreover, the (explicitly known) matrix $m_{\mathrm{f}}(u)$ is the contribution to the mass by the field, resulting in an "effective" total mass matrix $m_0(u)+m_{\mathrm{f}}(u)$. As has been shown in [6] (for the case of a scalar wave field rather than the full Maxwell equations), the effective equation to zero'th order $(m_0(u)+m_{\mathrm{f}}(u))\dot{u}=e\,(E_{\mathrm{ex}}(r)+u\wedge B_{\mathrm{ex}}(r))$ is Hamiltonian, and its solutions approximate well $\varepsilon q(\varepsilon^{-1}t)$, with $q(t)$ the solution of $(1)_\varepsilon$, $(2)_\varepsilon$ for $\varepsilon>0$ small. Nevertheless, including in (3) the first order term in ε is of fundamental importance since the Maxwell-Lorentz system models radiation reaction, i.e., the particle radiates and hence energy goes off to infinity in x-space. Therefore the effective equation that includes description of this effect is to have a slightly dissipative structure instead of being conservative.

However, a fundamental problem arises with interpreting the Lorentz-Dirac equation (3) as the effective equation of the Abraham-Lorentz system, since (3) is an ODE of *third* order in $r(t)$. It might be seen that for some data $(r_0,u_0,\dot{u}_0)\in(\mathbb{R}^3)^3$ the corresponding solutions tend to infinity like $\exp(\exp(t))$, the so-called run-away solutions; see [2, 12]. Therefore it is unclear which initial data lead to "physically relevant" well-behaved solutions, and in addition how to choose $\dot{u}_0$ for the comparison dynamics, since the full equations are of *second* order in $q(t)$ and hence provide only comparison initial conditions for position and velocity. There have been several attempts to solve this problem and to single out the "physically relevant" solutions [3, 4, 5, 9], but no complete (and rigorous) result has been known yet.

2 Main results

In this section we briefly describe our results on the problem described above. These have been obtained jointly with H. Spohn in [7, 8]. Firstly, to make rigorous the Taylor expansion of the self-force in (2) and to arrive at (3) up to an error $\mathcal{O}(\varepsilon^2)$, one has to have a priori bounds for velocity, acceleration, ..., up to $|\dddot{v}(t)|$; this is the most technical part of the proof and relies on suitable estimates on the group generated by the Maxwell operator.

Lemma 1. *For any solution of the Maxwell-Lorentz system $(1)_\varepsilon$, $(2)_\varepsilon$ that starts on the "soliton manifold" (existing for $E_{\mathrm{ex}}=B_{\mathrm{ex}}=0$ due to translation invariance) we have $\sup_{t\in\mathbb{R}}|v(t)|\le\bar{v}<1$. In addition, there exist constants $C>0$ and $\delta_0>0$ such that*

$$\sup_{t\in\mathbb{R}}|\dot{v}(t)|\le C\varepsilon,\quad \sup_{t\in\mathbb{R}}|\ddot{v}(t)|\le C\varepsilon^2,\quad \sup_{t\in\mathbb{R}}|\dddot{v}(t)|\le C\varepsilon^3$$

in case that $e\le\delta_0$. Here $\bar{v}$, C, and δ_0 depend only on bounds for the initial data, but not on ε.

Now consider (3) as a singular perturbation problem, written in the form $\dot{x}=f(x,y)$ and $\varepsilon\dot{y}=g(x,y,\varepsilon)$, with $x=(r,u)\in\mathbb{R}^6$ and $y=\dot{u}\in\mathbb{R}^3$, and functions f and g chosen according to (3) written as a system. Then for $\varepsilon=0$ there exists an invariant normally hyperbolic manifold $\mathcal{I}_0=\{(x,h_0(x)):x\in\mathbb{R}^6\}$. After some technical manipulations, see [7, 8], it can be verified that Fenichel's theorem [10] applies to yield for $\varepsilon>0$ small an invariant manifold

$\mathcal{I}_\varepsilon = \{(x, h_\varepsilon(x)) : x \in \mathbb{R}^6\}$ that is ε-close to $\mathcal{I}_0$, with h_ε being ε-close to h_0. It then could be shown that $\mathcal{I}_\varepsilon$ is *exactly* the set of "physically relevant" solutions to (3). E.g. $\dot{u}(t) \to 0$, $\ddot{u}(t) \to 0$ and $E_{\rm ex}(r(t)) + u(t) \wedge B_{\rm ex}(r(t)) \to 0$ as $t \to \infty$ for solutions on $\mathcal{I}_\varepsilon$, and conversely solutions to (3) with bounded $u(t)$, $\dot{u}(t)$, and $\ddot{u}(t)$, or with bounded $u(t)$ and $\dot{u}(t) \to 0$, have to be contained in $\mathcal{I}_\varepsilon$; see [7, 8] for details. In particular, the dynamics on $\mathcal{I}_\varepsilon$ does not have any run-away solution; this should be considered the true effective Abraham-Lorentz dynamics. It is easily accessible numerically, and the corresponding solutions are good approximations of solutions to the full system, as is expressed in the following theorem.

Theorem 1. *For any solution of the Maxwell-Lorentz system* $(1)_\varepsilon$, $(2)_\varepsilon$ *that starts on the "soliton manifold" the following holds. Let the charge* e *and* $\varepsilon \le \varepsilon_1$ *be sufficiently small (depending only on bounds for the data). At time* $\varepsilon t_\rho = 2\varepsilon R_\rho/(1-\bar{v})$ *we match the initial values,* $r(\varepsilon t_\rho) = \varepsilon q(t_\rho)$, $u(\varepsilon t_\rho) = v(t_\rho)$, *for the motion on the center manifold, i.e., the initial data for the comparison dynamics are* $(\varepsilon q(t_\rho), v(t_\rho), h_\varepsilon(\varepsilon q(t_\rho), v(t_\rho))) \in \mathcal{I}_\varepsilon$. *Then for every* $\tau > 0$ *there exists* $c(\tau) > 0$ *such that for all* $t \in [\varepsilon t_\rho, \varepsilon t_\rho + \tau]$ *we have the bounds*

$$|\varepsilon q(\varepsilon^{-1}t) - r(t)| \le c(\tau)\varepsilon, \quad |v(\varepsilon^{-1}t) - u(t)| \le c(\tau)\varepsilon, \quad |\varepsilon^{-1}\dot{v}(\varepsilon^{-1}t) - \dot{u}(t)| \le c(\tau)\varepsilon,$$

and

$$\Big|[\mathcal{E}_s(v(\varepsilon^{-1}t)) + e\phi_{\rm ex}(\varepsilon q(\varepsilon^{-1}t))] - [\mathcal{E}_s(u(t)) + e\phi_{\rm ex}(r(t))]\Big| \le c(\tau)\varepsilon^2$$

with $\mathcal{E}_s(v)$ *the energy of the soliton with velocity* $|v| < 1$. *Furthermore for* $t = \mathcal{O}(\varepsilon)$, *i.e.* $t \in [\varepsilon t_\rho, \varepsilon t_\rho + \varepsilon\tau]$, *the bound* $|\varepsilon q(\varepsilon^{-1}t) - r(t)| \le c(\tau)\varepsilon^3$ *and* $|v(\varepsilon^{-1}t) - u(t)| \le c(\tau)\varepsilon^2$ *is satisfied.*

The fact that e needs to be small is probably a technical condition that could be removed. Using the theorem it can also be shown that the adiabatic limit of the fields are given by the Liénard-Wiechert potentials of a moving point charge [11], and that the radiated energy agrees with Larmor's formula [11].

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