Adiabatic Limit for the Maxwell–Lorentz Equations

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Abstract

We consider the Abraham model of a rigid charge distribution coupled to the electromagnetic field and subject to, on the scale of the charge diameter, slowly varying external potentials. We prove that in the adiabatic limit the motion of the charged particle is governed by a effective Hamiltonian. To next order one has to add as a small correction the relativistically covariant radiation reaction. This third order equation has a repulsive center manifold and the true solution is well approximated by a trajectory on the center manifold. We also prove that in the adiabatic limit the fields are derived from the Liénard–Wiechert potentials of a moving point charge and that the radiated energy is given by Larmor's formula.

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1 Introduction

The Maxwell–Lorentz equations describe the motion of a charged particle coupled to its own electromagnetic field and possibly subject to external fields. We assume the particle to be relativistic, characterized by its bare mass m_{b} and charge e. To have the dynamics well defined one has to smoothen out the charge. For simplicity we adopt here the Abraham model of a rigid, velocity independent charge distribution [1, 15]. Taking units of space–time such that the speed of light $c = 1$ and unit of charge such that $\varepsilon_0 = 1 = \mu_0$, Maxwell's equations read

$$\frac{\partial}{\partial t} B(x,t) = -\nabla \wedge E(x,t), \quad \frac{\partial}{\partial t} E(x,t) = \nabla \wedge B(x,t) - e\rho(x - q(t))v(t) \tag{1}$$

together with the constraints

$$\nabla \cdot E(x,t) = e\rho(x - q(t)), \quad \nabla \cdot B(x,t) = 0. \tag{2}$$

They are coupled to the Lorentz equation,

$$\begin{aligned} \frac{d}{dt}\Big(m_{\mathrm{b}}\gamma(v(t))v(t)\Big) &= m_0(v(t))\dot v(t) \\ &= e \int d^3x\, \rho(x - q(t))\Big[(E(x,t) + E_{\mathrm{ex}}(x)) + v(t) \wedge (B(x,t) + B_{\mathrm{ex}}(x))\Big], \end{aligned} \tag{3}$$

where $v(t) = \dot q(t)$. Here $x \in \mathbb{R}^3$, $t \in \mathbb{R}$, $\gamma(v) = (1 - v^2)^{-1/2}$, E, B are the electric and magnetic fields, and $q(t) \in \mathbb{R}^3$ and $v(t) \in \mathcal{V} = \{v : |v| < 1\}$ are the position and the velocity of the charged particle, respectively. The velocity dependent (bare) mass $m_0(v)$ is defined as a 3×3 matrix through $m_0(v)\dot v = m_{\mathrm{b}}(\gamma \dot v + \gamma^3(v \cdot \dot v)v)$. The prescribed external fields, E_{ex}, B_{ex}, are are assumed to be time-independent and are derived from external potentials as

$$E_{\mathrm{ex}} = -\nabla \phi_{\mathrm{ex}}, \quad B_{\mathrm{ex}} = \nabla \wedge A_{\mathrm{ex}}.$$

These potentials can be fairly arbitrary, precise conditions will be stated below in (P). Finally, ρ is the charge distribution. It is normalized,

$$\int d^3x \rho(x) = 1, \tag{4}$$

and we require $\rho(x)$ to be smooth, radial, and compactly supported, i.e.,

$$\rho \in C_0^\infty(\mathbb{R}^3)\,, \quad \rho(x) = \rho_r(|x|)\,, \quad \rho(x) = 0 \quad \text{for} \quad |x| \ge R_\rho\,. \tag{C}$$

For an electron, R_ρ would be the classical electron radius which is defined by equating the rest energy with the electrostatic energy, i.e. $mc^2 = e^2/R_\rho$. In numbers one has $R_\rho = 3 \times 10^{-13}$cm, even smaller by a factor of 1836 for protons. External fields which can be generated in the laboratory vary slowly on that scale. Thus it seems natural, and physically mandatory, to study the motion of the charged particle under the condition that the external potentials ϕ_{ex}, A_{ex} vary slowly on the scale of R_ρ. This is the *adiabatic limit*. Formally we introduce a small dimensionless parameter ε, $\varepsilon \ll 1$, and require the potentials to be slowly varying as

$$\phi_{\rm ex}(\varepsilon x), \quad A_{\rm ex}(\varepsilon x). \tag{5}$$

In the Lorentz equation (3) the external fields then have to be substituted by $\varepsilon E_{\rm ex}(\varepsilon x)$, $\varepsilon B_{\rm ex}(\varepsilon x)$.

Let us first develop a qualitative picture for the motion of the charged particle, many details of which will be proved below. On a time scale $\mathcal{O}(R_\rho)$, i.e., $\mathcal{O}(R_\rho/c)$ with the speed of light $c = 1$, the forces are of order ε. Thus the particle moves on an essentially straight line trajectory with constant velocity. The electromagnetic fields adapt themselves correspondingly and in approximation one has a solution to (1) - (3) with $E_{\rm ex} = 0 = B_{\rm ex}$ travelling at constant velocity v. In analogy to nonlinear wave equations we call this solution a *charged soliton*. Depending on v it carries an energy $\mathcal{E}_s(v)$ and a momentum $\mathcal{P}_s(v)$, both of which are the sum of a mechanical and a field contribution. The interaction with the field modifies the bare energy–momentum relation, $E(p) = (m_{\rm b}^2 + p^2)^{1/2}$, to an effective energy–momentum relation $E_{\rm eff}(p) = \mathcal{E}_s(v(p))$, where $v(\mathcal{P}_s)$ is the inverse function to $\mathcal{P}_s(v)$. This relation will be computed in Section 2. We define the *microscopic* space–time scale by $q = \mathcal{O}(R_\rho)$, $t = \mathcal{O}(R_\rho)$.

On the *macroscopic* space–time scale, $q = \mathcal{O}(\varepsilon^{-1}R_\rho)$, $t = \mathcal{O}(\varepsilon^{-1}R_\rho)$, the charge responds to the external forces through the effective energy–momentum relation. Thus on this scale we have a Hamiltonian motion with a velocity–dependent mass $m(v) = m_0(v) + m_{\rm f}(v)$, where $m_{\rm f}(v)$ is the contribution from the field carried along with the charge. In a relativistic theory one would expect that

$$m_{\rm f}(v) = (m_{\rm e}/m_{\rm b})m_0(v), \tag{6}$$

where $m_{\rm e}$ is the electrostatic energy of the charge distribution,

$$m_{\rm e} = \frac{1}{2}e^2 \int d^3x d^3y \, (4\pi|x-y|)^{-1}\rho(x)\rho(y) = \frac{1}{2}e^2 \int d^3k \, |\hat{\rho}(k)|^2 k^{-2}.$$

However the Abraham model is nonrelativistic and therefore $m_{\rm f}(v)$ has a velocity dependence different from (6), cf. (33) below for the explicit expression. The effective Hamiltonian motion is then governed by

$$\frac{d}{dt}r(t) = u(t), \quad (m_0(u) + m_{\rm f}(u))\frac{d}{dt}u(t) = e\left(E_{\rm ex}(r(t)) + u(t) \wedge B_{\rm ex}(r(t))\right) \tag{7}$$

with (r, u) the position and velocity of the charge on the macroscopic scale.

Accelerated charges radiate at a rate roughly proportional to $\dot{v}(t)^2$, which corresponds to a loss $\int_0^t ds\, \dot{v}(s)^2$ in energy over the time span t. We have $|\dot{v}(t)| \cong \mathcal{O}(\varepsilon)$ and therefore on the macroscopic scale, $t = \mathcal{O}(\varepsilon^{-1})$, the energy loss is proportional to ε. Thus there must be dissipative corrections of order ε to the Hamiltonian motion (7). The structure of this radiation reaction was first guessed by Abraham [1, p. 123, Eq. (85)] in 1904 and derived more systematically in a famous paper by Dirac [4]. In our notation it reads

$$\begin{aligned}(m_0(u) + m_{\rm f}(u))\dot{u} \;&=\; e\,(E_{\rm ex}(r) + u \wedge B_{\rm ex}(r)) \\ &\quad + \varepsilon(e^2/6\pi)\left[\gamma^4(u\cdot\ddot{u})u + 3\gamma^6(u\cdot\dot{u})^2u + 3\gamma^4(u\cdot\dot{u})\dot{u} + \gamma^2\ddot{u}\right]\end{aligned} \tag{8}$$

with $u = \dot{r}$. The radiative reaction contains terms proportional to $\ddot{u}(t)$. Thus (8) has undesirable solutions, the runaway solutions, whose acceleration increases without limit. Dirac [4], reemphasized by Haag [7], proposed to eliminate such unphysical solutions by adding the asymptotic condition $\dot{u}(t) \to 0$ as $t \to \infty$. Of course, one would hope that the asymptotic condition is in fact a consequence of the equations of motion. More seriously, the instability of the solution flow to (8) renders it rather impractical in the sense that, unless one has an explicit solution, it is difficult to sort physical solutions from runaways.

As recognized in [5, 6, 11], equations of the form (8) have a repulsive center manifold $\mathcal{I}_\varepsilon$. For initial conditions away from $\mathcal{I}_\varepsilon$ the solution grows without bound, whereas on $\mathcal{I}_\varepsilon$ the acceleration $\dot{u}(t)$ remains bounded and the velocity $u(t)$ stays bounded away from 1. The main goal of our paper is to unravel an identical structure for the (nonrelativistic) Lorentz–Dirac equation (8) and to establish that for small ε the true solution of the Maxwell–Lorentz equations (1)-(3) is tracked with high precision by a solution of (8) on its center manifold. We will prove that the energy of the true solution and of the comparison solution on the center manifold of (8) differ only by order ϵ^2 on the macroscopic scale, where energy refers to the effective hamiltonian for (7). While this singles out the dissipative correction to (8), it leaves open whether the usual distance between these two solutions is of order ϵ^2.

Using results from geometric singular perturbation theory, the center manifold can be expanded in powers of ε. Up to order ε one obtains a second order differential equation, cf. Eq. (24) below, which has only stable solutions and describes the physically correct long–time behavior. Thus in the adiabatic limit the motion of the charged particle is governed by an effective second order differential equation and the associated electromagnetic fields are the Liénard–Wiechert fields of a point charge.

We outline the contents of our paper. In Section 2 we discuss the charged solitons and their energy and momentum. In Section 3 we state precise assumptions on the external potentials, explain the structure of the center manifold, and list our main results. Their proof uses a strategy which is similar to [10] but more involved. Thus one has to show that on the microscopic time scale the derivatives are bounded as $|\dot{q}(t)| < 1$, $|\ddot{q}(t)| \le C\varepsilon$, $|\dddot{q}(t)| \le C\varepsilon^2$, $|\ddddot{q}(t)| \le C\varepsilon^3$ uniformly in t. This is the most technical part of our paper and deferred to an appendix, Section 8. With these a priori bounds we are allowed to Taylor expand the self–force (Section 4) up to a uniformly bounded error of order ε^3 on the microscopic scale. The center manifold is discussed in Section 5. Finally in Sections 6 and 7 we determine the adiabatic limit of the fields and the energy radiated per unit time, which is in agreement with Larmor's formula, of course.

There is one result that came as a surprise to us and thus seems worth to be pointed out. As explained, since the Abraham model of a rigid charge is nonrelativistic, also the effective energy–momentum relation $E_{\text{eff}}(p)$ is nonrelativistic. However, the radiation reaction is relativistic and agrees with the result formally derived by Dirac. In covariant notation the radiation reaction term in the second line of (8) reads, reintroducing the velocity of light, c,

$$(e^2/6\pi c^3)(\ddot{u}^\mu - c^{-2}\dot{u}^\lambda \dot{u}_\lambda u^\mu),$$

compare with [12]. The radiation reaction apparently only knows about the total charge, but no finer details of the charge distribution.

2 Energy-momentum relation for a charged soliton

Throughout this section we consider the Maxwell–Lorentz equations with $E_{\rm ex} = 0 = B_{\rm ex}$. Since then the equations are invariant under spatial translations, one expects to have solutions travelling at constant velocity. In fact, for every $v \in \mathcal{V}$ there is a unique solution given by

$$S_{q,v}(t) = (E_v(x-q-vt), B_v(x-q-vt), q+vt, v), \quad q \in \mathbb{R}^3,\ |v|<1,\ t \in \mathbb{R}, \tag{9}$$

where

$$E_v(x) = -\nabla\phi_v(x) + v(v\cdot\nabla)\phi_v(x), \quad B_v(x) = -v \wedge \nabla\phi_v(x), \tag{10}$$

and $\hat\phi_v(k) = e\hat\rho(k)/[k^2 - (v\cdot k)^2]$, see [10]; the hat denotes Fourier transform. We call $S_{q,v} = S_{q,v}(0)$ the *charged soliton* with center $q \in \mathbb{R}^3$ and velocity $v \in \mathcal{V}$. If there are no external forces, the set of solitons is invariant under our dynamics.

The Maxwell–Lorentz equations can be rewritten in Hamiltonian form. Since the system is translation invariant, there is a conserved energy and momentum. In our variables these are given by

$$\begin{aligned}
\mathcal{H}_0(E,B,q,v) &= m_{\rm b}\gamma(v) + \frac{1}{2}\int d^3x\,[E^2(x) + B^2(x)],\\
\mathcal{P}_0(E,B,q,v) &= m_{\rm b}\gamma(v)v + \int d^3x\,[E(x)\wedge B(x)].
\end{aligned}$$

In particular, the charged soliton $S_{q,v}$ carries the energy $\mathcal{E}_s(v) = \mathcal{H}_0(S_{q,v})$ and the momentum $\mathcal{P}_s(v) = \mathcal{P}_0(S_{q,v})$. They are explicitly given by

$$\mathcal{E}_s(v) = m_{\rm b}\gamma(v) + m_{\rm e}\left[\frac{1}{|v|}\log\frac{1+|v|}{1-|v|} - 1\right], \tag{11}$$

and

$$\mathcal{P}_s(v) = m_{\rm b}\gamma(v)v + m_{\rm e}|v|^{-2}\left[\frac{1+v^2}{2|v|}\log\frac{1+|v|}{1-|v|} - 1\right]v. \tag{12}$$

We note that the soliton $S_{q,v}$ is also characterized as the minimizer of $\mathcal{H}_0$ at fixed total momentum $\mathcal{P}_0 = \mathcal{P}_s(v)$ under the constraints $\nabla\cdot E = e\rho(\cdot - q)$ and $\nabla\cdot B = 0$.

The map $\mathcal{P}_s : \mathcal{V} \to \mathbb{R}^3$ is one–to–one with inverse $v(\mathcal{P}_s)$. Thus the energy–momentum relation of the charged solitons is defined by

$$E_{\rm eff}(p) = \mathcal{E}_s(v(p)). \tag{13}$$

$E_{\rm eff}(p)$ determines the effective equations of motion (7) on the macroscopic scale. In the canonical variables q, p they are of Hamiltonian form with Hamiltonian

$$H(q,p) = E_{\rm eff}(p - eA_{\rm ex}(q)) + e\phi_{\rm ex}(q). \tag{14}$$

To check this, one only has to recall that $m(v) = (d\mathcal{P}_s/dv)(v)$ and $v(d\mathcal{P}_s/dv) = (d\mathcal{E}_s/dv)$, as can be verified from the explicit expressions (11) and (12).

If one expands (11), (12) for small velocities, then

$$\mathcal{E}_s(v) = m_{\rm b} + 2m_{\rm e} + (m_{\rm b} + (4m_{\rm e}/3))\frac{1}{2}v^2 + \mathcal{O}(v^4), \quad \mathcal{P}_s(v) = (m_{\rm b} + (4m_{\rm e}/3))v + \mathcal{O}(|v|^3). \tag{15}$$

Therefore in the small velocity limit the effective mass of the charged particle equals $m_{\rm b} + 4m_{\rm e}/3$.

3 Main results

We first transform (1)–(3) with external potentials (5) to the macroscopic space–time scale through

$$t = \varepsilon^{-1}t', \quad x = \varepsilon^{-1}x', \quad E(x,t) = \varepsilon^{3/2}E'(x',t'), \quad B(x,t) = \varepsilon^{3/2}B'(x',t'), \text{ and } q(t) = \varepsilon^{-1}q'(t').$$

The amplitudes of the transformed fields have been chosen such that the field energy remains invariant. There is little risk in omitting the primes, for simplicity. However, we indicate explicitly the ε–dependence of $q'(t')$ and denote it by $q^\varepsilon(t)$ with velocity $v^\varepsilon(t) = \dot{q}^\varepsilon(t)$. Then

$$\begin{aligned} &\frac{\partial}{\partial t}B(x,t) = -\nabla \wedge E(x,t), \quad \frac{\partial}{\partial t}E(x,t) = \nabla \wedge B(x,t) - \sqrt{\varepsilon}\, e\rho_\varepsilon(x - q^\varepsilon(t))v^\varepsilon(t), \\ &\dot{q}^\varepsilon(t) = v^\varepsilon(t), \quad m_0(v^\varepsilon(t))\dot{v}^\varepsilon(t) = [e(E_{\mathrm{ex}} + v^\varepsilon(t) \wedge B_{\mathrm{ex}}) * \rho_\varepsilon](q^\varepsilon(t)) \\ &\qquad + \sqrt{\varepsilon}\, e \int d^3x\, \rho_\varepsilon(x - q^\varepsilon(t))[E(x,t) + v^\varepsilon(t) \wedge B(x,t)] \end{aligned} \tag{16}$$

with the constraints

$$\nabla \cdot E(x,t) = \sqrt{\varepsilon}\, e\rho_\varepsilon(x - q^\varepsilon(t)), \quad \nabla \cdot B(x,t) = 0,$$

and $\rho_\varepsilon(x) = \varepsilon^{-3}\rho(\varepsilon^{-1}x)$. The energy function on the macroscopic scale is defined by

$$\mathcal{H}_{\mathrm{mac}}(E,B,q,v) = m_{\mathrm{b}}\gamma(v) + [e\phi_{\mathrm{ex}} * \rho_\varepsilon](q) + \frac{1}{2}\int d^3x\, [E^2(x) + B^2(x)].$$

For the external potentials we require $\phi_{\mathrm{ex}} \in C^3(\mathrm{I\!R}^3)$, $A_{\mathrm{ex}} \in C^3(\mathrm{I\!R}^3, \mathrm{I\!R}^3)$, and

$$|\nabla^k \phi_{\mathrm{ex}}|_\infty \le C \;\; (k = 0,1,2,3), \quad |\nabla^k A_{\mathrm{ex}}|_\infty \le C, \;\; (k = 1,2,3). \tag{P}$$

We could equally well treat the confining case, i.e., $\phi_{\mathrm{ex}}(q) \to \infty$ as $|q| \to \infty$, and $|\nabla^k \phi_{\mathrm{ex}}|_\infty \le C$ only for $k = 1,2,3$; cf. [11].

We note that in the rescaled coordinates, the Maxwell-Lorentz equations retain their form. Only the support of the charge distribution now shrinks propotional to ϵ and the total charge is $\sqrt{\varepsilon}e$. Of course, on the microscopic scale ε is a parameter appearing in the external potentials. The physical charge and electrostatic energy of the charge distribution remain fixed. In the more traditional point charge limit the charge distribution tends to a δ-function, which means that the electrostatic energy diverges. The particle and its comoving field can no longer respond to external forces. The only way to compensate is the to let the bare charge tend to $-\infty$ [4], which however results in instabilities [2].

Next, (16) has to be supplemented with initial data (E^0, B^0, q^0, v^0). If they have finite energy, then (16) has a unique (weak) solution global in time and the energy does not change along the solution trajectory. We refer to [10] and to [2], where also the smoothness of solutions in dependence on the smoothness of the initial data is discussed.

For the adiabatic limit it is natural to assume initially a charged soliton centered at q^0 with velocity v^0. With some extra effort we could deal with a more general class of initial data, but this issue will not be persued further. To define the initial data for (16) we only have to transform (10) to the macroscopic scale, which yields

$$
\begin{gathered}
q^\varepsilon(0)=q^0, \quad v^\varepsilon(0)=v^0, \\
E^0(x)=-\nabla\phi^\varepsilon_{v^0}(x-q^0)+(v^0\cdot\nabla\phi^\varepsilon_{v^0}(x-q^0))v^0, \quad B^0(x)=-v^0\wedge\nabla\phi^\varepsilon_{v^0}(x-q^0),
\end{gathered}
\tag{17}
$$

where

$$
\hat{\phi}^\varepsilon_v(k)=\sqrt{\varepsilon}e\,\hat{\rho}(\varepsilon k)[k^2-(k\cdot v)^2]^{-1}.
$$

Our first main result is an a priori estimate on the derivatives of $q^\varepsilon(t)$ uniformly in ε and t. It is proved in Appendix 8 and states

Lemma 3.1 *For any solution of (16) with initial conditions given by (17) we have*

$$
\sup_{t\in\mathbb{R}}|v^\varepsilon(t)|\le\bar{v}<1.
$$

In addition, there exist constants $C>0$ and $\delta_0>0$ such that

$$
\sup_{t\in\mathbb{R}}|\dot{v}^\varepsilon(t)|+\sup_{t\in\mathbb{R}}|\ddot{v}^\varepsilon(t)|+\sup_{t\in\mathbb{R}}|\dddot{v}^\varepsilon(t)|\le C
$$

provided $e\le\delta_0$. Here $\bar{v}$, C, and δ_0 depend only on bounds for the initial data, but not on ε.

With this result we are allowed to Taylor expand $q^\varepsilon(t)$ and $v^\varepsilon(t)$ in the self–force,

$$
F^\varepsilon_s(t)=\sqrt{\varepsilon}\,e\int d^3x\,\rho_\varepsilon(x-q^\varepsilon(t))[E(x,t)+v^\varepsilon(t)\wedge B(x,t)],
\tag{18}
$$

where $E(x,t)$ and $B(x,t)$ are to be inserted from the solution of the inhomogeneous Maxwell equations. $F^\varepsilon_s(t)$ will have then a contribution from the initial fields, which vanishes for $t\ge\varepsilon t_\rho:=2\varepsilon R_\rho/(1-\bar{v})$ using the compact support of ρ. The remaining contribution comes from the retarded fields and has the form

$$
\begin{aligned}
F^\varepsilon_s(t) \;=\; & -m_{\mathrm{f}}(v^\varepsilon(t))\dot{v}^\varepsilon(t)+\varepsilon(e^2/6\pi)\Big[\gamma^4(v^\varepsilon(t)\cdot\ddot{v}^\varepsilon(t))v^\varepsilon(t)+3\gamma^6(v^\varepsilon(t)\cdot\dot{v}^\varepsilon(t))^2v^\varepsilon(t) \\
& +3\gamma^4(v^\varepsilon(t)\cdot\dot{v}^\varepsilon(t))\dot{v}^\varepsilon(t)+\gamma^2\ddot{v}^\varepsilon(t)\Big]+\mathcal{O}(\varepsilon^2)
\end{aligned}
\tag{19}
$$

for small ε, where $\gamma=(1-v^\varepsilon(t)^2)^{-1/2}$ and the 3×3-matrix $m_{\mathrm{f}}(v)$ is defined in (33) below. Because $m(v)=(d\mathcal{P}_s/dv)(v)=m_0(v)+m_{\mathrm{f}}(v)$, we obtain from (16) and (19)

$$
\begin{aligned}
m(v^\varepsilon)\dot{v}^\varepsilon \;=\; & [e(E_{\mathrm{ex}}+v^\varepsilon\wedge B_{\mathrm{ex}})*\rho_\varepsilon](q^\varepsilon) \\
& +\varepsilon(e^2/6\pi)\Big[\gamma^4(v^\varepsilon\cdot\ddot{v}^\varepsilon)v^\varepsilon+3\gamma^6(v^\varepsilon\cdot\dot{v}^\varepsilon)^2v^\varepsilon+3\gamma^4(v^\varepsilon\cdot\dot{v}^\varepsilon)\dot{v}^\varepsilon+\gamma^2\ddot{v}^\varepsilon\Big]+\varepsilon^2f^\varepsilon(t).
\end{aligned}
\tag{20}
$$

Since $\rho_\varepsilon\to\delta_0$, the natural comparison dynamics to (20) is given by

$$
\begin{aligned}
m(u)\dot{u} \;=\; & e(E_{\mathrm{ex}}(r)+u\wedge B_{\mathrm{ex}}(r)) \\
& +\varepsilon(e^2/6\pi)\Big[\gamma^4(u\cdot\ddot{u})u+3\gamma^6(u\cdot\dot{u})^2u+3\gamma^4(u\cdot\dot{u})\dot{u}+\gamma^2\ddot{u}\Big],
\end{aligned}
\tag{21}
$$

with $\gamma = (1-u^2)^{-1/2}$.

Eqn. (21) can be rewritten as a singular perturbation problem. For this, we introduce

$$x = (r, u) \in \mathrm{IR}^3 \times \mathcal{V}, \quad y = \dot{u} \in \mathrm{IR}^3, \quad f(x, y) = (x_2, y) \in \mathcal{V} \times \mathrm{IR}^3, \quad \text{and}$$
$$g(x, y, \varepsilon) = a(x_2)^{-1}\Big((6\pi/e^2)[m(x_2)y - F_{\mathrm{ex}}(x)] - \varepsilon\,[3\gamma^6(x_2\cdot y)^2 x_2 + 3\gamma^4(x_2\cdot y)y]\Big),$$

where $\gamma = (1-x_2^2)^{-1/2}$, $F_{\mathrm{ex}}(x) = e(E_{\mathrm{ex}}(r) + u\wedge B_{\mathrm{ex}}(r))$, and $a(u)z = \gamma^4(u\cdot z)u + \gamma^2 z$, $z \in \mathrm{IR}^3$, with matrix inverse $a(u)^{-1}z = \gamma^{-2}[z - (u\cdot z)u]$. Thus (21) becomes

$$\dot{x} = f(x, y), \quad \varepsilon\dot{y} = g(x, y, \varepsilon). \tag{22}$$

For $\varepsilon = 0$, (22) has the invariant manifold

$$\mathcal{I}_0 = \{(x, y) : g(x, y, 0) = 0\} = \{(r, u, \dot{u}) : m(u)\dot{u} = F_{\mathrm{ex}}(r, u)\} = \{(x, h(x)) : x \in \mathrm{IR}^3 \times \mathcal{V}\}, \tag{23}$$

where $h(x) = m(x_2)^{-1}F_{\mathrm{ex}}(x)$. To find a global invariant manifold $\mathcal{I}_\varepsilon$ for (22) with $\varepsilon > 0$ small we apply the geometric singular perturbation theory, cf. [13, 8], which however requires to modify the functions $a(u)$ and $m(u)$ for $|u|$ near 1, due to the singularity at $|u| = 1$; see [11] for a similar approach. In fact, for the motion on $\mathcal{I}_\varepsilon$ such modification will play no role according to Lemma 3.1.

Let us explain in more detail the structure of the solution flow to (21). In (38) below, we will fix a small $\bar{\delta} = \bar{\delta}(\bar{v}) > 0$ satisfying some estimates; hence $\bar{\delta}$ is determined only through bounds for the initial data. We define

$$\mathcal{K}_{1-\bar{\delta}} = \mathrm{IR}^3 \times \{u \in \mathrm{IR}^3 : |u| \le 1 - \bar{\delta}\}$$

and continue $a(u)$ and $m(u)$ with their values at $|u| = 1-\bar{\delta}$ to the missing infinite strip $1-\bar{\delta} < |u| < 1$. With these modifications the basic assumptions (I), (II) from [13, p. 45] are satisfied, since $\mathcal{I}_0$ is normally hyperbolic (here: repulsive) at a rate independent of ε; cf. Lemma 5.1. Therefore there exist $\varepsilon_0 = \varepsilon_0(\bar{\delta}) > 0$ and a C^1-function $h(x, \varepsilon) = h_\varepsilon(x) : \mathrm{IR}^3 \times \mathcal{V} \times]0, \varepsilon_0] \to \mathrm{IR}^3$ such that for $\varepsilon \le \varepsilon_0$,

$$\mathcal{I}_\varepsilon = \{(x, h_\varepsilon(x)) : x \in \mathrm{IR}^3 \times \mathcal{V}\}$$

is forward invariant for the flow of (21) with the modified functions $a(u)$ and $m(u)$. Since the modified equation agrees with (21) in the interior of $\mathcal{K}_{1-\bar{\delta}}$, it follows that $\mathcal{I}_\varepsilon$ is locally invariant for the flow of (21), i.e. the solution of the modified equation is the solution to the original equation as long as it does not reach the boundary set $\{(x, h_\varepsilon(x)) = (r, u, h_\varepsilon(r, u)) : |u| = 1 - \bar{\delta}\}$. The flow for $\varepsilon = 0$ is governed by $\dot{x} = f(x, h(x))$, which agrees with (7). It then perturbs to $\dot{x} = f(x, h_\varepsilon(x))$ for $\varepsilon \le \varepsilon_0$, i.e.

$$\dot{r} = u, \quad \dot{u} = h_\varepsilon(r, u) = m(u)^{-1}F_{\mathrm{ex}}(r, u) + h_{1,\varepsilon}(r, u), \tag{24}$$

cf. [13, Thm. 2.1, Thm. 2.9], where

$$|h_\varepsilon(r, u)| \le c(\bar{\delta}) \quad \text{and} \quad |h_{1,\varepsilon}(r, u)| \le c(\bar{\delta})\varepsilon \tag{25}$$

for $(r, u) \in \mathrm{IR}^3 \times \mathcal{V}$ and $\varepsilon \in]0, \varepsilon_0]$.

The function $h_{1,\varepsilon}(r,u)$ can be expanded in ε. Then, up to errors of order ε^2, the motion on the center manifold is given by

$$\dot r = u, \qquad m(u)\dot u = e(E_{\rm ex}(r) + u\wedge B_{\rm ex}(r)) \\ + \varepsilon(e^2/6\pi)\, a(u)[u\cdot\nabla_r + h\cdot\nabla_u + 3\gamma^4(u\cdot h)]h \tag{26}$$

with $h = h(r,u) = em(u)^{-1}(E_{\rm ex}(r) + u\wedge B_{\rm ex}(r))$. Note that the error in passing from (21) to (26) is of the same order as the one neglected already in (21). Thus Eqn. (26) should be regarded as the true effective dynamics we have been searching for, including the correction due to radiation. We emphasize that (26) has *no* runaway solutions and can easily be implemented numerically.

Having then established all the assertions of Sections 4 and 5, we obtain the following main result.

Theorem 3.2 *Assume (P) for the potentials, and let the initial data $(E^0(x), B^0(x), q^0, p^0)$ be given by (17). Let the charge e and let $\varepsilon \le \varepsilon_1$ be sufficiently small, and introduce the center manifolds $\mathcal{I}_\varepsilon$ for the comparison dynamics (21) as explained above. At time $\varepsilon t_\rho = 2\varepsilon R_\rho/(1-\bar v)$ we match the initial values, $r(\varepsilon t_\rho) = q^\varepsilon(\varepsilon t_\rho)$, $u(\varepsilon t_\rho) = v^\varepsilon(\varepsilon t_\rho)$, for the motion on the center manifold, i.e., the initial data for the comparison dynamics are*

$$(q^\varepsilon(\varepsilon t_\rho), v^\varepsilon(\varepsilon t_\rho), h_\varepsilon(q^\varepsilon(\varepsilon t_\rho), v^\varepsilon(\varepsilon t_\rho))) \in \mathcal{I}_\varepsilon.$$

Then for every $\tau > 0$ there exists $c(\tau) > 0$ such that for all $t \in [\varepsilon t_\rho, \varepsilon t_\rho + \tau]$ we have the bounds

$$|q^\varepsilon(t) - r(t)| \le c(\tau)\varepsilon, \quad |v^\varepsilon(t) - u(t)| \le c(\tau)\varepsilon, \quad |\dot v^\varepsilon(t) - \dot u(t)| \le c(\tau)\varepsilon,$$

and

$$\Big|[\mathcal{E}_s(v^\varepsilon(t)) + e\phi_{\rm ex}(q^\varepsilon(t))] - [\mathcal{E}_s(u(t)) + e\phi_{\rm ex}(r(t))]\Big| \le c(\tau)\varepsilon^2$$

with $\mathcal{E}_s$ from (11). Furthermore for $t = \mathcal{O}(\varepsilon)$, i.e. $t \in [\varepsilon t_\rho, \varepsilon t_\rho + \varepsilon\tau]$, the bound

$$|q^\varepsilon(t) - r(t)| \le c(\tau)\varepsilon^3 \quad \textit{and} \quad |v^\varepsilon(t) - u(t)| \le c(\tau)\varepsilon^2$$

is satisfied.

Proof: We refer to [11, Section 5]. There the analogue of Theorem 3.2 is proved for the case of a scalar wave field, starting from a differential equation which has the same qualitative properties as (20). □

The smallness of the charge e, needed in the proof of Lemma 3.1, is presumably an artifact. However, the upper bound for e is determined only through bounds for the data in (17) and does not depend on other properties of the solutions. As a further weakness of Theorem 3.2, the radiation reaction correction of order ε in the comparison dynamics is required only in the short time bound and in the energy estimate. One would hope that Theorem 3.2 can be improved in the sense that the effective dynamics is ε^2 close to the true solution over some compact time interval.

Up to now we only discussed the adiabatic limit for the motion of the particle. Since

$$\lim_{\varepsilon\to 0} q^\varepsilon(t) = r(t) \quad \text{and} \quad \lim_{\varepsilon\to 0} v^\varepsilon(t) = u(t)$$

exist by Theorem 3.2, we can in addition infer the adiabatic limit for the electric and magnetic fields. We introduce $t_{\rm ret}$, the retarded time depending on x and t, as the unique solution of $t_{\rm ret} = t - |x - r(t_{\rm ret})|$ and denote $\hat{n}(x,t) = (x - r(t_{\rm ret}))/|x - r(t_{\rm ret})|$.

Theorem 3.3 *For the fields $E(t,x), B(t,x)$ from (16) we have for $x \neq r(t)$ the pointwise limits*

$$\lim_{\varepsilon\to 0} \frac{1}{\sqrt{\varepsilon}} E(x,t) = \bar{E}(x,t) \quad \textit{and} \quad \lim_{\varepsilon\to 0} \frac{1}{\sqrt{\varepsilon}} B(x,t) = \bar{B}(x,t),$$

where

$$\begin{aligned}\bar{E}(x,t) &= \frac{e}{4\pi|x - r(t_{\rm ret})|}\left[1 - \hat{n}(x,t)\cdot u(t_{\rm ret})\right]^{-3}\Big\{\hat{n}(x,t)\wedge([\hat{n}(x,t) - u(t_{\rm ret})]\wedge \dot{u}(t_{\rm ret}))\\ &\qquad + \frac{(1 - u^2(t_{\rm ret}))}{|x - r(t_{\rm ret})|}\left[\hat{n}(x,t) - u(t_{\rm ret})\right]\Big\}\end{aligned} \tag{27}$$

and $\bar{B}(x,t) = \hat{n}(x,t)\wedge\bar{E}(x,t)$. $\bar{E}$, $\bar{B}$ are the Liénard-Wiechert fields of a point charge [14].

4 Representation of the self-force

In this section we show that (19) holds. The calculations will be carried out on the original time scale of (1)-(3) to keep notation simpler. On this scale the self-force (18) reads

$$F_s(t) = e\int d^3x\,\rho(x - q(t))[E(x,t) + v(t)\wedge B(x,t)]. \tag{28}$$

¿From Maxwell's equations in (1), (2), cf. [10] and the arguments in the proof of Lemma 8.1 below, the fields are a sum of the initial and the retarded fields,

$$E(x,t) = E_{(0)}(x,t) + E_{(r)}(x,t) \quad \text{and} \quad B(x,t) = B_{(0)}(x,t) + B_{(r)}(x,t), \tag{29}$$

where

$$\hat{E}_{(0)}(k,t) = \cos|k|t\,\hat{E}(k,0) - i\frac{\sin|k|t}{|k|}k\wedge\hat{B}(k,0),$$

$$\hat{B}_{(0)}(k,t) = \cos|k|t\,\hat{B}(k,0) + i\frac{\sin|k|t}{|k|}k\wedge\hat{E}(k,0),$$

$$\hat{E}_{(r)}(k,t) = -\int_0^t ds\,\cos|k|(t-s)\,\hat{j}(k,s) + i\int_0^t ds\,\frac{\sin|k|(t-s)}{|k|}e\hat{\rho}(k,s)k, \tag{30}$$

$$\hat{B}_{(r)}(k,t) = -i\int_0^t ds\,\frac{\sin|k|(t-s)}{|k|}k\wedge\hat{j}(k,s), \tag{31}$$

with $\hat{\rho}(k,t) = \hat{\rho}(k)e^{ik\cdot q(t)}$ and $\hat{j}(k,t) = e\hat{\rho}(k)v(t)e^{ik\cdot q(t)}$. According to (29), we decompose $F_s(t)$ from (28) as

$$\begin{aligned}F_s(t) &= e\int d^3x\,\rho(x - q(t))[E_{(0)}(x,t) + v(t)\wedge B_{(0)}(x,t)]\\ &\quad + e\int d^3x\,\rho(x - q(t))[E_{(r)}(x,t) + v(t)\wedge B_{(r)}(x,t)]\\ &= F_{(0)}(t) + F_{(r)}(t).\end{aligned}$$

The following lemma states that only $F_{(r)}(t)$ contributes to the self-force.

Lemma 4.1 *For $t \geq t_\rho = 2R_\rho/(1-\bar{v})$ we have $F_{(0)}(t) = 0$.*

Proof: Let $G(t)$ denote the group generated by the free wave equation in $D^{1,2}(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$, with $D^{1,2}(\mathbb{R}^3) = \{\phi \in L^2(\mathbb{R}^3) : \nabla\phi \in L^2(\mathbb{R}^3, \mathbb{R}^3)\}$. Then by (17) and Fourier transform,

$$\begin{pmatrix} E_{(0)}(x,t) \\ \dot{E}_{(0)}(x,t) \end{pmatrix} = -e \int_{-\infty}^{0} ds\, [G(t-s)\Phi_E(\cdot - q^0 - v^0 s)](x)\,, \quad \Phi_E(x) = \begin{pmatrix} \rho(x)v^0 \\ \nabla\rho(x) \end{pmatrix},$$

and an analogous representation of $(B_{(0)}(x,t), \dot{B}_{(0)}(x,t))$, with Φ_E replaced through $\Phi_B(x) = (0, v^0 \wedge \nabla\rho(x))$; observe $E_{(0)}(x,0) = E^0(x)$ and $\dot{E}_{(0)}(x,0) = \nabla \wedge B^0(x)$ as well as $B_{(0)}(x,0) = B^0(x)$ and $\dot{B}_{(0)}(x,0) = -\nabla \wedge E^0(x)$. Hence both Φ_E and Φ_B have their supports in $\overline{B}_{R_\rho}(0)$. Therefore by Kirchhoff's formula applied to $[\ldots](x)$ it can be seen that $E_{(0)}(x,t) = 0 = B_{(0)}(x,t)$ for $|x - q^0| \leq t - R_\rho$. Substituting into the definition of $F_{(0)}(t)$ yields the claim, since $|x - q(t)| \leq R_\rho$ and $|x - q^0| \geq t - R_\rho$ implies $t \leq t_\rho$ by Lemma 3.1. $\square$

Lemma 4.2 *For $t \geq t_\rho$,*

$$F_{(r)}(t) = -m_{\mathrm{f}}(v)\dot{v} + (e^2/6\pi)\left[\gamma^4(v\cdot\ddot{v})v + 3\gamma^6(v\cdot\dot{v})^2 v + 3\gamma^4(v\cdot\dot{v})\dot{v} + \gamma^2\ddot{v}\right] + \mathcal{O}(\varepsilon^3) \tag{32}$$

with the 3×3-matrix

$$m_{\mathrm{f}}(v)z = m_{\mathrm{e}}\left[\left(\frac{\gamma^2}{v^4}(3-v^2) - \frac{3+v^2}{|v|^5}\,\mathrm{arth}|v|\right)(z\cdot v)v + \left(-|v|^{-2} + \frac{1+v^2}{|v|^3}\,\mathrm{arth}|v|\right)z\right], \tag{33}$$

denoting $\mathrm{arth}|v| = \frac{1}{2}\log\frac{1+|v|}{1-|v|}$.

Proof: Using Lemma 3.1 we are allowed to expand as

$$\begin{aligned} v(s) &= v - \dot{v}\tau + \frac{1}{2}\ddot{v}\tau^2 + \mathcal{O}(\varepsilon^3) \\ e^{ik\cdot[q(s)-q(t)]}\,v(s) &= e^{-i(k\cdot v)\tau}\Big(v - \dot{v}\tau + \frac{1}{2}\big[i(k\cdot\dot{v})v + \ddot{v}\big]\tau^2 - \frac{i}{6}\big[(k\cdot\ddot{v})v + 3(k\cdot\dot{v})\dot{v}\big]\tau^3 \\ &\qquad -\frac{1}{8}(k\cdot\dot{v})^2 v\tau^4\Big) + \mathcal{O}(\varepsilon^3), \end{aligned}$$

with $\tau = t - s$ and $v = v(t)$, etc. Introducing

$$I_p = \int_0^T d\tau \frac{\sin(|k|\tau)}{|k|} e^{-i(k\cdot v)\tau}\,\tau^p, \quad J_p = \int_0^T d\tau \cos(|k|\tau) e^{-i(k\cdot v)\tau}\,\tau^p, \quad p = 0,\ldots,4,$$

we obtain in Fourier transformed form by the definition of $F_{(r)}(t)$ and by (30), (31),

$$\begin{aligned}
&F_{(r)}(t)\\
&= -e^2\int d^3k\,|\hat\rho(k)|^2\Big(vJ_0-\dot vJ_1+\frac12\Big[i(k\cdot\dot v)v+\ddot v\Big]J_2-\frac{i}{6}\Big[(k\cdot\ddot v)v+3(k\cdot\dot v)\dot v\Big]J_3\\
&\qquad -\frac18(k\cdot\dot v)^2vJ_4\Big)\\
&\quad +e^2\int d^3k\,|\hat\rho(k)|^2\Big(i\Big[(1-v^2)k+(k\cdot v)v\Big]I_0+i\Big[(v\cdot\dot v)k-(k\cdot v)\dot v\Big]I_1\\
&\qquad +\frac12\Big[(v^2-1)(k\cdot\dot v)k-(k\cdot v)(k\cdot\dot v)v-i(v\cdot\ddot v)k+i(k\cdot v)\ddot v\Big]I_2\\
&\qquad +\frac16\Big[(1-v^2)(k\cdot\ddot v)k+(k\cdot v)(k\cdot\ddot v)v-3(v\cdot\dot v)(k\cdot\dot v)k+3(k\cdot v)(k\cdot\dot v)\dot v\Big]I_3\\
&\qquad +\frac{i}{8}(k\cdot\dot v)^2\Big[(v^2-1)k-(k\cdot v)v\Big]I_4\Big)+\mathcal{O}(\varepsilon^3). \qquad (34)
\end{aligned}$$

This formula holds for arbitrary $t,T\geq t_\rho$, since analogously to [9, Lemma 5.1] or [10, Lemma 3.2] it may be shown that for such t,T due to the compact support of ρ the $\int_0^t ds(\ldots)$-integrals can be changed to $\int_{t-T}^t ds(\ldots)$-integrals when (30) and (31) are inserted into $F_{(r)}(t)$.

To evaluate the terms in (34) as $T\to\infty$, we abbreviate the terms containing J_p as $\mathcal{J}_p$ and I_p as $\mathcal{I}_p$, respectively. We first consider the $\mathcal{I}_p$ and then turn to the $\mathcal{J}_p$.

$\mathcal{I}_0$-limit: Since $\int d^3k\,|\hat\rho(k)|^2\,kI_0\to 0$ as $T\to\infty$ by [9, Appendix A], we have $\mathcal{I}_0\to 0$.

$\mathcal{I}_1$-limit: As a consequence of the relation $\nabla_v I_p=-ikI_{p+1}$ we obtain

$$\mathcal{I}_1=-(v\cdot\dot v)\nabla_v\Big(\int d^3k\,|\hat\rho(k)|^2I_0\Big)+\dot v\,(v\cdot\nabla_v)\Big(\int d^3k\,|\hat\rho(k)|^2I_0\Big).$$

Then we may evaluate as $T\to\infty$

$$e^2\int d^3k\,|\hat\rho(k)|^2\,I_0\to e^2\int d^3k\,\frac{|\hat\rho(k)|^2}{k^2-(k\cdot v)^2}=2m_{\mathrm e}|v|^{-1}\,\mathrm{arth}|v|,$$

hence

$$e^2\,\mathcal{I}_1\to 2m_{\mathrm e}\Big[-(v\cdot\dot v)\Big(\frac{\gamma^2}{v^2}-|v|^{-3}\mathrm{arth}|v|\Big)v+\Big(\gamma^2-|v|^{-1}\mathrm{arth}|v|\Big)\dot v\Big].$$

$\mathcal{I}_2$-limit: From [9, Appendix A] and [10, Lemma 3.3] we know

$$\frac12e^2\int d^3k\,|\hat\rho(k)|^2\,(k\cdot\dot v)kI_2\to-\mu(v)\dot v,\qquad \int d^3k\,|\hat\rho(k)|^2I_1\to\gamma^2/4\pi,$$

as $T\to\infty$. Here

$$\mu(v)z=m_{\mathrm e}\Big[\Big(\frac{\gamma^2}{v^2}-|v|^{-3}\mathrm{arth}|v|\Big)\,z+\Big(\frac{\gamma^4}{v^4}(5v^2-3)+3|v|^{-5}\,\mathrm{arth}|v|\Big)\,(v\cdot z)v\Big]$$

for $|v|<1$. Thus

$$\begin{aligned}
e^2\mathcal{I}_2 &\to \gamma^{-2}\,\mu(v)\dot v + v\,(v\cdot\mu(v)\dot v) + \frac{1}{2}e^2\,(v\cdot\ddot v)\nabla_v(\gamma^2/4\pi) - \frac{1}{2}e^2\,\ddot v\,(v\cdot\nabla_v)(\gamma^2/4\pi)\\
&= m_{\mathrm{e}}\Big[\Big(\frac{\gamma^4}{v^4}(-v^4+6v^2-3) + \frac{3-v^2}{|v|^5}\mathrm{arth}|v|\Big)(v\cdot\dot v)v + \Big(|v|^{-2} - \frac{\gamma^{-2}}{|v|^3}\,\mathrm{arth}|v|\Big)\dot v\Big]\\
&\quad +(e^2/4\pi)\gamma^4\Big[(v\cdot\ddot v)v - v^2\ddot v\Big].
\end{aligned}$$

$\underline{\mathcal{I}_3\text{-limit:}}$ Making use of the relation $(\xi\cdot\nabla_v)\nabla_v I_1 = -k(k\cdot\xi)I_3$, we arrive at

$$\begin{aligned}
\mathcal{I}_3 &= -\frac{1}{6}\gamma^{-2}(\ddot v\cdot\nabla_v)\nabla_v\Big(\int d^3k\,|\hat\rho(k)|^2 I_1\Big) - \frac{1}{6}v\,(v\cdot\nabla_v)(\ddot v\cdot\nabla_v)\Big(\int d^3k\,|\hat\rho(k)|^2 I_1\Big)\\
&\quad +\frac{1}{2}(v\cdot\dot v)(\dot v\cdot\nabla_v)\nabla_v\Big(\int d^3k\,|\hat\rho(k)|^2 I_1\Big) - \frac{1}{2}\dot v\,(v\cdot\nabla_v)(\dot v\cdot\nabla_v)\Big(\int d^3k\,|\hat\rho(k)|^2 I_1\Big)\\
&\to (1/4\pi)\Big[-\frac{1}{3}\gamma^4(1+4\gamma^2)(v\cdot\ddot v) + 4\gamma^6(v\cdot\dot v)^2\Big]v - (1/\pi)\gamma^6 v^2(v\cdot\dot v)\dot v - (1/12\pi)\gamma^2\ddot v.
\end{aligned}$$

$\underline{\mathcal{I}_4\text{-limit:}}$ Observing $(\xi\cdot\nabla_v)^2\nabla_v I_1 = ik(k\cdot\xi)^2 I_4$ it follows that

$$\begin{aligned}
\mathcal{I}_4 &= -\frac{1}{8}(1-v^2)(\dot v\cdot\nabla_v)^2\nabla_v\Big(\int d^3k\,|\hat\rho(k)|^2 I_1\Big) - \frac{1}{8}v\,(v\cdot\nabla_v)(\dot v\cdot\nabla_v)^2\Big(\int d^3k\,|\hat\rho(k)|^2 I_1\Big)\\
&\to -(1/4\pi)\Big[\gamma^6\dot v^2 + 2\gamma^6(v\cdot\dot v)^2 + 6\gamma^8(v\cdot\dot v)^2\Big]v - (1/2\pi)\gamma^4(v\cdot\dot v)\dot v.
\end{aligned}$$

$\underline{\mathcal{J}_0\text{-limit:}}$ As in [9, Appendix A] it may be shown through direct calculation that as $T\to\infty$

$$\int d^3k\,|\hat\rho(k)|^2 J_0 \to i\int d^3k\,\frac{|\hat\rho(k)|^2}{k^2-(k\cdot v)^2}\,(k\cdot v) = 0, \tag{35}$$

the latter due to the symmetry of ρ. Hence $\mathcal{J}_0\to 0$.

$\underline{\mathcal{J}_1\text{-limit:}}$ Analogously to (35) one finds here that

$$\int d^3k\,|\hat\rho(k)|^2 J_1 \to -\int d^3k\,|\hat\rho(k)|^2\frac{k^2+(k\cdot v)^2}{[k^2-(k\cdot v)^2]^2} = -2e^{-2}m_{\mathrm{e}}\gamma^2.$$

Therefore $\mathcal{J}_1\to -2e^{-2}m_{\mathrm{e}}\gamma^2\dot v$.

$\underline{(\mathcal{J}_2+\mathcal{J}_3+\mathcal{J}_4)\text{-limit:}}$ By Lemma 4.3 below, $\int d^3k\,|\hat\rho(k)|^2 J_2 \to -\gamma^4/2\pi$ as $T\to\infty$. From this we obtain

$$\begin{aligned}
&\mathcal{J}_2+\mathcal{J}_3+\mathcal{J}_4\\
&= \frac{1}{2}v\,(\dot v\cdot\nabla_v)\Big(\int d^3k\,|\hat\rho(k)|^2 J_1\Big) - \frac{1}{2}\ddot v\Big(\int d^3k\,|\hat\rho(k)|^2 J_2\Big) - \frac{1}{6}v\,(\ddot v\cdot\nabla_v)\Big(\int d^3k\,|\hat\rho(k)|^2 J_2\Big)\\
&\quad -\frac{1}{2}\dot v\,(\dot v\cdot\nabla_v)\Big(\int d^3k\,|\hat\rho(k)|^2 J_2\Big) - \frac{1}{8}v\,(\dot v\cdot\nabla_v)^2\Big(\int d^3k\,|\hat\rho(k)|^2 J_2\Big)\\
&\to -2e^{-2}m_{\mathrm{e}}\gamma^4(v\cdot\dot v)v + (1/4\pi)\Big[\frac{4}{3}\gamma^6(v\cdot\ddot v) + \gamma^6\dot v^2 + 6\gamma^8(v\cdot\dot v)^2\Big]v\\
&\quad +(1/\pi)\gamma^6(v\cdot\dot v)\dot v + (1/4\pi)\gamma^4\ddot v.
\end{aligned}$$

Collecting all the terms together we get (32). $\square$

It remains to show

Lemma 4.3 *We have*

$$\int_0^\infty dt\, t^2 \int d^3k |\hat{\rho}(k)|^2 \cos|k|t\, e^{-i(k\cdot v)t} = -\gamma^4/2\pi.$$

Proof: Since through integration by parts,

$$J_p = \frac{\sin|k|T}{|k|} e^{-i(k\cdot v)T}\, T^p - pI_{p-1} - (v\cdot\nabla_v)I_{p-1}, \quad p\geq 1,$$

and since $\int d^3k\, |\hat{\rho}(k)|^2 I_1 \to \gamma^2/4\pi$, we only need to prove

$$\int d^3k\, |\hat{\rho}(k)|^2 \frac{\sin|k|T}{|k|} e^{-i(k\cdot v)T}\, T^2 \to 0, \quad T\to\infty,$$

for fixed $|v|<1$. After some transformations this expression may be written as

$$\frac{T}{|v|}\int d^3k\, \frac{|\hat{\rho}(k)|^2}{k^2}\,\sin(|k|T)\sin(|k||v|T) = (2\pi)^{-3/2}\frac{T}{|v|}\int d^3x\, \rho(x)\int d^3y\, \rho(y)\psi(x-y), \tag{36}$$

where $\hat{\psi}(k) = k^{-2}\sin(|k|T)\sin(|k||v|T)$. We claim that the integral in (36) vanishes for $T > 2R_\rho/(1-|v|)$. For this, according to (C) it suffices to show $\psi(x) = 0$ for $|x| \leq 2R_\rho$ and $T > 2R_\rho/(1-|v|)$. Explicitly we have through transformation to polar coordinates

$$(2\pi)^{3/2}\psi(x) = \frac{4\pi}{|x|}\int_0^\infty \frac{dR}{R}\,\sin(R|x|)\sin(RT)\sin(R|v|T).$$

With $\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt$,

$$\begin{aligned}\int_0^a \frac{dR}{R}\,\sin(R|x|)\sin(RT)\sin(R|v|T) \;&=\; \frac{1}{4}\mathrm{Si}(a[|x|-(1-|v|)T]) + \frac{1}{4}\mathrm{Si}(a[|x|+(1-|v|)T]) \\ &\quad -\frac{1}{4}\mathrm{Si}(a[|x|+(1+|v|)T]) - \frac{1}{4}\mathrm{Si}(a[|x|-(1+|v|)T]),\end{aligned}$$

and the latter tends to zero as $a\to\infty$, for $|x|\leq 2R_\rho$ and $T > 2R_\rho/(1-|v|)$, so we are done. $\square$

5 Dynamics on the center manifold

In this section we study the asymptotic properties of solutions to (21), or equivalently (22), on $\mathcal{I}_\varepsilon$. The results are quite similar to those in [11, Section 4], so we do not repeat all details.

Since $a(u)$ and $m(u)$ have been changed to a constant outside $\{u : |u| \leq 1-\bar{\delta}\}$, the following lemma shows that $\mathcal{I}_\varepsilon$ is repulsive in normal direction at some ε-independent rate.

Lemma 5.1 *The eigenvalues of* $D_y g(x,y,0) = (6\pi/e^2)a(x_2)^{-1}m(x_2)$ *are bounded below by a positive constant, uniformly in* $x = (r,u)$ *with* $r \in \mathbb{R}^3$ *and* $|u| \le 1-\delta$, *for all given* $\delta \in]0,1]$.

Proof: Analogously to [11, Lemma 4.1] one obtains the lower bound $(6\pi/e^2)(1-u^2)$. □

By means of a suitable Lyapunov function we can deduce much information on the asymptotic behaviour of solutions. The following result is straightforward.

Lemma 5.2 *Let*

$$G_\varepsilon(r,u,\dot{u}) = \mathcal{E}_s(u) + e\phi_{\rm ex}(r) - \varepsilon(e^2/6\pi)\,\gamma^4(u\cdot\dot{u}),$$

with $\mathcal{E}_s$ *from (11). Then*

$$\frac{d}{dt}\,G_\varepsilon(r,u,\dot{u}) = -\varepsilon(e^2/6\pi)\,[\gamma^4\dot{u}^2 + \gamma^6(u\cdot\dot{u})^2] \tag{37}$$

along solution trajectories $(r(t),u(t),\dot{u}(t))$ *of (21).*

The term on the right-hand side of (37) accounts for the energy radiated to infinity; see Section 7 below. Since $\inf_{q\in\mathbb{R}^3}\phi_{\rm ex}(q) > -\infty$ by (P), Lemma 5.2 implies through integration of (37) over $[0,\infty[$, analogously to [11, Thm. 4.3], the following

Theorem 5.3 *Let* (P) *hold for the potentials, and let a global solution* $(r(t),u(t))$ *of (21) be given with* $\sup_{t\ge 0}|u(t)| \le \bar{u}(\varepsilon) < 1$ *and* $\sup_{t\ge0}|\dot{u}(t)| \le c(\varepsilon)$, *for possibly* ε*-dependent constants* $\bar{u}(\varepsilon)$ *and* $c(\varepsilon)$. *Then*

$$\dot{u}(t)\to 0,\quad \ddot{u}(t)\to 0,\quad \text{and}\quad F_{\rm ex}(r(t),u(t))\to 0 \quad \text{as}\quad t\to\infty.$$

If also $\sup_{t\ge0}|r(t)| \le c(\varepsilon)$ *is known, then additionally*

$$u(t)\to 0 \quad \text{and}\quad \nabla\phi_{\rm ex}(r(t))\to 0 \quad \text{as}\quad t\to\infty.$$

The latter implies that the trajectory approaches a distinguished critical point of the electrostatic potential $\phi_{\rm ex}$ in case the set of critical points is discrete.

In Section 3 we modified the functions $a(u)$ and $m(u)$ to be constant outside $\{u : |u| \le 1-\bar{\delta}\}$ with some $\bar{\delta} = \bar{\delta}(\bar{v}) > 0$ small; we now give the precise definition of $\bar{\delta}$. Let

$$\bar{\delta} := \min\{(1-\bar{v})/2, (1-s_0)/2\}, \tag{38}$$

where, as a consequence of $\mathcal{E}_s(u)\to\infty$ for $|u|\to 1$,

$$s_0 = \sup\Big\{|u| : (r,u)\in\{(\tilde{r},\tilde{u}) : \mathcal{E}_s(\tilde{u}) + e\phi_{\rm ex}(\tilde{r}) \le c_0+1\} \text{ for some } r\in\mathbb{R}^3\Big\} < 1,$$

with c_0 chosen such that $\mathcal{K}_{\bar{v}} \subset \{(\tilde{r},\tilde{u}) : \mathcal{E}_s(\tilde{u}) + e\phi_{\rm ex}(\tilde{r}) \le c_0\}$, as is possible, since $\phi_{\rm ex}$ is bounded.

The next theorem summarizes properties of solutions of (21) resp. (22) on $\mathcal{I}_\varepsilon$, with the modified functions $a(u)$ and $m(u)$.

Theorem 5.4 *Assume* (P) *for the potentials. Then there exists* $\varepsilon_1 > 0$ *depending only on the bound* $\bar{v}$ *from Lemma 3.1 such that for* $\varepsilon \in]0,\varepsilon_1]$ *the following holds.*

(a) All solutions starting in points $(r,u,h_\varepsilon(r,u)) \in \mathcal{I}_\varepsilon$, $|u| \le \bar{v}$, stay away from the boundary $\{(r,u,h_\varepsilon(r,u)) : |u| = 1-\bar{\delta}\}$ for all future times.

(b) Solutions as in (a) exist globally, are solutions to (21), and remain on $\mathcal{I}_\varepsilon$ for all future times.

(c) Solutions as in (a) satisfy

$$\begin{aligned} \sup\{|u(t)| : t \in \mathrm{I\!R}, \varepsilon \in]0,\varepsilon_1]\} \le 1 - 2\bar{\delta} < 1\,, \quad \textit{and} \\ \sup\{|\dot{u}(t)| : t \in \mathrm{I\!R}, \varepsilon \in]0,\varepsilon_1]\} + \sup\{|\ddot{u}(t)| : t \in \mathrm{I\!R}, \varepsilon \in]0,\varepsilon_1]\} \le c(\bar{\delta}), \end{aligned} \tag{39}$$

and hence also

$$\dot{u}(t) \to 0, \quad \ddot{u}(t) \to 0, \quad \textit{and} \quad F_{\mathrm{ex}}(r(t),u(t)) \to 0 \quad \textit{as} \quad t \to \infty.$$

Proof: Analogously to Theorem 4.4 and Corollary 4.5 in [11]; the constant c_a has to be replaced by $(e^2/6\pi)\sup_{|u|\le\bar{v}}\gamma^4$. For the estimates in (39), it is important to use (25). □

Having seen that $\mathcal{I}_\varepsilon$ is a global forward invariant manifold for (21), we now address the converse problem and look for conditions which guarantee that some solution of (21) already has to lie on $\mathcal{I}_\varepsilon$. The following theorem rests on the fact that according to the normal hyperbolicity of $\mathcal{I}_0$, cf. Lemma 5.1, no trajectory can remain arbitrarily close to $\mathcal{I}_\varepsilon$ without being contained in $\mathcal{I}_\varepsilon$.

Theorem 5.5 *Suppose a family $(r^\varepsilon(t), u^\varepsilon(t))$, $\varepsilon \in]0,\varepsilon_2]$, of solutions to (21) is given such that*

$$\sup\{|u^\varepsilon(t)| : t \in \mathrm{I\!R}, \varepsilon \in]0,\varepsilon_2]\} \le \bar{u} < 1.$$

Additionally, assume that one of the following two conditions is satisfied.

(a) The bound

$$\sup\{|\dot{u}^\varepsilon(t)| : t \in \mathrm{I\!R}, \varepsilon \in]0,\varepsilon_2]\} + \sup\{|\ddot{u}^\varepsilon(t)| : t \in \mathrm{I\!R}, \varepsilon \in]0,\varepsilon_2]\} \le c_1$$

holds, or

(b) the asymptotic condition of Dirac and Haag, $\dot{u}^\varepsilon(t) \to 0$ as $t \to \infty$, holds for each $\varepsilon \in]0,\varepsilon_2]$.

Then the solutions have to lie on $\mathcal{I}_\varepsilon$, for sufficiently small ε.

Proof: We refer to [11, Propositions 4.6 and 4.7]. □

6 Adiabatic limit of the fields

We prove Theorem 3.3. First we need to derive suitable representation formulas for $E(x,t)$ and $B(x,t)$ from (16). Since $E_{(0)}$ solves the wave equation $\ddot{E}_{(0)} = \Delta E_{(0)}$, $E_{(0)}(x,0) = E^0(x)$, $\dot{E}_{(0)}(x,0) = \nabla \wedge B^0(x)$, we find from (17) that

$$\begin{aligned}
\frac{1}{\sqrt{\varepsilon}} E_{(0)}(x,t) &= -e \int_{-\infty}^{0} ds \, [G(t-s)\Phi_{E,\varepsilon}(\cdot - q^0 - v^0 s)]_1(x) \\
&= -e \int_{-\infty}^{0} ds \, \frac{1}{4\pi(t-s)^2} \int_{|y-x|=(t-s)} d^2y \, [((y-x)\cdot\nabla) j_\varepsilon^0(y,s) + j_\varepsilon^0(y,s)] \\
&\quad -e \int_{-\infty}^{0} ds \, \frac{1}{4\pi(t-s)} \int_{|y-x|=(t-s)} d^2y \, \nabla \rho_\varepsilon^0(y,s),
\end{aligned}$$

with $\Phi_{E,\varepsilon}(x) = (\rho_\varepsilon(x)v^0, \nabla\rho_\varepsilon(x))$, $j_\varepsilon^0(y,s) = \rho_\varepsilon(y - q^0 - v^0 s)v^0$, and $\rho_\varepsilon^0(y,s) = \rho_\varepsilon(y - q^0 - v^0 s)$, see also the proof of Lemma 4.1; $[\ldots]_1$ is the first component of $[\ldots]$. Similarly,

$$\frac{1}{\sqrt{\varepsilon}} B_{(0)}(x,t) = e \int_{-\infty}^{0} ds \, \frac{1}{4\pi(t-s)} \int_{|y-x|=(t-s)} d^2y \, \nabla \wedge j_\varepsilon^0(y,s).$$

Extending $q^\varepsilon(t)$ resp. $v^\varepsilon(t)$ to times $t \le 0$ by $q^0 + v^0 t$ resp. v^0, we hence obtain through inverse Fourier transform of (30), with $j_\varepsilon(y,s) = \rho_\varepsilon(y - q^\varepsilon(s))v^\varepsilon(s)$ and $\rho_\varepsilon(y,s) = \rho_\varepsilon(y - q^\varepsilon(s))$,

$$\begin{aligned}
\frac{1}{\sqrt{\varepsilon}} E(x,t) &= -e \int_{-\infty}^{t} ds \, \frac{1}{4\pi(t-s)^2} \int_{|y-x|=(t-s)} d^2y \, [((y-x)\cdot\nabla) j_\varepsilon(y,s) + j_\varepsilon(y,s)] \\
&\quad - \ e \int_{-\infty}^{t} ds \, \frac{1}{4\pi(t-s)} \int_{|y-x|=(t-s)} d^2y \, \nabla\rho_\varepsilon(y,s) \\
&= -e \int d^3y \, \Big(\frac{1}{4\pi|x-y|^2} \Big[v^\varepsilon(t - |x-y|) \, ((y-x)\cdot\nabla)\rho_\varepsilon(y - q^\varepsilon(t-|x-y|)) \\
&\qquad\qquad + \rho_\varepsilon(y - q^\varepsilon(t-|x-y|)) v^\varepsilon(t-|x-y|) \Big] \\
&\qquad + \frac{1}{4\pi|x-y|} \nabla\rho_\varepsilon(y - q^\varepsilon(t-|x-y|)) \Big).
\end{aligned} \tag{40}$$

In the same manner,

$$\frac{1}{\sqrt{\varepsilon}} B(x,t) = -e \int d^3y \, \frac{1}{4\pi|x-y|} \, v^\varepsilon(t-|x-y|) \wedge \nabla\rho_\varepsilon(y - q^\varepsilon(t-|x-y|)). \tag{41}$$

Now in both integrals (40) and (41) we introduce the transformation $z(y) = y - q^\varepsilon(t - |x-y|)$ with $\det(dy/dz) = [1 - v^\varepsilon(t-|x-y|)\cdot(x-y)/|x-y|]^{-1}$ and integrate by parts to remove the ∇'s from the $\rho_\varepsilon(z)$. Then we calculate the respective derivatives with respect to z and pass to the limit $\varepsilon \to 0$, using $\rho_\varepsilon(z) = \varepsilon^{-3}\rho(\varepsilon^{-1}z) \to \delta_0$ in the sense of distributions, according to (4). Moreover, in the limit $\varepsilon \to 0$ the equation $0 = y - r(t - |x-y|)$ for $z = 0$ is solved by $y = r(t_{\mathrm{ret}})$; in particular, $\det(dy/dz) \to [1 - \hat{n}(x,t)\cdot u(t_{\mathrm{ret}})]^{-1}$. Proceeding as indicated, a somewhat lengthy calculation then yields $\varepsilon^{-1/2}E(x,t) \to \bar{E}(x,t)$ and $\varepsilon^{-1/2}B(x,t) \to \bar{B}(x,t)$ with $\bar{E}(x,t)$ from (27) and $\bar{B}(x,t) = \hat{n}(x,t) \wedge \bar{E}(x,t)$. $\square$

7 Radiated energy

In this section we determine the energy radiated to infinity. For this, let

$$\begin{aligned}\mathcal{H}_{R,q^\varepsilon(t)}(t+R) &= m_b\gamma(v^\varepsilon(t+R)) + [e\phi_{\rm ex} * \rho_\varepsilon](q^\varepsilon(t+R)) \\ &\quad + \frac{1}{2}\int_{|x-q^\varepsilon(t)|\le R} d^3x\,[E^2(x,t+R)+B^2(x,t+R)]\end{aligned}$$

be the energy of (16) in a ball of radius R around the particle, at time $t+R$. Utilizing (16), this energy changes as

$$\begin{aligned}\frac{d}{dt}\Big(\mathcal{H}_{R,q^\varepsilon(t)}(t+R)\Big) &= \frac{1}{2}\int_{|x-q^\varepsilon(t)|=R} d^2x\,[E^2(x,t+R)+B^2(t,x+R)](n(x)\cdot v^\varepsilon(t)) \\ &\quad + \int_{|x-q^\varepsilon(t)|=R} d^2x\,E(x,t+R)\cdot[n(x)\wedge B(x,t+R)] \\ &= \frac{R^2}{2}\int_{|\omega|=1} d^2\omega\,[E^2(q^\varepsilon(t)+R\omega,t+R)+B^2(q^\varepsilon(t)+R\omega,t+R)](\omega\cdot v^\varepsilon(t)) \\ &\quad + R^2\int_{|\omega|=1} d^2\omega\,E(q^\varepsilon(t)+R\omega,t+R)\cdot[\omega\wedge B(q^\varepsilon(t)+R\omega,t+R)],\end{aligned}$$

where $n(x)$ is the outer normal. In this calculation we have used that

$$\int_{|x-q^\varepsilon(t)|>R} d^3x\,\rho_\varepsilon(x-q^\varepsilon(t+R))E(x,t+R)\cdot v^\varepsilon(t+R)=0$$

for R large, since $|x-q^\varepsilon(t+R)|\ge|x-q^\varepsilon(t)|-|q^\varepsilon(t+R)-q^\varepsilon(t)|\ge(1-\bar v)R\ge\varepsilon R_\rho$ according to Lemma 3.1. Thus by Theorem 3.3,

$$\begin{aligned}&\frac{1}{\varepsilon}\frac{d}{dt}\Big(\mathcal{H}_{R,q^\varepsilon(t)}(t+R)\Big)\longrightarrow \\ &\quad \frac{R^2}{2}\int_{|\omega|=1} d^2\omega\,[\bar E^2(r(t)+R\omega,t+R)+\bar B^2(r(t)+R\omega,t+R)](\omega\cdot u(t)) \\ &\quad + R^2\int_{|\omega|=1} d^2\omega\,\bar E(r(t)+R\omega,t+R)\cdot[\omega\wedge\bar B(r(t)+R\omega,t+R)]\end{aligned}$$

as $\varepsilon\to0$. In order to determine the amount of energy radiated from the particle to infinity, we take the limit $R\to\infty$ of the latter expression. For this, first observe that with $x=r(t)+R\omega$ and $\tau=t+R$ it follows that $t_{\rm ret}=t_{\rm ret}(x,\tau)=t$, since then $\tau-t_{\rm ret}=|x-r(t_{\rm ret})|$. Hence also $\hat n(x,\tau)=\omega$, and $\hat n(x,\tau)\wedge\bar E(x,\tau)=\bar B(x,\tau)$ thus yields

$$\begin{aligned}&\frac{1}{2}[\bar E^2(x,\tau)+\bar B^2(x,\tau)](\omega\cdot u(t))+\bar E(x,\tau)\cdot[\omega\wedge\bar B(x,\tau)] \\ &\quad = -(1-\omega\cdot u(t))\bar E^2(x,\tau)+(1-(\omega\cdot u(t))/2)(\omega\cdot\bar E(x,\tau))^2 \\ &\quad = -(1-\omega\cdot u(t))\bar E^2(x,\tau)+\mathcal{O}(R^{-4}). \qquad (42)\end{aligned}$$

In the last equation we have used that by (27)

$$\hat{n}(x,\tau)\cdot\bar{E}(x,\tau)=e\,\frac{(1-u^2(t_{\rm ret}))}{4\pi|x-r(t_{\rm ret})|^2}[1-\hat{n}(x,t)\cdot u(t_{\rm ret})]^{-2}=\mathcal{O}(R^{-2}).$$

Therefore (42) and (27) yield

$$\begin{aligned}
&\lim_{R\to\infty}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\,\frac{d}{dt}\Big(\mathcal{H}_{R,q^\varepsilon(t)}(t+R)\Big)\\
&= -\frac{e^2}{16\,\pi^2}\int_{|\omega|=1} d^2\omega\,\Big(-(1-u^2(t))[1-\omega\cdot u(t)]^{-5}(\omega\cdot\dot{u}(t))^2\\
&\qquad\qquad +2\,[1-\omega\cdot u(t)]^{-4}(\omega\cdot\dot{u}(t))(u(t)\cdot\dot{u}(t))+[1-\omega\cdot u(t)]^{-3}\dot{u}^2(t)\Big)\\
&= -(e^2/6\pi)[\gamma^4\dot{u}^2(t)+\gamma^6(u(t)\cdot\dot{u}(t))^2]=-(e^2/6\pi)\gamma^6[\dot{u}^2(t)-(u(t)\wedge\dot{u}(t))^2]
\end{aligned}$$

in agreement with (37) and Larmor's formula [14]. We remark that the same result is obtained if the limits $\varepsilon\to 0$ and $R\to\infty$ are interchanged. This can be shown as in [11, Section 7].

8 Appendix

We prove Lemma 3.1, and for this it will be more convenient to switch back to the original time scale of (1)-(3). Then Lemma 3.1 reads as

Lemma 3.1 *For any solution of (3) we have*

$$\sup_{t\in\mathbb{R}}|v(t)|\le\bar{v}<1.$$

In addition, there exist constants $C>0$ and $\delta_0>0$ such that

$$\sup_{t\in\mathbb{R}}|\dot{v}(t)|\le C\varepsilon,\quad \sup_{t\in\mathbb{R}}|\ddot{v}(t)|\le C\varepsilon^2,\quad \sup_{t\in\mathbb{R}}|\dddot{v}(t)|\le C\varepsilon^3$$

in case that $e\le\delta_0$ and that the trajectory starts on the soliton manifold. Here $\bar{v}$, C, and δ_0 do depend only on bounds for the initial data, but not on ε.

We split the proof into four subsections.

8.1 Bounding $|v(t)|$

By (P) we obtain

$$\mathcal{H}(E^0(x),B^0(x),q^0,v^0)\ge m_{\rm b}\gamma(v(t))+e\,\Big(\inf_{q\in\mathbb{R}^3}\phi_{\rm ex}(q)\Big),$$

since $\mathcal{H}(E(x,t),B(x,t),q(t),v(t))=[\phi_{\rm ex}(\varepsilon\cdot)*e\rho](q(t))+\mathcal{H}_0(E(x,t),B(x,t),q(t),v(t))$ is constant along solutions of (1)-(3); see [10]. Hence

$$\sup_{t\in\mathbb{R}}|v(t)|\le\bar{v}<1, \tag{43}$$

with $\bar{v}$ depending only on the data. For this estimate it was neither necessary that e be small nor that the trajectory started on the soliton manifold.

8.2 Bounding $|\dot{v}(t)|$

By (3) and since $S_{q,v}(t)$ is a solution with $\varepsilon = 0$,

$$\begin{aligned} \dot{v} &= em_0(v)^{-1}\Big[\varepsilon\int d^3x\,\rho(x-q)[E_{\rm ex}(\varepsilon x)+v\wedge B_{\rm ex}(\varepsilon x)] \\ &\qquad +\int d^3x\,\rho(x-q)\Big([E(x)-E_v(x-q)]+v\wedge[B(x)-B_v(x-q)]\Big)\Big], \end{aligned} \tag{44}$$

where $m_0(v)^{-1}z = m_{\rm b}{}^{-1}\gamma^{-1}(z-(v\cdot z)v)$, $z\in\mathbb{R}^3$, is the matrix inverse of $m_0(v)$. To estimate the second integral term, define

$$Z(x,t) = \begin{pmatrix} E(x,t)-E_{v(t)}(x-q(t)) \\ B(x,t)-B_{v(t)}(x-q(t)) \end{pmatrix}. \tag{45}$$

Then Maxwell equations and the relations $(v\cdot\nabla)E_v(x) = -\nabla\wedge B_v(x) + e\rho(x)v$, $(v\cdot\nabla)B_v(x) = \nabla\wedge E_v(x)$ imply

$$\dot{Z}(t) = \mathcal{A}Z(t) - f(t)\,,\quad \text{with}\quad \mathcal{A} = \begin{pmatrix} 0 & \nabla\wedge \\ -\nabla\wedge & 0\end{pmatrix} \quad\text{and}\quad f(x,t) = \begin{pmatrix} (\dot{v}(t)\cdot\nabla_v)E_v(x-q(t)) \\ (\dot{v}(t)\cdot\nabla_v)B_v(x-q(t))\end{pmatrix}. \tag{46}$$

The Maxwell operator $\mathcal{A}$ is the generator of a C^0-group $U(t)$, $t\in\mathbb{R}$, of isometries in $L^2(\mathbb{R}^3)^3\oplus L^2(\mathbb{R}^3)^3$, cf. [3, p. 435; (H2)]. Therefore we have the mild solution representation

$$Z(x,t) = [U(t)Z(\cdot,0)](x) - \int_0^t ds\,[U(t-s)f(\cdot,s)](x)\,. \tag{47}$$

Since the solution is supposed to start on the soliton manifold, $Z(0)=0$, so here the first term drops out. In the following lemma we state an explicit representation for $U(t-s)f(\cdot,s)$.

Lemma 8.1 *For given $f=(f_1,f_2)$ with $\nabla\cdot f_1 = 0$ and $\nabla\cdot f_2 = 0$ we have for $W(t,s,x) = (W_1(t,s,x),W_2(t,s,x)) = [U(t-s)f(\cdot,s)](x)$*

$$\begin{aligned} W_1(t,s,x) &= \frac{1}{4\pi(t-s)^2}\int_{|y-x|=(t-s)} d^2y\,\Big[(t-s)\nabla\wedge f_2(y,s)+f_1(y,s)+((y-x)\cdot\nabla)f_1(y,s)\Big], \quad (48)\\ W_2(t,s,x) &= \frac{1}{4\pi(t-s)^2}\int_{|y-x|=(t-s)} d^2y\,\Big[-(t-s)\nabla\wedge f_1(y,s)+f_2(y,s)+((y-x)\cdot\nabla)f_2(y,s)\Big]. \end{aligned}$$

Proof: We follow [10, Section A.1] and introduce the complex $C(x,t) = W_1(t,s,x)+iW_2(t,s,x)\in\mathbb{C}^3$. Then $\dot{C} = -i\,\nabla\wedge C$, thus $\dot{\hat{C}} = -k\wedge\hat{C} =: m(k)\hat{C}$, i.e., $\hat{C}(k,t) = \exp([t-s]m(k))\hat{C}(k,s)$. Since $m^2(k) = -k^2\mathrm{Id}+k\otimes k$ and $m^3(k) = -k^2m(k)$, etc., the exponential can be evaluated explicitly as $\exp(tm(k)) = \cos|k|t\,\mathrm{Id} + |k|^{-2}(1-\cos|k|t)(k\otimes k)+|k|^{-1}\sin|k|t\,m(k)$. Because $\hat{C}(k,s) = \hat{f}_1(k,s)+i\hat{f}_2(k,s)$ and $\nabla\cdot f_1 = 0 = \nabla\cdot f_2$, we obtain $(k\otimes k)\hat{C}(k,s) = (k\cdot\hat{C}(k,s))k = 0$, whence $\hat{C}(k,t) = [\cos|k|(t-s)\,\mathrm{Id}+|k|^{-1}\sin|k|(t-s)\,m(k)]\hat{C}(k,s)$. Taking into account that a general $f(x) = \int_{|y-x|=\tau}d^2y\,g(y)$ has $\hat{f}(k) = 4\pi\tau|k|^{-1}\sin|k|\tau\,\hat{g}(k)$, and $\widehat{\nabla\wedge f}(k) = -i\,k\wedge\hat{f}(k)$, this yields the claim. □

Corollary 8.2 *In the setting of Lemma 8.1, if in addition*

$$|f_1(y,s)|+|f_2(y,s)| \le C\frac{\alpha(s)}{(1+|y-q(s)|^n)} \quad \text{and} \quad |\nabla f_1(y,s)|+|\nabla f_2(y,s)| \le C\frac{\alpha(s)}{(1+|y-q(s)|^{n+1})}\,, \tag{49}$$

then

$$|W_1(t,s,x+q(t))|+|W_2(t,s,x+q(t))| \le C\,\frac{\alpha(s)}{1+(t-s)^n}\,, \quad |x|\le R_\rho\,.$$

Proof: We deal with e.g. the first component $W_1(t,s,x+q(t))$ for $|x|\le R_\rho$. Then for $y\in\mathbb{R}^3$ with $|y-(x+q(t))|=(t-s)$ we find $|y-q(s)|\ge(t-s)-R_\rho-|q(t)-q(s)|\ge(1-\bar v)(t-s)-R_\rho$ by (43). Hence we can distinguish cases e.g. $(t-s)\ge 1+2(1-\bar v)^{-1}R_\rho$ and $(t-s)\le\ldots$, and then use (49) in (48) for $(t-s)\ge\ldots$ to get $|W_1(t,s,x+q(t))|\le C(1+(t-s)^n)^{-1}\alpha(s)$ for $|x|\le R_\rho$. □

In particular, both the lemma and the corollary (with $n=2$ and $\alpha(s)=e|\dot v(s)|$) apply to f defined in (46). To see this, observe $\nabla\cdot B_v=0$, and $\nabla\cdot E_v=e\rho$ is independent of v, hence $\nabla\cdot f_1 = 0=\nabla\cdot f_2$. Concerning (49), we have $|\nabla_v E_v(x)|+|\nabla_v B_v(x)|\le C(|\nabla\phi_v(x)|+|\nabla_v\nabla\phi_v(x)|)$ and $|\nabla_v\nabla E_v(x)|+|\nabla_v\nabla B_v(x)|\le C(|\nabla^2\phi_v(x)|+|\nabla_v\nabla^2\phi_v(x)|)$. Through tedious, but straightforward calculation one can derive the following estimates on the derivatives of $\nabla\phi_v$, cf. [11]. These will also be used frequently in the sequel. We have

$$\begin{aligned}
|\nabla\phi_v(x)|+|\nabla_v\nabla\phi_v(x)|+|\nabla_v^2\nabla\phi_v(x)|+|\nabla_v^3\nabla\phi_v(x)| &\le Ce(1+|x|)^{-2}\,,\\
|\nabla^2\phi_v(x)|+|\nabla_v\nabla^2\phi_v(x)|+|\nabla_v^2\nabla^2\phi_v(x)|+|\nabla_v^3\nabla^2\phi_v(x)| &\le Ce(1+|x|)^{-3}\,,\\
|\nabla^3\phi_v(x)|+|\nabla_v\nabla^3\phi_v(x)|+|\nabla_v^2\nabla^3\phi_v(x)|+|\nabla_v^3\nabla^3\phi_v(x)| &\le Ce(1+|x|)^{-4}\,,\\
|\nabla^4\phi_v(x)|+|\nabla_v\nabla^4\phi_v(x)|+|\nabla_v^2\nabla^4\phi_v(x)|+|\nabla_v^3\nabla^4\phi_v(x)| &\le Ce(1+|x|)^{-5}\,,
\end{aligned} \tag{50}$$

for $x\in\mathbb{R}^3$ and $|v|\le\bar v<1$, with $\nabla=\nabla_x$. This implies that (49) holds as well. Consequently, since $U(t)$ is unitary, it follows from (47) that

$$\|Z(\cdot+q(t),t)\|_{L^2(\overline{B}_{R_\rho}(0))^3\oplus L^2(\overline{B}_{R_\rho}(0))^3} \le Ce\int_0^t ds\,\frac{|\dot v(s)|}{1+(t-s)^2}\,. \tag{51}$$

Therefore by (44) and assumptions (P) and (C), since $|m_0(v)^{-1}|\le C$,

$$|\dot v(t)|\le Ce\left(\varepsilon+e\int_0^t ds\,\frac{|\dot v(s)|}{1+(t-s)^2}\right).$$

Choosing e sufficiently small, this implies

$$\sup_{t\in\mathbb{R}}|\dot v(t)|\le C\varepsilon. \tag{52}$$

8.3 Bounding $|\ddot v(t)|$

Let $L(t)\phi = (v(t)\cdot\nabla_x)\phi + \partial_t\phi$ for functions $\phi=\phi(x,t)$. We also introduce the notation

$$\varphi(x,t) = E(x,t) - E_{v(t)}(x-q(t))\,, \quad \psi(x,t) = B(x,t) - B_{v(t)}(x-q(t))\,,$$

i.e. $Z=(\varphi,\psi)$, cf. (45). Using the explicit form of $m_0(v)^{-1}$ it follows from (52) that

$$\left|\frac{d}{dt}\, m_0(v(t))^{-1}\right| \le C|\dot v(t)| \le C\varepsilon.$$

Thus we obtain by differentiation of (44), invoking also the last step and (P),

$$|\ddot v(t)| \le C\Big(\varepsilon^2 + \Big|\dot v(t)\wedge\int d^3x\,\rho(x)\psi(x+q(t),t)\Big| + |M(t)|\Big),$$

where

$$M(t) = \int d^3x\,\rho(x)\Big[(L(t)\varphi)(x+q(t)) + v(t)\wedge(L(t)\psi)(x+q(t))\Big]. \tag{53}$$

Therefore by (52) and (51),

$$|\ddot v(t)| \le C\Big(\varepsilon^2 + |M(t)|\Big); \tag{54}$$

whence it remains to derive a bound for $M(t)$. For this, we introduce $\Sigma(x,t) = (L(t)Z)(x,t)$ so that

$$M(t) = \int d^3x\,\rho(x)[\Sigma_1(x+q(t),t) + v(t)\wedge\Sigma_2(x+q(t),t)].$$

Since generally $\frac{d}{dt}[L(t)\phi] = L(t)\dot\phi + (\dot v\cdot\nabla)\phi$, it follows from (46) that $\dot\Sigma = \mathcal{A}\Sigma - L(t)f + (\dot v\cdot\nabla)Z$. Therefore

$$\Sigma(x+q(t),t) = [U(t)\Sigma(0)](x+q(t)) + \int_0^t ds\,[U(t-s)((\dot v(s)\cdot\nabla)Z(\cdot,s) - L(s)f(\cdot,s))](x+q(t)), \tag{55}$$

the first term being zero as a consequence of $Z(0)=0$. We estimate both parts of the integral term separately.

What concerns the first, observe $\frac{d}{dt}(\nabla Z) = \mathcal{A}(\nabla Z) - \nabla f$ by (46), thus

$$\int_0^t ds\,[U(t-s)((\dot v(s)\cdot\nabla)Z(\cdot,s))](x+q(t)) = -\int_0^t ds\int_0^s d\tau\,[U(t-\tau)((\dot v(s)\cdot\nabla)f(\cdot,\tau))](x+q(t)), \tag{56}$$

since $\nabla Z(0)=0$. Now $\nabla\cdot(\nabla f) = \nabla(\nabla\cdot f) = 0$, hence we can use the estimates in (50) to see that Corollary 8.2 (with $n=3$) applies as above to yield by (52) the bound $C\varepsilon^2$ on the right-hand side of (56), for $|x|\le R_\rho$; observe that f has an additional $\dot v$ and that it was not necessary here to keep the e from the estimates in (50), since the bound $C\varepsilon^2$ is already fine.

Returning to the second part of the integral term in (55), we have

$$[L(s)f(\cdot,s)](x) = (\ddot v\cdot\nabla_v)\Phi_v(x-q) + (\dot v\cdot\nabla_v)^2\Phi_v(x-q)$$

with $q = q(s)$, etc., and $\Phi_v := (E_v, B_v)$. Since $\nabla\cdot E_v = e\rho$ and $\nabla\cdot B_v = 0$, we obtain $\nabla\cdot(L(s)f) = 0$. Thus by (50), Corollary 8.2 (with $n = 2$ and $\alpha(s) = |\dot v(s)|^2$ resp. $\alpha(s) = e|\ddot v(s)|$), and (52),

$$\Big|\int_0^t ds\,[U(t-s)(L(s)f(\cdot,s))](x+q(t))\Big| \le C\left(\varepsilon^2 + e\int_0^t ds\,\frac{|\ddot v(s)|}{1+(t-s)^2}\right), \quad |x|\le R_\rho.$$

Collecting together all the parts of $M(t)$, we finally arrive at

$$|M(t)| \le C\Big(\varepsilon^2 + e\int_0^t ds\,\frac{|\ddot v(s)|}{1+(t-s)^2}\Big).$$

Hence by (54) for e sufficiently small,

$$\sup_{t\in\mathbb{R}} |\ddot v(t)| \le C\varepsilon^2. \tag{57}$$

8.4 Bounding $|\dddot v(t)|$

The estimate $\sup_{t\in\mathbb{R}} |\dddot v(t)| \le C\varepsilon^3$ will be derived by differentiating (44) twice. Since we have

$$\Big|\frac{d^2}{dt^2}\,m_0(v(t))^{-1}\Big| \le C\Big(|\dot v(t)|^2 + |\ddot v(t)|\Big) \le C\varepsilon^2,$$

we can use the previous steps and (P) to find

$$|\dddot v| \le C\Big(\varepsilon^3 + \Big|\ddot v\wedge\int d^3x\,\rho(x)\psi(x+q,t)\Big| + \Big|\dot v\wedge\int d^3x\,[L(t)\psi](x+q)\Big| + |\dot M(t)|\Big).$$

By (52), (57), and the estimates of the two preceding sections, also

$$\Big|\ddot v\wedge\int d^3x\,\rho(x)\psi(x+q,t)\Big| + \Big|\dot v\wedge\int d^3x\,[L(t)\psi](x+q)\Big| \le C\varepsilon^3.$$

To derive the same bound for $\dot M(t)$, with $M(t)$ from (53), observe

$$\dot M(t) = \dot v\wedge\int d^3x\,[L(t)\psi](x+q) + \int d^3x\,\rho(x)\Big[(\mathcal{L}(t)\varphi)(x+q(t)) + v(t)\wedge(\mathcal{L}(t)\psi)(x+q(t))\Big], \tag{58}$$

where

$$\mathcal{L}(t)\phi = (\dot v(t)\cdot\nabla)\phi + (v(t)\cdot\nabla)^2\phi + 2(v(t)\cdot\nabla)\dot\phi + \ddot\phi$$

for a general $\phi = \phi(x,t)$; (58) results from $\frac{d}{dt}[(L(t)\phi)(x+q(t))] = (\mathcal{L}(t)\phi)(x+q(t))$. The first term on the right-hand side of (58) is again bounded by $C\varepsilon^3$, so we have to analyze the second. Since $\dot Z = \mathcal{A}Z - f$, we find the relation $\frac{d}{dt}(\mathcal{L}(t)Z) = \mathcal{A}(\mathcal{L}(t)Z) - \mathcal{L}(t)f + (\ddot v\cdot\nabla)Z + 2[(v\cdot\nabla)(\dot v\cdot\nabla)Z + (\dot v\cdot\nabla)\dot Z]$, and thus, since $\mathcal{L}(0)Z(0) = 0$,

$$\begin{aligned}
[\mathcal{L}(t)Z](x+q(t)) &= \int_0^t ds\,\Big[U(t-s)\Big(-\mathcal{L}(s)f(\cdot,s) + (\ddot v(s)\cdot\nabla)Z(\cdot,s) \\
&+ 2[(v(s)\cdot\nabla)(\dot v(s)\cdot\nabla)Z(\cdot,s) + (\dot v(s)\cdot\nabla)\dot Z(\cdot,s)]\Big)\Big](x+q(t)) \\
&=: T_1 + T_2 + T_3.
\end{aligned}$$

Estimate of T_2: This is the same term as (56), only with $\dot v(s)$ replaced by $\ddot v(s)$. Hence $|T_2| \le C\varepsilon^3$ for $|x| \le R_\rho$ by (57).

Estimate of T_1: By a direct calculation it may be shown that

$$[\mathcal{L}(s)f(\cdot,s)](x) = (\dddot v \cdot\nabla_v)\Phi_v(x-q) + 3(\dot v\cdot\nabla_v)(\ddot v\cdot\nabla_v)\Phi_v(x-q) + (\dot v\cdot\nabla_v)^3\Phi_v(x-q),$$

where $q = q(s)$, etc., and $\Phi_v = (E_v, B_v)$. Therefore in particular $\nabla\cdot(\mathcal{L}(s)f) = 0$, whence the estimates in (50), Corollary 8.2 with $n = 2$, (52), and (57) yield

$$|T_1| \le C\Big(\varepsilon^3 + e\int_0^t ds\,\frac{|\dddot v(s)|}{1+(t-s)^2}\Big), \quad |x| \le R_\rho.$$

Estimate of T_3: Let $P(t)\phi = (v(t)\cdot\nabla)\nabla\phi + \nabla\dot\phi$ for a general $\phi = \phi(x,t)$. Then $\frac{d}{dt}(P(t)Z) = \mathcal{A}(P(t)Z) - P(t)f + (\dot v\cdot\nabla)\nabla Z$ gives, since $P(0)Z(0) = 0$,

$$\begin{aligned} T_3 &= 2\int_0^t ds\,\dot v(s)\cdot[U(t-s)(P(s)Z(\cdot,s))](x+q(t)) \\ &= 2\int_0^t ds\,\dot v(s)\cdot\int_0^s d\tau\,\Big(U(t-\tau)[-P(\tau)f(\cdot,\tau) + (\dot v(\tau)\cdot\nabla)\nabla Z(\cdot,\tau)]\Big)(x+q(t)) \\ &=: T_{3,1} + T_{3,2}. \end{aligned}$$

What concerns $T_{3,1}$, we have

$$[P(\tau)f(\cdot,\tau)](x) = (\ddot v\cdot\nabla_v)\nabla\Phi_v(x-q) + (\dot v\cdot\nabla_v)^2\nabla\Phi_v(x-q)\,.$$

Hence the standard argument ($n = 3$ in Corollary 8.2) implies $|T_{3,1}| \le C\varepsilon^3$ for $|x| \le R_\rho$; observe the additional $\dot v(s)$ in the integral. Thus it remains to bound $T_{3,2}$. As $\frac{d}{dt}(\nabla^2 Z) = \mathcal{A}(\nabla^2 Z) - \nabla^2 f$, it follows from $\nabla^2 Z(0) = 0$ that

$$T_{3,2} = 2\int_0^t ds\int_0^s d\tau\int_0^\tau d\sigma\,[U(t-\sigma)(\dot v(s)\cdot\nabla)(\dot v(\tau)\cdot\nabla)f(\cdot,\sigma)](x+q(t))\,.$$

Since $\nabla\cdot(\nabla^2 f) = 0$, again Corollary 8.2 applies, this time with $n = 4$, and we obtain for $|x| \le R_\rho$

$$|T_{3,2}| \le C\varepsilon^3\int_0^t ds\int_0^s d\tau\int_0^\tau d\sigma\,\frac{1}{1+(t-\sigma)^4} \le C\varepsilon^3.$$

Summarizing all the estimates in this section results in

$$|\dddot v(t)| \le C\Big(\varepsilon^3 + e\int_0^t ds\,\frac{|\dddot v(s)|}{1+(t-s)^2}\Big).$$

Thus for e sufficiently small we get

$$\sup_{t\in\mathbb{R}}|\dddot v(t)| \le C\varepsilon^3,$$

and the proof of Lemma 3.1 is complete. $\square$

References

[1] Abraham M.: *Theorie der Elektrizität, Band II: Elektromagnetische Theorie der Strahlung*, Teubner, Leipzig 1905

[2] Bauer G. & Dürr D.: The Maxwell-Lorentz system of a rigid charge distribution, preprint

[3] Dautray R. & Lions J.-L.: *Mathematical Analysis and Numerical Methods for Science and Technology, Vol. 5: Evolution Problems I*, Springer, Berlin-Heidelberg-New York 1992

[4] Dirac P.A.M.: Classical theory of radiating electrons, *Proc. Royal Soc. London* **A 167**, 148-169 (1938)

[5] Carati A. & Galgani L.: Asymptotic character of the series of classical electrodynamics and an application to bremsstrahlung, *Nonlinearity* **6**, 905-914 (1993)

[6] Carati A., Delzanno P., Galgani L. & Sassarini J.: Nonuniqueness properties of the physical solutions of the Lorentz-Dirac equation, *Nonlinearity* **8**, 65-79 (1995)

[7] R. Haag: Die Selbstwechselwirkung des Elektrons, *Z. Naturforsch.* **10a**, 752-761 (1955)

[8] Jones Ch.: Geometric singular perturbation theory, in *Dynamical Systems, Proceedings, Montecatini Terme 1994*, Ed. Johnson R., LNM 1609, Springer, Berlin-New York 1995, pp. 44-118

[9] Komech A., Kunze M. & Spohn H.: Effective dynamics for a mechanical particle coupled to a wave field, to appear in *Comm. Math. Phys.*

[10] Komech A. & Spohn H.: Long-time asymptotics for the coupled Maxwell-Lorentz equations, to appear in *Comm. Partial Differential Equations*

[11] Kunze M. & Spohn H.: Radiation reaction and center manifolds, submitted to *SIAM J. Math. Anal.*

[12] Rohrlich F.: *Classical Charged Particles*, 2nd edition, Addison-Wesley, Reading, MA, 1990

[13] Sakamoto K.: Invariant manifolds in singular perturbation problems for ordinary differential equations, *Proc. Roy. Soc. Edinburgh Sect. A* **116**, 45-78 (1990)

[14] Scharf G.: *From Electrostatics to Optics*, Springer, Heidelberg 1994

[15] Yaghjian A.D.: *Relativistic Dynamics of a Charged Sphere*, Lecture Notes in Physics **m 11**, Springer, Berlin-New York 1992