Remarks on Boundedness of Semilinear Oscillators

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There has been much interest during the last years in boundedness questions linked to forced oscillator equations of the type

$$\ddot{x}+x+f(x)=p(t). \tag{1}$$

Here the case we are interested in is a bounded nonlinearity $f:\mathbb{R}\to\mathbb{R}$ for which the limits $f(\infty)=\lim_{x\to\infty}f(x)$ and $f(-\infty)=\lim_{x\to-\infty}f(x)$ both exist. Moreover, p is a 2π-periodic forcing, and we ask for sharp conditions which allow to decide whether all solutions of (1) are bounded in the $(x,\dot{x})$-phase plane.

First we are going to explain what kind of condition is to be expected, without discussing the regularity of f and p which is going to play an important role later. From an old paper of Lazer and Leech [2] it is known that if additionally $f(-\infty)\le f(x)\le f(\infty)$ on $\mathbb{R}$, then

$$\left|\int_0^{2\pi}p(t)e^{it}\,dt\right|<2[f(\infty)-f(-\infty)] \tag{2}$$

is a necessary and sufficient condition for periodic solutions to exist. Thus if (2) is violated, then there is no bounded solution of (1), since otherwise there must as well exist a periodic solution, by Massera's theorem, see [5]. Hence the natural question arises whether (2) is also sufficient for all solutions of (1) to be bounded.

This found a positive answer in the recent paper [4] of B. Liu, and the purpose of these notes is to explain this proof which does require only (2), but no smallness assumption on p as was needed before. The proof will also be simplified considerably using insights of R. Ortega that he explained during his lectures at the autumn school; see [7], also for much more references on the whole subject which are not repeated here. So the credits for the ideas in the present paper are exclusively due to these both persons. Nevertheless we had the feeling that writing a clear and short proof that makes use of the full technical machinery available at the moment could be useful, especially to non-specialists in this field.

At the end, boundedness of all solutions will be obtained by application of a suitable version of Moser's twist theorem. This theorem particularly will guarantee the existence of a sequence of invariant curves Γ_n for the Poincaré-map (time 2π return map), with "$\Gamma_n\to\infty$", i.e., every fixed ball in the phase plane is encircled by Γ_n for n large. Thus the time evolution up to time 2π of all trajectories starting inside and on Γ_n do form arbitrarily large invariant cylinders in $(x,\dot{x},t)$-geometry (periodically repeated in t-direction), thus confining the solutions to stay inside. See the nice figure in [7].

Before we can apply the twist theorem we have to go through a series of changes of variables. First, introducing the primitive $F(x)=\int_0^x f(\bar{x})\,d\bar{x}$ of f, (1) is the Hamiltonian system

$$\frac{dx}{dt}=\frac{\partial H_0}{\partial y},\ \frac{dy}{dt}=-\frac{\partial H_0}{\partial x},\quad \text{with}\quad H_0(x,y,t)=\frac{1}{2}(x^2+y^2)+F(x)-xp(t)\,. \tag{3}$$

It is more convenient to work with polar coordinates in the plane, which we take as $x=r^{1/2}\cos\theta$ and $y=r^{1/2}\sin\theta$ for $r>0$ and $\theta\in T^1$ (the unit circle). We obtain

$$\frac{dr}{dt}=\frac{\partial h}{\partial\theta},\ \frac{d\theta}{dt}=-\frac{\partial h}{\partial r},\quad \text{with}\quad h(r,\theta,t)=2H_0(x,y,t)=r+2F(r^{1/2}\cos\theta)-2r^{1/2}\cos\theta p(t);\tag{4}$$

the factor 2 is necessary to get the new system Hamiltonian, since $(r,\theta)\mapsto(x,y)$ has Jakobian determinant $\frac{1}{2}$.

The main new ingredient in the proof of Liu was performance of an average over the function f (resp. F), but for this purpose, the variable θ in (4) has to be interpreted as time, and time t takes the role of the new angle variable. This trick was first used in [3] and is also useful when f is not smooth but only continuous in x, e.g. $f(x)=\mathrm{sgn}(x)$. The change of variables $\theta\to t$ and $t\to\theta$ more precisely works as follows. Since f is bounded, $F(x)$ is linearly bounded. Hence for large and given $h>0$ and fixed $t,\theta\in T^1$, the equation

$$r+2F(r^{1/2}\cos\theta)-2r^{1/2}\cos\theta p(t)=h \tag{5}$$

has a unique solution $r=r(h,t,\theta)$, and this solution is of class $C^{n+1,m,n+1}$, if $f\in C^n$ and $p\in C^m$. Moreover, $r\to\infty$ as $h\to\infty$ uniformly in $t,\theta\in T^1$, and vice versa. Thus we arrive at the system

$$\frac{dh}{d\theta}=\frac{\partial r}{\partial t},\ \frac{dt}{d\theta}=-\frac{\partial r}{\partial h},\quad \text{with Hamiltonian}\quad r(h,t,\theta)\,. \tag{6}$$

Now we can pass to a system averaged in the new time $\theta\in T^1$. To do this, we introduce the notation $I(r,\theta)=2F(r^{1/2}\cos\theta)$ and $J(r)=\frac{1}{\pi}\int_0^{2\pi}F(r^{1/2}\cos\theta)d\theta$. In principle we would like just to replace $I(r,\theta)$ by $J(r)$ in the Hamiltonian e.g. in (4), but to again obtain a Hamiltonian system, also the radius and angle variables (h,t) have to be transformed in the right way to some (ρ,τ). This works by means of a so-called generating function S; see [1]. In general, if we let

$$\rho=h-\frac{\partial S}{\partial t},\ \tau=t+\frac{\partial S}{\partial\rho}\quad \text{and}\quad H(\rho,\tau,\theta)=r(h,t,\theta)-\frac{\partial S}{\partial\theta}\,,$$

then the new system for "radius" ρ and "angle" τ is again Hamiltonian. Taking $S(\rho,\theta)=\int_0^\theta[J(\rho)-I(\rho,\phi)]\,d\phi$ independent of τ, we hence have $\rho=h$ and arrive at

$$\frac{d\rho}{d\theta}=\frac{\partial H}{\partial\tau},\ \frac{d\tau}{d\theta}=-\frac{\partial H}{\partial\rho},\quad \text{with}\quad H(\rho,\tau,\theta)=r(h,t,\theta)+I(\rho,\theta)-J(\rho)\,. \tag{7}$$

Note that here $t=t(\rho,\tau,\theta)=\tau-\int_0^\theta[J'(\rho)-\frac{\partial I}{\partial\rho}(\rho,\phi)]\,d\phi$ depends on all three variables (ρ,τ,θ).

Up to now almost nothing happened, but we are already close to applying the twist theorem. Since we only have to find invariant curves with large "radius" ρ, we scale $\rho=\delta^{-2}u^{-1}$, where u belongs to some fixed interval $[1/a,a]$ (with $a>0$ to be determined later) and $\delta\to 0$. This finally yields in new variables (u,τ,θ) the system

$$\frac{du}{d\theta}=-\delta^2u^2\frac{\partial H}{\partial\tau}(\delta^{-2}u^{-1},\tau,\theta)\,,\quad \frac{d\tau}{d\theta}=-\frac{\partial H}{\partial\rho}(\delta^{-2}u^{-1},\tau,\theta)\,. \tag{8}$$

A reader familiar with those boundedness results might wonder why we didn't care about estimates on remainders up to this point. This is due to an idea of Ortega mentioned at the beginning that allows to improve the presentation of the technical part of those arguments. We intend to find invariant curves for the Poincaré-map of (7) at "$\rho=\infty$", corresponding to $\delta=0$ for (8). Thus we consider (8) as a differential equation in $z=(u,\tau)$ and time $\theta\in T^1$ that depends on some parameter δ, and we will derive the Taylor expansion in δ of the solution up to order one. The underlying abstract result about solutions to differential equations depending on some parameter $\delta\in[0,\delta_*]$ is

Lemma 1 *Let $\mathcal{F}:[0,\theta_*]\times D\times[0,\delta_*]\to\mathbb{R}^N$ be of class $C^{0,M+1,M+1}$, for some open connected $D\subset\mathbb{R}^N$. Assume there exists a compact $K\subset D$ such that for every initial value $z_0\in K$ the solution $z(\theta;z_0,\delta)$ of $z'=\mathcal{F}(\theta,z,\delta)$, $z(0)=z_0$, exists for all $\theta\in[0,\theta_*]$. Then*

$$z(\theta;z_0,\delta)=z(\theta;z_0,0)+\delta\frac{\partial z}{\partial\delta}(\theta;z_0,0)+\delta\mathcal{R}(\theta;z_0,\delta),\qquad \sup_{\theta\in[0,\theta_*]}\|\mathcal{R}(\theta;\cdot,\delta)\|_{C^M(K)}\to 0,\quad \delta\to 0^+.$$

Proof: Easily obtained with a Taylor expansion integral remainder term; see [7, Prop. 11], also for further notes. □

Thus for (8) we take $\theta_*=2\pi$, $N=2$, and

$$\mathcal{F}(\theta,(u,\tau),\delta)=\Big(-\delta^2u^2\frac{\partial H}{\partial\tau}(\delta^{-2}u^{-1},\tau,\theta),-\frac{\partial H}{\partial\rho}(\delta^{-2}u^{-1},\tau,\theta)\Big).$$

Also let $D=[1/a,a]\times[0,2\pi]$ and $K=[2/a,a-1]\times[0,2\pi]$. Then by the arguments to follow, solutions starting in K remain in D up to time $\theta=2\pi$ if δ is sufficiently small, since the limiting equation for u from (8) as $\delta\to 0$ is $\frac{du}{d\theta}=0$. Note also that $\mathcal{F}$ is of class $C^{n+1,n,m-1}$ in (θ,u,τ) for $f\in C^n$ and $p\in C^m$. Because later an estimate for the remainder $\mathcal{R}$ in C^5 is needed, we thus have to suppose $f\in C^6$ and $p\in C^7$. Then $\mathcal{F}$ is C^6 w.r. to $\delta\in]0,\delta_0]$, and we have to discuss now the regularity at $\delta=0$. This requires two technical lemmas under the additional assumption

$$x^kf^{(k)}(x)\to 0\quad\text{as}\quad |x|\to\infty,\quad k=1,\dots,6. \tag{9}$$

Lemma 2 *If (9) holds, then the derivatives of $\mathcal{F}$ w.r. to δ up to order six can be continuated to $\delta=0$.*

Proof: From (5) we obtain $(1+r^{-1/2}\cos\theta[f(x)-p(t)])\frac{\partial r}{\partial t}=2r^{1/2}\cos\theta\dot p(t)$, and therefore from $\frac{\partial H}{\partial\tau}=\frac{\partial r}{\partial t}$,

$$\lim_{\delta\to 0^+}\Big(-\delta^2u^2\frac{\partial H}{\partial\tau}(\delta^{-2}u^{-1},\tau,\theta)\Big)=0. \tag{10}$$

Similarly, $(1+r^{-1/2}\cos\theta[f(x)-p(\tau)])\frac{\partial r}{\partial h}=1$. Moreover, $\frac{\partial I}{\partial\rho}=\rho^{-1/2}\cos\theta f(x)=\mathcal{O}(\delta)$, and by Lemma 3 below, also $J'(\rho)=\mathcal{O}(\rho^{-1/2})=\mathcal{O}(\delta)$. Next, using (9), $\frac{\partial^2 I}{\partial\rho^2}=\mathcal{O}(\rho^{-3/2})$, and by Lemma 3, $J''(\rho)=\mathcal{O}(\rho^{-3/2})$. Therefore $\frac{\partial t}{\partial\rho}=\mathcal{O}(\rho^{-3/2})=\mathcal{O}(\delta^3)$, and hence $\frac{\partial H}{\partial\rho}=\frac{\partial r}{\partial h}+\frac{\partial r}{\partial t}\frac{\partial t}{\partial\rho}+\frac{\partial I}{\partial\rho}-J'(\rho)$ yields

$$\lim_{\delta\to 0^+}\Big(-\frac{\partial H}{\partial\rho}(\delta^{-2}u^{-1},\tau,\theta)\Big)=-1. \tag{11}$$

For the first derivative, we have $\frac{\partial}{\partial\delta}\Big(-\frac{\partial H}{\partial\rho}(\delta^{-2}u^{-1},\tau,\theta)\Big)=2\delta^{-3}u^{-1}\frac{\partial^2 H}{\partial\rho^2}(\delta^{-2}u^{-1},\tau,\theta)$, and

$$\frac{\partial^2 H}{\partial\rho^2}=\frac{\partial^2 r}{\partial h^2}+2\frac{\partial^2 r}{\partial h\partial t}\frac{\partial t}{\partial\rho}+\frac{\partial r}{\partial t}\frac{\partial^2 t}{\partial\rho^2}+\frac{\partial^2 r}{\partial t^2}\Big(\frac{\partial t}{\partial\rho}\Big)^2+\frac{\partial^2 I}{\partial\rho^2}-J''(\rho). \tag{12}$$

Differentiating (5) twice w.r. to h we get an expression for $\frac{\partial^2 r}{\partial h^2}$, turning out to behave as $\frac{\partial^2 r}{\partial h^2}\cong\frac{1}{2}\rho^{-3/2}\cos\theta[f(x)-p(t)]-\frac{1}{2}\rho^{-1}\cos^2\theta f'(x)$ for $\rho\to\infty$. The other principal parts in (12) are $\frac{\partial^2 I}{\partial\rho^2}=\frac{1}{2}\rho^{-1}\cos^2\theta f'(x)-\frac{1}{2}\rho^{-3/2}\cos\theta f(x)$ and $J''(\rho)\cong-\rho^{-3/2}\frac{1}{2\pi}[f(\infty)-f(-\infty)]$, cf. Lemma 3. Thus

$$\frac{\partial^2 r}{\partial h^2}+\frac{\partial^2 I}{\partial\rho^2}-J''(\rho)\cong\frac{1}{2}\rho^{-3/2}\Big(\frac{1}{\pi}[f(\infty)-f(-\infty)]-\cos\theta p(t)\Big).$$

What concerns the other terms in (12), with arguments analogous to the above it may be seen they are $\mathcal{O}(\rho^{-2})=\mathcal{O}(\delta^4)$. E.g., $\frac{\partial^2 r}{\partial h\partial t}\cong\rho^{-1/2}\cos\theta\dot{p}(t)$, hence $\frac{\partial^2 r}{\partial h\partial t}\frac{\partial t}{\partial\rho}=\mathcal{O}(\rho^{-1/2}\rho^{-3/2})=\mathcal{O}(\rho^{-2})$. Therefore we conclude from $\rho^{-3/2}=\delta^3u^{3/2}$ that

$$\lim_{\delta\to 0^+}\frac{\partial}{\partial\delta}\Big(-\frac{\partial H}{\partial\rho}(\delta^{-2}u^{-1},\tau,\theta)\Big)=u^{1/2}\Big(\frac{1}{\pi}[f(\infty)-f(-\infty)]-\cos\theta p(\tau)\Big). \tag{13}$$

Next, $\frac{\partial}{\partial\delta}\Big(-\delta^2u^2\frac{\partial H}{\partial\tau}(\delta^{-2}u^{-1},\tau,\theta)\Big)=2u^2[-\delta\frac{\partial H}{\partial\tau}(\delta^{-2}u^{-1},\tau,\theta)+\delta^{-1}u^{-1}\frac{\partial^2 H}{\partial\rho\partial\tau}(\delta^{-2}u^{-1},\tau,\theta)]$. Then $\frac{\partial H}{\partial\tau}=\frac{\partial r}{\partial t}\cong 2\rho^{1/2}\cos\theta\dot{p}(t)$ and $\frac{\partial^2 H}{\partial\rho\partial\tau}=\frac{\partial^2 r}{\partial h\partial t}+\frac{\partial^2 r}{\partial t^2}\frac{\partial t}{\partial\rho}\cong\rho^{-1/2}\cos\theta\dot{p}(t)+\mathcal{O}(\delta^2)$. This yields

$$\lim_{\delta\to 0^+}\frac{\partial}{\partial\delta}\Big(-\delta^2u^2\frac{\partial H}{\partial\tau}(\delta^{-2}u^{-1},\tau,\theta)\Big)=-2u^{3/2}\cos\theta\dot{p}(\tau). \tag{14}$$

The proof that the higher derivatives have a limit as $\delta\to 0^+$ is carried out in the same way, although the calculations are rather tedious (but elementary). $\square$

Lemma 3 *Under assumption (9) we have for* $J(\rho)=\frac{1}{\pi}\int_0^{2\pi}F(\rho^{1/2}\cos\theta)d\theta$

$$\lim_{\rho\to\infty}\Big(\frac{d^k}{d\rho^k}\rho^{1/2}\Big)^{-1}J^{(k)}(\rho)=\frac{2}{\pi}[f(\infty)-f(-\infty)],\quad k=0,\ldots,7.$$

Proof: We calculate $\rho^{3/2}J''(\rho)=-\frac{1}{2}\rho^{1/2}J'(\rho)+\frac{1}{4\pi}\rho^{1/2}\int_0^{2\pi}f'(\rho^{1/2}\cos\theta)\cos^2\theta\,d\theta$. As a consequence of $xf'(x)\to 0$ as $|x|\to\infty$, we infer from the dominated convergence theorem that the second term vanishes as $\rho\to\infty$. Hence for $k=2$ it is sufficient to prove the claim for $k=1$, and in a similar way the higher derivatives are handled. Concerning $k=1$, observe that again by the dominated convergence theorem

$$\begin{aligned}\rho^{1/2}J'(\rho) &= \frac{1}{2\pi}\Big(\int_0^{\pi/2}+\int_{\pi/2}^{3\pi/2}+\int_{3\pi/2}^{2\pi}\Big)f(\rho^{1/2}\cos\theta)\cos\theta\,d\theta\\ &\to \frac{1}{2\pi}\Big(\int_0^{\pi/2}f(\infty)\cos\theta\,d\theta+\int_{\pi/2}^{3\pi/2}f(-\infty)\cos\theta\,d\theta+\int_{3\pi/2}^{2\pi}f(\infty)\cos\theta\,d\theta\Big)\\ &= \frac{1}{\pi}[f(\infty)-f(-\infty)].\end{aligned}$$

Finally, the claim for $k=0$ follows from de L'Hospital's rule. $\square$

After this technical intermezzo we can return to the application of Lemma 1 to (8) and derive the expansion of the solutions of (8) in δ. With $z=(u,\tau)$ in the notation of Lemma 1 we conclude for an initial value $z_0=(u_0,\tau_0)$ from (10) and (11) that

$$u(\theta;z_0,0)=u_0\,,\quad \tau(\theta;z_0,0)=\tau_0-\theta\,.$$

Moreover, differentiating the right-hand sides of (8) w.r. to δ and setting $\delta=0$ yields by means of (14) and (13) that

$$\begin{aligned}\frac{\partial u}{\partial\delta}(\theta;z_0,0) &= -2u_0^{3/2}\int_0^\theta\cos\phi\,\dot p(\tau_0-\phi)\,d\phi,\\ \frac{\partial\tau}{\partial\delta}(\theta;z_0,0) &= u_0^{1/2}\int_0^\theta\Big(\frac{1}{\pi}[f(\infty)-f(-\infty)]-\cos\phi\,p(\tau_0-\phi)\Big)\,d\phi.\end{aligned}$$

Then application of Lemma 1 and also changing θ to $-\theta$ implies for the solution the expansion

$$\begin{aligned}\tau(\theta;z_0,\delta) &= \tau_0+\theta-\delta u_0^{1/2}\int_0^\theta\Big(\frac{1}{\pi}[f(\infty)-f(-\infty)]-\cos\phi\,p(\tau_0+\phi)\Big)\,d\phi+\delta\mathcal{R}_1(\theta;z_0,\delta)\,,\\ u(\theta;z_0,\delta) &= u_0+2\delta u_0^{3/2}\int_0^\theta\cos\phi\,\dot p(\tau_0+\phi)\,d\phi+\delta\mathcal{R}_2(\theta;z_0,\delta)\,,\end{aligned}$$

with $\|\mathcal{R}_i(\theta;\cdot,\delta)\|\to 0$ in $C^5(K)$ uniformly in θ as $\delta\to 0^+$. The δ-dependent Poincaré-map of (8) is $P:(\tau_0,u_0)\mapsto(\tau_1,u_1)=(\tau(2\pi;z_0,\delta),u(2\pi;z_0,\delta))$. Hence it takes the form

$$P_\delta:\quad \tau_1=\tau_0+2\pi+\delta l_1(\tau_0,u_0)+\delta\hat{\mathcal{R}}_1\,,\quad u_1=u_0+\delta l_2(\tau_0,u_0)+\delta\hat{\mathcal{R}}_2\,, \tag{15}$$

with

$$\begin{aligned}l_1(\tau_0,u_0) &= -u_0^{1/2}\Big(2[f(\infty)-f(-\infty)]-\int_0^{2\pi}\cos\theta\,p(\tau_0+\theta)\,d\theta\Big)=:-u_0^{1/2}\gamma(\tau_0)\,, \qquad (16)\\ l_2(\tau_0,u_0) &= 2u_0^{3/2}\int_0^{2\pi}\cos\theta\,\dot p(\tau_0+\theta)\,d\theta,\end{aligned}$$

and $\hat{\mathcal{R}}_i(\tau_0,u_0,\delta)\to 0$, C^5-uniformly in $(\tau_0,u_0)\in T^1\times K=T^1\times[2/a,a-1]$ as $\delta\to 0^+$. Now (15) does not have the standard form of a map with small twist, since the angular twist $l_1(\tau_0,u_0)$ depends both on "angle" τ_0 and "radius" u_0, and not only on u_0. There is, however, a version of the twist theorem by Ortega [6] that allows to deal with such situations. Recall that the fixed parameter $a>0$ is still free to be chosen appropriately, and we are interested in finding invariant curves of $P=P_\delta$ for all $\delta>0$ small, because this corresponds to $\rho\to\infty$. Applied to above situation, the appropriate twist theorem reads as follows.

Lemma 4 *Suppose that* $l_1(\tau_0,u_0)<0$ *and* $\frac{\partial l_1}{\partial u_0}(\tau_0,u_0)<0$ *for* $(\tau_0,u_0)\in T^1\times K$. *Moreover, assume* $l_1\in C^6(T^1\times K)$, $l_2\in C^5(T^1\times K)$, *and* $\hat{\mathcal{R}}_i\in C^{5,0}(T^1\times K\times[0,\delta_0])$. *Let there exist a function* $V\in C^6(T^1\times K)$ *such that* $\frac{\partial V}{\partial u_0}(\tau_0,u_0)>0$ *and*

$$l_1(\tau_0,u_0)\frac{\partial V}{\partial\tau_0}(\tau_0,u_0)+l_2(\tau_0,u_0)\frac{\partial V}{\partial u_0}(\tau_0,u_0)=0,\quad (\tau_0,u_0)\in T^1\times K. \tag{17}$$

Then, if there are $\bar a,\bar b$ *with* $2/a<\bar a<\bar b<a-1$ *and*

$$\overline{V}(2/a)<\underline{V}(\bar a)\le\overline{V}(\bar a)<\underline{V}(\bar b)\le\overline{V}(\bar b)<\underline{V}(a-1), \tag{18}$$

where $\overline{V}(u_0)=\max_{\tau_0\in T^1}V(\tau_0,u_0)$ *and* $\underline{V}(u_0)=\min_{\tau_0\in T^1}V(\tau_0,u_0)$, *then for δ sufficiently small,* P_δ *has an invariant curve.*

Note that P_δ, coming from a Hamiltonian system, has the intersection property, i.e., for every closed curve Γ in the cylinder $T^1\times K$ that is homotopic to $\{u_0=\text{const}\}$ we have $P_\delta(\Gamma)\cap\Gamma\neq\emptyset$.

The assumptions of Lemma 4 are now easily verified for (15), with $V(\tau_0,u_0)=\frac{u_0^{1/2}}{\gamma(\tau_0)}$, the main observation being the following, finally using (2). We have

$$\max_{\tau_0\in T^1}\int_0^{2\pi}\cos\theta\, p(\tau_0+\theta)\,d\theta=\left|\int_0^{2\pi}p(\theta)e^{i\theta}\,d\theta\right|, \tag{19}$$

as may be seen writing $p(\theta)=c_0+\sum_{k=1}^\infty(c_k\cos k\theta+d_k\sin k\theta)$ as a Fourier series. Indeed, the right-hand side equals $\pi\sqrt{c_1^2+d_1^2}$, whereas the left-hand side reads $\pi\max_{\tau_0\in T^1}[c_1\cos\tau_0-d_1\sin\tau_0]$. The latter expression can be made equal to the former by choosing the vector $(\cos\tau_0,\sin\tau_0)\in\mathbb{R}^2$ in the direction of $(c_1,d_1)\in\mathbb{R}^2$. Hence (19) is verified, and thus in particular $\min_{\tau_0\in T^1}\gamma(\tau_0)>0$ by (2); see (16). Consequently, $l_1<0$ and $\frac{\partial l_1}{\partial u_0}=\frac{l_1}{2u_0}<0$. Moreover, $\frac{\partial V}{\partial u_0}=-\frac{1}{2l_1}>0$. Then $\frac{\partial V}{\partial\tau_0}=\frac{l_2}{2l_1^2}$ yields (17). It remains to verify (18) for suitably chosen $a,\bar a$, and $\bar b$. Since $\underline{V}\le\overline{V}$, we have to check by definition of V that

$$\frac{(2/a)^{1/2}}{\min\gamma}<\frac{\bar a^{1/2}}{\max\gamma},\quad \frac{\bar a^{1/2}}{\min\gamma}<\frac{\bar b^{1/2}}{\max\gamma},\quad\text{and}\quad \frac{\bar b^{1/2}}{\min\gamma}<\frac{(a-1)^{1/2}}{\max\gamma},$$

but this is easy to satisfy, since we can take e.g. $\bar b=1$, $\bar a=\frac12\left(\frac{\min\gamma}{\max\gamma}\right)^2$, and a sufficiently large. Hence Lemma 4 applies to (15).

Having now found invariant curves of P_δ for all δ small, we can undo all the coordinate changes and conclude that there are arbitrarily large invariant cylinders in $(x,\dot x,t)$-geometry for (1). So we have completed the proof of the

Theorem. *Assume $p\in C^7$ is 2π-periodic, $f\in C^6$, and that (9) and (2) hold. Then all solutions of (1) are bounded.*

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