Boundedness and Unboundedness of Solutions for Reversible Oscillators at Resonance

Markus Kunze[1], Tassilo Küpper[1] & Bin Liu[2,*]

[1] Mathematisches Institut der Universität Köln
Weyertal 86, D - 50931 Köln, Germany
e-mail: mkunze@mi.uni-koeln.de
[2] School of Mathematical Sciences, Peking University
Beijing 100871, People's Republic of China
e-mail: bliu@pku.edu.cn

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Abstract

We consider the problem of boundedness of solutions for the oscillator

$$x'' + f(x)x' + n^2 x + \varphi(x) = p(t).$$

Since the system in general is non-Hamiltonian, we have to introduce reversibility assumptions to apply suitable twist theorems, for reversible maps with small twist. We then obtain a sharp result concerning the boundedness of solutions.

1 Introduction and Main Results

We investigate the nonlinear resonant oscillator

$$x'' + f(x)x' + n^2 x + \varphi(x) = p(t), \tag{1}$$

and we ask under what assumptions all solutions of (1) will be bounded in the (x, x')-phase plane for a 2π-periodic forcing p. There has been much interest in related problems, under various assumptions, starting with the papers of Littlewood [15, 16]. The first positive result was obtained by Morris in [27] for the seemingly simple $x'' + 2x^3 = p(t)$, and it was shown by means of KAM (Kolmogorov-Arnold-Moser) theory, more precisely by the twist theorem, that all solutions of the equation are bounded. This result relied on the fact that the nonlinearity $g(x) = 2x^3$ is "superlinear", i.e., $g(x)/x \to +\infty$ for $|x| \to +\infty$. Later several

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authors extended the method to incorporate much more general superlinear nonlinearities g; see [4, 5, 10, 12, 17, 29, 38]. Contrary to that, also systems have been found that allow for unbounded solutions, cf. [16, 26, 13, 14].

On the other hand, the study of $x''+g(x)=p(t)$ with a "sublinear" nonlinearity satisfying $g(x)/x\to 0$ and $\operatorname{sgn}(x)g(x)\to+\infty$ as $|x|\to+\infty$ is a relatively new field, and we only know of the papers [9, 25, 22].

The third main class of problems are those with linear growth at infinity, where the case $g(x)=n^2x+\varphi(x)$ has been settled in [31, 33] for special φ's, and in [19] in general. It moreover turned out that as well non-smooth oscillators like $g(x)=n^2x+\operatorname{sgn}(x)$ can be treated by the same method, see [8, 19], and there are also results on the unboundedness of solutions for linearly growing nonlinearities; cf. [36, 1, 6].

In the present paper we study the boundedness of the solutions to (1), i.e., we have $g(x,x')=f(x)x'+n^2x+\varphi(x)$ being no longer independent of x'. As a consequence, for $f\neq 0$ eq. (1) will no longer be a Hamiltonian system. This means that the classical twist theorem of Moser is not applicable, and to compensate for this, a possibility is to suppose that (1) is "reversible", i.e., the functions f, φ, and p are odd. Under these assumptions, the Poincaré map that results after a suitable series of transformations will be reversible as well, and hence we can try to rely on theorems that guarantee the existence of invariant curves for such reversible maps; see [28, 37]. However, we will see that the associated Poincaré map does not satisfy the usual assumptions of the corresponding twist theorems for reversible maps, rather it is a map with small twist. For such maps a useful variant of the twist theorem was proved by Ortega in [33] in the framework of area-preserving maps. Since our Poincaré map is induced by a reversible system and not a Hamiltonian system, there is however no reason to expect it to be area-preserving. Fortunately there is also a recent version of this small twist theorem for reversible systems [21], and this will allow us to prove the boundedness of all solutions to (1) under an appropriate condition relating φ and p. The existence of (quasi-) periodic solutions and Aubry-Mather sets for reversible systems is investigated in [3].

To state our hypotheses precisely, we introduce the anti-derivative

$$F(x)=\int_0^x f(\xi)\,d\xi,\quad x\in\mathbb{R},$$

of f.

We suppose that

(A1) f, φ, $p\in C^\infty(\mathbb{R})$.

(A2) f, φ, and p are odd functions, and $p(t+2\pi)=p(t)$.

(A3) We have

$$\lim_{x\to+\infty}\varphi(x)=:\varphi(+\infty)\in\mathbb{R}\quad\text{and}\quad\lim_{|x|\to\infty}x^6\varphi^{(6)}(x)=0.$$

(A4) We have

$$|x^kF^{(k)}(x)|\le M,\quad x\in\mathbb{R},\quad 0\le k\le 6, \tag{2}$$

for some constant $M>0$.

Remark 1 (a) Assumption $(A1)$ is not really necessary, a sufficient high order of differentiability for f, φ, and p would be enough.
(b) The function F is even by $(A2)$.
(c) Since φ is odd, it follows from $(A3)$ that also

$$\lim_{x\to-\infty}\varphi(x)=:\varphi(-\infty)=-\varphi(+\infty)$$

does exist.
(d) In addition, $(A3)$ implies

$$|x^k\varphi^{(k)}(x)|\le C,\quad 0\le k\le 6,\quad\text{and}\quad \lim_{|x|\to\infty}x^k\varphi^{(k)}(x)=0,\quad 1\le k\le 6;$$

see (15) below.
(e) Already the example $x''+xx'+x=0$ with unbounded solution $x(t)=-t$ shows that some condition on f or F is necessary; however, it is not clear to us whether (2) is sharp.

Now we are ready to state our main result on the boundedness of solutions to (1).

Theorem 1 *Assume* $(A1)$*-*$(A4)$ *holds, and moreover that*

$$4\,|\varphi(+\infty)|>\left|\int_0^{2\pi}p(t)\sin nt\,dt\right|. \tag{3}$$

Then for every solution x of (1) we have

$$\sup_{t\in\mathbb{R}}\Big(|x(t)|+|x'(t)|\Big)<+\infty.$$

To relate Theorem 1 to previously obtained results, note first that $\int_0^{2\pi}p(t)\cos nt\,dt=0$ by $(A2)$, whence $|\int_0^{2\pi}p(t)\sin nt\,dt|=|\int_0^{2\pi}p(t)\,e^{int}\,dt|$. Since $\varphi(-\infty)=-\varphi(+\infty)$, cf. Remark 1(c), we hence recover [19, Thm. 1] for $f=0$, but with the (unnecessary) additional assumption that φ and p be odd. For the case $f=0$ it is know that (3) is sharp, as is shown by the example

$$x''+x+\arctan x=A\sin t. \tag{4}$$

Here $\varphi(x)=\arctan x$ is odd and satisfies $(A3)$ with $\varphi(+\infty)=\frac{\pi}{2}$; cf. [19, p. 147]. But all solutions of (4) are unbounded, for every $A>2$, as follows from [1, 6]; see also Theorem 3 below.

Next we turn to the question when solutions of (1) will be unbounded, for general $f\neq 0$. If inequality (3) is reversed, we obtain the following theorem.

Theorem 2 *Suppose* $(A1)$*-*$(A4)$ *holds, and in addition*

$$4\,|\varphi(+\infty)|<\left|\int_0^{2\pi}p(t)\sin nt\,dt\right|. \tag{5}$$

Then (1) has unbounded solutions.

Remark 2 Condition (5) is sharp in the sense that it does not enforce *all* solutions to be unbounded. To give an example, consider $x''+x+\arctan x=p(t)$ with an odd and 2π-periodic $p\in C^\infty(\mathbb{R})$ such that a periodic solution x_* does exist. This can be achieved by ensuring $|\int_0^{2\pi} p(t)\sin t\,dt|<2\pi$, using the classical result of Lazer and Leach; see [11]. Then we define $R=\max_{t\in\mathbb{R}}|x_*(t)|$, choose any $a\in\mathbb{R}$, and select $\varphi\in C^\infty(\mathbb{R})$ odd satisfying $\varphi(x)=\arctan x$ for $|x|\le R$ and $\varphi(x)=a$ for $x\ge 2R$. We see that $(A3)$ holds with $\varphi(+\infty)=a$, since φ is finally constant, and the equation $x''+x+\varphi(x)=p(t)$ has the bounded (periodic) solution x_*. However, as a is arbitrary, we can in particular arrange for $4|a|<|\int_0^{2\pi} p(t)\sin t\,dt|$ to be verified, and hence by Theorem 2 it follows that bounded and unbounded solutions do co-exist in this example.

The point that all solutions are unbounded can be settled under an assumption sharper than (5).

Theorem 3 *Assume that f, φ, $p\in C(\mathbb{R})$, φ is bounded, p is 2π-periodic, and that*

$$4n|F|_\infty+2\left(\sup\varphi-\inf\varphi\right)<\left|\int_0^{2\pi} p(t)\,e^{int}\,dt\right|. \tag{6}$$

Then every solution of (1) is unbounded.

Note that (6) reads as

$$4n|F|_\infty+4\varphi(+\infty)<\left|\int_0^{2\pi} p(t)\,\sin nt\,dt\right|$$

in case that additionally p and φ are odd, $\lim_{x\to+\infty}\varphi(x)=:\varphi(+\infty)$ does exist, and $\varphi(-\infty)\le\varphi(x)\le\varphi(+\infty)$ for $x\in\mathbb{R}$. Hence under these hypotheses, reversing the inequality in (3) enforces all solutions to be unbounded, at least if the damping effects introduced by f are small enough.

2 Some transformations

We first rewrite (1) as

$$\left\{\begin{array}{rcl} x' &=& y-F(x),\\ y' &=& -x-\varphi(x)+p(t), \end{array}\right. \tag{7}$$

where here and throughout we set $n=1$ for simplicity. According to Remark 1(b), (7) is reversible under the transformation $(x,y)\mapsto(-x,y)$; see the Appendix (Section 5) for the notation and for further properties of reversible systems.

Passing to polar coordinates

$$x=r\sin\theta,\quad y=r\cos\theta,$$

after a short calculation (7) reads as

$$\left\{\begin{array}{rcl} r' &=& [p(t)-\varphi(r\sin\theta)]\cos\theta-F(r\sin\theta)\sin\theta,\\ \theta' &=& 1-[p(t)-\varphi(r\sin\theta)]\,r^{-1}\sin\theta-F(r\sin\theta)\,r^{-1}\cos\theta. \end{array}\right. \tag{8}$$

Next observe that

$$\Big|[p(t)-\varphi(r\sin\theta)]\,r^{-1}\sin\theta-F(r\sin\theta)\,r^{-1}\cos\theta\Big|\le C_0r^{-1} \tag{9}$$

for some $C_0>0$ according to our assumptions. In the following we are going to show that for the Poincaré map of a suitably transformed version of (8) there are invariant curves encircling $\{r=r_0\}$, for every $r_0>0$. Transformed back, this means that we may consider (8) assuming that $r(t)\ge 2C_0$ for all $t\in\mathbb{R}$ along a solution $t\mapsto(r(t),\theta(t))$. Therefore

$$\theta'(t)\ge\frac{1}{2},\quad t\in\mathbb{R},$$

by (8) means that $t\mapsto\theta(t)$ is globally invertible. Denoting $\theta\mapsto t(\theta)$ the inverse function, we see that $\theta\mapsto\big(r(t(\theta)),t(\theta)\big)$ solves

$$\left\{\begin{array}{rcl}\dfrac{dr}{d\theta}&=&g_1(r,t,\theta),\\ \dfrac{dt}{d\theta}&=&g_2(r,t,\theta),\end{array}\right. \tag{10}$$

where we continued to write $r(\theta)$ instead of $r(t(\theta))$; here g_1 and g_2 are defined as

$$\begin{aligned}g_1(r,t,\theta)&=\frac{[p(t)-\varphi(r\sin\theta)]\cos\theta-F(r\sin\theta)\sin\theta}{1-[p(t)-\varphi(r\sin\theta)]\,r^{-1}\sin\theta-F(r\sin\theta)\,r^{-1}\cos\theta},\\ g_2(r,t,\theta)&=\frac{1}{1-[p(t)-\varphi(r\sin\theta)]\,r^{-1}\sin\theta-F(r\sin\theta)\,r^{-1}\cos\theta}.\end{aligned}$$

Note that the above transformation results in exchanging the time and angle variable, since now θ plays the role of time in (10). In the framework of Hamiltonian systems, this trick is technically more difficult to use, as the resulting system still has to be Hamiltonian; see [12, 8, 18, 19] and the references therein. We also remark that (10) is reversible under the involution $(r,t)\mapsto(r,-t)$, since $g_1(r,-t,-\theta)=-g_1(r,t,\theta)$ and $g_2(r,-t,-\theta)=g_2(r,t,\theta)$ due to our symmetry assumptions.

To estimate subsequent error terms, we need to introduce some convenient notation.

Definition 1 *We say a function $g=g(r,t,\theta)$ is $O_n(r^{-j})$, if it is smooth in (r,t), continuous in θ, periodic of period 2π in both t and θ, and*

$$\left|r^{k+j}\,\frac{\partial^{k+l}}{\partial r^k\,\partial t^l}\,g\right|\le M,\quad 0\le k+l\le n,$$

for some constant $M>0$ and all (r,t,θ).

Moreover, we call a function $g=g(r,t,\theta)$ to be $o_n(r^{-j})$, if it is smooth in (r,t), continuous in θ, periodic of period 2π in both t and θ, and

$$\lim_{r\to+\infty}\left|r^{k+j}\,\frac{\partial^{k+l}}{\partial r^k\,\partial t^l}\,g\right|=0,\quad 0\le k+l\le n,$$

uniformly in (t,θ).

Thus e.g. $g=O_n(r^{-j})$ means that $g\cong r^{-j}$, any t-derivative up to order n does not change this estimate, and any further r-derivative results in improving the estimate by one power of r. In case $j=0$, we will write $O_n(1)$ and $o_n(1)$, respectively. From our assumptions, $C_0r^{-1}\le\frac{1}{2}$, and (9) it follows that

$$g_1, g_2\in O_6(1),$$

and (10) can be rewritten in the form

$$\left\{\begin{array}{rcl}\dfrac{dr}{d\theta} & = & [p(t)-\varphi(r\sin\theta)]\cos\theta-F(r\sin\theta)\sin\theta+O_6(r^{-1}),\\ \dfrac{dt}{d\theta} & = & 1+[p(t)-\varphi(r\sin\theta)]\,r^{-1}\sin\theta+F(r\sin\theta)\,r^{-1}\cos\theta+O_6(r^{-2}).\end{array}\right. \tag{11}$$

In addition, since (11) is the same as (10), the system is still reversible w.r. to $(r,t)\mapsto(r,-t)$. Here and in the sequel we often will not give detailed proofs of such error estimates, since they are completely analogous to those previously used in related papers. In principle, they follow by deriving the respective bounds for each constituent in a formula, and then utilize suitable general chain rules; see e.g. [19, Appendix], [8, Appendix], or [34, 7].

At the present stage of proof, it is not possible to employ a theorem on the existence of invariant curves for the Poincaré map of (11), as this map is not sufficiently close to a twist map (with small twist). Hence we first need to transform (11) further, and the next step in this direction is to bring the right-hand side of the first equation in (11) into the form $p(t)\cos\theta+O_6(r^{-1})$. The next lemma introduces the appropriate transformation.

Lemma 1 *Consider the transformation $(r,t)\mapsto(\lambda,t)$, where*

$$\lambda=r+S(r,\theta),\quad \textit{with}\quad S(r,\theta)=\int_0^\theta\Big[\varphi(r\sin\phi)\cos\phi+F(r\sin\phi)\sin\phi\Big]\,d\phi.$$

Then (11) is transformed into

$$\left\{\begin{array}{rcl}\dfrac{d\lambda}{d\theta} & = & p(t)\cos\theta+O_6(\lambda^{-1}),\\ \dfrac{dt}{d\theta} & = & 1+[p(t)-\varphi(\lambda\sin\theta)]\,\lambda^{-1}\sin\theta+F(\lambda\sin\theta)\,\lambda^{-1}\cos\theta+O_6(\lambda^{-2}).\end{array}\right. \tag{12}$$

The new system (12) is reversible w.r. to $(\lambda,t)\mapsto(\lambda,-t)$.

Proof: First note that $S(r,2\pi)=0$, as may be seen from substituting $\phi=2\pi-\vartheta$ and recalling Remark 1(b). Since the functions in the integrand of S are 2π-periodic in ϕ, this yields $S(r,\theta+2\pi)=S(r,\theta)$. Moreover, our assumptions show $S\in O_6(1)$. In particular,

$$\left|\frac{\partial S}{\partial r}(r,\theta)\right|\le Cr^{-1}$$

implies that $r\mapsto\lambda$ is a diffeomorphism from some $[r_0,+\infty[$ to an appropriate $[\lambda_0,+\infty[$, and in addition that $\lambda\cong r$. Hence (12) is obtained from (11) and the definition of S. Finally, that (12) is reversible w.r. to $(\lambda,t)\mapsto(\lambda,-t)$ follows from Lemma 6 in the Appendix, together with the observation that the transformation is invariant w.r. to this involution due to $S(r,-\theta)=S(r,\theta)$; see (52) below. □

Our next aim is to improve the second equation in (12) by averaging the term with φ over the (2π-periodic) time variable θ. We start with an auxiliary results concerning this average

$$J(\lambda)=\frac{1}{2\pi\lambda}\int_0^{2\pi}\varphi(\lambda\sin\phi)\sin\phi\, d\phi. \tag{13}$$

The following lemma provides information on the limiting behaviour of J.

Lemma 2 *For* $0\leq k\leq 6$ *we have*

$$\lim_{\lambda\to+\infty}\lambda^{k+1}J^{(k)}(\lambda)=(-1)^k k!\,\frac{2}{\pi}\,\varphi(+\infty). \tag{14}$$

Proof: We first show that

$$\lim_{x\to+\infty}x^k\varphi^{(k)}(x)=0,\quad 1\leq k\leq 6, \tag{15}$$

as was already stated in Remark 1(d). For this is suffices to prove

$$1\leq k\leq 5,\quad \lim_{x\to+\infty}x^{k+1}\varphi^{(k+1)}(x)=0\quad\Longrightarrow\quad \lim_{x\to+\infty}x^k\varphi^{(k)}(x)=0, \tag{16}$$

by assumption $(A3)$. To see (16), we have $|x^{k+1}\varphi^{(k+1)}(x)|\leq M$, $x\in[0,+\infty[$, for some constant $M>0$. This implies $\varphi^{(k)}(x)$ has a limit as $x\to+\infty$, since

$$|\varphi^{(k)}(x)-\varphi^{(k)}(\bar{x})|\leq\left|\int_{\bar{x}}^{x}|\varphi^{(k+1)}(\xi)|\,d\xi\right|\leq M\left|\int_{\bar{x}}^{x}\frac{d\xi}{\xi^{k+1}}\right|=Mk^{-1}\left|\frac{1}{x^k}-\frac{1}{\bar{x}^k}\right|$$

for $x,\bar{x}\in]0,+\infty[$, whence

$$\lim_{x\to+\infty}\varphi^{(k)}(x)=:a\in\mathbb{R}$$

does exist. Since φ is bounded (as $\varphi(+\infty)$ is finite), this is easily seen to yield $a=0$. Hence we may employ L'Hospital's rule to deduce

$$\lim_{x\to+\infty}x^k\varphi^{(k)}(x)=\lim_{x\to+\infty}\frac{\varphi^{(k)}(x)}{x^{-k}}=\lim_{x\to+\infty}\frac{\varphi^{(k+1)}(x)}{(-k)x^{-(k+1)}}=0$$

by assumption. This proves that (16) holds, and therefore we have as well shown (15).

Next, using the dominated convergence theorem we calculate

$$\begin{aligned}\lambda J(\lambda) &= \frac{1}{2\pi}\int_0^{\pi}\varphi(\lambda\sin\phi)\sin\phi\, d\phi+\frac{1}{2\pi}\int_{\pi}^{2\pi}\varphi(\lambda\sin\phi)\sin\phi\, d\phi\\ &\to \frac{\varphi(+\infty)}{2\pi}\int_0^{\pi}\sin\phi\, d\phi+\frac{\varphi(-\infty)}{2\pi}\int_{\pi}^{2\pi}\sin\phi\, d\phi=\frac{2}{\pi}\varphi(+\infty),\quad \lambda\to+\infty,\end{aligned} \tag{17}$$

due to Remark 1(c). It is moreover verified by induction that

$$J^{(k)}(\lambda)=-\frac{k}{\lambda}J^{(k-1)}(\lambda)+\frac{1}{2\pi\lambda^{k+1}}\int_0^{2\pi}\varphi^{(k)}(\lambda\sin\phi)\,(\lambda\sin\phi)^k\sin\phi\, d\phi,\quad k\geq 1.$$

Hence, as $\lambda\to+\infty$,

$$\lambda^{k+1}J^{(k)}(\lambda)=-k\,\lambda^k J^{(k-1)}(\lambda)+o(1),\quad 1\leq k\leq 6,$$

by (15) and the dominated convergence theorem. Together with (17) this implies (14). $\square$

Now we are ready to further transform system (12).

Lemma 3 *Introduce the change of variables* $(\lambda,t)\mapsto(\lambda,\tau)$*, where* $\tau=t+L(\lambda,\theta)$*, and*

$$L(\lambda,\theta)=\lambda^{-1}\int_0^\theta\Big[\varphi(\lambda\sin\phi)\sin\phi-\lambda J(\lambda)\Big]d\phi-\lambda^{-1}\int_0^\theta F(\lambda\sin\phi)\cos\phi\, d\phi.$$

Then (12) is transformed into

$$\begin{cases}\dfrac{d\lambda}{d\theta} &=& p(\tau)\cos\theta+O_6(\lambda^{-1}),\\ \dfrac{d\tau}{d\theta} &=& 1-J(\lambda)+p(\tau)\lambda^{-1}\sin\theta+O_5(\lambda^{-2}).\end{cases}\tag{18}$$

The new system (18) is reversible w.r. to $(\lambda,\tau)\mapsto(\lambda,-\tau)$*.*

Proof: The proof is similar to the one given for Lemma 1. By definition of J in (13), we have

$$\int_0^{2\pi}\Big[\varphi(\lambda\sin\phi)\sin\phi-\lambda J(\lambda)\Big]d\phi=0.\tag{19}$$

In addition, F being even implies

$$\int_0^{2\pi}F(\lambda\sin\phi)\cos\phi\, d\phi=0,\tag{20}$$

as we can set $\phi=\vartheta-\pi$ to find

$$\begin{aligned}\int_0^{2\pi}F(\lambda\sin\phi)\cos\phi\, d\phi &= -\int_\pi^{2\pi}F(\lambda\sin\vartheta)\cos\vartheta\, d\vartheta-\int_{2\pi}^{3\pi}F(\lambda\sin\vartheta)\cos\vartheta\, d\vartheta\\ &= -\int_0^{2\pi}F(\lambda\sin\vartheta)\cos\vartheta\, d\vartheta.\end{aligned}$$

Hence $L(\lambda,2\pi)=0$ by (19) and (20), and since once more the arguments are 2π-periodic in ϕ, this implies $L(\lambda,\theta+2\pi)=L(\lambda,\theta)$. As $J\in O_6(\lambda^{-1})$ by Lemma 2, it moreover follows that also $L\in O_6(\lambda^{-1})$ by our assumptions on φ and F, whence $\frac{\partial L}{\partial\lambda}=O_5(\lambda^{-2})$. In particular $(\lambda,t)\mapsto(\lambda,\tau)$ is a diffeomorphism for λ sufficiently large, and $\tau\cong t$. Moreover,

$$\frac{d\tau}{d\theta}=\frac{dt}{d\theta}+\frac{\partial L}{\partial\lambda}\frac{\partial\lambda}{\partial\theta}+\frac{\partial L}{\partial\theta},$$

and noting $\frac{\partial\lambda}{\partial\theta}=O_6(1)$ by (12), we see that (18) is a consequence of (12). Finally, (18) is reversible w.r. to $(\lambda,\tau)\mapsto(\lambda,-\tau)$ by Lemma 6 and (52) below, taking into account $L(\lambda,-\theta)=-L(\lambda,\theta)$. □

Next we intend to rewrite the term $J(\lambda)$ in the second equation of (18) in a more convenient way. For this purpose, we define

$$\eta(\lambda)=J(\lambda)-\frac{2}{\pi}\varphi(+\infty)\lambda^{-1}.\tag{21}$$

Lemma 4 *We have*

$$\eta=o_6(\lambda^{-1}).\tag{22}$$

Proof: It follows that

$$\eta^{(k)}(\lambda) = J^{(k)}(\lambda) - \frac{2}{\pi}\varphi(+\infty)(-1)^k k!\,\lambda^{-(k+1)},$$

and therefore, as $\lambda \to +\infty$,

$$\lambda^{k+1}\eta^{(k)}(\lambda) = \lambda^{k+1}J^{(k)}(\lambda) - \frac{2}{\pi}\varphi(+\infty)(-1)^k k! \to 0, \quad 0 \le k \le 6,$$

by (14) in Lemma 2. □

From (21) we find

$$J(\lambda) = \frac{2}{\pi}\varphi(+\infty)\lambda^{-1} + \eta(\lambda),$$

therefore (18) reads, using (22),

$$\left\{ \begin{array}{rcl} \dfrac{d\lambda}{d\theta} &=& p(\tau)\cos\theta + \lambda^{-1}O_6(1), \\ \dfrac{d\tau}{d\theta} &=& 1 - \dfrac{2}{\pi}\varphi(+\infty)\lambda^{-1} + p(\tau)\lambda^{-1}\sin\theta + \lambda^{-1}o_5(1). \end{array} \right. \tag{23}$$

We also recall that (23) is reversible w.r. to $(\lambda,\tau) \mapsto (\lambda,-\tau)$.

3 Invariant curves for the Poincaré map

In this section we will study the Poincaré map induced by (23). We first have to transform the unbounded λ-variable to a fixed domain of definition. To do so, we write

$$\lambda^{-1} = \varepsilon\varrho, \quad \varrho \in [\frac{1}{\gamma}, \gamma],$$

with $\varepsilon > 0$ small and $\gamma > 0$ a fixed parameter that will be determined later; thus $\varrho = O(1)$. Using the change of variables $(\lambda,\tau) \mapsto (\varrho,\tau)$ and $\frac{d\varrho}{d\theta} = -\varepsilon^{-1}\lambda^{-2}\frac{d\lambda}{d\theta}$, system (23) is recast into

$$\left\{ \begin{array}{rcl} \dfrac{d\varrho}{d\theta} &=& -\varepsilon\varrho^2 p(\tau)\cos\theta + \varepsilon^2 O_6(1), \\ \dfrac{d\tau}{d\theta} &=& 1 - \varepsilon\varrho\dfrac{2}{\pi}\varphi(+\infty) + \varepsilon\varrho p(\tau)\sin\theta + \varepsilon o_5(1). \end{array} \right. \tag{24}$$

Since the transformation $(\lambda,\tau) \mapsto (\varrho,\tau)$ is reversible w.r. to $G: (\lambda,\tau) \mapsto (\lambda,-\tau)$, so is (24), by Lemma 6 in Section 5.

To derive an expression for the corresponding Poincaré map, we make the ansatz

$$\varrho(\theta;\varrho_0,\tau_0) = \varrho_0 + \varepsilon F_1(\theta;\varrho_0,\tau_0,\varepsilon), \quad \tau(\theta;\varrho_0,\tau_0) = \tau_0 + \theta + \varepsilon F_2(\theta;\varrho_0,\tau_0,\varepsilon) \tag{25}$$

for the solution $(\varrho(\theta),\tau(\theta))$ of (24) with data $(\varrho(0),\tau(0)) = (\varrho_0,\tau_0)$, i.e., we have

$$F_1(0;\varrho_0,\tau_0,\varepsilon) = F_2(0;\varrho_0,\tau_0,\varepsilon) = 0, \tag{26}$$

and F_1, F_2 are defined through (25). Differentiating (25) and comparing to (24), we see that

$$\begin{cases} \dfrac{\partial F_1}{\partial \theta} &=& -(\varrho_0+\varepsilon F_1)^2 p(\tau_0+\theta+\varepsilon F_2)\cos\theta+\varepsilon O_6(1), \\ \dfrac{\partial F_2}{\partial \theta} &=& (\varrho_0+\varepsilon F_1)\Big(-\dfrac{2}{\pi}\varphi(+\infty)+p(\tau_0+\theta+\varepsilon F_2)\sin\theta\Big)+o_5(1). \end{cases}$$

From the defining formulas, it is not difficult to show that $F_1=O_6(1)$ and $F_2=o_5(1)$, whence

$$\begin{cases} F_1(2\pi;\varrho_0,\tau_0,\varepsilon) &=& -\varrho_0^2\displaystyle\int_0^{2\pi} p(\tau_0+\theta)\cos\theta\,d\theta+\varepsilon O_6(1), \\ F_2(2\pi;\varrho_0,\tau_0,\varepsilon) &=& \varrho_0\Big(-4\varphi(+\infty)+\displaystyle\int_0^{2\pi} p(\tau_0+\theta)\sin\theta\,d\theta\Big)+o_5(1) \end{cases} \tag{27}$$

by (26). Now note that due to periodicity of p, and since p is odd,

$$\begin{aligned} \int_0^{2\pi} p(\tau_0+\theta)\cos\theta\,d\theta &= \int_0^{2\pi} p(\theta)\cos(\theta-\tau_0)\,d\theta=\sin\tau_0\int_0^{2\pi} p(\theta)\sin\theta\,d\theta+\cos\tau_0\int_0^{2\pi} p(\theta)\cos\theta\,d\theta \\ &= \sin\tau_0\int_0^{2\pi} p(\theta)\sin\theta\,d\theta, \end{aligned}$$

and similarly

$$\int_0^{2\pi} p(\tau_0+\theta)\sin\theta\,d\theta=\cos\tau_0\int_0^{2\pi} p(\theta)\sin\theta\,d\theta.$$

It thus follows from (27) and $\varepsilon O_6(1)=o_6(1)$ that

$$\begin{cases} F_1(2\pi;\varrho_0,\tau_0,\varepsilon) &=& -\varrho_0^2\sin\tau_0\displaystyle\int_0^{2\pi} p(\theta)\sin\theta\,d\theta+o_6(1), \\ F_2(2\pi;\varrho_0,\tau_0,\varepsilon) &=& \varrho_0\Big(-4\varphi(+\infty)+\cos\tau_0\displaystyle\int_0^{2\pi} p(\theta)\sin\theta\,d\theta\Big)+o_5(1). \end{cases}$$

The Poincaré time-2π-map

$$P:\quad (\varrho_0,\tau_0)\mapsto(\varrho(2\pi;\varrho_0,\tau_0),\tau(2\pi;\varrho_0,\tau_0))=:(\varrho_1,\tau_1)$$

of (24) is well-defined for $\varepsilon\ll 1$ and hence has the form

$$P:\quad\begin{cases} \varrho_1 &=& \varrho_0-\varepsilon\varrho_0^2\sin\tau_0\displaystyle\int_0^{2\pi} p(\theta)\sin\theta\,d\theta+\varepsilon o_6(1), \\ \tau_1 &=& \tau_0+2\pi+\varepsilon\varrho_0\Big(-4\varphi(+\infty)+\cos\tau_0\displaystyle\int_0^{2\pi} p(\theta)\sin\theta\,d\theta\Big)+\varepsilon o_5(1), \end{cases} \tag{28}$$

for $(\varrho_0,\tau_0)\in[\frac{1}{\gamma},\gamma]\times S^1$. Observe P is reversible w.r. to the involution $(\varrho_0,\tau_0)\mapsto(\varrho_0,-\tau_0)$, since the systems of differential equations (24) is; see Definition 4 and Lemma 7 in the Appendix.

In order to complete the proof of Theorem 1, it is sufficient to show that for every $\varepsilon\ll 1$ the Poincaré map P from (28) has an invariant closed curve in the annulus $[\frac{1}{\gamma},\gamma]\times S^1$ which surrounds the circle $\varrho\equiv\frac{1}{\gamma}$. Usually, the existence of such invariant curves is obtained by means of Moser's twist theorem for reversible mappings, see [28, 37]. However, in its standard version, Moser's theorem is concerned with a map of the form

$$\left\{\begin{array}{lcl}\varrho_1 & = & \varrho_0+\dots,\\ \tau_1 & = & \tau_0+\omega+\varepsilon\varrho_0+\dots,\end{array}\right. \tag{29}$$

where ω is a fixed number, $\varepsilon>0$ is a small parameter, and the remaining terms indicated by dots are of order at least $o_5(\varepsilon)$ as $\varepsilon\to 0^+$. Comparing (29) to (28), we see that this result in its classical form does not apply to our Poincaré map P except if $\int_0^{2\pi}p(\theta)\sin\theta\,d\theta=0$.

In [33], see also [31, 32, 35], Ortega introduced a variant of the twist theorem for area-preserving maps with small twist. This result already proved useful in a number of papers, cf. e.g. [33, 19, 23]. Since the Poincaré map P from (28) does have small twist, it would be tempting to apply this result also in the present setting, but in general P will not be area-preserving (as normally is used to argue that P has the "intersection property"). However, fortunately a similar theorem is valid for reversible rather than area-preserving maps.

To state this result, let $A=[a,b]\times S^1$ denote a cylinder with universal cover $\mathcal{A}=[a,b]\times\mathbb{R}$. Consider a map

$$\bar{f}:\quad A\to\mathbb{R}\times S^1.$$

Suppose that

$$f:\quad \mathcal{A}\to\mathbb{R}\times\mathbb{R}$$

is a lift of $\bar{f}$ having the form $(\varrho_1,\tau_1)=f(\varrho_0,\tau_0)$, with

$$\left\{\begin{array}{lcl}\varrho_1 & = & \varrho_0+\varepsilon\,l_1(\varrho_0,\tau_0)+\varepsilon\,\varphi_1(\varrho_0,\tau_0),\\ \tau_1 & = & \tau_0+2\pi+\varepsilon\,l_2(\varrho_0,\tau_0)+\varepsilon\,\varphi_2(\varrho_0,\tau_0),\end{array}\right. \tag{30}$$

and the functions l_1, l_2, φ_1, and φ_2 satisfying

$$l_1,\varphi_1,\varphi_2\in C^5(\mathcal{A}),\quad l_2\in C^6(\mathcal{A}),\quad\text{and}\quad l_2>0\quad\text{as well as}\quad \frac{\partial l_2}{\partial\varrho_0}>0\quad\text{in}\quad\mathcal{A}. \tag{31}$$

In addition, we assume there is a function

$$I:\quad\mathcal{A}\to\mathbb{R}$$

which is 2π-periodic and even in τ_0 such that

$$I\in C^6(\mathcal{A}),\quad\text{and}\quad\frac{\partial I}{\partial\varrho_0}>0\quad\text{as well as}\quad l_1\frac{\partial I}{\partial\varrho_0}+l_2\frac{\partial I}{\partial\tau_0}=0\quad\text{in}\quad\mathcal{A}. \tag{32}$$

Define

$$I_{\max}(\varrho_0)=\max_{\tau_0}I(\varrho_0,\tau_0)\quad\text{and}\quad I_{\min}(\varrho_0)=\min_{\tau_0}I(\varrho_0,\tau_0),\quad \varrho_0\in[a,b].$$

Then we have the following variant of Ortega's result; cf. [21] for the proof.

Lemma 5 *Assume such I satisfying (32) does exist, and in addition that there are numbers $\tilde{a}$ and $\tilde{b}$ with*

$$a<\tilde{a}<\tilde{b}<b\quad\textit{and}\quad I_{\max}(a)<I_{\min}(\tilde{a})\le I_{\max}(\tilde{a})<I_{\min}(\tilde{b})\le I_{\max}(\tilde{b})<I_{\min}(b). \tag{33}$$

Then there are $\delta>0$ and $\Delta>0$ such that the following is true: suppose (30) and (31) hold for a lift f of $\bar{f}$ such that additionally

$$\|\varphi_1\|_{C^5(\mathcal{A})}+\|\varphi_2\|_{C^5(\mathcal{A})}<\delta \quad \text{and} \quad \varepsilon<\Delta,$$

and let f be reversible w.r. to the involution $G:(\varrho_0,\tau_0)\mapsto(\varrho_0,-\tau_0)$, i.e.,

$$f^{-1}=G\circ f\circ G \quad \text{in} \quad \mathcal{A};$$

see also Definition 4 below. Then $\bar{f}$ has an invariant curve Γ. Denoting $\mu(\Gamma,\varepsilon)\in S^1$ the rotation number of Γ, moreover $\lim_{\varepsilon\to 0^+}\mu(\Gamma,\varepsilon)=0$.

Remark 3 The change of variables $\tilde{\tau}=-\tau$, $\tilde{\varrho}=\varrho$, shows that (31) can be replaced by

$$l_1,\varphi_1,\varphi_2\in C^5(\mathcal{A}), \quad l_2\in C^6(\mathcal{A}), \quad \text{and} \quad l_2<0 \quad \text{as well as} \quad \frac{\partial l_2}{\partial\varrho_0}<0 \quad \text{in} \quad \mathcal{A}.$$

After this preparation, we can finally come to the

Proof of Theorem 1: By the preceding observations, and recalling the form of P from (28), the main task is to find the function $I:[\frac{1}{\gamma},\gamma]\times\mathbb{R}\to\mathbb{R}$ and the numbers $\tilde{a}$ and $\tilde{b}$; in particular, we let $a=\frac{1}{\gamma}$ and $b=\gamma$, and we remind that γ is still free to be chosen. Define

$$p^s=\int_0^{2\pi}p(t)\sin t\,dt.$$

By (3), we have

$$|\varphi(+\infty)|>\frac{1}{4}|p^s|, \tag{34}$$

and hence $\varphi(+\infty)\neq 0$. From (34) we deduce that

$$\operatorname{sgn}\Big(4\varphi(+\infty)-p^s\cos\tau_0\Big)=\operatorname{sgn}\Big(\varphi(+\infty)\Big), \quad \tau_0\in\mathbb{R}. \tag{35}$$

Let

$$I(\varrho_0,\tau_0)=\frac{\varrho_0}{4\varphi(+\infty)-p^s\cos\tau_0}\operatorname{sgn}\Big(\varphi(+\infty)\Big), \quad (\varrho_0,\tau_0)\in\mathcal{A},$$

and moreover

$$\beta=\max_{\tau_0\in\mathbb{R}}\Big|4\varphi(+\infty)-p^s\cos\tau_0\Big| \quad \text{and} \quad \alpha=\min_{\tau_0\in\mathbb{R}}\Big|4\varphi(+\infty)-p^s\cos\tau_0\Big|.$$

Then $0<\alpha\leq\beta$, whence

$$\gamma:=\left(2\frac{\beta}{\alpha}\right)^3>1, \quad \tilde{a}=\gamma^{-1/3}, \quad \text{and} \quad \tilde{b}=\gamma^{1/3} \tag{36}$$

are well-defined, in fact $\gamma\geq 8$. Then obviously $I\in C^\infty(\mathcal{A})$, and $\dfrac{\partial I}{\partial\varrho_0}>0$ in $\mathcal{A}$ according to (35). To verify (33), clearly $a<\tilde{a}<\tilde{b}<b$. Moreover, if we consider e.g. the case that

$\operatorname{sgn}\Big(\varphi(+\infty)\Big) > 0$, then we obtain $\beta = 4\varphi(+\infty) + |p^s|$ as well as $\alpha = 4\varphi(+\infty) - |p^s|$ by (35), and hence

$$I_{\max}(\varrho_0) = \frac{\varrho_0}{\alpha} \quad \text{and} \quad I_{\min}(\varrho_0) = \frac{\varrho_0}{\beta}.$$

Then the chain of inequalities to be satisfied in (33) reads as

$$\frac{a}{\alpha} < \frac{\tilde{a}}{\beta} \le \frac{\tilde{a}}{\alpha} < \frac{\tilde{b}}{\beta} \le \frac{\tilde{b}}{\alpha} < \frac{b}{\beta},$$

and this holds according to (36). The same argument applies in case that $\operatorname{sgn}\Big(\varphi(+\infty)\Big) < 0$, whence (33) indeed is verified.

Finally, to see that the last condition from (32) is met, note that we have

$$l_1(\varrho_0, \tau_0) = -\varrho_0^2 \sin\tau_0\, p^s \quad \text{and} \quad l_2(\varrho_0, \tau_0) = \varrho_0\Big(-4\varphi(+\infty) + \cos\tau_0\, p^s\Big) < 0,$$

cf. (28) and (30). Since also $\frac{\partial l_2}{\partial \varrho_0} < 0$ and $l_1 \frac{\partial I}{\partial \varrho_0} + l_2 \frac{\partial I}{\partial \tau_0} = 0$ in $\mathcal{A}$ by definition of I, we have shown that P satisfies the assumptions of Lemma 5, in the variant of Remark 3. Consequently, the Poincaré map P has an invariant closed curve for all $\varepsilon > 0$ small enough. This completes the proof of Theorem 1. □

4 Unbounded solutions

In this section we will give the proofs of Theorem 2 and Theorem 3.

Proof of Theorem 2: The idea of proof is related to [30, 20]: we will find a sector $\mathcal{D}$ in the right upper half-plane that is forward invariant for the Poincaré map, and moreover solutions are driven outward a fixed amount each time they enter this sector. We continue to assume $n = 1$, and w.l.o.g.

$$p^s = \int_0^{2\pi} p(t) \sin t\, dt > 4\varphi(+\infty) \ge 0, \tag{37}$$

the other cases are handled in the same way. Due to (37) we can choose $\bar{\tau}_0 \in]0, \frac{\pi}{2}]$ such that

$$p^s \cos\bar{\tau}_0 = 4\varphi(+\infty).$$

Hence

$$p^s \sin\bar{\tau}_0 =: \varepsilon_0 > 0,$$

and therefore we find $\delta_0 > 0$ with $[\bar{\tau}_0 - \delta_0, \bar{\tau}_0 + \delta_0] \subset]0, \pi[$ and

$$p^s \sin\tau_0 \ge \frac{1}{2}\varepsilon_0 \quad \text{for} \quad \tau_0 \in [\bar{\tau}_0 - \delta_0, \bar{\tau}_0 + \delta_0]. \tag{38}$$

By definition of $\bar{\tau}_0$, we may moreover select $\delta_1 \in]0, \frac{1}{2}\delta_0]$ such that

$$\begin{aligned} p^s \cos\tau_0 - 4\varphi(+\infty) &\le -\delta_1 &&: \quad \tau_0 \in [\bar{\tau}_0 + \tfrac{1}{2}\delta_0, \bar{\tau}_0 + \delta_0], \\ p^s \cos\tau_0 - 4\varphi(+\infty) &\ge \delta_1 &&: \quad \tau_0 \in [\bar{\tau}_0 - \delta_0, \bar{\tau}_0 - \tfrac{1}{2}\delta_0]. \end{aligned} \tag{39}$$

After this preliminary consideration, we return to (1), in the equivalent form

$$\left\{\begin{array}{rcl}\dfrac{d\lambda}{d\theta} & = & p(\tau)\cos\theta+\lambda^{-1}f_1(\lambda,\tau,\theta),\\ \dfrac{d\tau}{d\theta} & = & 1-\dfrac{2}{\pi}\varphi(+\infty)\lambda^{-1}+p(\tau)\lambda^{-1}\sin\theta+\lambda^{-1}f_2(\lambda,\tau,\theta).\end{array}\right. \tag{40}$$

see (23), with $f_1=O_6(1)$ and $f_2=o_5(1)$ as $\lambda\to+\infty$. Making the ansatz $\lambda(\theta)=\lambda_0+F_1$ and $\tau(\theta)=\tau_0+\theta+F_2$, compare to (25), it follows analogously to Section 3 that the Poincaré time-2π-map of (40) has the form

$$P:\quad\left\{\begin{array}{rcl}\lambda_1 & = & \lambda_0+p^s\sin\tau_0+\tilde{f}_1(\lambda_0,\tau_0),\\ \tau_1 & = & \tau_0+2\pi+\Big(p^s\cos\tau_0-4\varphi(+\infty)\Big)\lambda_0^{-1}+\lambda_0^{-1}\tilde{f}_2(\lambda_0,\tau_0),\end{array}\right. \tag{41}$$

where in particular

$$\lim_{\lambda\to+\infty}\tilde{f}_1(\lambda_0,\tau_0)=0=\lim_{\lambda\to+\infty}\tilde{f}_2(\lambda_0,\tau_0), \tag{42}$$

uniformly in $\tau_0\in S^1$.

At this point we could invoke [2, Prop. 2.1] to conclude the proof, since due to the form of P we can just set

$$\begin{array}{l}\mu_1(\tau_0)=p^s\cos\tau_0-4\varphi(+\infty),\quad \mu_2(\tau_0)=p^s\sin\tau_0,\\ F(\tau_0,\lambda_0)=\lambda_0^{-1}\tilde{f}_2(\lambda_0,\tau_0),\quad\text{and}\quad G(\tau_0,\lambda_0)=\tilde{f}_1(\lambda_0,\tau_0)\end{array}$$

in this result. However, for completeness we will give a separate explicit proof here.

By (42) we find

$$\Lambda\geq\max\Big\{1,8(4\varphi(+\infty)+p^s)\delta_0^{-1}\Big\} \tag{43}$$

such that

$$|\tilde{f}_1(\lambda_0,\tau_0)|\leq\frac{1}{4}\varepsilon_0\quad\text{and}\quad|\tilde{f}_2(\lambda_0,\tau_0)|\leq\frac{1}{2}\delta_1,\quad\lambda_0\geq\Lambda,\ \tau_0\in S^1. \tag{44}$$

We introduce the sector

$$\mathcal{D}=\Big\{(\lambda,\tau):\lambda\geq\Lambda,\ \|\tau-\bar{\tau}_0\|<\delta_0\Big\}$$

in the right upper half-plane, where $\|\tau-\bar{\tau}_0\|:=\min_{m\in\mathbf{Z}}|\tau-\bar{\tau}_0+2m\pi|$ corresponds to the distance on the circle. We are going to show that $\mathcal{D}$ is forward invariant for P, and that moreover all solutions starting in $\mathcal{D}$ will be unbounded.

To prove the invariance of $\mathcal{D}$, we fix $(\lambda_0,\tau_0)\in\mathcal{D}$ and denote $(\lambda_1,\tau_1)=P(\lambda_0,\tau_0)$. Then by (41), (38), and (44)

$$\lambda_1=\lambda_0+p^s\sin\tau_0+\tilde{f}_1(\lambda_0,\tau_0)\geq\lambda_0+\frac{1}{2}\varepsilon_0-|\tilde{f}_1(\lambda_0,\tau_0)|\geq\lambda_0+\frac{1}{4}\varepsilon_0>\Lambda, \tag{45}$$

since $\lambda_0\geq\Lambda$. Hence it remains to verify

$$\|\tau_1-\bar{\tau}_0\|<\delta_0. \tag{46}$$

By assumption, we know that $\|\tau_0-\bar{\tau}_0\|<\delta_0$. First consider the case that even $\|\tau_0-\bar{\tau}_0\|\leq\frac{1}{2}\delta_0$. Then by (41), (43), and (44)

$$\|\tau_1-\bar{\tau}_0\|\leq\|\tau_0-\bar{\tau}_0\|+\Big(|p^s\cos\tau_0|+4\varphi(+\infty)\Big)\lambda_0^{-1}+\lambda_0^{-1}|\tilde{f}_2(\lambda_0,\tau_0)|\leq\frac{1}{2}\delta_0+\frac{1}{8}\delta_0+\frac{1}{2}\delta_1<\delta_0.$$

On the other hand, if $\frac{1}{2}\delta_0 < \|\tau_0 - \bar{\tau}_0\| < \delta_0$, then we select $m_0 \in \mathbb{Z}$ such that

$$\|\tau_0 - \bar{\tau}_0\| = |\tau_0 - \bar{\tau}_0 - 2m_0\pi|,$$

and first analyze the sub-case $\tau_0 - \bar{\tau}_0 - 2m_0\pi > 0$, whence

$$\bar{\tau}_0 + 2m_0\pi + \frac{1}{2}\delta_0 < \tau_0 < \bar{\tau}_0 + 2m_0\pi + \delta_0.$$

We then get by means of (41), (39), and (44)

$$\begin{aligned}\tau_1 &= \tau_0 + 2\pi + \Big(p^s \cos\tau_0 - 4\varphi(+\infty)\Big)\lambda_0^{-1} + \lambda_0^{-1}\tilde{f}_2(\lambda_0,\tau_0) \\ &< \bar{\tau}_0 + 2(m_0+1)\pi + \delta_0 - \delta_1\lambda_0^{-1} + \frac{1}{2}\delta_1\lambda_0^{-1} < \bar{\tau}_0 + 2(m_0+1)\pi + \delta_0.\end{aligned} \tag{47}$$

Similarly,

$$\begin{aligned}\tau_1 &= \tau_0 + 2\pi + \Big(p^s \cos\tau_0 - 4\varphi(+\infty)\Big)\lambda_0^{-1} + \lambda_0^{-1}\tilde{f}_2(\lambda_0,\tau_0) \\ &> \bar{\tau}_0 + 2(m_0+1)\pi + \frac{1}{2}\delta_0 - \frac{1}{8}\delta_0 - \frac{1}{2}\delta_1 > \bar{\tau}_0 + 2(m_0+1)\pi + \frac{1}{8}\delta_0,\end{aligned} \tag{48}$$

and from (47) and (48) we conclude that (46) holds in this case. However, if $\tau_0 - \bar{\tau}_0 - 2m_0\pi \le 0$ and thus

$$\bar{\tau}_0 + 2m_0\pi - \delta_0 < \tau_0 < \bar{\tau}_0 + 2m_0\pi - \frac{1}{2}\delta_0,$$

then we may proceed analogously, utilizing the second estimate in (39). This completes the proof of (46), whence $P(\mathcal{D}) \subset \mathcal{D}$.

To verify that every solution starting in $\mathcal{D}$ is unbounded, let again $(\lambda_0,\tau_0) \in \mathcal{D}$, and denote $(\lambda_k,\tau_k) = P^k(\lambda_0,\tau_0)$, $k \ge 1$, its trajectory under P. According to (45) we have $\lambda_{k+1} \ge \lambda_k + \frac{1}{4}\varepsilon_0$, and therefore

$$\lambda_k \ge \lambda_0 + \frac{1}{4}\varepsilon_0 k \to \infty, \quad k \to \infty.$$

Consequently, for the solution starting in (x_0, x_0') corresponding to (λ_0,τ_0), we find

$$\sup_{t>0}\Big(|x(t)| + |x'(t)|\Big) = +\infty,$$

as was to be shown. $\square$

Finally we carry out the

Proof of Theorem 3 : The method is similar to [1, 6]. By rotating $\int_0^{2\pi} p(t)\, e^{int}\, dt \in \mathbb{C}$ to its absolute value, we can choose $\phi \in [0, 2\pi[$ such that

$$\int_0^{2\pi} p(t)\sin(nt+\phi)\, dt = \Big|\int_0^{2\pi} p(t)\, e^{int}\, dt\Big|.$$

We let

$$V(x,v) = v\sin\phi - nx\cos\phi + F(x)\sin\phi, \quad (x,v) \in \mathbb{R}^2,$$

and denote $(x(t),x'(t))=(x(t;x_0,v_0),x'(t;x_0,v_0))$ the solution to (1) with fixed data (x_0,v_0). Integration by parts twice reveals

$$\int_0^{2\pi}[x''+n^2x]\sin(nt+\phi)\,dt=[x'(2\pi)-x'(0)]\sin\phi-n[x(2\pi)-x(0)]\cos\phi.$$

On the other hand, by (1),

$$\begin{aligned}
&\int_0^{2\pi}[x''+n^2x]\sin(nt+\phi)\,dt\\
&=\int_0^{2\pi}[p(t)-\varphi(x)-f(x)x']\sin(nt+\phi)\,dt\\
&=\left|\int_0^{2\pi}p(t)\sin nt\,dt\right|-\int_0^{2\pi}\frac{d}{dt}F(x)\sin(nt+\phi)\,dt\\
&\quad-\int_{\{t\in[0,2\pi]:\sin(nt+\phi)\ge 0\}}\varphi(x)\sin(nt+\phi)\,dt-\int_{\{t\in[0,2\pi]:\sin(nt+\phi)<0\}}\varphi(x)\sin(nt+\phi)\,dt\\
&\ge\left|\int_0^{2\pi}p(t)\sin nt\,dt\right|-\Big[F(x(2\pi))-F(x(0))\Big]\sin\phi+n\int_0^{2\pi}F(x)\cos(nt+\phi)\,dt\\
&\quad-(\sup\varphi)\int_{\{t\in[0,2\pi]:\sin(nt+\phi)\ge 0\}}\sin(nt+\phi)\,dt-(\inf\varphi)\int_{\{t\in[0,2\pi]:\sin(nt+\phi)\ge 0\}}\sin(nt+\phi)\,dt\\
&=\left|\int_0^{2\pi}p(t)\sin nt\,dt\right|-2\Big(\sup\varphi-\inf\varphi\Big)\\
&\quad-\Big[F(x(2\pi))-F(x(0))\Big]\sin\phi+n\int_0^{2\pi}F(x)\cos(nt+\phi)\,dt.
\end{aligned}$$

Consequently, (6) implies

$$\begin{aligned}
&V(x(2\pi),x'(2\pi))-V(x(0),x'(0))\\
&=\left|\int_0^{2\pi}p(t)\sin nt\,dt\right|-2\Big(\sup\varphi-\inf\varphi\Big)+n\int_0^{2\pi}F(x)\cos(nt+\phi)\,dt\\
&\ge\left|\int_0^{2\pi}p(t)\sin nt\,dt\right|-2\Big(\sup\varphi-\inf\varphi\Big)-4n\,|F|_\infty=:\delta_0>0.
\end{aligned}$$

Iterating this argument and adding the resulting estimates, we see

$$V(x(2k\pi),x'(2k\pi))-V(x(0),x'(0))\ge\delta_0 k,\quad k\in\mathbb{N},$$

and this shows that the solution x is unbounded in the (x,x')-phase plane, since V maps bounded subsets of $\mathbb{R}^2$ onto bounded sets. $\Box$

5 Appendix: Reversible systems

We recall some definitions and facts related to reversible systems; see [37] and [24, Section 2] for more information and the proofs.

Definition 2 *Let $\Omega\subset\mathbb{R}^n$ be open, and $Z=Z(z,t):\Omega\times\mathbb{R}\to\mathbb{R}^n$ be continuous and 2π-periodic in the time variable t. Moreover, suppose $G:\mathbb{R}^n\to\mathbb{R}^n$ is an involution (i.e., G is a C^1-diffeomorphism such that $G^2=\mathrm{id}_{\mathbb{R}^n}$) satisfying $G(\Omega)=\Omega$. The system*

$$z'=Z(z,t)\tag{49}$$

is called reversible w.r. to G, if

$$DG(Gz)\,Z(Gz,-t) = -Z(z,t), \quad z\in\Omega, \quad t\in\mathbb{R}, \tag{50}$$

with DG denoting the Jacobian of G.

We are mostly interested in the special involution $G(x,y)=(-x,y)$ and $\bar{G}(x,y)=(x,-y)$ for $z=(x,y)\in\mathbb{R}^2$. Then condition (50) reads as

$$\text{for } G: \quad \begin{cases} Z_1(-x,y,-t) = Z_1(x,y,t) \\ Z_2(-x,y,-t) = -Z_2(x,y,t) \end{cases}, \quad \text{for } \bar{G}: \quad \begin{cases} Z_1(x,-y,-t) = -Z_1(x,y,t) \\ Z_2(x,-y,-t) = Z_2(x,y,t) \end{cases},$$

if we write $Z=(Z_1,Z_2)$.

The next definition and lemma is useful to show that a transformed version of a reversible system is again reversible.

Definition 3 *Let $T(\cdot,t)$ be an invertible transformation of Ω for every fixed t, and suppose G is an involution with $G(\Omega)=\Omega$. Then T is called G-invariant, if*

$$G(T(z,t)) = T(Gz,-t), \quad z\in\Omega, \quad t\in\mathbb{R}. \tag{51}$$

Returning to our special example $\bar{G}(x,y)=(x,-y)$ from above, (51) requires

$$\begin{cases} T_1(x,-y,-t) = T_1(x,y,t) \\ T_2(x,-y,-t) = -T_2(x,y,t) \end{cases}, \tag{52}$$

where $T=(T_1,T_2)$.

Lemma 6 *Assume that (49) is reversible w.r. to G. If a transformation $T(\cdot,t):\Omega\to\mathbb{R}^n$ is G-invariant, then the transformed system, i.e., the system satisfied by $\bar{z}(t)=T(z(t),t)$, is as well reversible w.r. to G.*

Finally, reversible systems will also lead to reversible Poincaré maps, in the following sense.

Definition 4 *Let $f:\Omega\to\mathbb{R}^n$ be a homeomorphism onto its image, and let G be an involution. Then f is called reversible w.r. to G on a set $\mathcal{D}\subset\Omega\cap f(\Omega)$ with $G(\mathcal{D})=\mathcal{D}$, if*

$$f^{-1} = G\circ f\circ G \quad in \quad \mathcal{D}.$$

Lemma 7 *Suppose the Cauchy problems associated with (49) are uniquely solvable, and let the time-2π-map P and its inverse P^{-1} be defined. Then P is reversible w.r. to G, provided that (49) is reversible w.r. to G.*

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