Bifurcation from the essential spectrum without sign condition on the nonlinearity

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Abstract

We study a problem from nonlinear optics that leads to an integro-differential equation which can be written as the abstract bifurcation problem $Au = \lambda Lu + \nabla\varphi(u)$. For such type of equation a general theorem on the bifurcation of solutions from the essential spectrum is proved. The hypothesis that φ be nonnegative (as is usually assumed) could be omitted and replaced by other conditions that allow to utilize the results for the particular example. We also indicate applications to semilinear elliptic equations.

1 Introduction and Main Results

In modern optical data transmission devices, often non-linear methods are used with the intention of compensating loss in the fiber by means of suitable non-linear effects. The transmission of such an optical signal can be modeled by the nonlinear Schrödinger equation

$$iA_z + d(z)A_{tt} + c(z)|A|^2 A = 0, \tag{1.1}$$

cf. [8, 10, 16, 17]. In (1.1) the complex-valued function $A = A(z,t)$ describes the envelope of the original electrical field, $z \in \mathbb{R}$ is the longitudinal coordinate along the fiber, and t denotes time. The dispersion $d(z) = \tilde{d}(z) + \langle d\rangle$ is periodic with mean value $\langle d\rangle$ over one compensation period, whereas the function $c(z)$ accounts for signal power oscillations. In case of a "lossless" model this function is well-approximated by the choice $c(z) \equiv 1$, as we will assume in the sequel. An important feature of the dispersion $d(z)$ (and hence also of the "local dispersion" $\tilde{d}(z)$) is that it is rapidly varying, thus resulting in rapid oscillations of e.g. the signal pulse width. However, there is an additional (slow) scale in the problem on which the solitary pulses just propagate according to some suitably averaged equation. Several papers derive and take advantage of such an averaged equation, see e.g. [13, 14, 15, 4, 5] and the references therein. For our purposes we follows [4, 5] and introduce the ansatz

$$A(z,t) = \int_{\mathbb{R}} q(z,\omega)\exp[-i\omega t - i\omega^2 R(z)]\,d\omega$$

with a new unknown function $q = q(z,\omega)$, where $R(z)$ is chosen such that $dR/dz = \tilde{d}$ and $\langle R\rangle = 0$. This transformation eliminates the rapidly oscillating part $\tilde{d}(z)$ of $d(z)$, and a formal averaging of the resulting equation for q yields, after some lengthy manipulation, the integro-differential equation

$$\begin{aligned} iq_z(z,\omega) &= \omega^2\langle d\rangle q(z,\omega) \\ &\quad - \int_{\mathbb{R}}\int_{\mathbb{R}} d\omega_1 d\omega_2\, \frac{\sin[\mu(\omega-\omega_1)(\omega-\omega_2)]}{\mu(\omega-\omega_1)(\omega-\omega_2)}\, q(z,\omega_1)q(z,\omega_2)q^*(z,\omega_1+\omega_2-\omega) \end{aligned} \tag{1.2}$$

for the unknown $q(z,\omega)$ in the spectral domain, cf. [4, 5]. Here q^* denotes the complex conjugate of q, and the characteristic length parameter $\mu > 0$ is related to the dispersion function $d(z)$; in particular, the case $\mu = 0$ corresponds to d being constant. This dispersion function $d(z)$ has to be taken a step function to make (1.2) a reasonable approximation; see [15]. Observe that for $\mu = 0$, (1.2) is just the nonlinear Schrödinger equation $iv_z = -\langle d\rangle v_{tt} - |v|^2 v$, where $\hat{v}(z,\omega) = q(z,\omega)$ is the Fourier transform of $v = v(z,t)$ w.r. to t. Making in (1.2) the ansatz $q(z,\omega) = u(\omega)e^{ikz}$ with $k \in \mathbb{R}$ and a real-valued "ground state" u, we finally arrive at the nonlinear eigenvalue problem

$$\Big(\omega^2\langle d\rangle + k\Big)u(\omega) = \int_{\mathbb{R}}\int_{\mathbb{R}} d\omega_1 d\omega_2\, \frac{\sin[\mu(\omega-\omega_1)(\omega-\omega_2)]}{\mu(\omega-\omega_1)(\omega-\omega_2)}\, u(\omega_1)u(\omega_2)u(\omega_1+\omega_2-\omega). \tag{1.3}$$

It is the aim of the present paper to investigate the solvability of (1.3), cf. also the corresponding discussion of a related problem in physical space (instead of Fourier space) that is given in [19, 7]. For simplicity we take $\langle d\rangle = 1$, set $\lambda = 1-k$, and denote the independent variable by x instead of ω. Then (1.3) becomes

$$(1+x^2)u(x) = \lambda u(x) + \Phi(u)(x), \tag{1.4}$$

with

$$\Phi(u)(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} dx_1 dx_2\, \frac{\sin[\mu(x-x_1)(x-x_2)]}{\mu(x-x_1)(x-x_2)}\, u(x_1)u(x_2)u(x_1+x_2-x), \quad x \in \mathbb{R}. \tag{1.5}$$

Dividing by $(1+x^2)$ in (1.4), it will be seen below that the resulting nonlinearity is a potential operator (in a suitable space of functions) with potential

$$\begin{aligned} \varphi(u) &= \frac{1}{4}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} dx_1 dx_2 dx_3 dx_4\, \frac{\sin[(\mu/2)(x_1^2+x_2^2-x_3^2-x_4^2)]}{(\mu/2)(x_1^2+x_2^2-x_3^2-x_4^2)} \\ &\qquad \times \delta(x_1+x_2-x_3-x_4)u(x_1)u(x_2)u(x_3)u(x_4). \end{aligned} \tag{1.6}$$

Thus (1.4) can be rewritten in the abstract (weak) form

$$u = \lambda L u + \nabla\varphi(u), \quad (\lambda, u) \in \mathbb{R}\times H \tag{1.7}$$

in some Hilbert space H, with a bounded linear operator L. There is a large literature on finding nontrivial solutions of problems of type (1.7), see the survey paper [12]. Since the self-adjoint (multiplication) operator $Su = (1 + x^2)u$ on $L^2(\mathrm{I\!R})$ in (1.4) has essential spectrum $\sigma_e(S) = [1, \infty[= \sigma(S)$ (spectrum of S), a natural attempt is to look for solutions when $\lambda < 1$ and to decide whether $\lambda = 1$, the lowest point of the spectrum, is a bifurcation point of nontrivial solutions. However, the results known thus far for bifurcation from the lowest point of the spectrum mainly deal with the situation that $\varphi(u) \geq 0$ for $u \in H$, i.e., a nonnegative potential. For the particular problem introduced above, with potential given by (1.6), it was nevertheless quite unclear that this assumption be satisfied. Therefore the demand arose to relax the hypotheses on φ accordingly, and surprisingly this could be achieved in sufficient generality to easily cover (1.7) without determining whether $\varphi(u) \geq 0$ for all u is indeed valid. The corresponding general theorem will be formulated and proved in Section 2, whereas Section 3 contains the application to the concrete problem. Thus we have shown that $k = 0$ is a bifurcation point to the right for nontrivial solutions of (1.3), in agreement with known numerical results. Moreover, in Section 3 we will also verify that for every $k > 0$ in fact a corresponding nontrivial solution does exist.

It later turned out, however, that $\varphi(u) \geq 0$ for all u holds for the particular potential φ from (1.6), cf. Section 3.4. Nevertheless, since to validate this assertion it is necessary to be quite familiar with the dispersion-managed optical fiber problems, although in the end our abstract result does not give something new here, it can at least be considered a welcome simplification.

A further important example to which general bifurcation results can be applied are semilinear elliptic equations of the form

$$-\Delta u + V(x)u = \lambda u + a(x)|u|^{p-2}u, \quad x \in \mathrm{I\!R}^N, \tag{1.8}$$

for certain potentials V such that the linearized equation $-\Delta u + V(x)u = 0$ does have only essential spectrum. We then explain in Section 4, utilizing the abstract result from Section 2, how the usual assumption $a(x) \geq 0$, $x \in \mathrm{I\!R}^N$ (guaranteeing the non-negativity of the corresponding potential) can be modified to cover as well some classes of sign-changing functions a. It should be noted that several results on bifurcation from the essential spectrum without sign-condition on a have already been obtained in the one-dimensional case with $V = 0$, i.e.,

$$-u'' = \lambda u + a(x)|u|^{p-2}u, \quad x \in \mathrm{I\!R}, \quad \lim_{|x|\to\infty} u(x) = 0,$$

under various different sets of assumptions on a and p, see e.g. [1], improving on the earlier [9, 11].

2 An abstract bifurcation theorem

We discuss existence of solutions to the equation

$$Au = \lambda Lu + N(u), \quad (\lambda, u) \in \mathrm{I\!R} \times H, \tag{2.1}$$

in a Hilbert space H whose norm and inner product are denoted by $|\cdot|$ and $\langle \cdot, \cdot \rangle$, respectively. We assume $u = 0$ is a solution for all $\lambda \in \mathrm{I\!R}$, and we give conditions that $(\Lambda, 0) \in \mathrm{I\!R} \times H$, with a certain $\Lambda \in \mathrm{I\!R}$, is a bifurcation point from this trivial solution. By definition this means that there is a sequence $(\lambda_n, u_n) \in \mathbb{R} \times H$ of solutions to (2.1) such that $u_n \neq 0$, $\lambda_n \to \Lambda$, and $u_n \to 0$ in H. We call $(\Lambda, 0)$ a bifurcation point to the left in case that $\lambda_n < \Lambda$ for $n \in \mathbb{N}$. Moreover we denote $B(H)$ the bounded linear operators on H, and I stands for the identity on H.

Our hypotheses are as follows.

(H1) $A, L \in B(H)$, $A^* = A$, $L^* = L$, and $\langle Lu, u\rangle > 0$ for $u \neq 0$.

(H2) The "nonlinearity" $N : H \to H$ satisfies $N(u) = o(|u|)$ as $|u| \to 0$, and there exists $\varphi \in C^1(H, \mathbb{R})$ such that $\nabla\varphi = N$, that is, $\nabla\varphi(u)v = \langle N(u), v\rangle$ for $u, v \in H$. Without loss of generality we assume $\varphi(0) = 0$.

(H3) There exist $\lambda_0 \in \mathbb{R}$ and $\nu > 0$ such that $A - \lambda_0 L \geq \nu I$, i.e., $\langle (A - \lambda_0 L)u, u\rangle \geq \nu |u|^2$ for $u \in H$.

(H4) φ is weakly sequentially continuous.

(H5) If $\varphi(u) > 0$ and $t \geq 1$, then $\varphi(tu) \geq t^2\varphi(u)$.

(H6) Let

$$\Lambda = \inf\left\{ \frac{\langle Au, u\rangle}{\langle Lu, u\rangle} : u \in H, u \neq 0\right\}.$$

Then $\Lambda \geq \lambda_0$ by $(H3)$, and we assume the following.

(H6a) There exists $r_0 > 0$ such that $\mathcal{M}(r) < \frac{1}{2}\Lambda r^2$ for $r \in]0, r_0]$, where

$$\mathcal{M}(r) = \inf\{J(u) : u \in S(r)\}, \quad \text{with} \quad S(r) = \{u \in H : \langle Lu, u\rangle = r^2\}$$

and $J(u) = \frac{1}{2}\langle Au, u\rangle - \varphi(u)$.

(H6b) There exist $C > 0$, $n \in \mathbb{N}$, and $a_i \in [0, 1[$, $b_i \geq 1 - a_i$, $1 \leq i \leq n$, such that for $r \in]0, r_0]$

$$\varphi(u) \leq C \sum_{i=1}^{n} \langle (A - \Lambda L)u, u\rangle^{a_i} \langle Lu, u\rangle^{b_i} = C \sum_{i=1}^{n} \langle (A - \Lambda L)u, u\rangle^{a_i} r^{2b_i}, \quad u \in S(r).$$

Now we are ready to state

Theorem 2.1 *Let conditions $(H1) - (H6)$ hold. Then $(\Lambda, 0)$ is a bifurcation point to the left. More precisely, for every $r \in]0, r_0]$ there exist a solution $(\lambda_r, u_r) \in \mathbb{R} \times H$ of (2.1) such that $u_r \in S(r)$ is a minimizer of J, that is, $J(u_r) = \mathcal{M}(r)$. Moreover, $0 < |u_r| \leq Cr$ and $0 < \Lambda - \lambda_r \leq C|u_r|^{-1}|N(u_r)|$.*

The assumptions and the theorem are much the same as in [12, Thm. 11], up to one important difference: We did not assume $\varphi(u)$ to be nonnegative for all $u \in H$; cf. [12, (K), p. 413]. As noted in the introduction, this hypothesis would be in question for the particular example we have in mind. One motivation to have $\varphi(u) \geq 0$ is the following. Taking $\varphi = 0$ (corresponding to the linearized problem $Au = \lambda Lu$), we find $\mathcal{M}_{\varphi=0}(r) = \frac{1}{2}\Lambda r^2$, and thus $\mathcal{M}(r) \leq \frac{1}{2}\Lambda r^2$ for φ with $\varphi(u) \geq 0$. Whence the assumption $\mathcal{M}(r) < \frac{1}{2}\Lambda r^2$ comes naturally; see (b) in [12, Thm. 11]. However, in concrete examples one might often have the chance to verify directly the key estimate $\mathcal{M}(r) < \frac{1}{2}\Lambda r^2$ for small $r > 0$, since this only amounts to finding some particular $u_0 \in S(r)$ with the property $\frac{1}{2}\langle Au_0, u_0\rangle < \frac{1}{2}\Lambda r^2 + \varphi(u_0)$. We also note that [12, (K), p. 413] contained the condition $\langle N(u), u\rangle \geq 2\varphi(u)$ which (for general φ) is equivalent to $\varphi(tu) \geq t^2\varphi(u)$ for $t \geq 1$. This could be relaxed to $(H5)$, as is particularly satisfied for φ being homogeneous of degree $k \geq 2$, without a condition on the sign of φ.

Proof of Theorem 2.1: The proof is along standard lines, see [12, Thm. 11], but nevertheless we include some details to explain where and how the modified assumptions enter. First we need to show that J is bounded below on $S(r)$, for every $r \in]0, r_0]$. We introduce the positive, linear, and selfadjoint $\tilde{A} = A - \Lambda L$ and estimate

$$J(u) \geq \frac{1}{2}\langle \tilde{A}u, u\rangle + \frac{1}{2}\Lambda r^2 - C\sum_{i=1}^{n} \langle \tilde{A}u, u\rangle^{a_i} r^{2b_i}$$

for $u \in S(r)$ by $(H6b)$. Due to $a_i \in [0, 1[$, the map $z \mapsto \frac{1}{2}z - C\sum_{i=1}^{n} z^{a_i} r^{2b_i}$ is bounded below for $z \in [0, \infty[$, and hence $\mathcal{M}(r)$ is finite. Next we show that the infimum is attained. Let $(u_n) \subset S(r)$ be a minimizing sequence, that is, $J(u_n) \to \mathcal{M}(r)$. Set

$$\tilde{J}(u) = \frac{1}{2}\langle \tilde{A}u, u\rangle - \varphi(u) = J(u) - \frac{1}{2}\Lambda\langle Lu, u\rangle, \quad \tilde{\mathcal{M}}(r) = \inf\{\tilde{J}(u) : u \in S(r)\} = \mathcal{M}(r) - \frac{1}{2}\Lambda r^2.$$

By $(H6a)$, and since $u_n \in S(r)$ and $J(u_n) \to \mathcal{M}(r) < \frac{1}{2}\Lambda r^2$, we may assume that $\tilde{J}(u_n) \leq 0$ for all $n \in \mathbb{N}$. Next note that

$$u \in S(r),\ \tilde{J}(u) \leq 0 \quad \Longrightarrow \quad \nu|u|^2 \leq C\sum_{i=1}^{n} r^{2b_i/(1-a_i)} + (\Lambda - \lambda_0)r^2. \tag{2.2}$$

To see this, by $(H3)$ for $u \in S(r)$ we have that $\nu|u|^2 \leq \langle (A - \lambda_0 L)u, u\rangle = \langle \tilde{A}u, u\rangle + (\Lambda - \lambda_0)r^2$, and moreover in case $\tilde{J}(u) \leq 0$ by $(H6b)$

$$\frac{1}{2}\langle \tilde{A}u, u\rangle \leq \varphi(u) \leq C\sum_{i=1}^{n} \langle \tilde{A}u, u\rangle^{a_i} r^{2b_i}.$$

Hence for some index $1 \leq i_0 \leq n$ we must have $1/(2Cn) \leq \langle \tilde{A}u, u\rangle^{-(1-a_{i_0})} r^{2b_{i_0}}$, and this proves (2.2). Since $\tilde{J}(u_n) \leq 0$, in particular (u_n) is bounded by (2.2), and therefore without loss of generality $u_n \rightharpoonup u_*$ (weak convergence) in H, for some $u_* \in H$. Then

$$J(u_n) = \frac{1}{2}\langle Au_n, u_n\rangle - \varphi(u_n) \geq \frac{1}{2}\Lambda\langle Lu_n, u_n\rangle - \varphi(u_n) = \frac{1}{2}\Lambda r^2 - \varphi(u_n)$$

together with $(H4)$ and $(H6a)$ yields $\frac{1}{2}\Lambda r^2 > \mathcal{M}(r) \geq \frac{1}{2}\Lambda r^2 - \varphi(u_*)$, and therefore $\varphi(u_*) > 0$. Consequently, $u_* \neq 0$ by $(H2)$. In general, if $P \in B(H)$ is nonnegative and self-adjoint, then the Cauchy-Schwarz inequality shows that $u \mapsto \langle Pu, u\rangle$ is convex, and hence lower semicontinuous for the weak topology as it is continuous; see [6, Thm. 4, p. 14]. This remark applies to both $\tilde{A}$ and L, and hence firstly $0 < \langle Lu_*, u_*\rangle \leq \liminf_{n\to\infty}\langle Lu_n, u_n\rangle = r^2$. Thus $t = \dfrac{r}{\langle Lu_*, u_*\rangle^{1/2}} \geq 1$ and $tu_* \in S(r)$. Secondly, by $(H4)$,

$$\tilde{J}(u_*) \leq \liminf_{n\to\infty} \tilde{J}(u_n) = \liminf_{n\to\infty}(J(u_n) - \frac{1}{2}\Lambda r^2) = \mathcal{M}(r) - \frac{1}{2}\Lambda r^2 = \tilde{\mathcal{M}}(r).$$

As we want to apply $(H5)$ with u_* and $t \geq 1$, it is now clear that this assumption is needed only for u with $\varphi(u) > 0$. It follows that

$$\tilde{\mathcal{M}}(r) \leq \tilde{J}(tu_*) = \frac{1}{2}t^2\langle \tilde{A}u_*, u_*\rangle - \varphi(tu_*) \leq t^2\left(\frac{1}{2}\langle \tilde{A}u_*, u_*\rangle - \varphi(u_*)\right) = t^2\tilde{J}(u_*) \leq t^2\tilde{\mathcal{M}}(r). \tag{2.3}$$

Because $\tilde{\mathcal{M}}(r) = \mathcal{M}(r) - \frac{1}{2}\Lambda r^2 < 0$ we obtain that $t = 1$, hence $u_* \in S(r)$, and thus by (2.3), $\mathcal{M}(r) = \tilde{\mathcal{M}}(r) + \frac{1}{2}\Lambda r^2 = \tilde{J}(u_*) + \frac{1}{2}\Lambda r^2 = J(u_*)$. Thus u_* is a minimizer. Since this argument applies to any $r \in]0, r_0]$, for each such r we find $\lambda_r \in \mathbb{R}$ with

$$Au_r = \lambda_r Lu_r + N(u_r), \tag{2.4}$$

by the Lagrange multiplier rule. Therefore (λ_r, u_r) is a solution to (2.1), and also $u_r \neq 0$ as a consequence of $\langle Lu_r, u_r\rangle = r^2$. Moreover, the construction of u_r shows $\varphi(u_r) > 0$, $J(u_r) = \mathcal{M}(r)$, and $\tilde{J}(u_r) \leq \tilde{\mathcal{M}}(r) < 0$. Thus $|u_r| \leq Cr$ by (2.2), since $2b_i/(1-a_i) \geq 2$. From (2.4) we obtain

$$\Lambda r^2 = \Lambda\langle Lu_r, u_r\rangle \leq \langle Au_r, u_r\rangle = \lambda_r r^2 + \langle N(u_r), u_r\rangle,$$

and therefore $(\Lambda - \lambda_r)r^2 \leq \langle N(u_r), u_r\rangle$. By $(H5)$, $\varphi(u_r + \varepsilon u_r) - \varphi(u_r) \geq (2\varepsilon + \varepsilon^2)\varphi(u_r)$ for $\varepsilon > 0$. Dividing by ε and letting ε tend to zero we conclude $\langle N(u_r), u_r\rangle = \nabla\varphi(u_r)u_r \geq 2\varphi(u_r)$. From $(H6a)$ and (2.4) it then follows that

$$\frac{1}{2}\Lambda r^2 > \mathcal{M}(r) = J(u_r) = \frac{1}{2}\lambda_r r^2 + \frac{1}{2}\langle N(u_r), u_r\rangle - \varphi(u_r) \geq \frac{1}{2}\lambda_r r^2.$$

Therefore $\lambda_r < \Lambda$, and $(\Lambda - \lambda_r)|u_r|^2 \leq C^2(\Lambda - \lambda_r)r^2 \leq C^2\langle N(u_r), u_r\rangle$ finally yields $\Lambda - \lambda_r \leq C|u_r|^{-1}|N(u_r)| \to 0$ as $r \to 0^+$. □

3 Existence of ground states

In this section we shall consider (1.3) from the introduction, in the form (1.4). We wish to prove the existence of solutions to (1.4) for all $\lambda < 1$, corresponding to $k = 1 - \lambda > 0$ for the original problem (1.3). Furthermore we want to show that $\lambda = 1$ is a bifurcation point to the left from the trivial solution $u = 0$ in (1.4). The restriction to $\lambda < 1$ comes from the fact that the multiplication operator $(Su)(x) = (1 + x^2)u(x)$ on $L^2(\mathbb{R})$ has purely continuous spectrum $[1, \infty[$; see [18]. We do not know whether there are solutions over the continuous spectrum, but the numerical observations e.g. in [15] suggest that this is not the case. Note also that $S : D(S) = \{u \in L^2(\mathbb{R}) : (1+x^2)u \in L^2(\mathbb{R})\} \to L^2(\mathbb{R})$ is non-negative, unbounded, and self-adjoint. To deal with such problems, a weak solution approach often is most convenient. Following the reduction in [12, Sect. 3], see also the references therein, (1.4) should be considered on the "form domain" $D(S^{1/2})$ which is here, up to equivalence of norms,

$$L_1^2 = L_1^2(\mathbb{R}) = \left\{u \in L^2 : |u|_{L_1^2}^2 = \int (1 + x^2)u^2(x)\,dx < \infty\right\}.$$

When no domain of integration is indicated, it is always understood that the integral is taken over $\mathbb{R}$, and we often abbreviate $L^p = L^p(\mathbb{R})$. The space $H = L_1^2$ is a Hilbert space with inner product $\langle u,v\rangle = \int(1+x^2)u(x)v(x)\,dx$, and L^2_1 is isomorphic to $W^{1,2}(\mathbb{R})$ through Fourier transform. The weak formulation of (1.4) then is

$$u = \lambda Lu + N(u), \quad (Lu)(x) = \frac{1}{1+x^2}\,u(x) \quad \text{and} \quad N(u)(x) = \frac{1}{1+x^2}\,\Phi(u)(x). \tag{3.1}$$

In Section 3.2 we will prove that (3.1) has a solution for all $\lambda < 1$. Then Section 3.3 contains the application of Theorem 2.1 to show that $\lambda = 1$ is a bifurcation point from the left. We start with a preliminary section containing some technical results concerning L^2_1 and the properties of Φ and N; in particular it will be seen that N is a potential operator.

3.1 Some technical preliminaries

Lemma 3.1 *We have the embedding $L^2_1 \subset L^p$ for $p \in]\frac{2}{3}, 2]$.*

Proof: Fix $p \in]\frac{2}{3}, 2[$. By Hölder's inequality with exponents $(2/p)$ and $2/(2-p)$,

$$|u|^p_{L^p} = \int \frac{(1+|x|)^p}{(1+|x|)^p}|u(x)|^p\,dx \le \Big(\int \frac{dx}{(1+|x|)^{2p/(2-p)}}\Big)^{1-p/2}\,\Big(\int (1+|x|)^2 u^2(x)\,dx\Big)^{p/2}.$$

Since $2p/(2-p) > 1$, this gives the claim. $\square$

Next we show that the best "Sobolev constant" of the embedding $L^2_1 \subset L_2$ equals unity, which is the lowest point of the essential spectrum of S.

Lemma 3.2 *Let*

$$\Lambda = \inf\Big\{\frac{|u|_{L^2_1}}{|u|_{L^2}} : u \in L^2_1, u \neq 0\Big\}.$$

Then $\Lambda = 1$.

Proof: Clearly $\Lambda \ge 1$. On the other hand, for $u = \mathbf{1}_{[-\varepsilon,\varepsilon]}$ we find $\Lambda \le \frac{|u|_{L^2_1}}{|u|_{L^2}} = \sqrt{1+\frac{1}{3}\varepsilon^2} \to 1$, $\varepsilon \to 0$. $\square$

Lemma 3.3 *The interpolation inequality*

$$|u|_{L^1} \le C|u|^{1/2}_{L^2}|u|^{1/2}_{L^2_1}, \quad u \in L^2_1,$$

holds, with $C = 2\sqrt{2}$.

Proof: Let $M = |u|_{L^2_1}/|u|_{L^2}$. Then by Hölder's inequality

$$\begin{aligned} |u|_{L^1} &= \int_{-M}^{M}|u(x)|\,dx + \int_{|x|>M}\frac{(1+x^2)^{1/2}}{(1+x^2)^{1/2}}|u(x)|\,dx \\ &\le (2M)^{1/2}|u|_{L^2} + \Big(\int_{|x|>M}\frac{dx}{1+x^2}\Big)^{1/2}|u|_{L^2_1} \le (2M)^{1/2}|u|_{L^2} + (2/M)^{1/2}|u|_{L^2_1}. \end{aligned}$$

This completes the proof. $\square$

Next we are going to show that φ from (1.6) is a potential for N. Recall from (3.1) that

$$N(u)(x) = \frac{1}{1+x^2}\,\Phi(u)(x), \quad \text{hence} \quad \langle Nu, v\rangle = \int \Phi(u)(x)v(x)\,dx, \quad u,v \in L^2_1, \tag{3.2}$$

with Φ from (1.5).

Lemma 3.4 *We have that $\varphi \in C^1(L^2_1, \mathbb{R})$, with*

$$\nabla\varphi(u)v = \int \Phi(u)(x)v(x)\,dx = \langle N(u), v\rangle, \quad u, v \in L^2_1.$$

Proof: An elementary calculation shows

$$\begin{aligned}
&\int \Phi(u)(x)v(x)\,dx \\
&= \int\!\!\int\!\!\int dx_1dx_2dx_3 \,\frac{\sin[\mu(x_3-x_1)(x_3-x_2))]}{\mu(x_3-x_1)(x_3-x_2)}\, u(x_1)u(x_2)u(x_1+x_2-x_3)v(x_3) \\
&= \frac{1}{4}\Big(\int\!\!\int\!\!\int\!\!\int dx_1dx_2dx_3dx_4\, \frac{\sin[(\mu/2)(x_1^2+x_2^2-x_3^2-x_4^2)]}{(\mu/2)(x_1^2+x_2^2-x_3^2-x_4^2)}\,\delta(x_1+x_2-x_3-x_4) \\
&\qquad \times \Big[v(x_1)u(x_2)u(x_3)u(x_4) + u(x_1)v(x_2)u(x_3)u(x_4) \\
&\qquad\quad + u(x_1)u(x_2)v(x_3)u(x_4) + u(x_1)u(x_2)u(x_3)v(x_4)\Big]\Big)
\end{aligned}$$

since all four terms equal $\int \Phi(u)(x)v(x)\,dx$. A straight-forward estimate then yields for fixed $u \in L^2_1$

$$\begin{aligned}
\left|\varphi(u+v) - \varphi(u) - \int \Phi(u)(x)v(x)\,dx\right| &\le C\Big(|u|^2_{L^1}|v|^2_{L^1} + |u|_{L^1}|v|^3_{L^1} + |v|^4_{L^1}\Big) \\
&\le C(u)(1+|v|_{L^2_1})^2|v|^2_{L^2_1}
\end{aligned}$$

by Lemma 3.1. Hence $\nabla\varphi(u)v = \int \Phi(u)(x)v(x)\,dx = \langle N(u), v\rangle$. To show that the derivative $\nabla\varphi : L^2_1 \to B(L^2_1)$ is continuous, we note that for $u, \bar{u}, v \in L^2_1$

$$\begin{aligned}
|\nabla\varphi(u)v - \nabla\varphi(\bar{u})v| &\le 2|u|_{L^1}|u|_{L^2}|u-\bar{u}|_{L^1}|v|_{L^2} + |u|^2_{L^1}|u-\bar{u}|_{L^2}|v|_{L^2} \\
&\le C|u|^2_{L^2_1}|u-\bar{u}|_{L^2_1}|v|_{L^2_1}.
\end{aligned}$$

Hence $|\nabla\varphi(u) - \nabla\varphi(\bar{u})| \le C|u|^2_{L^2_1}|u-\bar{u}|_{L^2_1}$, that is, $\nabla\varphi$ even is locally Lipschitz. □

Remark 3.5 For later reference we note that $\varphi(tu) = t^4\varphi(u)$ for $t \in \mathbb{R}$ and $u \in L^1_2$, and therefore also $\nabla\varphi(u)u = 4\varphi(u)$.

The next lemma discusses the behaviour of Φ and φ, respectively, w.r. to weak convergence.

Lemma 3.6 *If $u_n \rightharpoonup u$ in L^2_1 then $\Phi(u_n) \rightharpoonup \Phi(u)$ in L^2 and $\varphi(u_n) \to \varphi(u)$.*

Proof: For the first claim let $v \in L^2$ and define

$$w_n(x_1,x_2) = \int dx\, \frac{\sin[\mu(x-x_1)(x-x_2)]}{\mu(x-x_1)(x-x_2)}\, u_n(x_1+x_2-x)v(x), \quad (x_1,x_2) \in \mathbb{R}^2.$$

Since $|\sin(z)/z| \le 1$, we obtain $|w_n(x_1,x_2)| \le |u_n|_{L^2}|v|_{L^2} \le C|v|_{L^2}$, and moreover for

$$\phi(x) = \frac{\sin[\mu(x_2-x)(x_1-x)]}{\mu(x_2-x)(x_1-x)} v(x_1+x_2-x)$$

we have $\phi \in L^2$. As in particular $u_n \rightharpoonup u$ in L^2 we obtain $w_n(x_1,x_2) \to w(x_1,x_2)$ a.e. in $\mathbb{R}^2$, with w defined analogously to w_n, having u in place of u_n. Let

$$\tilde{w}_n(x_1,x_2) = \frac{w_n(x_1,x_2)}{(1+x_1^2)^{1/2}(1+x_2^2)^{1/2}} \quad \text{and} \quad \tilde{w}(x_1,x_2) = \frac{w(x_1,x_2)}{(1+x_1^2)^{1/2}(1+x_2^2)^{1/2}}.$$

Then $\tilde{w}_n(x_1,x_2) \to \tilde{w}(x_1,x_2)$ a.e. in $\mathbb{R}^2$ and

$$|\tilde{w}_n(x_1,x_2) - \tilde{w}(x_1,x_2)|^2 \le C(1+x_1^2)^{-1}(1+x_2^2)^{-1} \in L^1(\mathbb{R}^2).$$

Hence the dominated convergence theorem yields $\tilde{w}_n \to \tilde{w}$ strongly in $L^2(\mathbb{R}^2)$. Define $\tilde{u}_n(x) = (1+x^2)^{1/2}u_n(x)$ and $\tilde{u}(x) = (1+x^2)^{1/2}u(x)$, and we also introduce $\tilde{U}_n(x_1,x_2) = \tilde{u}_n(x_1)\tilde{u}_n(x_2)$ as well as $\tilde{U}(x_1,x_2) = \tilde{u}(x_1)\tilde{u}(x_2)$. Then $u_n \rightharpoonup u$ in L_1^2 implies $\tilde{u}_n \rightharpoonup \tilde{u}$ in L^2, and this in turn shows $\tilde{U}_n \rightharpoonup \tilde{U}$ in $L^2(\mathbb{R}^2)$. To verify the last statement, it is enough to remark that the convergence only has to be tested on functions ϕ in $\mathcal{D}(\mathbb{R}^2)$. As finite linear combinations of separating functions of type $\phi(x_1,x_2) = \phi_1(x_1)\phi_2(x_2)$, $\phi_1,\phi_2 \in \mathcal{D}(\mathbb{R})$, are dense in $\mathcal{D}(\mathbb{R}^2)$, we indeed obtain $\tilde{U}_n \rightharpoonup \tilde{U}$ in $L^2(\mathbb{R}^2)$. But then

$$\begin{aligned} \int \Phi(u_n)(x)v(x)\,dx &= \int\int dx_1dx_2\, w_n(x_1,x_2)u_n(x_1)u_n(x_2) \\ &= \int\int dx_1dx_2\, \tilde{w}_n(x_1,x_2)\tilde{U}_n(x_1,x_2) \\ &\to \int\int dx_1dx_2\, \tilde{w}(x_1,x_2)\tilde{U}(x_1,x_2) = \int \Phi(u)(x)v(x)\,dx \end{aligned}$$

as $n \to \infty$. The second claim $\varphi(u_n) \to \varphi(u)$ is verified similarly. □

We also need an estimate on φ.

Lemma 3.7 *We have that*

$$|\varphi(u)| \le C|u|_{L^2}^3 |u|_{L_1^2} \le C|u|_{L_1^2}^4, \quad u \in L_1^2.$$

Proof: By definition of φ in (1.6) we obtain

$$|\varphi(u)| \le \frac{1}{4}\int\int\int dx_1dx_2dx_3\, |u(x_1)||u(x_2)||u(x_1+x_2-x_3)| \le \frac{1}{4}|u|_{L^1}^2|u|_{L^2}^2,$$

hence Lemma 3.3 gives the first estimate, whereas the second follows from Lemma 3.1. □

Next we show that the nonlinearity N in (3.1) is of higher order near $u=0$.

Lemma 3.8 *We have that*

$$|N(u)|_{L_1^2} \le C|u|_{L_1^2}^3, \quad u \in L_1^2.$$

Proof: First note that for a.e. $x \in \mathbb{R}$

$$|\Phi(u)(x)| \le \int\int dx_1dx_2\, |u(x_1)||u(x_2)||u(x_2-[x-x_1])| = \Big(|u| * |u| * |u(-\cdot)|\Big)(x), \tag{3.3}$$

with $u(-\cdot)(x) = u(-x)$. Thus by Young's inequality, see e.g. [3, p. 205],

$$|\Phi(u)|_{L^\infty} \le |u|_{L^1}\Big||u| * |u(-\cdot)|\Big|_{L^\infty} \le |u|_{L^1}|u|_{L^2}^2 \le C|u|_{L_1^2}^3,$$

the latter by Lemma 3.1. Therefore $|N(u)|_{L_1^2}^2 = \displaystyle\int \frac{1}{1+x^2}(\Phi(u)(x))^2\,dx \le C|u|_{L_1^2}^6$ as was to be shown. □

Corollary 3.9 *The estimate*

$$|\Phi(u)|_{L^2} \le C|u|^3_{L^2_1}, \quad u \in L^2_1,$$

holds.

Proof: We apply Young's inequality twice in (3.3) and obtain

$$|\Phi(u)|_{L^2} \le \Big||u| * |u| * |u(-\cdot)|\Big|_{L^2} \le |u|_{L^{4/3}}\Big||u| * |u(-\cdot)|\Big|_{L^{4/3}} \le |u|_{L^{4/3}}|u|^2_{L^{8/7}}.$$

Thus we can use Lemma 3.1. $\square$

Finally we derive an estimate that will be needed in the sequel to establish, on the one hand, $(H6a)$, and, on the other hand, to show that a certain functional possesses a "mountain-pass geometry".

Lemma 3.10 *There exists $\varepsilon_0 > 0$ such that for $\varepsilon \in]0, \varepsilon_0]$*

$$\varphi(e_0^\varepsilon) \ge \frac{2}{3}\varepsilon^3, \quad \textit{where} \quad e_0^\varepsilon = \mathbf{1}_{[-\varepsilon,\varepsilon]}.$$

Proof: Fix $\beta_0 > 0$ such that $S(z) = \frac{\sin[\mu z]}{\mu z} \ge \frac{1}{2}$ for $|z| \le \beta_0$. Since $x_1^2 + x_2^2 + x_3^2 - (x_1 + x_2 - x_3)^2 = 2(x_3 - x_1)(x_2 - x_3)$ we may rewrite

$$\varphi(e_0^\varepsilon) = \frac{1}{4}\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon} dx_1 dx_2 dx_3\, S((x_3 - x_1)(x_3 - x_2))\, \mathbf{1}_{[-\varepsilon,\varepsilon]}(x_1 + x_2 - x_3).$$

Let $\varepsilon_0 = \frac{1}{2}\sqrt{\beta_0}$. For $\varepsilon \in]0, \varepsilon_0]$ and $x_1, x_2, x_3 \in [-\varepsilon, \varepsilon]$ we then obtain $|(x_3 - x_1)(x_3 - x_2)| \le 4\varepsilon^2 \le \beta_0$ and thus

$$\varphi(e_0^\varepsilon) \ge \frac{1}{8}\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon} dx_1 dx_2 dx_3\, \mathbf{1}_{[x_1+x_2-\varepsilon, x_1+x_2+\varepsilon]}(x_3).$$

Now $\int_{-\varepsilon}^{\varepsilon} dx_3 \mathbf{1}_{[z-\varepsilon,z+\varepsilon]}(x_3) = \mathbf{1}_{\{|z|\le 2\varepsilon\}}(z)(2\varepsilon - |z|)$, whence

$$\varphi(e_0^\varepsilon) \ge \frac{1}{8}\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon} dx_1 dx_2\, (2\varepsilon - |x_1 + x_2|) = \frac{1}{8}\Big(8\varepsilon^3 - \int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon} dx_1 dx_2\, |x_1 + x_2|\Big).$$

Since the latter integral may be evaluated as $\frac{8}{3}\varepsilon^3$, the proof is complete. $\square$

3.2 Existence of solutions for all $\lambda < 1$

In this section we will show that for every $\lambda < \Lambda = 1$ fixed there exists a nontrivial solution to (3.1). Define

$$I(u) = \frac{1}{2}|u|^2_{L^2_1} - \frac{1}{2}\lambda\langle Lu, u\rangle - \varphi(u), \quad u \in L^2_1.$$

Then $I \in C^1(L^2_1, \mathbb{R})$, and the critical points of I are the solutions to (3.1), by Lemma 3.4. The first step is to prove that I has a strict local minimum at $u = 0$.

Lemma 3.11 *There exist constants $\delta_0, \delta_1 > 0$ such that $I(u) \geq \delta_1 |u|^2_{L^2_1}$ for $|u|_{L^2_1} \leq \delta_0$.*

Proof: Since $\langle Lu, u\rangle = |u|^2_{L^2}$, we obtain from Lemma 3.2 and Lemma 3.7 that

$$I(u) \geq \frac{1}{2}|u|^2_{L^2_1} - \frac{1}{2}\lambda |u|^2_{L^2_1} - C|u|^4_{L^2_1} = \Big(\frac{1}{2}(1-\lambda) - C|u|^2_{L^2_1}\Big)|u|^2_{L^2_1}.$$

As $\lambda < 1$ the claim is obvious. □

Next we show that I attains negative values outside large balls.

Lemma 3.12 *Fix some $\varepsilon \in]0, \varepsilon_0]$ and define $u_0 = e_0^\varepsilon$, cf. Lemma 3.10. Then $I(\rho u_0) \to -\infty$ for $\rho \to \infty$.*

Proof: By Remark 3.5 and Lemma 3.10 we can estimate

$$I(\rho u_0) = \frac{1}{2}\rho^2\Big(|u_0|^2_{L^2_1} - \lambda |u_0|^2_{L^2}\Big) - \rho^4 \varphi(u_0) \leq \frac{1}{2}\rho^2\Big(|u_0|^2_{L^2_1} - \lambda |u_0|^2_{L^2}\Big) - \rho^4\alpha_0,$$

with $\alpha_0 = 2\varepsilon^3/3 > 0$. Therefore $I(\rho u_0) \to -\infty$ as $\rho \to \infty$. □

Since $I(0) = 0$, Lemma 3.11 and Lemma 3.12 can be summarized as to saying that the functional I has a "mountain pass geometry". According to [2, Thm. 1.1.4] there exists a sequence $(u_n) \subset L^2_1$ such that

$$I(u_n) \to c > 0, \quad \nabla I(u_n) \to 0, \quad n \to \infty, \tag{3.4}$$

with $c = \inf_{g\in\Gamma} \sup_{u \in g([0,1])} I(u)$, where Γ is the class of continuous paths $g : [0,1] \to L^2_1$ connecting $u = 0$ to $u = \rho_0 u_0$. Here $\rho_0 > 0$ is chosen such that $I(\rho_0 u_0) < 0$. Using Remark 3.5 we obtain $\nabla I(u)u = |u|^2_{L^2_1} - \lambda |u|^2_{L^2} - 4\varphi(u)$ for $u \in L^2_1$. From this it follows by definition of I that, on the one hand,

$$\frac{1}{2}\nabla I(u)u = \frac{1}{2}[2I(u) + 2\varphi(u)] - 2\varphi(u) = I(u) - \varphi(u). \tag{3.5}$$

On the other hand,

$$\nabla I(u)u = |u|^2_{L^2_1} - \lambda|u|^2_{L^2} + 4I(u) - 2|u|^2_{L^2_1} + 2\lambda |u|^2_{L^2} = 4I(u) - |u|^2_{L^2_1} + \lambda |u|^2_{L^2}.$$

Utilizing the latter relation we can estimate

$$(1-\lambda)|u_n|^2_{L^2_1} \leq |u_n|^2_{L^2_1} - \lambda |u_n|^2_{L^2} = 4I(u_n) - \nabla I(u_n)u_n \leq 4(c+1) + |u_n|_{L^2_1}$$

for large n. Consequently (u_n) is bounded, and therefore without loss of generality $u_n \rightharpoonup u$ for some $u \in L^2_1$. We shall show $u \neq 0$ and that u is a critical point of I. Firstly, by (3.4), (3.5), and Lemma 3.6

$$2c = 2c - 0 \longleftarrow 2I(u_n) - \nabla I(u_n)u_n = 2\varphi(u_n) \longrightarrow 2\varphi(u), \quad n \to \infty.$$

Whence $\varphi(u) = c > 0$ excludes the possibility $u = 0$. Moreover, for $v \in L^2_1 \subset L^2$ we find $\langle Lu_n, v\rangle = \langle u_n, Lv\rangle \to \langle u, Lv\rangle = \langle Lu, v\rangle$, since in particular $Lv \in L^2_1$. In addition, $\langle N(u_n), v\rangle = \int \Phi(u_n)(x)v(x)dx \to \int \Phi(u)(x)v(x)dx = \langle N(u), v\rangle$ by (3.2) and Lemma 3.6. Consequently,

$$0 \longleftarrow \nabla I(u_n)v = \langle u_n, v\rangle - \lambda\langle Lu_n, v\rangle - \langle N(u_n), v\rangle \longrightarrow \langle u, v\rangle - \lambda \langle Lu, v\rangle - \langle N(u), v\rangle,$$

and hence $\nabla I(u) = 0$, that is, u is a nontrivial critical point of I. Thus we have shown

Theorem 3.13 *For every $\lambda < 1$ there exists a nontrivial solution $u \in L^2_1$ of (3.1).*

In fact we have found L^2–solutions of (1.4) and hence of the original problem (1.3).

Corollary 3.14 *The solutions from Theorem 3.13 satisfy (1.4) in L^2.*

Proof: Since (3.1) holds, we have $(1+x^2)u(x) = \lambda u(x) + \Phi(u)(x)$ a.e. in $\mathbb{R}$. Due to Corollary 3.9, $\Phi(u) \in L^2$, and therefore $\lambda u + \Phi(u) \in L^2$. Hence also $(1+x^2)u$ in L^2. □

3.3 Bifurcation from the lowest point of the essential spectrum

To apply Theorem 2.1 to (3.1), we first note that $(H1)$ holds, with $A = I$. Moreover, by Lemma 3.8 and Lemma 3.4, $(H2)$ is satisfied as well. Since $\langle Lu, u\rangle = |u|^2_{L^2}$, see (3.1), we can choose e.g. $\lambda_0 = 0$ and $\nu = 1$ to get $(H3)$. Next, $(H4)$ is part of the assertion of Lemma 3.6, whereas $(H5)$ is clear from the fact that φ is homogeneous of degree four; see Remark 3.5. To verify $(H6a)$, let $r_0 = \sqrt{2\varepsilon_0} > 0$, with ε_0 from Lemma 3.10. As $\Lambda = 1$ by Lemma 3.2, the remarks following Theorem 2.1 show that it suffices to prove $u_0^r \in S(r)$ and

$$\frac{1}{2}|u_0^r|^2 < \frac{1}{2}r^2 + \varphi(u_0^r), \quad \text{with} \quad u_0^r = e_0^{r^2/2} = \mathbf{1}_{[-r^2/2, r^2/2]};$$

cf. Lemma 3.10. For this, we have $|u_0^r|^2_{L^2} = r^2$ and $|u_0^r|^2_{L^2_1} = \int_{-r^2/2}^{r^2/2}(1+x^2)dx = r^2 + \frac{1}{12}r^6$. Hence Lemma 3.10 yields

$$\frac{1}{2}r^2 + \varphi(u_0^r) \geq \frac{1}{2}r^2 + \frac{2}{3}\left(\frac{r^2}{2}\right)^3 = \frac{1}{2}r^2 + \frac{1}{12}r^6 > \frac{1}{2}\left(r^2 + \frac{1}{12}r^6\right) = \frac{1}{2}|u_0^r|^2_{L^2_1},$$

and therefore $(H6a)$ holds. Finally, as what concerns $(H6b)$, we have $|\varphi(u)| \leq C|u|^3_{L^2}|u|_{L^2_1}$ for $u \in L^2_1$ by Lemma 3.7. Next observe that

$$\begin{aligned} |u|^3_{L^2}|u|_{L^2_1} &= \langle u, u\rangle^{1/2}\langle Lu, u\rangle^{3/2} = \Big[\langle (I-L)u, u\rangle + \langle Lu, u\rangle\Big]^{1/2}\langle Lu, u\rangle^{3/2} \\ &\leq \langle (I-L)u, u\rangle^{1/2}\langle Lu, u\rangle^{3/2} + \langle Lu, u\rangle^2. \end{aligned}$$

Thus we may set $n = 2$, $a_1 = 1/2$, $b_1 = 3/2$, $a_2 = 0$, and $b_2 = 2$ to obtain $(H6b)$. Therefore Theorem 2.1 indeed applies.

Theorem 3.15 *With $\Lambda = 1$ the point $(\Lambda, 0) \in \mathbb{R} \times H$ is a bifurcation point to the left for (3.1). There exists $r_0 > 0$ and for each $r \in]0, r_0]$ a solution (λ_r, u_r) of (3.1) such that $0 < |u_r|_{L^2_1} \leq Cr$ and $0 < 1 - \lambda_r \leq Cr^2$.*

Note that the last estimate follows from Theorem 2.1 and Lemma 3.8. Analogously to Corollary 3.14, the (λ_r, u_r) in fact are solutions to (1.4).

3.4 Positivity of φ

As mentioned in the introduction, in fact $\varphi(u) \geq 0$ for all $u \in L_1^2$ is satisfied for

$$\begin{aligned}\varphi(u) \;&=\; \frac{1}{4}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} d\omega_1 d\omega_2 d\omega_3 d\omega_4 \,\frac{\sin[(\mu/2)(\omega_1^2+\omega_2^2-\omega_3^2-\omega_4^2)]}{(\mu/2)(\omega_1^2+\omega_2^2-\omega_3^2-\omega_4^2)} \\ &\qquad \times \delta(\omega_1+\omega_2-\omega_3-\omega_4)u(\omega_1)u(\omega_2)u(\omega_3)u(\omega_4).\end{aligned}$$

We sketch a (formal) argument which was explained to us by V. Zharnitsky. Let $T(t)$ be the solution operator to

$$iv_t + d(t)v_{xx} = 0,$$

i.e., $v(t,x) = [T(t)v_0](x)$, with $d(t)$ the dispersion function that is assumed to be 1-periodic. We have the representation

$$[T(t)v_0](x) = (2\pi)^{-1}\int_{\mathbb{R}} d\omega\, e^{-i\omega x - i\omega^2 R(t)}\int_{\mathbb{R}} dy\, e^{i\omega y} v_0(y),$$

where $(dR/dt)(t) = d(t)$. For a prescribed real-valued u we choose v_0 such that $u(\omega) = (2\pi)^{-1}\int_{\mathbb{R}} dy\, e^{i\omega y} v_0(y)$ and calculate

$$\begin{aligned}&\int_0^1 dt \int_{\mathbb{R}} dx \left|[T(t)v_0](x)\right|^4 \\ &= \int_0^1 dt \int_{\mathbb{R}} dx \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} d\omega_1 d\omega_2 d\omega_3 d\omega_4\, e^{i(\omega_1+\omega_2-\omega_3-\omega_4)x}\, e^{i(\omega_1^2+\omega_2^2-\omega_3^2-\omega_4^2)R(t)} \\ &\qquad \times u(\omega_1)u(\omega_2)u(\omega_3)u(\omega_4). \end{aligned} \tag{3.6}$$

To evaluate the dx-integral we use $\int_{\mathbb{R}} e^{i\omega x}\, dx = 2\pi\delta(\omega)$, and for the dt-integral one observes $\int_0^1 e^{i\omega R(t)}\, dt = \frac{\sin[(\mu/2)\omega]}{(\mu/2)\omega}$ for a certain $\mu > 0$ related to $d(t)$, if $d(t)$ is chosen to be a step function of type $d(t) = d_1$ in $[0,l]$ and $d(t) = d_2$ in $[l,1]$; cf.[15]. Thus (3.6) yields

$$0 \leq \int_0^1 dt \int_{\mathbb{R}} dx \left|[T(t)v_0](x)\right|^4 = 2\pi\varphi(u),$$

as desired.

4 Applications to semilinear elliptic equations

In this section we indicate how Theorem 2.1 can be used to prove the bifurcation of solutions from the lowest point of the essential spectrum for semilinear elliptic equations with a nonlinearity that may change sign.

We consider (1.8) from the introduction, i.e.,

$$-\Delta u(x) + V(x)u(x) = \lambda u(x) + a(x)|u(x)|^{p-2}u(x), \quad x \in \mathbb{R}^N.$$

The corresponding weak formulation can be cast in the form of (3.1), in the Hilbert space $H = H^1 = H^1(\mathbb{R}^N)$. Here

$$\langle Au, v\rangle = \int \Big\{V(x)u(x)v(x) + \nabla u(x)\cdot\nabla v(x)\Big\}\, dx, \quad \langle Lu, v\rangle = \int u(x)v(x)\, dx,$$
$$\text{and} \quad \varphi(u) = \frac{1}{p}\int a(x)|u(x)|^p\, dx;$$

see [12, p. 401 & Sect. 6]. In this section, integrals with no domain of integration indicated are taken over $\mathbb{R}^N$. The main purpose here is to exemplify that the non-negativity condition usually imposed on $a(x)$ can be relaxed. As we want to discuss only the simplest case, we make the following assumptions.

(A1) $V \in L^\infty(\mathbb{R}^N)$.

(A2) $|x|^2 V \in L^\infty(\mathbb{R}^N)$ and $|V_-|_{L^{N/2}(\mathbb{R}^N)} < \frac{N-2}{2(N-1)}$, with $V_-(x) = \max\{0, -V(x)\}$.

(A3) $N \geq 3$ and $2 < p < 2 + \frac{4}{N} - \frac{2\tau}{N}$, with τ from $(A6)$ below.

(A4) $a \in L^\infty(\mathbb{R}^N)$.

(A5) $\operatorname{ess\,sup}\{|a(x)| : |x| \geq n\} \to 0$ as $n \to \infty$.

(A6) Let $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x_1 \geq 0\}$. Then there exist $\tau \in [0, 2[$ and $\sigma \in]\tau, p + N[$ such that

$$a(x) \geq K_1 |x|^{-\tau}, \quad x \in \mathbb{R}^N_+, \quad \text{and} \quad |a(x)| \leq K_2 |x|^{-\sigma}, \quad x \in \mathbb{R}^N \setminus \mathbb{R}^N_+,$$

for some constants $K_1, K_2 > 0$.

Condition $(A6)$ was only designed to allow for an easy example of how the relaxed conditions on $a(x)$ may look like, there is broad room for modifications and variants. The assumption $N \geq 3$ was included to avoid distinguishing cases for $N = 1, 2$ or $N \geq 3$. Note also that by $(A3)$ we are ensured that $p < 2 + \frac{4}{N} < 2^* = \frac{2N}{N-2}$.

Theorem 4.1 *Assume $(A1) - (A6)$ holds. Then $(u, \lambda) = (0, 0) \in H \times \mathbb{R}$ is a bifurcation point to the left for (1.8).*

Proof: The hypotheses $(H1)$ and $(H2)$ of Theorem 2.1 are satisfied. Moreover, $(H3)$ holds for any $\lambda_0 < \inf V$, with $\nu = \min\{1, \inf V - \lambda_0\} > 0$, cf. [12, p. 427]. Using $(A3)$, $(A4)$, and $(A5)$ it is possible to verify $(H4)$, as p is subcritical, cf. [12, p. 430]. Since φ is homogeneous of degree p, in particular $(H5)$ is valid. Due to $(A1)$ and $(A2)$ it is found that $\Lambda = 0$, and $(A1)$, $(A2)$, and $(A3)$ show $(H6b)$ holds as is explained in [12, p. 429]. So it remains to check $(H6a)$. This amounts to finding, for all $r > 0$ small, an $u_0 \in H$ (that may depend on r) with

$$|u_0|_{L^2} = r \quad \text{and} \quad \frac{1}{2}\langle A u_0, u_0 \rangle < \varphi(u_0). \tag{4.1}$$

For this we consider, following [12], for $\alpha > 0$ the family of functions $u_\alpha(x) = v(\alpha x)$, $x \in \mathbb{R}^N$, where $v(y) = |y| e^{-|y|}$. Then $|u_\alpha|_2^2 = C_1 \alpha^{-N}$ and $|\nabla u_\alpha|_2^2 = C_2 \alpha^{2-N}$ for appropriate constants $C_1, C_2 > 0$. In addition, by $(A2)$

$$\int V(x) u_\alpha^2(x)\, dx \leq C \int |x|^{-2} |v(\alpha x)|^2\, dx \leq C_3 \alpha^{2-N}$$

for some $C_3 > 0$. Therefore

$$\frac{1}{2}\langle A u_\alpha, u_\alpha \rangle \leq C_4 \alpha^{2-N}. \tag{4.2}$$

To derive a lower bound on $\varphi(u_\alpha)$, we obtain from $(A6)$

$$
\begin{aligned}
\varphi(u_\alpha) &= \frac{1}{p}\int_{\mathbb{R}^N_+} a(x)|u_\alpha(x)|^p\,dx + \frac{1}{p}\int_{\mathbb{R}^N\setminus\mathbb{R}^N_+} a(x)|u_\alpha(x)|^p\,dx \\
&\geq \frac{K_1}{p}\int_{\mathbb{R}^N_+} |x|^{-\tau}|v(\alpha x)|^p\,dx - \frac{K_2}{p}\int_{\mathbb{R}^N\setminus\mathbb{R}^N_+} |x|^{-\sigma}|v(\alpha x)|^p\,dx \\
&= C_5\alpha^{\tau-N} - C_6\alpha^{\sigma-N}
\end{aligned}
\tag{4.3}
$$

by setting $y=\alpha x$ and observing $x\in\mathbb{R}^N_+$ iff $y\in\mathbb{R}^N_+$.

Let $r>0$ be fixed and define $u_0 = t_\alpha u_\alpha$, with $t_\alpha = rC_1^{-1/2}\alpha^{N/2}$ and $\alpha>0$ still to be chosen. Then $|u_0|_2 = t_\alpha|u_\alpha|_2 = r$, and according to (4.2) and (4.3) it follows that

$$
\begin{aligned}
\varphi(u_0) - \frac{1}{2}\langle Au_0,u_0\rangle &= t_\alpha^p\varphi(u_\alpha) - \frac{1}{2}t_\alpha^2\langle Au_\alpha,u_\alpha\rangle \geq t_\alpha^p\Big(C_5\alpha^{\tau-N} - C_6\alpha^{\sigma-N}\Big) - \frac{C_4}{2}t_\alpha^2\alpha^{2-N} \\
&\geq C_7r^p\alpha^{\tau-N+Np/2} - C_8r^p\alpha^{\sigma-N+Np/2} - C_9r^2\alpha^2 \\
&= \alpha^{\tau-N+Np/2}\Big(C_7r^p - C_8r^p\alpha^{\sigma-\tau} - C_9r^2\alpha^{2-\tau+N-Np/2}\Big)
\end{aligned}
$$

for all $\alpha>0$. Due to $(A6)$, $\sigma>\tau$, and $(A3)$ implies $2-\tau+N-Np/2>0$. Thus we can fix $\alpha>0$ sufficiently small to see (4.1) is satisfied for the corresponding u_0. Consequently $(H6a)$ holds, and Theorem 2.1 applies to yield the claim. □

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