Analyse Mathématique/Mathematical Analysis

B.V. solutions to evolution problems with time-dependent domains

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Abstract - We obtain solutions with bounded variation of evolution problems governed by time-dependent maximal monotone operators, generalizing both Moreau's sweeping process [1] and Péralba's work [2].

Solutions v.b. de problèmes d'évolution avec domaine variable

Résumé - On obtient des solutions à variation bornée de problèmes d'évolution gouvernés par des opérateurs maximaux monotones dépendant du temps. On généralise ainsi le processus de rafle de J.J. Moreau [1] et les travaux de Péralba [2].

Titre(s) courant(s): Problèmes d'évolution v.b./B.V. evolution problems

Version française abrégée - Le but de cette Note est l'étude du problème d'évolution

$$-\frac{du}{dr}(t)\in A(t)u(t) \tag{1}$$

dr-presque partout dans l'intervalle $[0,T]$, où $A(t):D(A(t))\subset H\to 2^H$ est un opérateur maximal monotone dans un espace de Hilbert H. La mesure positive dr sur $[0,T]$ est la mesure de Stieltjes d'une fonction non-décroissante et continue à droite $r:[0,T]\to[0,\infty[$. On suppose que la variation de $A(\cdot)$ est contrôlée par dr, au sens suivant:

$$\operatorname{dis}(A(t),A(s))\le r(t)-r(s),\quad 0\le s\le t\le T, \tag{2}$$

où $\operatorname{dis}(\cdot,\cdot)$ est la pseudo-métrique définie par Vladimirov [3]

$$\operatorname{dis}(A,B)=\sup\left\{\frac{\langle y-\bar y,\bar x-x\rangle}{1+|y|+|\bar y|}:x\in D(A),y\in Ax,\bar x\in D(B),\bar y\in B\bar x\right\}, \tag{3}$$

si A et B sont des opérateurs maximaux monotones avec domaines $D(A)$ et $D(B)$ respectivement. On note $A^\circ(t,x)$ l'élément de norme minimale dans $A(t)x$ et on suppose que

$$|A^\circ(t,x)|\le c(t)\,(1+|x|)\quad \text{pour}\quad t\in[0,T],\,x\in D(A(t)), \tag{4}$$

où $c\in L^1([0,T];dr)$ est fixé. On obtient le résultat d'existence suivant:

THÉORÈME. - *Pour chaque $t\in[0,T]$, on se donne un opérateur maximal monotone $A(t):D(A(t))\to 2^H$ de telle façon que (2) et (4) sont valables. Alors pour tout $u_0\in D(A(0))$, le problème d'évolution (1) avec condition initiale $u(0)=u_0$ a une solution à variation bornée $u:[0,T]\to H$. De façon précise, $u(t)\in D(A(t))$ pour chaque $t\in[0,T]$, et on peut écrire $du=u'dr$ avec une densité $u'\in L^1([0,T],dr;H)$ qui satisfait $-u'(t)\in A(t)u(t)$ dr-presque partout.*

Ce théorème possède une variante locale: si pour tout $R>0$ il existe $M_R>0$ tel que

$$|A^\circ(t,x)|\leq M_R \quad \text{pour} \quad t\in[0,T] \quad \text{et} \quad x\in D(A(t)) \quad \text{avec} \quad |x|\leq R,$$

alors il existe une solution unique sur un intervalle $[0,T_1]$ avec $T_1\in]0,T]$ suffisamment petit.

Si $A(t)x=N_{C(t)}x$, cône normal au convexe fermé $C(t)$ au point $x\in C(t)$, on sait [3] que la "distance" entre les opérateurs égale la distance de Hausdorff d_H entre les ensembles en question, c'est-à-dire $\mathrm{dis}(A(t),A(s))=d_H(C(t),C(s))$. Il s'ensuit que la rafle par un convexe mobile ayant variation bornée est un cas particulier de notre théorème.

On traite aussi le cas absolument continu, c'est-à-dire, où l'on suppose

$$\mathrm{dis}(A(t),A(s))\leq|a(t)-a(s)| \quad \text{pour} \quad t,s\in[0,T]$$

avec $a\in W^{1,1}([0,T])$. On obtient alors une solution absolument continue pour le problème

$$-\frac{du}{dt}(t)\in A(t)u(t) \quad dt-p.p.$$

Si on pose $A(t)=\partial\varphi_t$, où les fonctions $x\mapsto\varphi_t(x)=\varphi(t,x)$ sont convexes s.c.i., on généralise aussi les résultats de Péralba [2]. En effet, si les polaires φ_t^* vérifient

$$\varphi_t^*(x)\leq\varphi_s^*(x)+|a(t)-a(s)| \quad \text{pour} \quad 0\leq s\leq t\leq T, \quad x\in H,$$

alors on a $\mathrm{dis}(\partial\varphi_t,\partial\varphi_s)\leq|a(t)-a(s)|$.

Les preuves peuvent être généralisées de façon à résoudre les problèmes non homogènes correspondants, c'est-à-dire $-(du/dr)(t)\in A(t)u(t)+f(t)$ resp. $-(du/dt)(t)\in A(t)u(t)+f(t)$, où f est dr-intégrable resp. Lebesgue-intégrable. Les preuves complètes sont données dans [4].

1. Introduction and Main Results. - In this Note, we deal with the evolution problem

$$-\frac{du}{dr}(t)\in A(t)u(t) \quad dr-\text{a.e. in }[0,T], \tag{1.1}$$

where $A(t):D(A(t))\subset H\to 2^H$ is a maximal monotone operator (mmop) in a Hilbert space H for every $t\in[0,T]$, and the dependence $t\mapsto A(t)$ is - in some sense - of bounded variation. To be more precise, we assume that

$$\mathrm{dis}(A(t),A(s))\leq r(t)-r(s), \quad 0\leq s\leq t\leq T, \tag{1.2}$$

where $r:[0,T]\to[0,\infty)$ is right-continuous and nondecreasing. Moreover, $\mathrm{dis}(\cdot,\cdot)$ is the pseudo-metric for mmops, introduced in [3], and given through formula (3) for two mmops A and B with domains $D(A)$ and $D(B)$, respectively. We let also denote $A^\circ(t,x)$ the element of minimal norm of $A(t)x$ and suppose that for some $c\in L^1([0,T];dr)$

$$|A^\circ(t,x)|\leq c(t)(1+|x|) \quad \text{for} \quad t\in[0,T],\, x\in D(A(t)). \tag{1.3}$$

We then obtain the following

THEOREM. - *Suppose that for every $t \in [0,T]$ a maximal monotone operator $A(t): D(A(t)) \to 2^H$ is given such that (1.2) and (1.3) hold. Then for every $u_0 \in D(A(0))$ the evolution problem (1.1) with initial condition $u(0)=u_0$ has a unique solution of bounded variation $u:[0,T] \to H$. To be more precise, $u(t) \in D(A(t))$ for every $t \in [0,T]$, and we have $du = u'dr$ with density $u' \in L^1([0,T],dr;H)$ such that $-u'(t) \in A(t)u(t)$ dr-a.e.*

If one is contented with local solutions, it is also possible to assume instead of (1.3) only that for all $R>0$ there is an $M_R>0$ such that

$$|A^\circ(t,x)| \le M_R \quad \text{for} \quad t \in [0,T] \quad \text{and} \quad x \in D(A(t)) \quad \text{with} \quad |x| \le R.$$

Then one can prove the existence of a unique b.v. solution of (1.1) on a sufficiently small interval $[0,T_1] \subset [0,T]$ with $T_1 > 0$.

We also consider the case that $t \mapsto A(t)$ is absolutely continuous, i.e.

$$\operatorname{dis}(A(t),A(s)) \le |a(t)-a(s)| \quad \text{for all} \quad t,s \in [0,T]$$

with some $a \in W^{1,1}([0,T])$. Then, if again (1.3) is assumed to hold, we obtain the existence of a unique a.c. solution of

$$-\frac{du}{dt}(t) \in A(t)u(t) \quad dt-a.e. \quad \text{on} \quad [0,T].$$

If we let $A(t) = \partial\varphi_t$, where the functions $x \mapsto \varphi_t(x) = \varphi(t,x)$ are convex and l.s.c., it may be seen that we also generalize Péralba's results [2]. In fact, if we assume that the polars φ_t^* satisfy

$$\varphi_t^*(x) \le \varphi_s^*(x) + |a(t)-a(s)| \quad \text{for} \quad 0 \le s \le t \le T, \quad x \in H,$$

then it follows that $\operatorname{dis}(\partial\varphi_t, \partial\varphi_s) \le |a(t)-a(s)|$.

We note that our proofs can also be generalized to obtain solutions for the corresponding inhomogeneous problems $-(du/dr)(t) \in A(t)u(t) + f(t)$ resp. $-(du/dt)(t) \in A(t)u(t) + f(t)$ for dr-integrable resp. Lebesgue-integrable $f:[0,T] \to H$.

In particular, our results apply to the case $A(t)x = N_{C(t)}x$, the normal cone to a moving closed convex set $t \mapsto C(t)$ for $x \in C(t)$. Since here $A^\circ(t,x) = 0$ and $\operatorname{dis}(A(t),A(s)) = d_H(C(t),C(s))$, the Hausdorff distance between $C(t)$ and $C(s)$ (cf. [3]), we thus have generalized the corresponding well-known results concerning the "classical" sweeping process.

2. PROOF OF THE THEOREM. - Here we just sketch the proof of existence of a solution to (1.1) on $[0,T]$, under the assumptions (1.2) and (1.3) with $c(t) \equiv c$. Concerning more details, and also for the local variant, the a.c. case, and the inhomogeneous problems, the reader is referred to [4].

We first carry out a complete discretization of the problem, i.e. we fix a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_n \searrow 0$ and partitions $0 = t_0^n < t_1^n < \cdots < t_{k_n}^n = T$ of $[0,T]$ such that

$$|t_{i+1}^n - t_i^n| + dr(]t_i^n, t_{i+1}^n]) \le \varepsilon_n \quad \text{for} \quad i = 0,\ldots,k_n-1. \tag{1.4}$$

These partitions can be obtained analogously to [1, Prop. 3b] by considering the measure $dt + dr$. We define $\delta_{i+1}^n = dr(]t_i^n, t_{i+1}^n]) \ge 0$ for $i = 0,\ldots,k_n-1$, and a sequence of right-continuous (rc) step-functions $u_n:[0,T] \to H$ through

$$u_n(t) = u_0 \quad \text{for} \quad t \in [0,t_1^n[\quad \text{and} \quad u_n(t) = u_i^n \quad \text{for} \quad i \ge 1, \quad t \in [t_i^n, t_{i+1}^n[, \tag{1.5}$$

where for $i=0,\dots,k_n-1$

$$u_{i+1}^n = J_{i+1}^n(u_i^n) \quad \text{with} \quad J_{i+1}^n x = \left(I+\delta_{i+1}^n A(t_{i+1}^n)\right)^{-1} x \quad \text{for} \quad x\in H \quad \text{if} \quad \delta_{i+1}^n>0 \tag{1.6}$$

and $J_{i+1}^n = I$ if $\delta_{i+1}^n = 0$. Moreover, we let $u_n(T)=u_{k_n}^n$. By means of the (u_n), following [1] we also define rc approximants $v_n:[0,T]\to H$ through

$$v_n(t)=\frac{r(t)-r(t_i^n)}{r(t_{i+1}^n)-r(t_i^n)}(u_{i+1}^n-u_i^n)+u_i^n \quad \text{for} \quad t\in[t_i^n,t_{i+1}^n[$$

and $v_n(T)=u_{k_n}^n$. Note that here we use the convention $\frac{0}{0}=0$, i.e. $v_n(t)=u_i^n=u_{i+1}^n$ in $[t_i^n,t_{i+1}^n[$ in case that $\delta_{i+1}^n=0$. Now it follows analogously to [1, Lemma 1] that

$$dv_n = v_n'\,dr \quad \text{with} \quad v_n'(t)=\frac{u_{i+1}^n-u_i^n}{r(t_{i+1}^n)-r(t_i^n)} \quad \text{for} \quad t\in]t_i^n,t_{i+1}^n] \tag{1.7}$$

and $v_n'(0)=0$. Consequently, if we let $\varphi_n:[0,T]\to[0,T]$ be given through $\varphi_n(t)=t_{i+1}^n$ for $t\in]t_i^n,t_{i+1}^n]$ and $\varphi_n(0)=0$, then (1.5), (1.6), and (1.7) just say that

$$-v_n'(t)\in A\Big(\varphi_n(t),v_n(\varphi_n(t))\Big) \quad \text{for} \quad dr-\text{a.e. } t\in[0,T]. \tag{1.8}$$

We intend to obtain a solution of (1.1) by passing to a suitable limit in (1.8), and therefore we first have to derive uniform bounds. For that, the following lemma is needed.

LEMMA 1. - *Let A,B be mmops. Then with $J_\lambda^B=(I+\lambda B)^{-1}$ for $\lambda>0$ and $x\in D(A)$ we have*

$$|x-J_\lambda^B x|\le \lambda|A^\circ x|+\operatorname{dis}(A,B)+\sqrt{\lambda(1+|A^\circ x|)\operatorname{dis}(A,B)}.$$

Since $\operatorname{dis}(A(t_i^n),A(t_{i+1}^n))\le\delta_{i+1}^n$ by (1.2), Lemma 1 together with (1.6) can be used to obtain by means of (1.3) and a discrete version of Gronwall's lemma

LEMMA 2. - *There exists a constant $K>0$ such that for all $n\in\mathbb{N}$ and $i=0,\dots,k_n$ resp. $i=0,\dots,k_n-1$*

$$|u_i^n|\le K \quad \textit{resp.} \quad |u_{i+1}^n-u_i^n|\le K\,dr(]t_i^n,t_{i+1}^n]). \tag{1.9}$$

In particular,

$$\sup_{n\in\mathbb{N}}|u_n|_\infty\le K \quad \textit{and} \quad \sup_{n\in\mathbb{N}}\operatorname{var}(u_n)=\sup_{n\in\mathbb{N}}\left(\sum_{i=0}^{k_n-1}|u_{i+1}^n-u_i^n|\right)\le K\,dr([0,T]) \tag{1.10}$$

as well as

$$|u_n(t)-u_n(s)|\le K\Big(dr(]s,t])+\varepsilon_n\Big) \quad \textit{for} \quad n\in\mathbb{N} \quad \textit{and} \quad 0\le s\le t\le T. \tag{1.11}$$

Hence, because of $|u_n-v_n|_\infty\le\varepsilon_n$, (1.10), (1.9) and (1.7), there is a subsequence (again indexed by $n\in\mathbb{N}$) and a bv function $u:[0,T]\to H$ such that $v_n(t)\to u(t)$ weakly for every $t\in[0,T]$ (cf. [5, Thm. 0.2.1]) and $v_n'\to u'$ weakly in $L^2([0,T],H;dr)$, where u' denotes a density of du w.r.t. dr, which exists, since $|u(t)-u(s)| \leq K(r(t)-r(s))$ for $0 \leq s \leq t \leq T$ by (1.11).

One can show next that $v_n \to u$ uniformly strongly on $[0,T]$ (in fact, only pointwise strong convergence would be needed in the sequel). To see this, we note that the definition of dis, (1.8), (1.9), (1.7), and (1.2) imply for $t \in]0,T]$ and $n,m \in \mathbb{N}$

$$\Big\langle v_n(\varphi_n(t)) - v_m(\varphi_m(t)), v_n'(t) - v_m'(t) \Big\rangle \leq (1+2K)\Big(|r(\varphi_n(t)) - r(t)| + |r(\varphi_m(t)) - r(t)|\Big).$$

By relating to the left-hand side of this estimate and by using the rc of r we thus can obtain

$$\Big\langle v_n(t) - v_m(t), v_n'(t) - v_m'(t) \Big\rangle \leq \Delta_{n,m}(t) \tag{1.12}$$

with functions $\Delta_{n,m}$ being bounded uniformly in n,m and $t \in [0,T]$, and satisfying $\Delta_{n,m}(t) \to 0$ as $n,m \to \infty$ for every $t \in [0,T]$. Let $w = v_n - v_m$. Then w is rcbv with $dw = dv_n - dv_m$ and $\dfrac{dw}{dr} = v_n' - v_m'$. Hence we may apply Moreau's formula $dw^2 \leq 2w^+ \cdot dw^+$ (cf. [6, (2)] or [5, p.9]) and (1.12) to find

$$\frac{1}{2}|v_n(t) - v_m(t)|^2 = \frac{1}{2}\int_{]0,t]} dw^2 \leq \int_{]0,t]} w^+ \cdot dw^+ \leq \int_{]0,t]} \langle v_n - v_m, v_n' - v_m' \rangle \, dr \leq \int_{[0,T]} \Delta_{n,m}(s)\, dr(s).$$

Summing up, by Lebesgue's dominated convergence theorem and by the previous results we therefore have obtained

LEMMA 3. - *We have* $|v_n - u|_\infty \to 0$ *and* $v_n' \to u'$ *weakly in* $L^2([0,T],H;dr)$ *as* $n \to \infty$.

To show that u is a solution, a main point is to prove the inclusion (1.1). Since every $A(t)$ is monotone, in particular for dr-a.e. $t \in [0,T]$ and all $\xi \in D(A(\varphi_n(t)))$

$$\Big\langle v_n'(t), v_n(\varphi_n(t)) - \xi \Big\rangle \leq \Big\langle A(\varphi_n(t), \xi), \xi - v_n(\varphi_n(t)) \Big\rangle. \tag{1.13}$$

Because it can be seen that also $u(t) \in D(A(t))$ on $[0,T]$, the application of the following results will be useful. We remark that Lemma 4 is [7, Prop. 2.7], and Lemma 5 is a particular case of a more general result, closely related to one given in [8].

LEMMA 4. - *Let* A *be a mmop,* $x \in \overline{D(A)}$ *and* $y \in H$ *such that* $\langle A^\circ \eta - y, \eta - x \rangle \geq 0$ *for* $\eta \in D(A)$. *Then* $x \in D(A)$ *and* $y \in Ax$.

LEMMA 5. - *Let* A_n *be a mmop for* $n \in \mathbb{N}$ *such that* $\operatorname{dis}(A_n, A) \to 0$ *for some mmop* A. *Suppose further that* $|A_n^\circ x| \leq c(1+|x|)$ *for some* $c > 0$, *all* $n \in \mathbb{N}$ *and* $x \in D(A_n)$. *Then for every* $\eta \in D(A)$ *there exists a sequence* (ξ_n) *with* $\xi_n \in D(A_n)$, $\xi_n \to \eta$ *and* $A_n^\circ \xi_n \to A^\circ \eta$.

Thus, by Lemma 4, it is enough to prove

$$\langle u'(t), u(t) - \eta \rangle \leq \langle A^\circ(t,\eta), \eta - u(t) \rangle \tag{1.14}$$

for $t \in [0,T]$ outside some suitable dr-nullset and for $\eta \in D(A(t))$. To do this, we apply Lemma 5 and use (1.13) together with Lemma 3 to obtain $\limsup_{n\to\infty} \langle v_n'(t), u(t) - \eta \rangle \leq \langle A^\circ(t,\eta), \eta - u(t) \rangle$.

By Lemma 3 we can use Mazur's theorem to pass to suitable convex combinations of the v_n' which converge strongly in $L^2([0,T],H;dr)$, and if we choose a subsequence of these convex combinations which converges pointwisely strongly to u', we finally arrive at (1.14). This completes the outline of the proof of the theorem. □

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