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On the Period of Periodic Motions of a Particle in a Scalar Wave Field

It is shown that a periodic solution of the nonlinear infinite-dimensional Hamiltonian system describing a particle in a scalar wave field can have only certain discrete periods that are related to zeroes of the Fourier transform of the function that realizes the coupling.

1. Introduction

For the motion of a charged particle $q(t) \in \mathbb{R}^3$ subject to its self-generated field, a simplified model is governed by the equations

$$\begin{aligned} \dot{q}(t) &= \frac{p(t)}{\sqrt{1+p(t)^2}}, \qquad & \dot{p}(t) &= -\nabla V(q(t)) + \int d^3y\, \phi(y,t) \nabla\rho(y-q(t)), \\ \dot{\phi}(x,t) &= \pi(x,t), \qquad & \dot{\pi}(x,t) &= \Delta\phi(x,t) - \rho(x-q(t)), \end{aligned} \tag{1}$$

with $x \in \mathbb{R}^3$ and $t \in \mathbb{R}$. Here $\dot{q}(t)$ and $p(t) = (1-\dot{q}(t)^2)^{-1/2}\,\dot{q}(t) \in \mathbb{R}^3$ are the velocity and the momentum of the particle, respectively, and $\phi(x,t) \in \mathbb{R}$ denotes the (scalar) field. If the wave equation $\ddot{\phi} = \Delta\phi - \rho(x-q(t))$ in (1) were replaced by the full Maxwell equations for electric and magnetic fields $E(x,t), B(x,t) \in \mathbb{R}^3$, then (1) would be the more general and well-known Maxwell-Lorentz model, also called Abraham model; see [8] for an up-to date review. An important role is played by the function ρ in (1), since it accounts for considering the particle a charged sphere with charge ρ rather than a point particle: If one formally sets $\rho = \delta_0$, then the wave equation $\ddot{\phi} = \Delta\phi - \delta(x-q(t))$ means that there is only charge at the point $x = q(t)$ where the particle sits. For ρ we assume

$$\rho \in C_0^\infty(\mathbb{R}^3), \quad \rho(x) = 0 \quad \text{for} \quad |x| \geq R_\rho, \quad \rho(x) = \rho_r(|x|), \tag{C}$$

to be satisfied.

Our goal is to investigate the long-time behaviour of (1) as a dynamical system on the state-space $\mathcal{E} = D^{1,2}(\mathbb{R}^3) \oplus \mathbb{R}^3 \oplus L^2(\mathbb{R}^3) \oplus \mathbb{R}^3$, with the canonical norm, where $D^{1,2}(\mathbb{R}^3) = \{\phi \in L^6(\mathbb{R}^3) : |\nabla\phi(x)| \in L^2(\mathbb{R}^3)\}$. Assuming here and in the sequel that $V \in C^2(\mathbb{R}^3)$ with $\inf_{q\in\mathbb{R}^3} V(q) > -\infty$, it is known from [4; Lemma 2.1] that for data $Y_0 = (\phi_0, q_0, \pi_0, p_0) \in \mathcal{E}$ there exists a unique global (weak) solution $t \mapsto Y(t) \in \mathcal{E}$ of (1), $Y(t) = (\phi(\cdot,t), q(t), \pi(\cdot,t), p(t))$, such that $Y(0) = Y_0$.

It became apparent that for the dynamics of (1) a particularly important role seems to be played by the Fourier transform

$$\hat{\rho}(k) = (2\pi)^{-3/2} \int d^3x\, e^{-i\,k\cdot x} \rho(x), \quad k \in \mathbb{R}^3,$$

of ρ. In case that $\hat{\rho}(k) \neq 0$ for all $k \in \mathbb{R}^3$, it was also shown it [4] that solutions of (1) for *any* data $Y_0 \in \mathcal{E}$ (with a certain decay at $|x| \to \infty$) will approach the set of all equilibria of (1), in the topology on $\mathcal{E}$ generated by the seminorms induced by restriction of integrals over $\{x \in \mathbb{R}^3 : |x| \leq R\}$, $R > 0$. This corresponds to the fact that the particle is accelerated by the potential V and hence radiates energy in x-space. Therefore, if a fixed ball $\{x \in \mathbb{R}^3 : |x| \leq R\}$ is considered, then all energy should be dissipated from this ball as $t \to \infty$.

Contrary to this satisfactory answer, from a rigorous viewpoint the situation is much less clear if $\hat{\rho}$ admits zeroes, for simplicity we suppose

$$\hat{\rho}(k) = 0 \quad \text{for} \quad k \in \mathbb{R}^3,\ |k| = a > 0, \quad \hat{\rho}(k) \neq 0, \quad |k| \neq a; \tag{2}$$

note that since ρ is radially symmetric, so is $\hat{\rho}$. In this case, linearizing (1) about a fixed point, it can be seen that the linearized system possesses periodic solutions, of period $T = 2\pi/a$. It has been proved in [5] that, due to the coupling to the continuous spectrum, these "linearized periodic solutions" do not generate periodic solutions of the non-linear problem, in a certain sense. This somehow excludes the existence of periodic solutions to (1) close to equilibrium points, but it remained open whether there are periodic solutions at all, e.g., with large energy.

In the present note we will prove that if a periodic solution to (1) does exist, then necessarily it is of period $T = (2\pi/a)n$ for some $n \in \mathbb{N}$. More precisely, we show

Theorem 1. *Assume (C) and (2) are satisfied, and let* $Y(t) = (\phi(\cdot,t), q(t), \pi(\cdot,t), p(t))$ *be a solution to (1) with data* $Y_0 = (\phi_0, q_0, \pi_0, p_0) \in \mathcal{E}$ *such that for some* $R_0 > 0$ *the functions* $\phi_0(x)$ *and* $\pi_0(x)$ *are* C^2 *and* C^1*-differentiable, respectively, outside the ball* $\{x \in \mathbb{R}^3 : |x| \le R_0\}$, *and for* $|x| \to \infty$

$$|\phi^0(x)| + |x| \Big(|\nabla\phi_0(x)| + |\pi_0(x)|\Big) + |x|^2 \Big(|\nabla\nabla\phi_0(x)| + |\nabla\pi_0(x)|\Big) = \mathcal{O}(|x|^{-\sigma}) \tag{3}$$

holds, with some $\sigma > 1/2$. *Then, if* $q(\cdot)$ *is periodic of period* $T > 0$ *and non-constant, we already must have* $T = (2\pi/a)n$ *for some* $n \in \mathbb{N}$.

Remark 1. *(i) Since (1) is a nonlinear infinite-dimensional Hamiltonian system, the fact that there should be periodic solutions of only discrete periods seems to be somewhat unlikely. Typically one would expect periodic solutions of "very many" periods, cf. e.g. [2]. Perhaps this indicates that there are no periodic solutions at all, and that every trajectory has to tend to the set of all equilibria in the long-time limit.*

(ii) For the choice $\rho(x) \cong \delta(|x| - b)$, *which means that the charge is uniformly distributed over the sphere* $|x| = b$, *an analogous (but quite formally derived) result has been obtained in [7]; see also the remarks in [6; p. 937]. For this* ρ *the existence of a periodic solution is claimed in [1], but the argument is non-rigorous. Note, however, that in this case (C) is not satisfied, and* $\hat{\rho}(k) \cong \frac{\sin kb}{kb}$.

(iii) The decay assumption (3) is the one already used in [4].

2. Proof of Theorem 1

The following result from [4; Prop. 3.1] will play the main role in the proof.

Lemma 1. *In the situation described in Theorem 1,*

$$\int_0^\infty dt \int_{|\omega|=1} d^2\omega \, |R_\omega(t)|^2 < \infty, \quad \textit{with} \quad R_\omega(t) = \int d^3x \, \rho(x - q(t + \omega\cdot x)) \frac{\omega \cdot \ddot{q}(t+\omega\cdot x)}{(1 - \omega\cdot\dot{q}(t+\omega\cdot x))^2}.$$

This estimate obtains from a careful analysis of how much energy does leave a ball $\{x \in \mathbb{R}^3 : |x| \le R\}$ as $t \to \infty$, and it reflects the fact that there is energy dissipated in quantitative terms. Note that due to the a priori bound $\sup_{t\in\mathbb{R}} |\dot{q}(t)| = \sup_{t\in[0,T]} |\dot{q}(t)| \le \bar{v} < 1$, see [4], we always have $|\omega\cdot\dot{q}(t+\omega\cdot x)| \le \bar{v}$ for $|\omega| = 1$, whence $R_\omega(t)$ is well-defined.

Now we can proceed to show Theorem 1.

Proof. Since q is T–periodic, so are $\dot{q}$ and $\ddot{q}$. Whence $R_\omega(\cdot)$ is T-periodic as well, for each $|\omega| = 1$, and thus the same holds for $t \mapsto \int_{|\omega|=1} d^2\,\omega |R_\omega(t)|^2$. Utilizing Lemma 1 we find that this is only possible in case that $R_\omega(t) = 0$ for all $|\omega| = 1$ and $t \in [0,T]$. To rewrite $R_\omega(t)$ appropriately, we introduce the 1D version $\rho_{(1)}(\xi) = \int_{\mathbb{R}^2} \rho(x_1, x_2, \xi)\, dx_1 dx_2$ of ρ, with $\xi \in \mathbb{R}$. We perform a change of variables

$$\mathbb{R}^3 \ni x \mapsto \Big(\omega_1\cdot(x - \varphi(\omega\cdot x)),\, \omega_2\cdot(x - \varphi(\omega\cdot x)),\, \omega\cdot x\Big) = z \in \mathbb{R}^3,$$

with $\varphi(\xi) = q(t+\xi) \in \mathbb{R}^3$, and $\omega_1, \omega_2 \in \mathbb{R}^3$ are chosen in such a way that $\{\omega_1, \omega_2, \omega\}$ is an orthonormal base of $\mathbb{R}^3$. Then it may be calculated that

$$|\det(\frac{dz}{dx})| \equiv 1 \quad \text{and} \quad |x - \varphi(\omega\cdot x)| = |(z_1, z_2, z_3 - \omega\cdot\varphi(z_3))|,$$

hence

$$\rho(x - q(t+\omega\cdot x)) = \rho(z_1, z_2, z_3 - \omega\cdot\varphi(z_3))$$

due to (C). Thus we obtain

$$R_\omega(t)=\int d^1\xi\,\rho_{(1)}(\xi-\omega\cdot q(t+\xi))\frac{\omega\cdot\ddot q(t+\xi)}{(1-\omega\cdot\dot q(t+\xi))^2}=0,\quad |\omega|=1,\quad t\in[0,T].$$

Writing $r(t)=\omega\cdot q(t)$, this yields, due to $\rho_{(1)}(-\xi)=\rho_{(1)}(\xi)$,

$$\begin{aligned}0 &= \int d^1\xi\,\rho_{(1)}(\xi-r(t+\xi))\frac{\ddot r(t+\xi)}{(1-\dot r(t+\xi))^2}=\int d^1\xi\,\rho_{(1)}(t-[\xi-r(\xi)])\frac{\ddot r(\xi)}{(1-\dot r(\xi))^2}\\ &= \int d^1\theta\,\rho_{(1)}(t-\theta)\frac{\ddot r(\xi(\theta))}{(1-\dot r(\xi(\theta)))^3},\end{aligned}$$

the latter through the substitution $\theta=\xi-r(\xi)$, $\frac{d\theta}{d\xi}=1-\dot r(\xi)$; recall that $|\dot r(\xi)|\le\bar v<1$, hence $\xi\mapsto\theta(\xi)$ is a diffeomorphism with inverse $\theta\mapsto\xi(\theta)$. Since this is a convolution, we can summarize that

$$\Big(\rho_{(1)}*g_\omega\Big)(t)=0,\quad |\omega|=1,\quad t\in[0,T],\quad\text{with}\quad g_\omega(\theta)=\frac{\omega\cdot\ddot q(\xi(\theta))}{(1-\omega\cdot\dot q(\xi(\theta)))^3},\tag{4}$$

see also [4]. Due to the assumed periodicity we may as well write "$t\in\mathbb{R}$" instead of "$t\in[0,T]$". According to (2),

$$\hat\rho_{(1)}(k)=0\quad\text{for}\quad k\in\mathbb{R},\ |k|=a>0,\quad \hat\rho_{(1)}(k)\neq 0,\quad |k|\neq a.\tag{5}$$

From (4) and (5) we want to conclude that $T=(2\pi/a)n$ for some $n\in\mathbb{N}$. We will use the following lemma, the proof of which is certainly known, but since we did't find a precise reference we will include some details.

L e m m a 2. *If $\rho_{(1)}\in C_0^\infty(\mathbb{R})$ satisfies (5), and if $g\in C_b(\mathbb{R})$ is such that*

$$\Big(\rho_{(1)}*g\Big)(t)=0,\quad t\in\mathbb{R},$$

then either $g\equiv 0$ or $g(t)=A\sin(at+\theta)$, $t\in\mathbb{R}$, for some $A,\theta\in\mathbb{R}$.

P r o o f. (of Lemma 2). Define the finite measure $\mu(A)=\int_A\rho_{(1)}(\xi)\,d\xi$ for $A\in\mathcal{B}(\mathbb{R})$, the Borel σ-algebra on $\mathbb{R}$. Then $\rho_{(1)}*g\equiv 0$ can be rewritten as $\mu*g\equiv 0$. With the notation $\sigma(g)=\bigcap_{f\in N(g)}\{k\in\mathbb{R}:\hat f(k)=0\}$, where $N(g)=\{f\in L^1(\mathbb{R};\mathbb{C}):f*g\equiv 0\}$, it is thus known from [3; Thm. 15.4.6(vii)] that

$$\sigma(g)\subset\{k\in\mathbb{R}:\hat\mu(k)=0\}=\{k\in\mathbb{R}:\hat\rho_{(1)}(k)=0\}=\{-a,+a\},$$

the latter by (5). Hence there are not many possibilities for the spectrum $\sigma(g)$ of g. If $\sigma(g)=\emptyset$, then $g\equiv 0$ by [3; Thm. 15.4.7]. In case that $\sigma(g)=\{-a\}$, $\sigma(g)=\{a\}$, or $\sigma(g)=\{-a,a\}$, it follows from [3; Thm. 15.4.17(iii)] that g has the form $g(t)=\alpha e^{-iat}+\beta e^{iat}$ for some $\alpha,\beta\in\mathbb{C}$. Since g is real-valued, we arrive at

$$g(t)=[\mathrm{Re}\,\alpha+\mathrm{Re}\,\beta]\cos(at)+[\mathrm{Im}\,\beta-\mathrm{Im}\,\alpha]\sin(at)=A\sin(at+\theta),\quad t\in\mathbb{R},$$

for suitable A,θ. This completes the proof of Lemma 2.

To continue with the proof of Theorem 1, we now can apply Lemma 2 to each $g_\omega(\cdot)$ from (4). Hence for each $|\omega|=1$ we find $A_\omega,\theta_\omega\in\mathbb{R}$ such that

$$\frac{\omega\cdot\ddot q(\xi(\theta))}{(1-\omega\cdot\dot q(\xi(\theta)))^3}=g_\omega(\theta)=A_\omega\sin(a\theta+\theta_\omega),\quad\theta\in\mathbb{R}.$$

Setting $\theta=\theta(\xi)$, the inverse of $\theta\mapsto\xi(\theta)$, we infer

$$\frac{\omega\cdot\ddot q(t)}{(1-\omega\cdot\dot q(t))^3}=A_\omega\sin(a[t-\omega\cdot q(t)]+\theta_\omega),\quad t\in\mathbb{R},\tag{6}$$

where we wrote t instead of ξ. To deduce further information from this implicit ODE for q, we re-introduce $u(t)=t-\omega\cdot q(t)$, i.e., $u=\theta$. Then (6) reads as

$$\ddot u+A_\omega\sin(au+\theta_\omega)\dot u^3=0,\quad t\in\mathbb{R}.$$

Writing this as $\dot{u} = z$, $\dot{z} = -A_\omega \sin(au + \theta_\omega) z^3$, it follows that

$$\frac{du}{dz} = -\frac{1}{A_\omega \sin(au + \theta_\omega) z^2}.$$

Hence integration reveals that for all $|\omega| = 1$ there exists $C_\omega \in \mathbb{R}$ such that

$$\omega \cdot q(t) = A_\omega \sin(a[t - \omega \cdot q(t)] + \theta_\omega) + C_\omega, \quad t \in \mathbb{R}. \tag{7}$$

This finally gives the claim of the theorem, since due to T-periodicity of q for all $t \in \mathbb{R}$

$$\begin{aligned} A_\omega \sin(aT + a[t - \omega \cdot q(t)] + \theta_\omega) &= A_\omega \sin(a[t + T - \omega \cdot q(t+T)] + \theta_\omega) = \omega \cdot q(t+T) - C_\omega \\ &= \omega \cdot q(t) - C_\omega = A_\omega \sin(a[t - \omega \cdot q(t)] + \theta_\omega). \end{aligned} \tag{8}$$

From (7) we conclude that it is not possible to have $A_\omega = 0$ for all $|\omega| = 1$, as in this case

$$\frac{4\pi}{3}\, q(t) = \int_{|\omega|=1} d^2\omega\, (\omega \cdot q(t))\omega = \int_{|\omega|=1} d^2\omega\, C_\omega\, \omega$$

is independent of t, i.e., q is constant. Thus for one $|\omega| = 1$ with $A_\omega \neq 0$ it follows from (8) that

$$\sin(aT + \bar{u}(t)) = \sin(\bar{u}(t)), \quad t \in \mathbb{R},$$

with $\bar{u}(t) = a[t - \omega \cdot q(t)] + \theta_\omega$, $t \in \mathbb{R}$. Since $\bar{u} : \mathbb{R} \to \mathbb{R}$ is onto, we finally find

$$\sin(aT + \tau) = \sin(\tau), \quad \tau \in \mathbb{R},$$

hence $T = (2\pi/a)n$ for some $n \in \mathbb{N}$ as was to be shown.

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3. References

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