Simplified models of electromagnetic and gravitational radiation damping

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Abstract

In previous work the authors analysed the global properties of an approximate model of radiation damping for charged particles. This work is put into context and related to the original motivation of understanding approximations used in the study of gravitational radiation damping. It is examined to what extent the results obtained previously depend on the particular model chosen. Comparisons are made with other models for gravitational and electromagnetic fields. The relation of the kinetic model for which theorems were proved to certain many-particle models with radiation damping is exhibited.

1 Introduction

The study of gravitational radiation in general relativity relies in an essential way on sophisticated approximation methods. It is important to understand the relation between the approximate models and the exact theory. Many of the issues arising in this context have recently been surveyed by Blanchet [3]. The relation between the exact and approximate equations is relatively straightforward. What is much more difficult is to establish a relation between solutions of the two sets of equations.

A standard approach to the theory of gravitational radiation (see e.g. [5], [7], [3]) is to combine post-Newtonian and post-Minkowskian approximations via matching. There are some mathematical results available on these individual types of approximation. It has been shown that under suitable conditions the post-Minkowskian approximations are asymptotic [8] but divergent [19]. The post-Newtonian approximations are more difficult to handle. A general mathematical framework for formulating the relevant questions was set up in [20]. It was shown how the well-known divergences of higher order post-Newtonian approximations could be understood within that approach. This framework also provided a basis for showing that the post-Newtonian expansion is asymptotic at the lowest order (Newtonian) level [21]. It is reasonable to expect that analogous results could be proved for the 1PN and 2PN levels but this has not been done yet.

None of the results mentioned above prove anything about matching the two approximations and until this can be done the possibilities of understanding anything about radiation (even at the quadrupole level) in a rigorous way are very limited. In the following these limits will not be removed. A more modest goal is pursued, which is to understand some of the effective models for radiation themselves while postponing the question of their relation to the exact equations. Radiation comes in at the 2.5PN level, but there are hybrid models where radiation at that level is combined with a description of matter and non-radiative gravitational fields at the Newtonian or 1PN level. These models will be referred to as (0+2.5)PN and (1+2.5)PN models.

For purposes of comparison it is valuable to consider the case of charged matter emitting electromagnetic radiation as an analogue of matter emitting gravitational radiation. In this case the post-Minkowskian expansion becomes trivial due to the linearity of the Maxwell field while there is a non-trivial analogue of the post-Newtonian expansion. In the electromagnetic case radiation first appears at the 1.5PN level and it is possible to define (0+1.5)PN and (1+1.5)PN models. These will be referred to as the Coulomb and Darwin cases respectively.

In [15] the authors proved theorems on the global existence and asymptotic behaviour of solutions of a certain model of the (0+1.5)PN type in the electromagnetic case. The purpose of the present paper is to put this mathematical result into a wider context and to discuss its ramifications. For technical details of the proof the reader is referred to [15].

A general mathematical setting for expansions of PN type is as follows. Matter is described by a function $f(\lambda)$ depending on a parameter λ belonging to an interval $(0, \lambda_0]$. The field (gravitational or electromagnetic) is described by a function $g(\lambda)$. The idea is to consider a one-parameter family of solutions which become non-relativistic in the limit $\lambda \to 0$ in some appropriate sense. The parameter λ corresponds to c^{-2} where c is the speed of light. More precisely, the given functions f and g describe a one-parameter family of physical systems represented in suitable parameter-dependent units. Then c is to be interpreted as the numerical value of the speed of light in the given units. A more conventional physical description of the post-Newtonian expansion would say, for instance, that velocities of the matter are small with respect to the speed of light. The connection between the two descriptions is that to say that the matter velocity is of order unity and the speed of light large in one system of units is equivalent to saying that the matter velocity is small and the speed of light has a fixed value in a different (fixed) system of units.

In our discussion of expansions for charged matter we will start with the following formal expressions:

$$E = E_0 + \lambda^{1/2}E_1 + \lambda E_2 + \lambda^{3/2}E_3 + \dots, \tag{1}$$
$$B = B_0 + \lambda^{1/2}B_1 + \lambda B_2 + \lambda^{3/2}B_3 + \dots, \tag{2}$$
$$\rho = \rho_0 + \lambda^{1/2}\rho_1 + \lambda \rho_2 + \lambda^{3/2}\rho_3 + \dots, \tag{3}$$
$$j = j_0 + \lambda^{1/2}j_1 + \lambda j_2 + \lambda^{3/2}j_3 + \dots. \tag{4}$$

The use of expansions in half-integral powers of a parameter λ ensures that the notation fits with the usual terminology of post-Newtonian expansions. If we write $\lambda = c^{-2}$ then the Maxwell equations are:

$$\mathrm{div}E = 4\pi\rho, \quad \mathrm{curl}E = -\lambda^{1/2}\partial_t B, \tag{5}$$
$$\mathrm{div}B = 0, \quad \mathrm{curl}B = \lambda^{1/2}\partial_t E + 4\pi\lambda^{1/2}j. \tag{6}$$

Substituting the expansions into the equations and comparing coefficients gives a sequence of equations for these coefficients. Note that $\mathrm{div}B_0 = 0$ and $\mathrm{curl}B_0 = 0$. It follows that assuming fields defined on $\mathbf{R}^3$ and vanishing at infinity $B_0 = 0$. This condition will be assumed from now on. The first equations for coefficients of E of the sequence (ordering by powers of λ) are $\mathrm{div}E_0 = 4\pi\rho_0$ and $\mathrm{curl}E_0 = 0$. The second equation implies that E_0 is the gradient of a function U and then the first equation implies that $\Delta U = 4\pi\rho_0$. Thus the Poisson equation of electrostatics is recovered, with U being the electrostatic potential.

Next a remark will be made concerning the coefficients of the odd half-integer powers in the expansion of E, ρ and j and the integer powers in the expansion of B. These can be consistently set to zero. In fact if the coefficients belonging to this class are set to zero in the matter quantities then (assuming fields defined on $\mathbf{R}^3$ and vanishing at infinity) the coefficients of the fields belonging to this class automatically vanish. In this section it will be assumed that all these coefficients are identically zero. They will, however, be important in the later section 3. In a pure post-Newtonian expansion like the above they play no essential role and if they are maintained they just lead to 'shadow equations' as discussed in [20].

Consider next B_1. It satisfies the equations $\mathrm{div}B_1 = 0$ and $\mathrm{curl}B_1 = \partial_t E_0 + 4\pi j_0$. Combining these equations and using the fact that $\mathrm{curl}E_0 = \mathrm{div}B_1 = 0$ gives $\Delta B_1 = -4\pi\mathrm{curl}j_0$. Thus B_1 solves a Poisson equation with a source determined by the lowest order coefficient in the expansion of the current. It is easily shown, using the asymptotic conditions assumed above, that the solution of this Poisson equation satisfies the original first order equations. Next combining the equations $\mathrm{div}E_2 = 4\pi\rho_2$ and $\mathrm{curl}E_2 = -\partial_t B_1$ and substituting for $\mathrm{curl}B_1$ leads to the equation

$$\Delta E_2 = \partial_t^2 E_0 + 4\pi(\nabla\rho_2 + \partial_t j_0). \tag{7}$$

Assuming that the matter variables have compact support for each fixed value of t the second term on the right hand side of this equation also has compact support and leads to no difficulties in solving this Poisson equation. It remains to discuss the first term, which is not compactly supported. Assume that the matter sources satisfy the continuity equation $\partial_t\rho + \mathrm{div}j = 0$. Then the total charge, $\int \rho(x)dx$, is independent of time and the first term on the right hand side of the Poisson equation for E_2 is $O(r^{-3})$. It follows (for example from results in the appendix of [20]) that there is a solution for E_2 which vanishes at infinity and is unique in a suitable function space. The case of B_3 is more problematic. According to the assumptions made up to now it should satisfy the Poisson equation:

$$\Delta B_3 = \partial_t^2 B_1 - 4\pi \mathrm{curl} j_2. \tag{8}$$

Once again the second term is unproblematic, but the first needs to be looked at more closely since it might have a contribution which only falls off like r^{-1} as $r \to \infty$. This contribution is made by the second time derivative of the quantity $\int j_0(x)dx$. Using the continuity equation and partial integration the latter quantity can be rewritten as $-(d/dt)\int x\rho_0(x)dx$. This is (up to sign) the first time derivative of the dipole moment $D(t)$ of the charge distribution at the Coulomb level. Thus the coefficient which is problematic is d^3D/dt^3 and this is a quantity which comes up in the description of radiation damping. What has happened here, roughly speaking, is that with B_3 which is the coefficient in a term of order $\lambda^{3/2}$ we have reached the level at which radiation damping plays a role and the naive post-Newtonian expansion breaks down. A correct analysis using matching would show that the asymptotic conditions to be imposed on B_3 are different from the vanishing of that quantity at infinity. With a right hand side of the given form the Poisson integral will diverge and it is not to be expected that the Poisson equation for B_3 will have a solution which vanishes at infinity. In any case it does not have a solution in the kind of weighted spaces introduced in [20]. Thus the expansion here breaks down at the 1.5PN level in much the same way as the post-Newtonian expansion for gravity breaks down at the 3PN level (as shown in [20]).

Adopting terminology adapted to that used in [10] we call a solution of the equations for E_0 the Coulomb approximation, a solution of the equations for E_0 and B_1 the quasi-electrostatic approximation, and a solution of the equations for E_0, B_1 and E_2 the Darwin approximation.

In order to throw some more light on the particular form of the expansions chosen it is useful to reexpress the Maxwell equations in four-dimensional form. The Maxwell equations in Minkowski space are $\partial_\alpha F^\alpha{}_\beta = 4\pi j_\beta$ and $\partial_\alpha F_{\beta\gamma} + \partial_\gamma F_{\alpha\beta} + \partial_\beta F_{\gamma\alpha} = 0$. Here $F^{\alpha\beta}$ is the field tensor and j^α is the four-current. The expansions above can be written in terms of one-parameter families of these quantities. Since we are interested in the Newtonian limit it is useful to introduce a family of metrics $f_{\alpha\beta}(\lambda)$ as in [20] by $f_{00} = -1$, $f_{0a} = 0$, $f_{ab} = \lambda\delta_{ab}$. All these metrics are flat and in fact represent the Minkowski metric in different units. As basic quantities we take one-parameter families $F^\alpha{}_\beta(\lambda)$ and $j^\alpha(\lambda)$ defined for $\lambda > 0$ and assume that they have limits for $\lambda \to 0$. The tensor $f_{\alpha\beta}(\lambda)$ also has a limit for $\lambda \to 0$ but the limit is not a Lorentz metric. In general indices will be raised and lowered using the metric $f_{\alpha\beta}$ and its inverse.

Define $E^a = F^a{}_0$, $B^1 = \lambda^{-1/2}F^2{}_3$, $B^2 = \lambda^{-1/2}F^3{}_1$, $B^3 = \lambda^{-1/2}F^1{}_2$. The last three relations can also be expressed in terms of the spatial volume form associated to f, call it ϵ_f. Using this we can write $B^a = (\epsilon_f)^a{}_b{}^c F^b{}_c$. In general situations will be considered where ${F^a}_0$ has a non-zero limit and then the same is true of E^a. On the other hand it will be assumed that ${F^a}_b$ is $O(\lambda)$ as $\lambda \to 0$ and then B^a is automatically $O(\lambda^{1/2})$. Define $\rho = j^0$ and take the current in the three-dimensional form of the Maxwell equations to have components j^a. With these definitions the three- and four-dimensional forms of the Maxwell equations are equivalent and the expansions of E, B and j above correspond to expanding ${F^\alpha}_\beta$ in integral powers of λ, starting with the power zero, and assuming that the first coefficient in the expansion is zero when both α and β are spatial. The correctness of this statement depends on the fact that the coefficients of the odd half-integer powers in the expansion of E, ρ and j and the integer powers in the expansion of B have been set to zero. Radiation damping comes in at order $\lambda^{3/2}$ and this is not allowed for by the expansion in integral powers of λ. However as long as we are doing an expansion of post-Newtonian type it is the case that if j^α contains only integral powers of λ the same is true of ${F^\alpha}_\beta$.

In the practical applications of post-Newtonian approximations it is usual to expand the fields but not the matter quantities. This point has been discussed briefly in section 6 of [20]. What this means in effect is that we are not talking about functions which solve a certain system of equations exactly but rather about parameter-dependent families of functions which satisfy certain equations up to an error of the order of a power of the parameter. Related to this is the fact that the equations themselves are often only fixed up to an error of a certain order. In reality we are dealing with equivalence classes of equations. This point will be illustrated by comparing the Darwin model introduced by Degond and Raviart in [10] with the Darwin approximation introduced above. Degond and Raviart split the electric field into transverse and longitudinal parts and set the transverse part of E to zero in the Maxwell equation containing $\mathrm{curl}B$ to obtain the Darwin model. We claim that at the level of what we call the Darwin approximation the expansions of the Darwin model of [10] and the full Maxwell equations are identical. Thus at that level both sets of equations are in the same equivalence class. In terms of our expansion the transverse part of E is that arising from the equations for $\mathrm{curl}E_n$ while the equation containing $\mathrm{curl}B$ gives rise to the equations for $\mathrm{curl}B_n$. The only coefficient B_n which occurs at the level of the Darwin approximation is B_1, in the equation $\mathrm{curl}B_1$ the only contribution from the electric field comes from E_0, and E_0 has no transverse part. Thus on this level the passage from the full Maxwell equations to the Darwin model has no effect.

Up to this point we have concentrated on the fields and said little about the matter. The equations of motion for charged particles contain the Lorentz force $F_L = E+\lambda^{1/2}v\times B$. In terms of the expansion coefficients this gives $E_0+\lambda(E_2+v\times B_1)$ as the contribution up to order λ. The most common choices of matter models when discussing approximation schemes are point particles or a perfect fluid. In general relativity the concept of a point particle is very problematic and can itself only be defined in an approximate sense. A perfect fluid is better, but when it comes to obtaining mathematically rigorous results it also has its problems. On the one hand there is the problem of the formation of shocks, which is always a danger when considering solutions of the Euler equations on a long time scale. When dealing with radiation we do wish to consider the evolution of the system on a long time scale. Once shocks are formed we enter a regime where little control is possible with known mathematical techniques. The other problem concerns the question of fluid bodies. In studying radiation we want to consider isolated systems. The easiest thing would be to take the fluid to have compact support. Unfortunately the Cauchy problem for solutions of the Euler equations with compactly supported initial density is poorly understood, even locally in time. There has been some progress, since Wu [24] proved a local existence theorem for the incompressible Euler equations with free boundary in the irrotational case. Christodoulou and Lindblad [6] proved estimates for the case with rotation which will hopefully lead to an analogous result in that situation. However much remains to be understood in that area. It might be thought that replacing compactly supported initial data with data which fall off at infinity might simplify the Cauchy problem for a fluid body but nobody has managed to take advantage of this yet.

In Section 2 models of radiation which combine expansions at PN levels which belong to different powers of the expansion parameter ('hybrid models') are discussed. After reviewing the results of [15] on one model of this type for electromagnetic radiation we discuss various issues relating to the possibility of generalizing those results to related models for electromagnetism and gravitation. In the model of [15] the matter is described by kinetic theory. Section 3 of this paper relates the kinetic model of [15] to the underlying dynamics of a many-particle system with radiation reaction, thus providing a better justification for the kinetic model which was originally introduced in an ad hoc manner. The paper concludes with a brief outlook.

2 Hybrid models

By a hybrid model we mean a model where an expansion of post-Newtonian type up to a certain order is combined with effects of radiation which are formally of higher order in the expansion parameter λ. Some variants of this procedure were already mentioned in the introduction. The model studied in [15] was for the case of the electromagnetic field with a kinetic description of the matter at the Newtonian level and dipole radiation at the 1.5PN level. It was inspired by a model of Blanchet, Damour and Schäfer [4] for the case of the gravitational field with a hydrodynamic description of matter at either the Newtonian or 1PN level and quadrupole radiation at the 2.5PN level. The replacement of the fluid by kinetic theory was motivated by the difficulties with a mathematical treatment of fluids mentioned in the introduction. The replacement of the gravitational by the electromagnetic field was motivated by simplicity. The extent to which the particular choice of model in [15] was essential in obtaining results will be discussed below.

In [4] an initial model with higher order (in fact fifth order) time derivatives which could not be expected to have a good initial value formulation was modified by two methods for reducing the order of the time derivatives. The resulting model is not equivalent to the original one but is in the same equivalence class in a sense already indicated. In other words the equations of the two models agree up to terms which are regarded as being of higher order at the given level of approximation. One method of reduction is to do an appropriate change of matter variables. The other is to substitute the equations of motion into the undesirable time derivatives and to discard terms which arise from this which are formally of higher order. In [4] the first method is used to remove two time derivatives and the second to remove the remaining three. Thus we could call this a (0+2.5)PN or (1+2.5)PN model with 3+2 reduction of time derivatives. The model of [15] is a (0+1.5)PN model with 2+1 reduction of time derivatives.

The results of [15] will now be stated. For reasons explained in that paper it is necessary to have species of particles with different charge to mass ratios in order to get an interesting result. The particular choice made was to have two species of particles with unit mass and charges of unit magnitude and opposite sign. The phase space densities of these two species of particles are described by functions f^+ and f^-. These functions satisfy the Vlasov equation with a force term which is the sum of a Coulomb term resulting from a potential U and a radiation reaction term. The primitive form of the latter term is (up to sign) $\epsilon d^3 D/dt^3$ where

$$D(t) = \int x\,(\rho^+(t,x) - \rho^-(t,x))\,dx \tag{9}$$

is the dipole moment of the charge distribution and ϵ is a small constant. The charge distribution itself is defined to be

$$\rho(t,x) = \rho^+(t,x) - \rho^-(t,x) = \int f^+(t,x,v) - f^-(t,x,v)\,dv. \tag{10}$$

The primitive form is reduced by procedures of the type already mentioned in such a way that the equations for f^+ and f^- are of the form

$$\partial_t f^+ + (v + \epsilon D^{[2]}(t))\cdot\nabla_x f^+ + \nabla U\cdot\nabla_v f^+ = 0, \tag{11}$$

$$\partial_t f^- + (v - \epsilon D^{[2]}(t))\cdot\nabla_x f^- - \nabla U\cdot\nabla_v f^- = 0, \tag{12}$$

where

$$D^{[2]}(t) = \int \nabla U(t,x)(\rho^+(t,x) + \rho^-(t,x))\,dx. \tag{13}$$

The full system of equations for U and f is called the VPD system (Vlasov-Poisson with damping). Define the total energy $\mathcal{E}$ of the system by

$$\mathcal{E}(t) = \mathcal{E}_{\mathrm{kin}}(t) + \mathcal{E}_{\mathrm{pot}}(t), \tag{14}$$

with

$$\begin{aligned}
\mathcal{E}_{\mathrm{kin}}(t) &= \frac{1}{2}\int\int |v|^2(f^+(t,x,v) + f^-(t,x,v))\,dxdv, \quad \text{and} \\
\mathcal{E}_{\mathrm{pot}}(t) &= \frac{1}{8\pi}\int |\nabla U(t,x)|^2\,dx,
\end{aligned} \tag{15}$$

denoting kinetic and potential energy, respectively. With these preliminaries in hand we can state the main theorem of [15].

Theorem 1 *If $f_0^\pm$ are smooth initial data with compact support for the VPD system, then there is a unique smooth solution $f^\pm$ of VPD for $t \geq 0$ with data $f^\pm(t=0) = f_0^\pm$. The energy evolves according to*

$$\dot{\mathcal{E}}(t) = -\epsilon \left|D^{[2]}(t)\right|^2. \tag{16}$$

Moreover, the following estimates hold for $t \in [0, \infty[$.

(a) $\|\rho^\pm(t)\|_p \leq C(1+t)^{-\frac{3(p-1)}{2p}}$ *for* $p \in [1, \frac{5}{3}]$;

(b) $\|\nabla U(t)\|_p \leq C(1+t)^{-\frac{5p-3}{7p}}$ *for* $p \in [2, \frac{15}{4}]$;

(c) $|D^{[2]}(t)| \leq C(1+t)^{-\frac{8}{7}}$.

Here $\| \|_p$ denotes the L^p norm.

Thus we obtain a global in time existence theorem and statements about how various quantities decay as $t \to \infty$. The theorem was proved by adapting methods previously applied to the usual Vlasov-Poisson system by Lions and Perthame [18] (global existence) and Illner and Rein [13] (asymptotic decay).

Next we consider whether similar results can be obtained for other hybrid models. First we give the equations for some of these models. Starting with the primitive form of the equations for charged particles described by the Vlasov equation at the (0+1.5)PN level, a 1+2 reduction can be carried out. This means introducing new variables in the characteristic equations by $\tilde{X}^\pm = X^\pm \mp \epsilon \dot{D}(t)$ and $\tilde{V}^\pm = V^\pm \mp \epsilon \ddot{D}(t)$ with the result that the new characteristic equations are $\dot{\tilde{X}}^\pm = \tilde{V}^\pm$ and $\dot{\tilde{V}}^\pm = -\nabla U(t, \tilde{X}^\pm \pm \epsilon \dot{D}(t))$. The Vlasov equation with these characteristics has the form

$$\partial_t f^\pm + v \cdot \nabla_x f^\pm - \nabla U(t, x \pm \epsilon \dot{D}(t)) \cdot \nabla_v f^\pm = 0. \tag{17}$$

Then the other method must be applied to replace $\dot{D}(t)$ by $D^{[1]}(t)$, where

$$D^{[1]}(t) = \int\int v(f^+ - f^-) dx dv,$$

cf. section 1 of [15]. This defines the model. There is no obvious obstacle to proving local existence for this case. The other hybrid model obtained from the same starting point is that resulting from a 3+0 reduction. In that case the Vlasov equation is of the form

$$\partial_t f^\pm + v \cdot \nabla_x f^\pm \pm (\nabla U(t, x) + \epsilon D^{[3]}(t)) \cdot \nabla_v f^\pm = 0 \tag{18}$$

where

$$D^{[3]}(t) = \int\int \nabla \dot{U}(f^+ + f^-) dx dv + \int\int \nabla\nabla U \cdot v(f^+ + f^-) dx dv. \tag{19}$$

In this case proving local existence may be more difficult, due to the second derivatives of U occurring in the Vlasov equation. There is a loss of one derivative compared to the 2+1 and 1+2 reductions.

The consequences of the lost derivative will now be examined in some more detail. In order to get a local existence theorem the orders of differentiability of the different unknowns must fit together properly. Suppose that U is k times differentiable. Then f, and hence ρ will be $k-2$ times differentiable. Thus, in order that a match be obtained, two derivatives must be gained when solving the Poisson equation. It is well known (see e.g. [11]) that this is not true in the context of pointwise differentiability properties. It is true when regularity is measured in Hölder or Sobolev spaces [11]. Thus in order to get a local existence theorem it is either necessary to work with spaces of that kind or to see that at some point a derivative can be gained in comparison to the naive counting just given. This appears as a borderline case for local existence.

In the case of the 2+1 reduction studied in [15] a simple and convenient formula for the energy loss of the system due to emission of radiation was obtained. For the 3+0 and 1+2 reductions things seem to be more complicated. In the model considered in [15] the rate of change of the total energy is $-\epsilon|D^{[2]}(t)|^2$, which is manifestly negative. The corresponding expression for the 3+0 reduction is $\epsilon D^{[1]}(t)\cdot D^{[3]}(t)$, which is neither manifestly negative nor obviously related to the other expression. On a formal level one link can be seen. Suppose that we replace the quantities $D^{[n]}(t)$ by the true derivatives $D^{(n)}(t)$. Then we have the identity that

$$D^{(1)}(t)\cdot D^{(3)}(t)=-|D^{(2)}(t)|^2+\partial_t(D^{(1)}(t)\cdot D^{(2)}(t)). \tag{20}$$

If we suppose that the function $D(t)$ has a suitable oscillatory behaviour then the last term will average to zero so that in an average sense the left hand side is equal to the first term on the right hand side. To turn this formal argument into a rigorous one would require at the very least a lot more information than was obtained in [15]. The relation to the energy loss in the 1+2 reduction is even less clear.

Consider next a model with gravitation and kinetic theory at the (0+2.5)PN level. In comparison to the electromagnetic case the main difference is that the third derivative of the dipole moment is replaced by the fifth derivative of the quadrupole moment. The primitive form of the radiation reaction term is (up to sign) $\epsilon(d^5Q_{ab}/dt^5)x^a$, where the quadrupole moment is given by

$$Q_{ab}(t)=\int\Big(x_ax_b-\frac{1}{3}|x|^2\delta_{ab}\Big)\rho(t,x)\,dx. \tag{21}$$

The fifth order derivative is reduced to the third order derivative by the transformation (cf. equation (10) in [15])

$$\tilde{X}=X-\epsilon Q^{(3)}(t)\cdot X, \tag{22}$$

$$\tilde{V}=V-\epsilon Q^{(4)}(t)\cdot X+\epsilon Q^{(3)}(t)\cdot V. \tag{23}$$

The transformed Vlasov equation is

$$\partial_t f + (v - 2\epsilon Q^{(3)}(t) \cdot v) \cdot \nabla_x f$$
$$-[\nabla U(t, x + \epsilon Q^{(3)}(t)) - \epsilon Q^{(3)}(t) \cdot \nabla U(t,x)] \cdot \nabla_v f = 0. \tag{24}$$

In doing this transformation terms which are formally of order ϵ^2 have been discarded. The next step in the 3+2 reduction of the gravitational case is to replace the time derivative $Q^{(3)}$ by the reduced quantity $Q^{[3]}$. This gives rise to second derivatives of the potential as in the 3+0 reduction of the electromagnetic case. To write the expression for the reduced time derivative $Q^{[3]}$ in a compact form it is useful to introduce the following operation on tensors

$$\mathrm{STF}(T_{ab}) = (1/2)(T_{ab} + T_{ba}) - (1/3)\delta^{cd} T_{cd} \delta_{ab}. \tag{25}$$

It takes the symmetric trace-free part of a given tensor. Then

$$Q^{[3]}(t) = \mathrm{STF}\left(2 \int\int [(v \cdot \nabla\nabla U + \nabla \dot{U}) \otimes x + 3\nabla U \otimes v] f(t,x,v)\, dx dv\right). \tag{26}$$

This may be compared with equation (6.6) of [4] where a corresponding equation is given for the hydrodynamic case. Two of the terms above can easily be seen to correspond to terms in the equation of [4]. The sum of the remaining terms in the latter equation correspond to the term above involving $\dot{U}$ via the equations of motion. There does not seem to be a useful way to eliminate the time derivative of the potential in the case where the matter is described by kinetic theory.

As in the 3+0 reduction in the electromagnetic case the second derivatives may make proving local existence difficult. Note that a 4+1 reduction would lead to the appearance of third derivatives of the potential, the loss of two derivatives compared the model treated in [15] and, presumably, to ill-posedness of the evolution equations.

3 Comparison with N-particle systems

The VPD model considered above has been introduced as an effective model for electromagnetic radiation. The purpose of this section is to see how this model relates to the effective equations for N-particle systems of Abraham-Lorentz type [16, 17, 22] in the limit $N \to \infty$. As before we assume the fields E and B do satisfy the Maxwell equations (5), (6), and they are expanded as in (1), (2) in $\lambda^{1/2} = c^{-1}$. Contrary to section 1, however, we do not suppose a priori that the coefficients of the odd half-integer powers in the expansion of E, ρ and j and the integer powers in the expansion of B are set to zero.

3.1 The Darwin approximation

In section 1 we have already seen that the first few equations the coefficients have to satisfy are $B_0 = 0$ and

$$\Delta E_0 = 4\pi \nabla \rho_0, \quad \Delta B_1 = -4\pi\, \mathrm{curl} j_0,$$
$$\Delta E_1 = 4\pi \nabla \rho_1, \quad \Delta E_2 = 4\pi \nabla \rho_2 + \partial_t(\partial_t E_0 + 4\pi j_0).$$

We intend to derive an explicit approximation of the Lorentz force $F_L = E + \lambda^{1/2} v \times B$ up to order $\mathcal{O}(\lambda) = \mathcal{O}(c^{-2})$, given through

$$F_L = E_0 + \lambda^{1/2} E_1 + \lambda(E_2 + v \times B_1) + \mathcal{O}(\lambda^{3/2}). \tag{27}$$

Suppressing the time argument, clearly

$$E_0(x) = 4\pi\Delta^{-1}(\nabla\rho_0)(x) = \int \frac{x-y}{|x-y|^3}\,\rho_0(y)\,dy, \tag{28}$$

with the analogous formula for E_1 by replacing ρ_0 by ρ_1. Moreover,

$$\begin{aligned} B_1(x) &= -4\pi\Delta^{-1}(\mathrm{curl} j_0)(x) = -\int \frac{x-y}{|x-y|^3} \times j_0(y)\,dy \\ &= -\int \frac{x-y}{|x-y|^3} \times j(y)\,dy + \mathcal{O}(\lambda^{1/2}). \end{aligned} \tag{29}$$

To determine E_2, we note that $\partial_t \rho_0 + \mathrm{div} j_0 = 0$ in view of $\mathrm{div} E_0 = 4\pi\rho_0$ and $\mathrm{curl} B_1 = \partial_t E_0 + 4\pi j_0$, whence

$$\Delta(\partial_t E_0) = 4\pi\nabla(\partial_t\rho_0) = -4\pi\nabla(\mathrm{div} j_0).$$

Accordingly, E_2 satisfies

$$\begin{aligned} E_2 &= 4\pi\Delta^{-1}(\nabla\rho_2) + \partial_t\,\Delta^{-1}\Big(\partial_t E_0 + 4\pi j_0\Big) \\ &= 4\pi\Delta^{-1}(\nabla\rho_2) + 4\pi\partial_t\Big(-\Delta^{-2}(\nabla\mathrm{div} j_0) + \Delta^{-1} j_0\Big). \end{aligned}$$

Here we follow the device of [4, p. 296] how to solve iterated Poisson equations, and observing that $X(x) = \int |x-y|\,\sigma(y)\,d^3y$ is the solution to $\Delta^2 X = -8\pi\sigma$, after some calculation the more explicit form

$$\begin{aligned} E_2(x) &= 4\pi\Delta^{-1}(\nabla\rho_2)(x) - \frac{1}{2}\partial_t\Bigg(\int \frac{1}{|x-y|}\,j_0(y)\,dy \\ &\qquad + \int \frac{1}{|x-y|^3}\,(x-y)\cdot j_0(y)\,(x-y)\,dy\Bigg) \end{aligned} \tag{30}$$

is found. Conversely, it may be verified directly that this function E_2 obeys $\mathrm{div} E_2 = 4\pi\rho_2$ and $\mathrm{curl} E_2 = -\partial_t B_1$. Taking into account (28), (29), and (30), we have obtained an explicit representation for F_L from (27) up to order $\mathcal{O}(\lambda)$.

To make the connection with the $N \to \infty$ limit of a system of N particles we need to rewrite E_2 further and make use of the fact that this description is not unique since terms of order $\mathcal{O}(\lambda^{3/2})$ can be added or subtracted. Adopting the terminology from section 1 this means that we look for a more convenient representative in the same equivalence class of equations. From now on we assume that $f = f_0 + \lambda^{1/2} f_1 + \lambda f_2 + \ldots$ satisfies the relativistic Vlasov equation

$$\partial_t f + v\cdot\nabla_x f + (E + \lambda^{1/2}\, v \times B)\cdot\nabla_p f = 0 \tag{31}$$

coupled to the Maxwell equations (5), (6). Here $p \in \mathbf{R}^3$ denotes the momentum, thus $v = (1+\lambda p^2)^{-1/2} p$ for the velocity if we consider particles of unit mass. Therefore $v = p + \mathcal{O}(\lambda)$ implies that

$$0 = \partial_t f_0 + p \cdot \nabla_x f_0 + E_0 \cdot \nabla_p f_0 + \mathcal{O}(\lambda^{1/2}) \tag{32}$$

i.e., to lowest order $f \cong f_0$ is a solution of the usual Vlasov-Poisson system with electric field E_0 obeying $\Delta E_0 = 4\pi \nabla \rho_0$. Hence we can invoke (32) and $j_0 = \int v f_0 \, dp$ to write

$$\begin{aligned} \partial_t j_0 &= -\int p\,(p \cdot \nabla_x f_0 + E_0 \cdot \nabla_p f_0)\, dp + \mathcal{O}(\lambda^{1/2}) \\ &= -\int p(p \cdot \nabla_x f_0)\, dp + \int E_0 f_0 \, dp + \mathcal{O}(\lambda^{1/2}). \end{aligned} \tag{33}$$

Next we explicitly perform the ∂_t-differentiation in (30) and substitute the approximate expression for $\partial_t j_0$ from (33). We obtain after some simplification, including $E_0 = 4\pi \Delta^{-1}(\nabla \rho) + \mathcal{O}(\lambda^{1/2})$ and $f_0 = f + \mathcal{O}(\lambda^{1/2})$, that

$$\begin{aligned} E_2(x) &= 4\pi \Delta^{-1}(\nabla \rho_2)(x) - 2\pi \int\!\!\int \frac{1}{|x-y|}\, \Delta^{-1}(\nabla\rho)(y)\, f(y,p)\, dy\, dp \\ &\quad - 2\pi \int\!\!\int \frac{x-y}{|x-y|^3}\,(x-y)\cdot \Delta^{-1}(\nabla\rho)(y)\, f(y,p)\, dy\, dp \\ &\quad + \frac{1}{2}\int\!\!\int \left(\frac{p^2}{|x-y|^3} - \frac{3(p\cdot(x-y))^2}{|x-y|^5}\right)(x-y)\, f(y,p)\, dy\, dp \\ &\quad + \mathcal{O}(\lambda^{1/2}). \end{aligned} \tag{34}$$

In (27) the field E_2 carries the factor λ. Whence we can as well use E_2 in its form (34) rather than (30) to obtain a valid approximation of F_L up to order $\mathcal{O}(\lambda)$. In particular, (34) is more convenient to compare this approximation to what comes out from the associated particle model. In [16] it was show that the dynamics of N particles coupled to their self-generated Maxwell field can be described over long times by means of the effective equations

$$\frac{d}{dt} T_{\mathrm{kin}}(v_\alpha) = G_\alpha^{\mathrm{CD}}(q, v, \dot v) + \mathcal{O}(\lambda^{3/2}), \quad 1 \le \alpha \le N, \tag{35}$$

where T_{kin} is the kinetic energy, and the force G_α^{CD} is given by

$$\begin{aligned} &G_\alpha^{\mathrm{CD}}(q,v,\dot v) \\ &= e_\alpha \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N} e_\beta \frac{\xi_{\alpha\beta}}{|\xi_{\alpha\beta}|^3} + \lambda e_\alpha \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N} e_\beta \Bigg(-\frac{1}{2|\xi_{\alpha\beta}|}\, \dot v_\beta - \frac{(\dot v_\beta \cdot \xi_{\alpha\beta})}{2|\xi_{\alpha\beta}|^3}\, \xi_{\alpha\beta} \\ &\quad + \frac{v_\beta^2}{2|\xi_{\alpha\beta}|^3}\, \xi_{\alpha\beta} - \frac{3(v_\beta\cdot\xi_{\alpha\beta})^2}{2|\xi_{\alpha\beta}|^5}\, \xi_{\alpha\beta} - \frac{(v_\alpha \cdot v_\beta)}{|\xi_{\alpha\beta}|^3}\, \xi_{\alpha\beta} + \frac{(v_\alpha\cdot\xi_{\alpha\beta})}{|\xi_{\alpha\beta}|^3}\, v_\beta \Bigg). \end{aligned} \tag{36}$$

Here particle α has position $q_\alpha(t) \in \mathbf{R}^3$ and velocity $v_\alpha(t) = \dot{q}_\alpha(t)$. Moreover, $\xi_{\alpha\beta} = q_\alpha - q_\beta$. Note that in [16, Lemma 3.5], eq. (35) is formulated on a different scale by means of a dimensionless parameter $\epsilon \ll 1$. In this units, the error was $\mathcal{O}(\epsilon^{7/2})$, and to pass to (35) it is first necessary to multiply by ϵ^{-2} to undo the transformation, and then $\epsilon \cong \lambda$, whence the error term is of order $\mathcal{O}(\lambda^{3/2})$. In addition, in [16] the charge distribution ρ and the current j in the Maxwell equations are considered without the factor 4π. Hence to adjust to (5), (6) the expression for $G_\alpha^{\rm CD}$ had to be multiplied by 4π. We also remark that (35) holds in the limit of slow particles which are far apart, cf. [16]. The equation $\frac{d}{dt} T_{\rm kin}(v_\alpha) = G_\alpha^{\rm CD}(q, v, \dot{v})$ is governed by a suitable Lagrangian, consisting of a Coulomb and a Darwin contribution, cf. [14, Ch. 12.6] and [16, Eq. (1.1)]. Since the Abraham model considered in [16] is only semi-relativistic, the particular form of $T_{\rm kin}$ cannot be expected to agree with the corresponding expression resulting from the Vlasov-Maxwell system through expansion in powers of $\lambda^{1/2}$; see [2] for a recent improved model. However, in the formal limit $N \to \infty$ the force $G_\alpha^{\rm CD}$ coincides with $F_L \cong E_0 + \lambda^{1/2} E_1 + \lambda(E_2 + v \times B_1)$, using E_2 from (34). To see this, we fix one particle α, i.e., its position $x \cong q_\alpha \in \mathbf{R}^3$ and its velocity $v \cong v_\alpha \in \mathbf{R}^3$. Setting $e_\beta = 1$ for all β corresponding to (31), then

$$\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N} \frac{\xi_{\alpha\beta}}{|\xi_{\alpha\beta}|^3} \stackrel{N\to\infty}{\longrightarrow} \int\int \frac{x-y}{|x-y|^3} f(x,p)\, dy\, dp = 4\pi\Delta^{-1}(\nabla\rho)(x)$$

results in the Coulomb part; note that the corresponding expression in F_L is given by $E_0 + \lambda^{1/2} E_1 + \lambda 4\pi\Delta^{-1}(\nabla\rho_2) = 4\pi\Delta^{-1}(\nabla\rho) + \mathcal{O}(\lambda^{3/2})$. In addition, we obtain

$$\begin{aligned}
&\lambda \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N} \left(-\frac{(v_\alpha\cdot v_\beta)}{|\xi_{\alpha\beta}|^3}\,\xi_{\alpha\beta} + \frac{(v_\alpha\cdot\xi_{\alpha\beta})}{|\xi_{\alpha\beta}|^3}\,v_\beta \right) \\
&\stackrel{N\to\infty}{\longrightarrow} \quad \lambda \int\int \frac{dy\,dp}{|x-y|^3}\, v\times(p\times(x-y))\, f(x,p) = \lambda\, v\times B_1(x) + \mathcal{O}(\lambda^{3/2}),
\end{aligned}$$

cf. (29). To deal with the remaining part of (36) we have to reexpress the $\dot{v}_\beta$ through lower order terms. Since $\lambda(-\frac{1}{2|\xi_{\alpha\beta}|}\,\dot{v}_\beta - \frac{(\dot{v}_\beta\cdot\xi_{\alpha\beta})}{2|\xi_{\alpha\beta}|^3}\,\xi_{\alpha\beta}) = \mathcal{O}(\lambda)$, we can simply substitute

$$\dot{v}_\beta = m_\beta \dot{v}_\beta = \sum_{\substack{\nu=1\\ \nu\neq\beta}}^{N} \frac{\xi_{\beta\nu}}{|\xi_{\beta\nu}|^3} + \mathcal{O}(\lambda) \stackrel{N\to\infty}{\longrightarrow} 4\pi\Delta^{-1}(\nabla\rho)(y) + \mathcal{O}(\lambda)$$

without changing the validity of (35). Hence it follows that

$$\lambda \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N} \left(-\frac{1}{2|\xi_{\alpha\beta}|}\,\dot{v}_\beta - \frac{(\dot{v}_\beta\cdot\xi_{\alpha\beta})}{2|\xi_{\alpha\beta}|^3}\,\xi_{\alpha\beta} + \frac{v_\beta^2}{2|\xi_{\alpha\beta}|^3}\,\xi_{\alpha\beta} - \frac{3(v_\beta\cdot\xi_{\alpha\beta})^2}{2|\xi_{\alpha\beta}|^5}\,\xi_{\alpha\beta} \right)$$

$$\stackrel{N\to\infty}{\longrightarrow} \lambda \int\int \Big(-\frac{2\pi}{|x-y|}\,\Delta^{-1}(\nabla\rho)(y) - 2\pi\frac{x-y}{|x-y|^3}\,\Delta^{-1}(\nabla\rho)(y)\cdot(x-y)$$
$$+\frac{p^2}{2|x-y|^3}\,(x-y) - \frac{3(p\cdot(x-y))^2}{2|x-y|^5}\,(x-y)\Big)\, f(y,p)\,dy\,dp + \mathcal{O}(\lambda^{3/2}),$$

in agreement with (34).

To summarize, up to the Darwin order the expansion in powers $\lambda^{1/2}$ of the Vlasov-Maxwell system does agree with the infinite particle number limit of the particle system's effective dynamics.

3.2 Approximation to the order of radiation reaction

The purpose of this section is to push the expansion of the Lorentz force one step further to include the terms of order $\mathcal{O}(\lambda^{3/2})$, i.e., we consider

$$F_L = E_0 + \lambda^{1/2}E_1 + \lambda(E_2 + v\times B_1) + \lambda^{3/2}(E_3 + v\times B_2) + \mathcal{O}(\lambda^2), \qquad (37)$$

cf. (27). The equations to be solved for the coefficients E_3 and B_2 are $\mathrm{div}E_3 = 4\pi\rho_3$, $\mathrm{curl}E_3 = -\partial_t B_2$, $\mathrm{div}B_2 = 0$, and $\mathrm{curl}B_2 = \partial_t E_1 + 4\pi j_1$, and this leads to

$$\Delta E_3 = 4\pi\nabla\rho_3 + \partial_t(\partial_t E_1 + 4\pi j_1), \quad \Delta B_2 = -4\pi\,\mathrm{curl} j_1.$$

The solution of the latter equation is analogous to that of (29) and just contributes to the expansion of j. Concerning E_3, it could be treated just as E_2 was, obtaining a solution which vanished at infinity. There would be no trace of radiation reaction and this solution is not the appropriate one for obtaining an approximation of a solution of the Maxwell equations at the 1.5PN level. The equations of the pure post-Newtonian expansion of section 2 do not contain information about damping in this case. They need to be supplemented by asymptotic conditions different from the condition of vanishing at infinity. In the case of the Maxwell equations this additional information could be obtained by doing expansions of the fundamental solution different from the post-Newtonian one.

As in the previous section 3.1, the connection can be made between this result and the $N\to\infty$ limit of a system of N particles. In [17] it was proved that to the order of radiation reaction the particle dynamics is governed over long times by

$$\frac{d}{dt}\,T_{\mathrm{kin}}(v_\alpha) = G_\alpha^{\mathrm{CD}}(q,v,\dot v) + G_\alpha^{\mathrm{RR}}(q,v,\dot v) + \mathcal{O}(\lambda^2), \quad 1\le\alpha\le N,$$

cf. (35), with the new $\mathcal{O}(\lambda^{3/2})$-term

$$G_\alpha^{\mathrm{RR}}(q,v,\dot v) = \frac{e_\alpha}{12\pi}\lambda^{3/2}\sum_{\substack{\beta,\beta'=1\\ \beta\neq\beta'}}^{N} e_\beta e_{\beta'}\left(\frac{e_\beta}{m_\beta}-\frac{e_{\beta'}}{m_{\beta'}}\right)\Big(\frac{1}{|q_\beta-q_{\beta'}|^3}(v_\beta-v_{\beta'}) - \frac{3(q_\beta-q_{\beta'})\cdot(v_\beta-v_{\beta'})}{|q_\beta-q_{\beta'}|^5}\,(q_\beta-q_{\beta'})\Big) \qquad (38)$$

accounting for the radiation reaction. We do assume now that there are two species of particles of equal unit mass and of total number $N = 2M$, such that $e_\alpha = e = 1$ for $\alpha = 1, \dots, M$ and $e_\alpha = -e = -1$ for $\alpha = M+1, \dots, 2M$. The respective particle number densities are denoted by f^+ and f^-. Decomposing the $\sum_{\beta,\beta'=1,\beta\neq\beta'}^{N}$ in (38) accordingly, we see that

$$\begin{aligned} G_\alpha^{\mathrm{RR}}(q,v,\dot v) \quad &\overset{N\to\infty}{\longrightarrow} \quad \frac{1}{3\pi}\lambda^3 \int\!\!\int dx_1\, dp_1 \int\!\!\int dx_2\, dp_2\, f^+(x_1,p_1) f^-(x_2,p_2) \\ &\cdot\left(\frac{1}{|x_1-x_2|^3}(p_1-p_2) - \frac{3(x_1-x_2)\cdot(p_1-p_2)}{|x_1-x_2|^5}\,(x_1-x_2)\right). \quad (39) \end{aligned}$$

On the other hand, recalling the dipole moment $D(t)$ from (9) and $D^{[2]}(t)$ from (13) we have

$$\dot D(t) = \int\!\!\int p\,(f^+ - f^-)\,dx\,dp, \qquad \ddot D(t) = D^{[2]}(t) + \mathcal{O}(\epsilon).$$

Differentiating the latter relation w.r.t. time, and using the explicit form of the electric field ∇U obtained from $\Delta U = 4\pi\rho$ in conjunction with the Vlasov equations (11) and (12) for f^+ and f^-, we obtain through explicit evaluation that

$$\begin{aligned} &\dddot D(t) \\ &= \int\!\!\int\!\!\int\!\!\int dx_1\, dp_1\, dx_2\, dp_2\, (p_2\cdot\nabla_{x_2})\frac{x_1-x_2}{|x_1-x_2|^3} \\ &\qquad \cdot(f^+(x_2,p_2) - f^-(x_2,p_2))\,(f^+(x_1,p_1) + f^-(x_1,p_1)) \\ &\quad + \int\!\!\int dx_1\, dp_1\, (p_1\cdot\nabla)\nabla U\,(f^+ + f^-) + \mathcal{O}(\epsilon) \\ &= \int\!\!\int\!\!\int\!\!\int dx_1\, dp_1\, dx_2\, dp_2 \left(\frac{1}{|x_1-x_2|^3}(p_1-p_2)\right. \\ &\qquad \left. - \frac{3(x_1-x_2)\cdot(p_1-p_2)}{|x_1-x_2|^5}\,(x_1-x_2)\right) \\ &\qquad \cdot(f^+(x_2,p_2) - f^-(x_2,p_2))\,(f^+(x_1,p_1) + f^-(x_1,p_1)) + \mathcal{O}(\epsilon). \end{aligned}$$

Now we observe that using the change of variables $x_1 \leftrightarrow x_2$ and $p_1 \leftrightarrow p_2$ it is clear that the term containing $f^+(x_2,p_2)f^+(x_1,p_1)$ as well as the one containing $f^-(x_2,p_2)f^-(x_1,p_1)$ does vanish, whereas the other two terms are the same and add up. Consequently we finally find

$$\begin{aligned} \dddot D(t) \quad = \quad & 2\int\!\!\int\!\!\int\!\!\int dx_1\, dp_1\, dx_2\, dp_2 \left(\frac{1}{|x_1-x_2|^3}(p_1-p_2)\right. \\ & \left. - \frac{3(x_1-x_2)\cdot(p_1-p_2)}{|x_1-x_2|^5}\,(x_1-x_2)\right) f^+(x_1,p_1) f^-(x_2,p_2). \end{aligned}$$

Comparing this result to (39) we see that the VPD model considered in section 2 is a quite reasonable one to describe electromagnetic radiation reaction.

4 Outlook

In this paper various models of radiation damping and their mutual relations have been discussed. Potential difficulties in establishing mathematical results about these models have been pointed out. In the models of section 1 the force on a particle is determined instantaneously by the distribution of all particles. These were self-contained models of continuum mechanics. Another type of model for radiation damping has been studied in the literature. In that case the description of matter is a schematic one (one or more oscillators) while the field satisfies a wave equation. Examples are a model of Burke [5] and one of Aichelburg and Beig [1] whose mathematical properties were further studied in [12] and [23]. It would be interesting to make connections between the latter models and those discussed in the present paper.

The central open question concerning the models for electromagnetic radiation damping in section 2 is that of proving that solutions of these models approximate solutions of the Vlasov-Maxwell system in some appropriate sense. The results of section 3 may help to give indications how results of this kind should be formulated and proved. However all that was done up to now was to point out a formal similarity between the equations. If a complete rigorous analysis of the case of the Vlasov-Maxwell system (or even of a model such as that of [5]) were obtained then it could serve as a basis for an attack on the much harder case of radiation damping for the Einstein equations.

The notion of radiation damping has a central function in making predictions about the emission of gravitational waves by physical sources on the basis of general relativity. Thus it enters essentially into the theoretical background for gravitational wave physics and efforts to put it on a firm mathematical basis are amply motivated. We hope that the results of this paper, including those which pinpoint what is not known, open an avenue for progress towards a rigorous treatment of gravitational radiation damping.

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