Post-Coulombian Dynamics at Order c^{-3}

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Abstract

We study the dynamics of N charges interacting with the Maxwell field. If their initial velocities are small compared to the velocity of light, c, then in lowest order their motion is governed by the static Coulomb Lagrangian. We investigate higher order corrections with an explicit control on the error terms. The Darwin correction, order $|v/c|^2$, has been proved previously. In this contribution we obtain the dissipative corrections due to radiation damping, which are of order $|v/c|^3$ relative to the Coulomb dynamics. If all particles have the same charge-to-mass ratio, the dissipation would vanish at that order.

1 Introduction

Experimental general relativity is at the edge of taking the lead in ultraprecision, surpassing even the famous measurements of the anomalous magnetic moment of the electron [4]. The best-studied test case is provided by the Hulse-Taylor binary pulsar PSR 1913+16, which consists of two neutron stars, each roughly of one solar mass and with a radius of 10 km. The stars are a distance 2.6×10^6 km apart and revolve around their common center of mass with a period of about 7h 45min at a speed of $|v/c| \cong 10^{-3}$ (with c the velocity of light). One of the stars rotates around its own body axis and emits a precisely pulsed radio wave which can be detected on earth, thus providing an indirect measurement of the orbital motion [8, 21].

On the theoretical side one has to solve Einstein's equations with matter such that the mass is well-concentrated in the two neutron stars. Since $|v/c| \ll 1$, a natural strategy is to expand the metric in powers of $|v/c|$. The zero order contribution corresponds to the non-relativistic limit, where the stars move on the Kepler orbits of the Newtonian theory of gravity. Therefore higher order corrections are commonly called "post-Newtonian" and they are counted in powers of $|v/c|^2$. To first post-Newtonian order, $|v/c|^2$, the motion of the binary pulsar is governed by additional velocity dependent forces, and this order is followed by corrections, order $|v/c|^4$, which are still of conservative (Hamiltonian) nature. Damping through the emission of gravitational waves appears at order $|v/c|^5$, which roughly means a correction of order 10^{-15} relative to the Kepler orbit. The predictions of the theory and the observed minute shrinking of the orbit agree within 0.3%. It is even claimed that improved experimental devices would yield a precision of order $|v/c|^{11}$, see [22]. At present theoretical studies of 3.5 post-Newtonian dynamics are available, cf. [5] and the references therein.

From a mathematical point of view one would like to establish that the true orbit of the neutron stars, as governed by Einstein's equations, is well-approximated by the solution of the effective second order differential equation at the appropriate post-Newtonian order. While order zero has been accomplished in [16] for asymptotically flat geometry, any further progress seems difficult at this point. In fact, since in the relativistic context there is no sufficiently general theory for the existence of solutions, even the notion of a true orbit is somewhat vague. Therefore in the present paper we propose to investigate a very similar, but considerably less involved problem where the neutron stars are replaced by charges and the gravitational field is replaced by the Maxwell field. Of course the physics is then completely different, but as a theoretical problem most qualitative features are maintained with the welcome simplification that the equations for the electromagnetic field are linear, in contrast to Einstein's equations. Moreover the matter field can be modeled through a rigid charge distribution. For a few charges a realization as clean as the Hulse-Taylor pulsar seems to be difficult. For a single charge in a Penning trap the radiation damping is measured through the shrinking amplitude of the cyclotron mode [4, 20], while two charges of opposite sign will rapidly form a neutral atom which is governed by the laws of quantum mechanics. Despite this fact we believe that our approach will improve on the understanding of how radiation damping emerges from a fully microscopic Hamiltonian system for the interaction of matter with a wave field.

The dynamical system we will study consists of Maxwell's equations for the electromagnetic field,

$$c^{-1}\frac{\partial}{\partial t}B(x,t) = -\nabla \wedge E(x,t), \quad c^{-1}\frac{\partial}{\partial t}E(x,t) = \nabla \wedge B(x,t) - c^{-1}j(x,t), \tag{1.1}$$

$$\nabla \cdot E(x,t) = \rho(x,t), \quad \nabla \cdot B(x,t) = 0, \tag{1.2}$$

and the Lorentz force equations for the charges,

$$\frac{d}{dt}\Big(m_{\mathrm{b}\alpha}\gamma_\alpha v_\alpha(t)\Big) = e_\alpha\Big(E_\varphi(q_\alpha(t),t) + c^{-1}\, v_\alpha(t)\wedge B_\varphi(q_\alpha(t),t)\Big), \quad \alpha = 1,\dots,N. \tag{1.3}$$

Our above example corresponds to two charges, but a general number $N \geq 1$ will make little difference, except that some solutions of the Coulomb dynamics may fail to exist globally in time. Concerning our notation, E is the electric and B the magnetic field. Moreover $q_\alpha(t)$ denotes the position of particle α, and its velocity is $v_\alpha(t) = \dot{q}_\alpha(t)$. The particle has (bare) mass $m_{\mathrm{b}\alpha}$, charge e_α, and a relativistic kinetic energy with $\gamma_\alpha = (1-(v_\alpha/c)^2)^{-1/2}$. Each particle carries a rigid charge distribution as given by the form factor φ, which we assume to be smooth, radial, compactly supported, and normalized, i.e.,

$$0 \leq \varphi \in C_0^\infty(\mathrm{I\!R}^3)\,, \quad \varphi(x) = \varphi_r(|x|)\,, \quad \varphi(x) = 0 \ \text{ for } \ |x| \geq R_\varphi\,, \quad \int d^3x\,\varphi(x) = 1\,. \tag{C}$$

Then the charge and current densities generated by the charges are given by

$$\rho(x,t) = \sum_{\alpha=1}^N e_\alpha\varphi(x - q_\alpha(t)) \quad \text{and} \quad j(x,t) = \sum_{\alpha=1}^N e_\alpha\varphi(x-q_\alpha(t))v_\alpha(t), \tag{1.4}$$

which determine the source terms in Maxwell equations and thereby couple (1.1), (1.2), and (1.3). The functions E_φ and B_φ in (1.3) are the fields smeared out by φ, i.e., we introduce $E_\varphi(x,t) = \int \varphi(x-x')E(x',t)d^3x'$ and $B_\varphi(x,t) = \int \varphi(x-x')B(x',t)d^3x'$. The coupled equations (1.1), (1.2), (1.3), and (1.4) define the Abraham model.

Another variant of interest is to have large N and to use a fluid-like description for the particles in terms of a distribution function $f_\alpha(q,v,t)$ for species α, which is then governed by the Liouville equation corresponding to (1.3). Together with (1.1), (1.2), and the continuum analogue of (1.4) one arrives at the Vlasov-Maxwell system. In the limit of small velocities (corresponding to $c\to\infty$) this system is well approximated by the Vlasov-Poisson equations [7, 19]. Corrections due to radiation damping are studied in [12].

For charges interacting with the radiation field, as modeled by Abraham, the zero order effective dynamics is just the Coulomb dynamics, with the Darwin term appearing as the first post-Coulombian (1PC) correction. Both are conveniently summarized through the Lagrangian function

$$\begin{aligned}\mathcal{L}_{\mathrm{D}}(r,u) \;&=\; \sum_{\alpha=1}^N\Big(\frac{1}{2}m_\alpha u_\alpha^2 + \frac{1}{8c^2}\,m_\alpha^* u_\alpha^4\Big) - \frac{1}{2}\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^N \frac{e_\alpha e_\beta}{4\pi|r_\alpha - r_\beta|}\\ &\quad + \frac{1}{4c^2}\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^N \frac{e_\alpha e_\beta}{4\pi|r_\alpha-r_\beta|}\Big(u_\alpha\cdot u_\beta + |r_\alpha-r_\beta|^{-2}(u_\alpha\cdot[r_\alpha-r_\beta])(u_\beta\cdot[r_\alpha-r_\beta])\Big).\end{aligned} \tag{1.5}$$

Here $r = (r_1,\dots,r_N)$ and $u = (u_1,\dots,u_N)$ denote position and velocity of the particles in the approximating system, to notationally distinguish them from the "true" positions and velocities as governed by (1.1), (1.2), (1.3), and (1.4).

In (1.5) the kinetic energy is necessarily expanded in $|v/c|$. For a relativistic particle with rest mass m_0 we have the kinetic energy $T(v) = m_0c^2(1-\gamma^{-1}) \simeq m_0c^2\big(\frac{1}{2}(v/c)^2 + \frac{1}{8}(v/c)^4 + \mathcal{O}((v/c)^6)\big)$. Within the Abraham model the situation is slightly more complicated. Firstly, the comoving

Coulomb field carries some inertia and the bare kinetic energy is renormalized to an effective kinetic energy T_{eff}. In addition, since the charge distribution is rigid in the given rest frame, T_{eff} cannot be of relativistic form. There is a recent proposal for a relativistic model of extended charges coupled to the radiation field, which necessarily also includes their inner rotation, cf. [1]. If one would carry out the small velocity expansion for such a fully relativistic model, T_{eff} has to be relativistic. Instead, for the Abraham model one obtains

$$m_\alpha = m_{\mathrm{b}\alpha} + \frac{4}{3}\, e_\alpha^2 m_{\mathrm{f}} \quad \text{and} \quad m_\alpha^* = m_{\mathrm{b}\alpha} + \frac{16}{15}\, e_\alpha^2 m_{\mathrm{f}}, \quad \text{with} \quad m_{\mathrm{f}} = \frac{1}{2c^2} \int d^3k\, |\hat{\varphi}(k)|^2 k^{-2}. \tag{1.6}$$

The task ahead is to improve the Lagrangian effective equations of motion

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}_{\mathrm{D}}}{\partial u_\alpha}\Big) = \frac{\partial \mathcal{L}_{\mathrm{D}}}{\partial r_\alpha}, \quad \alpha = 1, \ldots, N, \tag{1.7}$$

associated to $\mathcal{L}_{\mathrm{D}}$ from 1PC to 1.5PC. This cannot be a mere addition of extra terms to $\mathcal{L}_{\mathrm{D}}$, since at 1.5PC the charges loose energy through dipole radiation, which must be reflected by dissipative contributions appearing in the dynamics. A formal expansion in $|v/c|$, some details of which are explained in Section 3 below, yields the next-order approximate equations of motion

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}_{\mathrm{D}}}{\partial u_\alpha}\Big) = \frac{\partial \mathcal{L}_{\mathrm{D}}}{\partial r_\alpha} + \frac{e_\alpha}{6\pi c^3} \sum_{\beta=1}^{N} e_\beta \ddot{u}_\beta, \quad \alpha = 1, \ldots, N. \tag{1.8}$$

The presence of the $\ddot{u}_\beta$-terms means that the dimension of the phase space is increased from $6N$ for (1.7) to $9N$ for (1.8). As in the case of a single particle, the equations of motion at 1.5PC are of third order and admit unphysical runaway solutions with velocities which grow exponentially fast in time. To obtain the effective dynamics free from runaway solutions, the standard (formal) practice, also used in the analogous general relativity setting, is to regard the $\ddot{u}_\beta$-terms in (1.8) as a small perturbation. Therefore one may think of differentiating (the explicit form of) Eq. (1.7) with respect to t and of substituting the resulting expression for $\ddot{u}_\beta$ back into (1.8), at the same time dropping all higher than second derivatives of the r_α's. The resulting equation is

$$\begin{aligned} \frac{d}{dt}\Big(\frac{\partial \mathcal{L}_{\mathrm{D}}}{\partial u_\alpha}\Big) &= \frac{\partial \mathcal{L}_{\mathrm{D}}}{\partial r_\alpha} + \frac{e_\alpha}{12\pi c^3} \sum_{\substack{\beta,\beta'=1\\ \beta\neq\beta'}}^{N} \frac{e_\beta e_{\beta'}}{4\pi}\Big(\frac{e_\beta}{m_\beta} - \frac{e_{\beta'}}{m_{\beta'}}\Big) \\ &\qquad \times \Big[\frac{1}{|r_\beta - r_{\beta'}|^3}(u_\beta - u_{\beta'}) - \frac{3(r_\beta - r_{\beta'})\cdot(u_\beta - u_{\beta'})}{|r_\beta - r_{\beta'}|^5}(r_\beta - r_{\beta'})\Big] \end{aligned} \tag{1.9}$$

for $\alpha = 1, \ldots, N$. Thus our goal is to prove that the true solution q_α, v_α is well-approximated by a solution of (1.9), up to errors of order $|v/c|^4$. We note in passing that (1.9) implies there is no radiation damping at 1.5PC in case the charge-to-mass ratios e_β/m_β are independent of β.

The substitution described above looks like a magic trick, since there is no a priori reason to expect that the terms dropped in the end are really of higher order. However, as has been recognized for some time, cf. [13, 14], the substitution can be justified rigorously in the framework of singular (geometric) perturbation theory. The basic observation is that in (1.8), transformed to the appropriate dimensionless scale, the highest derivatives $\ddot{u}_\beta$ carry a small prefactor. The solution flow then admits for a repulsive manifold which can be addressed as center-like, since it contains the true effective dynamics. On this center manifold there is slow motion corresponding to the physical relevant solutions. For initial conditions away from the center manifold the solution trajectory runs off to infinity exponentially fast. Hence (1.9) has to be interpreted as the lowest order approximation to the motion on the center manifold. There is one caveat, however: the matrix of coefficients for the $\ddot{u}$-term, i.e., the map $(\ddot{u}_1, \dots, \ddot{u}_N) \mapsto ((e_\alpha/6\pi c^3)\sum_{\beta=1}^N e_\beta \ddot{u}_\beta)_{1\leq\alpha\leq N}$, is not invertible. Such a case does not seem to be covered by the standard geometric theory of singular perturbations, and we therefore had to supply the missing pieces.

In [15] we proved that the motion according to the Darwin Lagrangian well approximates the true orbit to this order, provided that the initial electromagnetic field minimizes the field energy at ρ, j given through the initial positions and velocities of the charges, cf. (2.5) below. As pointed out by V. Imaikin we handled the initial slip somewhat loosely. This is no problem at order 0PC, however at order 1PC the desired precision requires us in fact to adjust the data for the charges at some later time (rather than $t = 0$) which is short on the Coulomb time scale but long on the microscopic time scale, but still the bounds on the time-derivatives of the solution are needed starting from $t = 0$. We use the occasion to supply a complete proof, see Lemma 3.3 and Lemma 3.4 in Section 3.

On a formal level, without control of the error term, 2PC has been computed by Barker, O'Connell [2, 3] and by Damour, Schäfer [6]. Their starting point is the Wheeler-Feynman action, truncated at $1/c^4$, with point-like charges. The equations of motion are for the charges only and derive from a higher order Lagrangian. Working out the same order for the Abraham model, which non-rigorously could be handled with little extra effort, one would obtain additional terms reflecting the finite size of the charge distribution.

These formal expansions assume implicitly that for $1 \leq \alpha \leq N$

$$|v_\alpha(t) - u_\alpha(t)| \leq \text{const.}\, |v/c|^5 \tag{1.10}$$

at 2PC, say. Here v_α is the true solution and u_α is the 2PC approximate solution, with $|v|$ denoting some average initial velocity. The approximation is supposed to be valid over many Coulomb periods. There are two difficulties associated with (1.10). Firstly, one has to specify for which initial conditions this estimate holds, and secondly $v_\alpha(t)$ depends as well on the initial data for the Maxwell field. Does (1.10) mean that for generic initial data of the Abraham model there is *some* solution of the comparison dynamics such that (1.10) holds? In fact we are not able to come even close to (1.10). At 1.5PC we will prove a precise estimate in the radial direction of the form

$$|v_\alpha(t)^2 - u_\alpha(t)^2| \leq \text{const.}\, |v/c|^4,$$

for $\alpha = 1, \dots, N$, cf. (3.23) below, whereas for the phase we obtain

$$|v_\alpha(t) - u_\alpha(t)| \leq \text{const.}\, |v/c|^3.$$

Thus on the shell of constant kinetic energy our rigorous estimate is not improved as compared to 1PC. It might be the case that our method is not powerful enough to distinguish such fine details. But even on a theoretical physics level it would be of interest to better understand the precise claim hidden behind (1.10).

In the present paper we will establish the 1.5PC approximation, where the main effort goes into a control of the error terms. From our analysis a rather general pattern (presumably valid to any order) emerges. (i) At each order higher derivatives in t and higher powers of $|r_i - r_j|^{-1}$ do appear, as dictated by their dimension. For example, at 1.5PC a term like $\ddot{u}_\alpha$ is dimensionally admitted. Of course each term comes with a prefactor which has to be computed from the Abraham model and which also might vanish, usually because of symmetry. (ii) From step (i) in the effective equations of motion necessarily higher time derivatives come up and the phase space for the approximate dynamics is of dimension larger than $6N$. However, since in the appropriate dimensionless form the higher time derivatives carry a small prefactor, the solution flow has a $6N$ dimensional repulsive center manifold. The motion on this center manifold can be approximated by a second order equation which is the desired comparison dynamics at the given PC approximation.

2 Scales

As in any multiscale problem we have to first identify the relevant scale parameter. The dynamical equations (1.1), (1.2), (1.3), and (1.4) are written on the microscopic scale for which distance is measured in units of R_φ, the radius of the charge distribution from (C), and time is measured in units of $t_\varphi = R_\varphi/c$. On this scale we require the particles to be far apart initially, which means

$$|q_\alpha(0) - q_\beta(0)| \cong \varepsilon^{-1} R_\varphi \quad (\alpha \neq \beta), \tag{2.1}$$

and this defines the dimensionless parameter $\varepsilon > 0$, $\varepsilon \ll 1$. It will be part of the proof to verify that (2.1) is preserved in the course of time. In addition we require $|v_\alpha(0)|/c$ to be small. To find the right order in ε, let us for the moment assume $|v_\alpha(0)| = \varepsilon^\gamma c$ with $\gamma > 0$ to be determined. To see the changes due to self-action and mutual interaction we have to follow the dynamics over long times of some order $t = \varepsilon^{-\delta} t_\varphi$. Over this time span we obtain a change in position $\Delta q = \varepsilon^\gamma c\, \varepsilon^{-\delta} t_\varphi = \varepsilon^{\gamma-\delta} R_\varphi$, and this should equal $\Delta q = \varepsilon^{-1} R_\varphi$ in view of (2.1). Anticipating that the force is proportional to the squared inverse distance, multiplying with the dimensionally right factor we find for the change in velocity that $\Delta v = (R_\varphi^3/t_\varphi^2)(\Delta q)^{-2} t = \varepsilon^{2-\delta} c$, which should be of order $\varepsilon^\gamma c$, by assumption. Solving for γ and δ we find $\gamma = \frac{1}{2}$ as well as $\delta = \frac{3}{2}$. Thus we require that initially

$$|v_\alpha(0)| \cong \sqrt{\varepsilon} c,$$

and we have to consider times of order

$$t \cong \varepsilon^{-3/2} t_\varphi \,. \tag{2.2}$$

Again it will be part of our proof to see that over the time span (2.2) the velocities remain of order $\sqrt{\varepsilon} c$. In passing, note that for the Hulse-Taylor pulsar $\varepsilon \cong 10^{-6}$.

Having settled the initial conditions for the charges we turn to the electromagnetic field. The initial field is assumed to have finite energy, i.e.,

$$\frac{1}{2} \int \left(|E(x,0)|^2 + |B(x,0)|^2 \right) d^3x < \infty \,.$$

Our picture is that in the neighborhood of the charges through radiation the electromagnetic field very rapidly (in a time of order $\varepsilon^{-1} t_\varphi$) reaches a state of minimal energy at the given constraint due to the presence of the charges, the positions of which have been changing only little on the Coulomb scale during this time span. In [11] such a behavior has been established for somewhat simpler situations. Here we concentrate on longer times and merely assume an initial electromagnetic field of low energy. For a charge at constant velocity v the comoving electric and magnetic fields are

$$E_v(x) = -\nabla \phi_v(x) + c^{-2} (v \cdot \nabla \phi_v(x)) v \quad \text{and} \quad B_v(x) = -c^{-1}\, v \wedge \nabla \phi_v(x), \tag{2.3}$$

where ϕ_v is defined through its Fourier transform

$$\hat{\phi}_v(k) = e\, \hat{\varphi}(k) / [k^2 - c^{-2} (k \cdot v)^2], \tag{2.4}$$

with $e = e_\alpha$ for $v = v_\alpha$. Such fields we call charge solitons (centered at zero with velocity v). If $q^0 = (q_1^0, \dots, q_N^0)$ denotes the initial positions and $v^0 = (v_1^0, \dots, v_N^0)$ the initial velocities, then we choose the initial fields as linear superposition of the form

$$E(x,0) = E^0(x) = \sum_{\alpha=1}^{N} E_{v_\alpha^0}(x - q_\alpha^0) \quad \text{and} \quad B(x,0) = B^0(x) = \sum_{\alpha=1}^{N} B_{v_\alpha^0}(x - q_\alpha^0). \tag{2.5}$$

We can think of these fields as generated by the charges which have been forced to move freely as $q_\alpha(t) = q_\alpha^0 + v_\alpha^0 t$ for $-\infty < t \le 0$.

To summarize, we consider a situation where the charges are far apart (on the scale R_φ) and move slowly (on the scale c) over long times (on the scale t_φ). On a mathematical level this means that the initial conditions and the time span under consideration are ε-dependent. As a consequence, also in the comparison dynamics the initial conditions are ε-dependent. Here ε is merely a convenient device to order terms according to their magnitude. In particular, nPC translates to the order ε^{2+n}. An equivalent approach, which will not be used here, would be to rewrite the equations of motion on the Coulomb scale. Then the initial conditions are approximately ε-independent. However, the evolution equations pick up some ε-dependence, except for the pure Coulomb dynamics which is scale invariant. Through the transformation to the Coulomb scale the order would be reduced by a factor ε^2, and nPC would correspond to order ε^n.

3 Main results

In the following we use units for which $c = 1$. We first summarize the dynamics under consideration. The Maxwell equations are

$$\frac{\partial}{\partial t} B(x,t) = -\nabla \wedge E(x,t), \quad \frac{\partial}{\partial t} E(x,t) = \nabla \wedge B(x,t) - \sum_{\alpha=1}^{N} \rho_\alpha(x - q_\alpha(t)) v_\alpha(t), \tag{3.1}$$

with the constraints

$$\nabla \cdot E(x,t) = \sum_{\alpha=1}^{N} \rho_\alpha(x - q_\alpha(t)), \quad \nabla \cdot B(x,t) = 0. \tag{3.2}$$

Here we have introduced the shorthand notation

$$\rho_\alpha = e_\alpha \varphi,$$

and φ is assumed to satisfy (C). The Lorentz force equation is

$$\frac{d}{dt}\Big(m_{\mathrm{b}\alpha} \gamma_\alpha v_\alpha(t)\Big) = \int d^3x\, \rho_\alpha(x - q_\alpha(t)) \Big[E(x,t) + v_\alpha(t) \wedge B(x,t)\Big], \quad 1 \le \alpha \le N, \tag{3.3}$$

where $\gamma_\alpha = \left(1 - v_\alpha^2\right)^{-1/2}$. The initial conditions for the electric and magnetic field are given by (2.5), and for the initial positions $q_\alpha^0 = q_\alpha(0)$ resp. the initial velocities $v_\alpha(0) = v_\alpha^0$ we require

$$C_1 \varepsilon^{-1} \le |q_\alpha^0 - q_\beta^0| \le C_2 \varepsilon^{-1} \quad (\alpha \ne \beta), \tag{3.4}$$

for some constants $C_1, C_2 > 0$, as well as

$$|v_\alpha^0| \le C_3 \sqrt{\varepsilon} \tag{3.5}$$

with $C_3 > 0$.

To order 0PC the comparison system is governed by the Coulomb dynamics

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}_{\rm C}}{\partial u_\alpha}\Big) = \frac{\partial \mathcal{L}_{\rm C}}{\partial r_\alpha}, \quad \alpha = 1, \dots, N, \tag{3.6}$$

with Coulomb Lagrange function

$$\mathcal{L}_{\rm C}(r,u) = \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha u_\alpha^2 - \frac{1}{2} \sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{N} \frac{e_\alpha e_\beta}{4\pi |r_\alpha - r_\beta|},$$

where $r = (r_1, \dots, r_N)$ and $u = (u_1, \dots, u_N)$. Because of the Coulomb singularity the solution to (3.6) may exist only for a finite time, either because two particles collide or since one particle is being expelled to infinity. To formalize this, for given data $(\bar r_\alpha^0, \bar u_\alpha^0) = (\varepsilon q_\alpha^0, \varepsilon^{-1/2} v_\alpha^0)$ we denote $\tau_{\rm C} \in]0,\infty]$ the first time for which either $\lim_{t\to\tau_{\rm C}^-} |\bar r_\alpha(t) - \bar r_\beta(t)| = 0$ for some $\alpha \neq \beta$ or $\lim_{t\to\tau_{\rm C}^-} |\bar r_\alpha(t)| = \infty$ for some α holds for the corresponding solution $(\bar r(t), \bar u(t))$ of (3.6) with data $(\bar r_\alpha^0, \bar u_\alpha^0)$.

Our first step is to conclude from (3.4) and (3.5) that if under the Coulomb dynamics (3.6) there is no collision/expulsion, the same holds for the full system.

Lemma 3.1 *Let the initial data for the Abraham model satisfy (3.4), (3.5), and (2.5). We take*

$$\bar r_\alpha^0 = \varepsilon q_\alpha^0 \quad \text{and} \quad \bar u_\alpha^0 = \varepsilon^{-1/2} v_\alpha^0, \quad \alpha = 1, \dots, N, \tag{3.7}$$

as data for (3.6). Moreover we fix $\delta_0 \in]0, \tau_{\rm C}[$ and $T_0 > 0$. Then there exists a constant $C_* > 0$ such that

$$C_* \varepsilon^{-1} \leq \inf_{t\in[0,\min\{\tau_{\rm C}-\delta_0, T_0\}\varepsilon^{-3/2}]} |q_\alpha(t) - q_\beta(t)| \quad (\alpha \neq \beta). \tag{3.8}$$

See Section 9 for the proof. In the following we write

$$T = \min\{\tau_{\rm C} - \delta_0, T_0\}. \tag{3.9}$$

Next we recall a result which has been obtained in [15, Lemma 2.1].

Lemma 3.2 *Let the initial data for the Abraham model satisfy (3.4), (3.5), and (2.5). Moreover, assume that*

$$C_* \varepsilon^{-1} \leq \inf_{t\in[0,T\varepsilon^{-3/2}]} |q_\alpha(t) - q_\beta(t)| \quad (\alpha \neq \beta), \tag{3.10}$$

with $T > 0$ from (3.9) (or any other T). Then there exist constants $C^*, C_v > 0$ such that

$$\sup_{t\in[0,T\varepsilon^{-3/2}]} |q_\alpha(t) - q_\beta(t)| \leq C^* \varepsilon^{-1} \quad (\alpha \neq \beta), \tag{3.11}$$

and for $\alpha = 1, \dots, N$

$$\sup_{t\in[0,T\varepsilon^{-3/2}]} |v_\alpha(t)| \leq C_v \sqrt{\varepsilon} \tag{3.12}$$

are satisfied. In particular, $\sup_{t\in[0,T\varepsilon^{-3/2}]} |v_\alpha(t)| \leq \bar v < 1$ for some $\bar v$. Moreover, there is $C > 0$ and $\bar e > 0$ such that for $\alpha = 1, \dots, N$ we have

$$\sup_{t\in[0,T\varepsilon^{-3/2}]} |\dot v_\alpha(t)| \leq C\varepsilon^2, \tag{3.13}$$

provided that $|e_\alpha| \leq \bar e$, $\alpha = 1, \dots, N$. In (3.11), (3.12), and (3.13), the constants C and $\bar e$ do depend only on T and the bounds for the initial data, but not on ε.

In the situation described in Lemma 3.2 we set

$$t_0 = 4(R_\varphi + C^* \varepsilon^{-1}). \tag{3.14}$$

The bounds from Lemma 3.2 also lead to a bound on the $|q_\alpha(t)|$, since for $\varepsilon \in]0,1]$ we have

$$\begin{aligned} |q_\alpha(t)| &\le |q_\alpha(t) - q_\alpha^0| + |q_\alpha^0| \le C_v \sqrt{\varepsilon} t + |q_\alpha^0| \\ &\le \Big(C_v T + \max_{1\le\alpha\le N} |q_\alpha^0| \Big) \varepsilon^{-1} =: C_q\, \varepsilon^{-1}, \quad t \in [0, T\varepsilon^{-3/2}]. \end{aligned} \tag{3.15}$$

It is moreover possible to establish an a priori estimate for the $\ddot{v}_\alpha(t)$. Defining

$$\tau_{**} = (C_*/8)\varepsilon^{-1}, \tag{3.16}$$

we have the following result.

Lemma 3.3 *Under the assumptions of Lemma 3.2, including the smallness hypothesis $|e_\alpha| \le \bar{e}$, $\alpha = 1, \ldots, N$, there exists a constant $C > 0$ such that for $\alpha = 1, \ldots, N$ we have*

$$\sup_{t\in[0,T\varepsilon^{-3/2}]} |\ddot{v}_\alpha(t)| \le C\varepsilon^{5/2} \quad \textit{and} \quad \sup_{t\in[\tau_{**},T\varepsilon^{-3/2}]} |\ddot{v}_\alpha(t)| \le C\varepsilon^{7/2}.$$

We remark that τ_{**} could be replaced by any other time of order $\mathcal{O}(\varepsilon^{-1})$. For the analogous estimate in [15] we claimed that the expressions resulting from data terms at time $t = 0$ all vanish, however actually some of them do not. A bound of order $\mathcal{O}(\varepsilon^{7/2})$ cannot hold starting from time $t = 0$, but only after some initial time slip of order ε^{-1}, which is still good enough for our purposes. The full argument for Lemma 3.3 is not given, since it is superseded by the proof of Lemma 3.4 below, where a corresponding problem arises.

In order to expand the dynamics up to the order of radiation reaction, we need to have control of one further t-derivative. Thus a main issue will be to verify the following lemma.

Lemma 3.4 *Under the assumptions of Lemma 3.2, including $|e_\alpha| \le \bar{e}$, $\alpha = 1, \ldots, N$, there is a constant $C > 0$ such that for $\alpha = 1, \ldots, N$ we have*

$$\sup_{t\in[6\tau_{**},T\varepsilon^{-3/2}]} |\dddot{v}_\alpha(t)| \le C\varepsilon^5.$$

The rather technical proof is given in Section 7. It turned out that the principal method used in [13, 14, 15] needed to be improved in a substantial manner in order to be applied here as well, which is mainly due to the "bad" decay properties of solutions to wave equations in the vicinity of the light-cone.

Using Lemma 3.4 as a key ingredient, it will then follow from Lemma 4.6 that

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}_{\rm D}}{\partial v_\alpha}\Big) = \frac{\partial \mathcal{L}_{\rm D}}{\partial q_\alpha} + \frac{e_\alpha}{6\pi} \sum_{\beta=1}^N e_\beta \ddot{v}_\beta + \mathcal{O}(\varepsilon^4), \quad \alpha = 1, \ldots, N, \tag{3.17}$$

where $\mathcal{L}_{\rm D}$ is the Darwin Lagrangian from (1.5) which governs the system up to order $\mathcal{O}(\varepsilon^3)$, cf. [15]. As comparison effective dynamics we hence introduce (1.8).

Let us next define the Darwin energy

$$\begin{aligned}\mathcal{H}_{\mathrm{D}}(r,u) &= \sum_{\alpha=1}^{N}\Big(\frac{1}{2}m_\alpha u_\alpha^2+\frac{3}{8}m_\alpha^* u_\alpha^4\Big)+\frac{1}{2}\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{N}\frac{e_\alpha e_\beta}{4\pi|r_\alpha-r_\beta|}\\ &\quad+\frac{1}{4}\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{N}\frac{e_\alpha e_\beta}{4\pi|r_\alpha-r_\beta|}\Big(u_\alpha\cdot u_\beta+\frac{u_\alpha\cdot(r_\alpha-r_\beta)}{|r_\alpha-r_\beta|^2}\,u_\beta\cdot(r_\alpha-r_\beta)\Big).\end{aligned} \tag{3.18}$$

If $\mathcal{H}_{\mathrm{D}}$ is corrected by the Schott term, i.e., if we let

$$\mathcal{H}_{\mathrm{RR}}(r,u,\dot u)=\mathcal{H}_{\mathrm{D}}(r,u)-\sum_{\alpha,\beta=1}^{N}\frac{e_\alpha e_\beta}{6\pi}\,u_\alpha\cdot\dot u_\beta,$$

then it can be verified that along solutions of (1.8) this “energy” is decreasing. More precisely one obtains

$$\frac{d}{dt}\mathcal{H}_{\mathrm{RR}}=-\frac{1}{6\pi}\Big(\sum_{\alpha=1}^{N}e_\alpha\dot u_\alpha\Big)^2. \tag{3.19}$$

Due to the presence of runaway solutions in (1.8), the data for (1.8) in the data space $\mathbb{R}^{9N}=\mathbb{R}^{3N}\times\mathbb{R}^{3N}\times\mathbb{R}^{3N}$ leading to physically reasonable solutions have to be singled out. In Section 5 we will accordingly construct a kind of center manifold $\mathcal{I}_\varepsilon$ for (1.8) on which the true effective dynamics takes place and which is locally invariant for (1.8), in the sense specified in Theorem 5.1. It will moreover turn out that the true effective dynamics (on the center manifold) of solutions to the full system is approximately described by a second order system of ODEs, cf. (3.22) below. Note that the existence of runaway solutions does not contradict (3.19), since $\mathcal{H}_{\mathrm{RR}}$ in general may be indefinite due to the Schott term $\sum_{\alpha,\beta=1}^{N}\frac{e_\alpha e_\beta}{6\pi}\,u_\alpha\cdot\dot u_\beta$.

Our main result is the following theorem.

Theorem 3.5 *Assume the data (q_α^0,v_α^0), $\alpha=1,\ldots,N$, and $E^0(x)$ and $B^0(x)$ are such that (3.4), (3.5), and (2.5) are verified. Define τ_{C}, δ_0, and T_0 as in Lemma 3.1, and introduce T through (3.9). Then Lemma 3.1 implies the existence of $C_*>0$ such that (3.8) holds, and this in turn yields the existence of suitable constants such that the bounds from Lemmas 3.2, 3.3, and 3.4 are satisfied, provided that $|e_\alpha|\le\bar e$, $1\le\alpha\le N$. Moreover, t_0 from (3.14) is defined.*

In this basic setup we denote

$$K_\varepsilon=\Big\{(r,u)\in\mathbb{R}^{3N}\times\mathbb{R}^{3N}:|r|\le 4C_q\varepsilon^{-1},\ |u|\le 4C_v\sqrt{\varepsilon}\Big\},$$

with C_q from (3.15) and C_v from (3.12), respectively. Then there exists $\varepsilon_1>0$ and for each $\varepsilon\in]0,\varepsilon_1]$ a C^4-function $h_\varepsilon:K_\varepsilon\to\mathbb{R}^{3N}$ with the following property. Consider the true solution $(q_\alpha(t),v_\alpha(t))$ resulting from (1.4), (3.1), (3.2), and (3.3), and the solution $(r_\alpha(t),u_\alpha(t))$ of (1.8) with data

$$r_\alpha(t_0)=q_\alpha(t_0),\quad u_\alpha(t_0)=v_\alpha(t_0),\quad \textit{and}\quad \dot u_\alpha(t_0)=h_\varepsilon\big(q_\alpha(t_0),v_\alpha(t_0)\big), \tag{3.20}$$

for $1\le\alpha\le N$. Then $(r_\alpha(t),u_\alpha(t))$ exists at least for $t\in[t_0,T\varepsilon^{-3/2}]$, and the estimates

$$|q_\alpha(t)-r_\alpha(t)|\le C\sqrt{\varepsilon},\quad |v_\alpha(t)-u_\alpha(t)|\le C\varepsilon^2,\quad |\dot v_\alpha(t)-\dot u_\alpha(t)|\le C\varepsilon^{7/2}, \tag{3.21}$$

hold for $t \in [t_0, T\varepsilon^{-3/2}]$. Moreover, along such solutions of the effective equation (1.8) we have

$$\begin{aligned}\frac{d}{dt}\Big(\frac{\partial\mathcal{L}_{\rm D}}{\partial u_\alpha}\Big) &= \frac{\partial\mathcal{L}_{\rm D}}{\partial r_\alpha} + \frac{e_\alpha}{12\pi}\sum_{\substack{\beta,\beta'=1\\ \beta\neq\beta'}}^{N} \frac{e_\beta e_{\beta'}}{4\pi}\Big(\frac{e_\beta}{m_\beta} - \frac{e_{\beta'}}{m_{\beta'}}\Big) \\ &\qquad \times\Big[\frac{1}{|\xi_{\beta\beta'}|^3}(u_\beta - u_{\beta'}) - \frac{3}{|\xi_{\beta\beta'}|^5}\,\xi_{\beta\beta'}\cdot(u_\beta - u_{\beta'})\,\xi_{\beta\beta'}\Big] + \mathcal{O}(\varepsilon^{9/2}) \quad (3.22)\end{aligned}$$

for $\alpha = 1, \ldots, N$ and $t \in [t_0, T\varepsilon^{-3/2}]$; here we set $\xi_{\alpha\beta} = q_\alpha(t) - q_\beta(t)$. In addition, with $\mathcal{H}_{\rm D}$ from (3.18) we obtain

$$\Big|\mathcal{H}_{\rm D}(q(t), v(t)) - \mathcal{H}_{\rm D}(r(t), u(t))\Big| \le C\varepsilon^3, \quad t \in [t_0, T\varepsilon^{-3/2}]. \quad (3.23)$$

Remarks 3.6 (a) In fact for every $k \in \mathbb{N}$ with $k \ge 4$ it can be achieved that h_ε is of class C^k, with ε_1 possibly having to be decreased further.

(b) The functions h_ε do depend only on the input constants C_1, C_2, C_3, $\tau_{\rm C}$, δ_0, and T_0.

4 Expansion of the Lorentz force term

Due to the bound on $\ddot{v}_\alpha$ from Lemma 3.4 it is possible to rigorously expand the Lorentz force

$$F_\alpha(t) = \int d^3x\, \rho_\alpha(x - q_\alpha(t))[E(x,t) + v_\alpha(t)\wedge B(x,t)] \quad (4.1)$$

up to the order of radiation reaction. In view of (3.1) and (3.2) we have

$$E(x,t) = E^{(0)}(x,t) + E^{(r)}(x,t) \quad \text{and} \quad B(x,t) = B^{(0)}(x,t) + B^{(r)}(x,t),$$

with

$$\begin{aligned}\hat{E}^{(0)}(k,t) &= \cos|k|t\,\hat{E}(k,0) - i\,\frac{\sin|k|t}{|k|}\,k\wedge\hat{B}(k,0),\\ \hat{B}^{(0)}(k,t) &= \cos|k|t\,\hat{B}(k,0) + i\,\frac{\sin|k|t}{|k|}\,k\wedge\hat{E}(k,0),\\ \hat{E}^{(r)}(k,t) &= -\int_0^t ds\,\cos|k|(t-s)\,\hat{j}(k,s) + i\int_0^t ds\,\frac{\sin|k|(t-s)}{|k|}\,\hat{\rho}(k,s)k,\\ \hat{B}^{(r)}(k,t) &= -i\int_0^t ds\,\frac{\sin|k|(t-s)}{|k|}\,k\wedge\hat{j}(k,s),\end{aligned}$$

where $j(x,t)$ and $\rho(x,t)$ are given by (1.4), with $c = 1$. Accordingly we write $F_\alpha(t)$ from (4.1) as

$$\begin{aligned}F_\alpha(t) &= \int d^3x\,\rho_\alpha(x - q_\alpha(t))[E^{(0)}(x,t) + v_\alpha(t)\wedge B^{(0)}(x,t)]\\ &\quad + \int d^3x\,\rho_\alpha(x - q_\alpha(t))[E^{(r)}(x,t) + v_\alpha(t)\wedge B^{(r)}(x,t)]\\ &=: F_\alpha^{(0)}(t) + F_\alpha^{(r)}(t). \quad (4.2)\end{aligned}$$

With $t_0 = 4(R_\varphi + C^*\varepsilon^{-1})$ from (3.14) we first recall from [15, Lemma 3.1] the following result concerning $F_\alpha^{(0)}(t)$.

Lemma 4.1 *For $t \in [t_0, T\varepsilon^{-3/2}]$ we have $F^{(0)}_\alpha(t) = 0$.*

Applying the Fourier transform and noting $\rho_\alpha = e_\alpha \varphi$, it is moreover seen that the contribution $F^{(r)}_\alpha(t)$ to $F_\alpha(t)$ resulting from the retarded parts of the fields is

$$F^{(r)}_\alpha(t) = e_\alpha^2 F^{(r)}_{\alpha\alpha}(t) + \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N} e_\alpha e_\beta F^{(r)}_{\alpha\beta}(t), \tag{4.3}$$

where

$$\begin{aligned} F^{(r)}_{\alpha\beta}(t) \;&=\; \int_0^t ds \int d^3k\, |\hat\varphi(k)|^2 e^{-ik\cdot[q_\alpha(t)-q_\beta(s)]} \Big\{ -\cos|k|(t-s)\, v_\beta(s) + i\,\frac{\sin|k|(t-s)}{|k|}\, k \\ &\qquad -i\,\frac{\sin|k|(t-s)}{|k|}\, v_\alpha(t)\wedge(k\wedge v_\beta(s))\Big\}, \end{aligned} \tag{4.4}$$

for $\alpha, \beta = 1, \ldots, N$.

4.1 Expansion of the self-force $F^{(r)}_{\alpha\alpha}(t)$

For $t \in [t_0, T\varepsilon^{-3/2}]$ we have

$$\begin{aligned} F^{(r)}_{\alpha\alpha}(t) \;&=\; \int_0^\infty d\tau \int d^3k\, |\hat\varphi(k)|^2 e^{-i(k\cdot v_\alpha)\tau} \Big(1 + \frac{i}{2}(k\cdot \dot v_\alpha)\tau^2 - \frac{i}{6}(k\cdot \ddot v_\alpha)\tau^3\Big) \\ &\quad\times \Big\{ -\cos|k|\tau\, [v_\alpha - \dot v_\alpha \tau + \frac12\, \ddot v_\alpha \tau^2] + i\,\frac{\sin|k|\tau}{|k|}\, k \\ &\qquad -i\,\frac{\sin|k|\tau}{|k|}\, v_\alpha \wedge (k\wedge[v_\alpha - \dot v_\alpha\tau])\Big\} + \mathcal{O}(\varepsilon^4), \end{aligned} \tag{4.5}$$

with $v_\alpha = v_\alpha(t)$, etc. The proof of this formula will not be elaborated. It proceeds similar to the proof of Lemma 4.3 below, cf. also [10, 14]. Arguing formally, we utilize

$$e^{-ik\cdot[q_\alpha(t)-q_\beta(s)]} \;\cong\; e^{-i(k\cdot v_\alpha)\tau}\Big(1 + \frac{i}{2}(k\cdot \dot v_\alpha)\tau^2 - \frac{i}{6}(k\cdot \ddot v_\alpha)\tau^3\Big) + \mathcal{O}(\varepsilon^4), \tag{4.6}$$

$$v_\alpha(s) \;\cong\; v_\alpha - \dot v_\alpha \tau + \frac12\, \ddot v_\alpha \tau^2 + \mathcal{O}(\varepsilon^5), \tag{4.7}$$

in (4.4) with $\alpha = \beta$, where $\tau = t - s$. To make this argument rigorous, it is necessary to observe that $\int_0^t ds(\ldots) = \int_{t-\bar t}^t ds(\ldots) = \int_0^{\bar t} d\tau(\ldots)$ for any $t, \bar t \geq \frac{2R_\varphi}{1-\bar v}$, as it follows in case that $\alpha = \beta$ from condition (C) by transforming (4.4) back to space variables, cf. Lemma 4.3. Thus we can start with the expression for $F^{(r)}_{\alpha\alpha}(t)$ obtained from (4.4) and first replace $\int_0^t ds(\ldots)$ by $\int_0^{\bar t_0} d\tau(\ldots)$ for a fixed $\bar t_0 \geq \frac{2R_\varphi}{1-\bar v}$. Then we use (4.6), (4.7), the key point being that now $\tau = t - s \in [0, \bar t_0]$ is of order $\tau = \mathcal{O}(1)$, whence e.g. $\mathcal{O}(\ddot v_\alpha \tau^3) = \mathcal{O}(\ddot v_\alpha) = \mathcal{O}(\varepsilon^5)$ in (4.7). Finally we observe that in the resulting expression $\int_0^{\bar t_0} d\tau(\ldots) + \mathcal{O}(\varepsilon^4)$ the same argument once more can be employed to yield that $\int_0^{\bar t_0} d\tau(\ldots)$ equals $\int_0^{\bar t} d\tau(\ldots)$, for any other $\bar t \geq \frac{2R_\varphi}{1-\bar v}$. Accordingly we arrive at (4.5), taking the limit $\bar t \to \infty$.

With the notation

$$I_p = \int_0^{\bar t} d\tau\, \frac{\sin(|k|\tau)}{|k|} e^{-i(k\cdot v_\alpha)\tau}\, \tau^p, \quad J_p = \int_0^{\bar t} d\tau \cos(|k|\tau) e^{-i(k\cdot v_\alpha)\tau}\, \tau^p, \quad p \in \mathbb{N}_0,$$

we may then reformulate (4.5) as

$$
\begin{aligned}
F^{(r)}_{\alpha\alpha}(t) &= \lim_{\bar t\to\infty}\Big(-\int d^3k\,|\hat\varphi(k)|^2\Big\{v_\alpha J_0 - \dot v_\alpha J_1 + \frac{i}{2}(k\cdot\dot v_\alpha)v_\alpha J_2 + \frac12\,\ddot v_\alpha J_2\Big\} \\
&\qquad + \int d^3k\,|\hat\varphi(k)|^2\Big\{ i\,[(1-v_\alpha^2)k + (k\cdot v_\alpha)v_\alpha]I_0 + i\,[(v_\alpha\cdot\dot v_\alpha)k - (k\cdot v_\alpha)\dot v_\alpha]I_1 \\
&\qquad\qquad -\frac12(k\cdot\dot v_\alpha)[(1-v_\alpha^2)k+(k\cdot v_\alpha)v_\alpha]I_2 + \frac16(k\cdot\ddot v_\alpha)kI_3\Big\}\Big) + \mathcal{O}(\varepsilon^4). \\
&=: F^{(r)}_{\alpha\alpha,\mathrm{old}}(t) + F^{(r)}_{\alpha\alpha,\mathrm{new}}(t) + \mathcal{O}(\varepsilon^4), \qquad (4.8)
\end{aligned}
$$

with

$$
F^{(r)}_{\alpha\alpha,\mathrm{new}}(t) = \lim_{\bar t\to\infty}\Big(-\frac12\,\ddot v_\alpha\int d^3k\,|\hat\varphi(k)|^2 J_2 + \frac16\int d^3k\,|\hat\varphi(k)|^2\,(k\cdot\ddot v_\alpha)kI_3\Big).
$$

Note that due to the fact that only terms up to order $\mathcal{O}(\varepsilon^4)$ have to be taken into account some expressions appearing in (4.5) have dropped out when passing to (4.8). Compared to the expansion of $F^{(r)}_{\alpha\alpha}(t)$ up to order $\mathcal{O}(\varepsilon^{7/2})$ in [15, (3.6)], we have picked up the two additional terms denoted $F^{(r)}_{\alpha\alpha,\mathrm{new}}(t)$, which both contain $\ddot v_\alpha$. From [15, (3.7), (3.8)] we know that

$$
F^{(r)}_{\alpha\alpha,\mathrm{old}}(t) = -\Big(\frac43 + \frac{8}{15}\,v_\alpha^2\Big)m_{\mathrm f}\dot v_\alpha - \frac{16}{15}\,m_{\mathrm f}(v_\alpha\cdot\dot v_\alpha)v_\alpha + \mathcal{O}(\varepsilon^4), \qquad (4.9)
$$

where $m_{\mathrm f} = \frac12\int d^3k\,|\hat\varphi(k)|^2 k^{-2}$, cf. (1.6). To evaluate the contribution of the new term, we recall from [14, Lemma 4.3] that

$$
\lim_{\bar t\to\infty}\int d^3k\,|\hat\varphi(k)|^2 J_2 = \int_0^\infty d\tau\tau^2\int d^3k\,|\hat\varphi(k)|^2\cos(|k|\tau)e^{-i(k\cdot v_\alpha)\tau} = -\frac{1}{2\pi}\,\gamma_\alpha^4,
$$

where $\gamma_\alpha = (1-v_\alpha^2)^{-1/2}$. In addition, $(\xi\cdot\nabla_v)\nabla_v I_1 = -(k\cdot\xi)kI_3$ for $\xi\in\mathbb{R}^3$, and also

$$
\lim_{\bar t\to\infty}\int d^3k\,|\hat\varphi(k)|^2 I_1 = \frac{1}{4\pi}\,\gamma_\alpha^2
$$

due to [14, p. 637]. It follows that

$$
\begin{aligned}
F^{(r)}_{\alpha\alpha,\mathrm{new}}(t) &= \frac{1}{4\pi}\,\gamma_\alpha^4\ddot v_\alpha - \frac{1}{24\pi}\,(\ddot v_\alpha\cdot\nabla_v)\nabla_v\gamma_\alpha^2 = \frac{1}{4\pi}\,\gamma_\alpha^4\ddot v_\alpha - \frac{1}{12\pi}\Big(\gamma_\alpha^4\ddot v_\alpha + 4\gamma_\alpha^6(v_\alpha\cdot\ddot v_\alpha)v_\alpha\Big) \\
&= \frac{1}{6\pi}\,\ddot v_\alpha + \mathcal{O}(\varepsilon^4), \qquad (4.10)
\end{aligned}
$$

the latter by expanding $\gamma_\alpha^4 = 1+\mathcal{O}(\varepsilon)$. We can summarize (4.9) and (4.10) in the following lemma.

Lemma 4.2 *For $t\in[t_0, T\varepsilon^{-3/2}]$ we have*

$$
F^{(r)}_{\alpha\alpha}(t) = -\Big(\frac43 + \frac{8}{15}\,v_\alpha^2\Big)m_{\mathrm f}\dot v_\alpha - \frac{16}{15}\,m_{\mathrm f}(v_\alpha\cdot\dot v_\alpha)v_\alpha + \frac{1}{6\pi}\,\ddot v_\alpha + \mathcal{O}(\varepsilon^4).
$$

4.2 Expansion of the interaction force $F^{(r)}_{\alpha\beta}(t)$, $\alpha\neq\beta$

We return to (4.4) and consider $F^{(r)}_{\alpha\beta}(t)$ for $\alpha\neq\beta$. The main difference to Section 4.1 results from the fact that now $\xi_{\alpha\beta} = q_\alpha(t) - q_\beta(t) = \mathcal{O}(\varepsilon^{-1})$, cf. Lemma 3.2. This point in conjunction with Lemma 3.4 also plays the key role in the proof of the following technical lemma whose proof is postponed to Section 8.

Lemma 4.3 *Let $1 \le \alpha, \beta \le N$, $\alpha \neq \beta$. For $t \in [t_0, T\varepsilon^{-3/2}]$ we have*

$$
\begin{aligned}
(a) \quad & -\int_0^t ds \int d^3k\, |\hat\varphi(k)|^2 e^{-ik\cdot[q_\alpha(t)-q_\beta(s)]} \cos|k|(t-s)\, v_\beta(s) \\
&= -\int_0^\infty d\tau \int d^3k\, |\hat\varphi(k)|^2 e^{-ik\cdot\xi_{\alpha\beta}} \cos|k|\tau \Big\{ v_\beta - i\tau(k\cdot v_\beta)v_\beta - \tau\dot v_\beta + \frac{i}{2}\tau^2(k\cdot\dot v_\beta)v_\beta \\
&\qquad -\frac{1}{2}\tau^2(k\cdot v_\beta)^2 v_\beta + i\tau^2(k\cdot v_\beta)\dot v_\beta + \frac{1}{2}\tau^2\ddot v_\beta \Big\} + \mathcal{O}(\varepsilon^4), \\
(b) \quad & i\int_0^t ds \int d^3k\, |\hat\varphi(k)|^2 e^{-ik\cdot[q_\alpha(t)-q_\beta(s)]} \frac{\sin|k|(t-s)}{|k|} k \\
&= i\int_0^\infty d\tau \int d^3k\, |\hat\varphi(k)|^2 e^{-ik\cdot\xi_{\alpha\beta}} \frac{\sin|k|\tau}{|k|} k \Big\{ 1 - ik\cdot\Big[\tau v_\beta - \frac{1}{2}\tau^2\dot v_\beta + \frac{1}{6}\tau^3\ddot v_\beta\Big] \\
&\qquad -\frac{1}{2}\Big[\tau^2(k\cdot v_\beta)^2 - \tau^3(k\cdot v_\beta)(k\cdot\dot v_\beta)\Big] + \frac{i}{6}\tau^3(k\cdot v_\beta)^3 \Big\} + \mathcal{O}(\varepsilon^4), \\
(c) \quad & (-i)\int_0^t ds \int d^3k\, |\hat\varphi(k)|^2 e^{-ik\cdot[q_\alpha(t)-q_\beta(s)]} \frac{\sin|k|(t-s)}{|k|} v_\alpha(t)\wedge(k\wedge v_\beta(s)) \\
&= (-i)\int_0^\infty d\tau \int d^3k\, |\hat\varphi(k)|^2 e^{-ik\cdot\xi_{\alpha\beta}} \frac{\sin|k|\tau}{|k|} v_\alpha \wedge \Big(k\wedge\Big\{v_\beta - \tau\dot v_\beta - i\tau(k\cdot v_\beta)v_\beta\Big\}\Big) \\
&\quad + \mathcal{O}(\varepsilon^4).
\end{aligned}
$$

Here $v_\alpha = v_\alpha(t)$, etc., and $\xi_{\alpha\beta} = q_\alpha(t) - q_\beta(t)$.

These expressions will be substituted back into (4.4). To simplify notation, we introduce for $p \in \mathbb{N}_0$

$$
A_p := \int_0^\infty d\tau \int d^3k\, |\hat\varphi(k)|^2 e^{-ik\cdot\xi_{\alpha\beta}} \frac{\sin|k|\tau}{|k|}\tau^p = (4\pi)^{-1} \int\int d^3x d^3y\, \varphi(x)\varphi(y)|\xi_{\alpha\beta}+x-y|^{p-1}
$$

and

$$
\begin{aligned}
B_p &:= \int_0^\infty d\tau \int d^3k\, |\hat\varphi(k)|^2 e^{-ik\cdot\xi_{\alpha\beta}} \cos(|k|\tau)\,\tau^p \\
&= (-p)(4\pi)^{-1}\int\int d^3x d^3y\, \varphi(x)\varphi(y)|\xi_{\alpha\beta}+x-y|^{p-2} = (-p)A_{p-1}.
\end{aligned}
$$

Hence it follows from Lemma 4.3 that for $\alpha \neq \beta$ and $t \in [t_0, T\varepsilon^{-3/2}]$ we have

$$
\begin{aligned}
F^{(r)}_{\alpha\beta}(t) &= -v_\beta B_0 - v_\beta(v_\beta\cdot\nabla_\xi)B_1 + \dot v_\beta B_1 + \frac{1}{2}(\dot v_\beta\cdot\nabla_\xi)B_2 v_\beta - \frac{1}{2}(v_\beta\cdot\nabla_\xi)^2 B_2 v_\beta + (v_\beta\cdot\nabla_\xi)B_2\dot v_\beta \\
&\quad -\frac{1}{2}B_2\ddot v_\beta - \nabla_\xi A_0 - (v_\beta\cdot\nabla_\xi)\nabla_\xi A_1 + \frac{1}{2}(\dot v_\beta\cdot\nabla_\xi)\nabla_\xi A_2 - \frac{1}{2}(v_\beta\cdot\nabla_\xi)^2\nabla_\xi A_2 \\
&\quad -\frac{1}{6}(\ddot v_\beta\cdot\nabla_\xi)\nabla_\xi A_3 + \frac{1}{2}(v_\beta\cdot\nabla_\xi)(\dot v_\beta\cdot\nabla_\xi)\nabla_\xi A_3 - \frac{1}{6}(v_\beta\cdot\nabla_\xi)^3\nabla_\xi A_3 \\
&\quad +v_\alpha\wedge(\nabla_\xi A_0\wedge v_\beta) - v_\alpha\wedge(\nabla_\xi A_1\wedge\dot v_\beta) + v_\alpha\wedge(\nabla_\xi\wedge(v_\beta\cdot\nabla_\xi)A_1 v_\beta) + \mathcal{O}(\varepsilon^4) \\
&= -v_\beta(v_\beta\cdot\nabla_\xi)B_1 + \dot v_\beta B_1 - \frac{1}{2}B_2\ddot v_\beta - \nabla_\xi A_0 + \frac{1}{2}(\dot v_\beta\cdot\nabla_\xi)\nabla_\xi A_2 - \frac{1}{2}(v_\beta\cdot\nabla_\xi)^2\nabla_\xi A_2 \\
&\quad -\frac{1}{6}(\ddot v_\beta\cdot\nabla_\xi)\nabla_\xi A_3 + \frac{1}{2}(v_\beta\cdot\nabla_\xi)(\dot v_\beta\cdot\nabla_\xi)\nabla_\xi A_3 - \frac{1}{6}(v_\beta\cdot\nabla_\xi)^3\nabla_\xi A_3 \\
&\quad +(v_\alpha\cdot v_\beta)\nabla_\xi A_0 - v_\beta(v_\alpha\cdot\nabla_\xi)A_0 + \mathcal{O}(\varepsilon^4),
\end{aligned}
\tag{4.11}
$$

where in the last reduction we have used that $A_1 = (4\pi)^{-1}$, $B_2 = -(2\pi)^{-1}$, and $B_0 = 0$, hence in particular $\nabla_\xi A_1 = \nabla_\xi B_2 = 0$. We rewrite (4.11) as

$$F^{(r)}_{\alpha\beta}(t) = F^{(r)}_{\alpha\beta,\mathrm{old}}(t) + F^{(r)}_{\alpha\beta,\mathrm{new}}(t) + \mathcal{O}(\varepsilon^4), \tag{4.12}$$

with

$$F^{(r)}_{\alpha\beta,\mathrm{new}}(t) = -\frac{1}{2}B_2\ddot{v}_\beta - \frac{1}{6}(\ddot{v}_\beta\cdot\nabla_\xi)\nabla_\xi A_3 + \frac{1}{2}(v_\beta\cdot\nabla_\xi)(\dot{v}_\beta\cdot\nabla_\xi)\nabla_\xi A_3 - \frac{1}{6}(v_\beta\cdot\nabla_\xi)^3\nabla_\xi A_3 \tag{4.13}$$

being the new radiation reaction contribution compared to our expansion up to order $\mathcal{O}(\varepsilon^3)$ in [15]. According to [15, Section 3.2] we have

$$F^{(r)}_{\alpha\beta,\mathrm{old}}(t) = g_{\alpha\beta}(t) + \mathcal{O}(\varepsilon^4), \tag{4.14}$$

where

$$\begin{aligned} g_{\alpha\beta}(t) \;\; = \;\; & \frac{\xi_{\alpha\beta}}{4\pi|\xi_{\alpha\beta}|^3} - \frac{1}{8\pi|\xi_{\alpha\beta}|}\,\dot{v}_\beta - \frac{(\dot{v}_\beta\cdot\xi_{\alpha\beta})}{8\pi|\xi_{\alpha\beta}|^3}\,\xi_{\alpha\beta} + \frac{v_\beta^2}{8\pi|\xi_{\alpha\beta}|^3}\,\xi_{\alpha\beta} - \frac{3(v_\beta\cdot\xi_{\alpha\beta})^2}{8\pi|\xi_{\alpha\beta}|^5}\,\xi_{\alpha\beta} \\ & - \frac{(v_\alpha\cdot v_\beta)}{4\pi|\xi_{\alpha\beta}|^3}\,\xi_{\alpha\beta} + \frac{(v_\alpha\cdot\xi_{\alpha\beta})}{4\pi|\xi_{\alpha\beta}|^3}\,v_\beta, \quad \text{with} \quad \xi_{\alpha\beta} = q_\alpha(t) - q_\beta(t). \end{aligned} \tag{4.15}$$

We recall that for proving (4.14) the most difficult part is to show that

$$\left|\nabla_\xi A_0 + \frac{\xi_{\alpha\beta}}{4\pi|\xi_{\alpha\beta}|^3}\right| = \frac{1}{4\pi|\xi_{\alpha\beta}|^2}\left|\int\!\!\int d^3x\,d^3y\,\varphi(x)\varphi(y)\left(\frac{\vec{n}+\frac{x-y}{|\xi_{\alpha\beta}|}}{|\vec{n}+\frac{x-y}{|\xi_{\alpha\beta}|}|^3} - \vec{n}\right)\right| = \mathcal{O}(\varepsilon^4),$$

with $\vec{n} = \xi_{\alpha\beta}/|\xi_{\alpha\beta}|$. However, this turns out to work well by expanding $\psi(R) = (\vec{n}+R)/|\vec{n}+R|^3 = \vec{n} + R - 3(\vec{n}\cdot R)\vec{n} + \mathcal{O}(\varepsilon^2)$, where $R = (x-y)/|\xi_{\alpha\beta}| = \mathcal{O}(\varepsilon)$ for $|x|, |y| \le R_\varphi$, and by noting that $\int\!\int d^3x\,d^3y\,\varphi(x)\varphi(y)(x-y) = 0$. Therefore we only have to consider the new part $F^{(r)}_{\alpha\beta,\mathrm{new}}(t)$ from (4.13). To begin with, we recall that $B_2 = -2A_1 = -(2\pi)^{-1}$. The following lemma deals with the remaining terms.

Lemma 4.4 *For $\alpha \neq \beta$ and $t \in [0, T\varepsilon^{-3/2}]$ the following holds.*

(a) $\frac{1}{6}(\ddot{v}_\beta\cdot\nabla_\xi)\nabla_\xi A_3 = \frac{1}{12\pi}\,\ddot{v}_\beta$, *and*

(b) $(v_\beta\cdot\nabla_\xi)(\dot{v}_\beta\cdot\nabla_\xi)\nabla_\xi A_3 = 0 = (v_\beta\cdot\nabla_\xi)^3\nabla_\xi A_3$.

Proof: We have $A_3 = (4\pi)^{-1}\int\!\int d^3x d^3y\,\varphi(x)\varphi(y)|\xi_{\alpha\beta}+x-y|^2$, and therefore

$$\nabla_\xi A_3 = \frac{1}{2\pi}\int\!\!\int d^3x d^3y\,\varphi(x)\varphi(y)\Big(\xi_{\alpha\beta}+x-y\Big) = \frac{1}{2\pi}\,\xi_{\alpha\beta},$$

the latter in view of $\int\!\int d^3x d^3y\,\varphi(x)\varphi(y)(x-y) = 0$ by the symmetry of φ, cf. condition (C). □

Turning back to $F^{(r)}_{\alpha\beta,\mathrm{new}}(t)$ from (4.13) we hence have shown that for $\alpha \neq \beta$ and $t \in [t_0, T\varepsilon^{-3/2}]$ the estimate

$$F^{(r)}_{\alpha\beta,\mathrm{new}}(t) = \frac{1}{4\pi}\,\ddot{v}_\beta - \frac{1}{12\pi}\,\ddot{v}_\beta + \mathcal{O}(\varepsilon^4) = \frac{1}{6\pi}\,\ddot{v}_\beta + \mathcal{O}(\varepsilon^4)$$

holds. In view of (4.12) and (4.14) we arrive at the following lemma.

Lemma 4.5 *For $\alpha \neq \beta$ and $t \in [t_0, T\varepsilon^{-3/2}]$ we have*

$$F^{(r)}_{\alpha\beta}(t) = g_{\alpha\beta}(t) + \frac{1}{6\pi}\,\ddot{v}_\beta + \mathcal{O}(\varepsilon^4),$$

with $g_{\alpha\beta}(t)$ given by (4.15).

4.3 Summary of the expansion

From (4.1), (4.2), and Lemma 4.1 we see that $F_\alpha(t) = F_\alpha^{(r)}(t)$ for $t \in [t_0, T\varepsilon^{-3/2}]$. Thus (4.3), Lemma 4.2, and Lemma 4.5 can be summarized as follows. For $t \in [t_0, T\varepsilon^{-3/2}]$ the Lorentz force $F_\alpha(t)$ from (4.1) allows for the representation

$$\begin{aligned} F_\alpha(t) &= -\Big(\frac{4}{3} + \frac{8}{15}\, v_\alpha^2\Big) e_\alpha^2 m_{\rm f} \dot v_\alpha - \frac{16}{15}\, e_\alpha^2 m_{\rm f} (v_\alpha \cdot \dot v_\alpha) v_\alpha + G_\alpha(q, v, \dot v) + \frac{e_\alpha}{6\pi} \sum_{\beta=1}^N e_\beta \ddot v_\beta + \mathcal{O}(\varepsilon^4), \\ G_\alpha(q, v, \dot v) &= \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^N e_\alpha e_\beta\, g_{\alpha\beta}(t) \\ &= \frac{e_\alpha}{4\pi} \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^N e_\beta \bigg(\frac{\xi_{\alpha\beta}}{|\xi_{\alpha\beta}|^3} - \frac{1}{2|\xi_{\alpha\beta}|}\, \dot v_\beta - \frac{(\dot v_\beta \cdot \xi_{\alpha\beta})}{2|\xi_{\alpha\beta}|^3}\, \xi_{\alpha\beta} + \frac{v_\beta^2}{2|\xi_{\alpha\beta}|^3}\, \xi_{\alpha\beta} - \frac{3(v_\beta \cdot \xi_{\alpha\beta})^2}{2|\xi_{\alpha\beta}|^5}\, \xi_{\alpha\beta} \\ &\qquad\qquad - \frac{(v_\alpha \cdot v_\beta)}{|\xi_{\alpha\beta}|^3}\, \xi_{\alpha\beta} + \frac{(v_\alpha \cdot \xi_{\alpha\beta})}{|\xi_{\alpha\beta}|^3}\, v_\beta \bigg), \end{aligned} \tag{4.16}$$

with $t_0 = 4(R_\varphi + C^*\varepsilon^{-1})$, cf. (3.14), and $\xi_{\alpha\beta} = q_\alpha(t) - q_\beta(t)$, $v_\alpha = v_\alpha(t)$, $v_\beta = v_\beta(t)$, etc., and moreover $m_{\rm f} = \frac{1}{2} \int d^3k\, |\hat\varphi(k)|^2 k^{-2}$.

Expanding $\gamma_\alpha = 1 + \frac{1}{2} v_\alpha^2 + \mathcal{O}(\varepsilon^2)$ and $\gamma_\alpha^3 = 1 + \mathcal{O}(\varepsilon)$ in the Lorentz equation $\frac{d}{dt}(m_{{\rm b}\alpha}\gamma_\alpha v_\alpha) = m_{{\rm b}\alpha}(\gamma_\alpha \dot v_\alpha + \gamma_\alpha^3 (v_\alpha \cdot \dot v_\alpha) v_\alpha) = F_\alpha(t)$, cf. (3.3), and recalling $m_\alpha = m_{{\rm b}\alpha} + \frac{4}{3} e_\alpha^2 m_e$ as well as $m_\alpha^* = m_{{\rm b}\alpha} + \frac{16}{15} e_\alpha^2 m_e$, we thus have deduced the following main lemma.

Lemma 4.6 *For $1 \le \alpha \le N$ and $t \in [t_0, T\varepsilon^{-3/2}]$ we have*

$$M_\alpha(v_\alpha) \dot v_\alpha = G_\alpha(q, v, \dot v) + \frac{e_\alpha}{6\pi} \sum_{\beta=1}^N e_\beta \ddot v_\beta + \mathcal{O}(\varepsilon^4), \tag{4.17}$$

where G_α is given by (4.16), and $M_\alpha(v)$ is the (3×3)-matrix $M_\alpha(v)(z) = (m_\alpha + \frac{1}{2} m_\alpha^* v^2) z + m_\alpha^* (v \cdot z) v$ for $v, z \in \mathbb{R}^3$.

Note that (4.17) agrees with (3.17), as may be verified through explicit calculation. Introducing the transformation

$$\begin{aligned} &\bar q_\alpha(\tau) = \varepsilon q_\alpha(\varepsilon^{-3/2}\tau), \quad \bar v_\alpha(\tau) = \varepsilon^{-1/2} v_\alpha(\varepsilon^{-3/2}\tau), \\ &\dot{\bar v}_\alpha(\tau) = \varepsilon^{-2} \dot v_\alpha(\varepsilon^{-3/2}\tau), \quad \ddot{\bar v}_\alpha(\tau) = \varepsilon^{-7/2} \ddot v_\alpha(\varepsilon^{-3/2}\tau), \end{aligned}$$

we arrive at the following version of Lemma 4.6. Here and henceforth we drop the overbar for simplicity, and τ will also be denoted by t.

Lemma 4.7 *For $1 \le \alpha \le N$ and $t \in [\varepsilon^{3/2} t_0, T]$ we have*

$$M_\alpha(v_\alpha, \varepsilon) \dot v_\alpha = G_\alpha(q, v, \dot v, \varepsilon) + \varepsilon^{3/2}\, \frac{e_\alpha}{6\pi} \sum_{\beta=1}^N e_\beta \ddot v_\beta + \mathcal{O}(\varepsilon^2), \tag{4.18}$$

where $G_\alpha(q,v,\dot v,\varepsilon)\in \mathbb{R}^3$ *is defined as*

$$\begin{aligned} G_\alpha(q,v,\dot v,\varepsilon) &= \frac{e_\alpha}{4\pi}\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^N e_\beta\Big(\frac{\xi_{\alpha\beta}}{|\xi_{\alpha\beta}|^3}-\frac{\varepsilon}{2|\xi_{\alpha\beta}|}\,\dot v_\beta-\frac{\varepsilon(\dot v_\beta\cdot\xi_{\alpha\beta})}{2|\xi_{\alpha\beta}|^3}\,\xi_{\alpha\beta}+\frac{\varepsilon v_\beta^2}{2|\xi_{\alpha\beta}|^3}\,\xi_{\alpha\beta}-\frac{3\varepsilon(v_\beta\cdot\xi_{\alpha\beta})^2}{2|\xi_{\alpha\beta}|^5}\,\xi_{\alpha\beta} \\ &\qquad -\frac{\varepsilon(v_\alpha\cdot v_\beta)}{|\xi_{\alpha\beta}|^3}\,\xi_{\alpha\beta}+\frac{\varepsilon(v_\alpha\cdot\xi_{\alpha\beta})}{|\xi_{\alpha\beta}|^3}\,v_\beta\Big), \end{aligned} \tag{4.19}$$

and $M_\alpha(v,\varepsilon)$ *is the* (3×3)*-matrix given by*

$$M_\alpha(v,\varepsilon)(z)=(m_\alpha+\frac{\varepsilon}{2}m_\alpha^* v^2)z+\varepsilon m_\alpha^*(v\cdot z)v,\quad v,z\in\mathbb{R}^3. \tag{4.20}$$

We remark that on the scale utilized in Lemma 4.7, all quantities $\xi_{\alpha\beta}$, v_α, $\dot v_\alpha$ and $\ddot v_\alpha$ are of the order $\mathcal{O}(1)$.

5 Construction of the center manifold

In this section we are going to construct a kind of (locally) invariant manifold of solutions to the effective system

$$M_\alpha(u_\alpha,\varepsilon)\dot u_\alpha = G_\alpha(r,u,\dot u,\varepsilon)+\varepsilon^{3/2}\,\frac{e_\alpha}{6\pi}\sum_{\beta=1}^N e_\beta\ddot u_\beta,\quad 1\le\alpha\le N, \tag{5.1}$$

i.e., to (4.18) without the error term $\mathcal{O}(\varepsilon^2)$. We define

$$K_0=\Big\{(r,u)\in\mathbb{R}^{3N}\times\mathbb{R}^{3N}:|r|\le 4C_q,\ |u|\le 4C_v\Big\}, \tag{5.2}$$

with C_q and C_v from (3.15) and (3.12), respectively. We are going to prove the following theorem concerning (5.1).

Theorem 5.1 *For every* $k\in\mathbb{N}$ *with* $k\ge 4$ *there exists* $\varepsilon_1>0$ *and a* C^k*-function* $\hat h:[0,\varepsilon_1]\times K_0\to\mathbb{R}^{3N}$ *such that*

$$\mathcal{I}_\varepsilon=\Big\{(r,u,\dot u):\ \dot u=\hat h_\varepsilon(r,u),\ (r,u)\in K_0\Big\}\subset\mathbb{R}^{3N}\times\mathbb{R}^{3N}\times\mathbb{R}^{3N} \tag{5.3}$$

is locally invariant for (5.1), in the following sense. Consider given data $(r(\tau_0),u(\tau_0),\dot u(\tau_0))\in\mathbb{R}^{3N}\times\mathbb{R}^{3N}\times\mathbb{R}^{3N}$ *for (5.1) such that*

$$\begin{aligned} &\dot u(\tau_0)=\hat h_\varepsilon\big(r(\tau_0),u(\tau_0)\big),\quad |r(\tau_0)|\le 2C_q,\quad |u(\tau_0)|\le 2C_v,\quad \text{and}\\ &(C_*/2)\le|r_\alpha(\tau_0)-r_\beta(\tau_0)|\le 2C^*\ \ (\alpha\neq\beta), \end{aligned}$$

hold. Then the corresponding solution $(r(t),u(t),\dot u(t))$ *of (5.1) with this data at* $t=\tau_0$ *exists at least until* $\tau_1>\tau_0$ *and satisfies*

$$\dot u(t)=\hat h_\varepsilon\big(r(t),u(t)\big),\quad t\in[\tau_0,\tau_1], \tag{5.4}$$

where $T\ge\tau_1>\tau_0$ *denotes the longest time such that*

$$|r(t)|\le 3C_q,\quad |u(t)|\le 3C_v,\quad \text{and}\quad (C_*/3)\le|r_\alpha(t)-r_\beta(t)|\le 3C^*\ \ (\alpha\neq\beta), \tag{5.5}$$

are valid simultaneously, for $t \in [\tau_0, \tau_1]$. Moreover for $t \in [\tau_0, \tau_1]$ we also have

$$\sum_{\beta=1}^{N} e_\beta \ddot{u}_\beta = \frac{1}{2} \sum_{\substack{\beta,\beta'=1 \\ \beta \neq \beta'}}^{N} \frac{e_\beta e_{\beta'}}{4\pi} \left(\frac{e_\beta}{m_\beta} - \frac{e_{\beta'}}{m_{\beta'}} \right) \Big[\frac{1}{|\xi_{\beta\beta'}|^3} (u_\beta - u_{\beta'}) - \frac{3}{|\xi_{\beta\beta'}|^5} \xi_{\beta\beta'} \cdot (u_\beta - u_{\beta'})\, \xi_{\beta\beta'} \Big] + \mathcal{O}(\varepsilon), \tag{5.6}$$

recall $\xi_{\beta\beta'} = r_\beta - r_{\beta'}$. The constant $C > 0$ defining $\mathcal{O}(\varepsilon)$, i.e., $|\ldots| \leq C\varepsilon$, does depend only on the input constants C_q, C_v, C_*, C^*, and T, but not on $\tau_1 \leq T$.

In order to build the center-like manifold described in the theorem we cannot specify particular data, whence we are forced to a priori smoothing out the Coulomb singularity and to introduce a bound for u_α. Thus rather than with (5.1) we will be dealing with the regularized problem

$$M_\alpha^{\mathrm{reg}}(u_\alpha, \varepsilon)\dot{u}_\alpha = G_\alpha^{\mathrm{reg}}(r, u, \dot{u}, \varepsilon) + \varepsilon^{3/2} \frac{e_\alpha}{6\pi} \sum_{\beta=1}^{N} e_\beta \ddot{u}_\beta, \quad 1 \leq \alpha \leq N, \tag{5.7}$$

where $M_\alpha^{\mathrm{reg}}(u_\alpha, \varepsilon)$ and $G_\alpha^{\mathrm{reg}}(r, u, \dot{u}, \varepsilon)$ are obtained from $M_\alpha(u_\alpha, \varepsilon)$ and $G_\alpha(r, u, \dot{u}, \varepsilon)$ by replacing all u_α by u_α^{reg} and all $\xi_{\alpha\beta}$ by $\xi_{\alpha\beta}^{\mathrm{reg}}$, respectively, with

$$u_\alpha^{\mathrm{reg}} = \chi_1(|u_\alpha|)u_\alpha \quad \text{and} \quad \xi_{\alpha\beta}^{\mathrm{reg}} = \chi_2(|\xi_{\alpha\beta}|)\xi_{\alpha\beta}. \tag{5.8}$$

Here $\chi_1 : [0, \infty[\to [0, 1]$ is a smooth function such that $\chi_1(s) = 1$ for $s \in [0, 3C_v]$ and $\chi_1(s) = 0$ for $s \in [4C_v, \infty[$, whereas $\chi_2 :]0, \infty[\to [0, \infty[$ is smooth and such that $\chi_2(s)s = s$ for $s \in [C_*/3, 3C^*]$ as well as $\chi_2(s)s \in [C_*/4, 4C^*]$ for $s \in [0, \infty[$; the constants C_*, C^*, and C_v are those appearing in Lemma 3.2. We also note that

$$\left| u_\alpha^{\mathrm{reg}} \right| \leq 4C_v \quad \text{and} \quad C_*/4 \leq \left| \xi_{\alpha\beta}^{\mathrm{reg}} \right| \leq 4C^*. \tag{5.9}$$

To rewrite (5.7), we introduce the linear map

$$P : (\mathbb{R}^3)^N \to (\mathbb{R}^3)^N, \quad Pz = \left(\frac{e_\alpha}{6\pi} \sum_{\beta=1}^{N} e_\beta z_\beta \right)_{1 \leq \alpha \leq N} \quad \text{for} \quad z = (z_1, \ldots, z_N) \in (\mathbb{R}^3)^N.$$

Then (5.7) reads as

$$\varepsilon^{3/2} P\ddot{u} = M^{\mathrm{reg}}(u, \dot{u}, \varepsilon) - G^{\mathrm{reg}}(r, u, \dot{u}, \varepsilon), \tag{5.10}$$

with

$$M^{\mathrm{reg}}(u, \dot{u}, \varepsilon) = \Big(M_\alpha^{\mathrm{reg}}(u_\alpha, \varepsilon)\dot{u}_\alpha \Big)_{1 \leq \alpha \leq N} \in (\mathbb{R}^3)^N, \quad \text{and} \tag{5.11}$$

$$G^{\mathrm{reg}}(r, u, \dot{u}, \varepsilon) = \Big(G_\alpha^{\mathrm{reg}}(r, u, \dot{u}, \varepsilon) \Big)_{1 \leq \alpha \leq N} \in (\mathbb{R}^3)^N. \tag{5.12}$$

At this point we observe that (5.10) does not present a singular perturbation problem of standard form

$$\dot{x} = f(x, y), \quad \varepsilon\dot{y} = g(x, y, \varepsilon),$$

where we think of $x = (r, u) \in \mathbb{R}^{3N} \times \mathbb{R}^{3N}$ and $y = \dot{u} \in \mathbb{R}^{3N}$, due to the presence of P which has $\dim(\mathrm{range}(P)) = 3$ (assuming that $e_\alpha \neq 0$ for all α). However, (5.10) can be suitably transformed and cast in standard form, but with different variables. To see this, we first diagonalize P by means of the matrix $A : (\mathbb{R}^3)^N \to (\mathbb{R}^3)^N$,

$$Az = \Big(e_\alpha z_1\Big)_{1\le\alpha\le N} + \Big(e_2 z_2, e_3 z_3 - e_1 z_2, e_4 z_4 - e_2 z_3, \dots, e_N z_N - e_{N-2} z_{N-1}, -e_{N-1} z_N\Big) \tag{5.13}$$

for $z = (z_1, \dots, z_N) \in (\mathbb{R}^3)^N$, being composed of the eigenvectors of P as columns; note the eigenvalues are $\lambda = e^2 := \sum_{\alpha=1}^N e_\alpha^2$ (3 times) and $\lambda = 0$ ($3N-3$ times). Then

$$A^t z = \Big(\sum_{\alpha=1}^N e_\alpha z_\alpha, e_2 z_1 - e_1 z_2, e_3 z_2 - e_2 z_3, \dots, e_N z_{N-1} - e_{N-1} z_N\Big), \tag{5.14}$$

and it can be verified that

$$A^t P A z = \frac{e^4}{6\pi}\Big(z_1, 0, \dots, 0\Big),$$

as a consequence of $\sum_{\beta=1}^N e_\beta (Az)_\beta = e^2 z_1$. Hence we can introduce the equivalent variables

$$r = A\underline{\mathrm{r}}, \quad u = A\underline{\mathrm{u}}, \quad \dot{u} = A\underline{\dot{\mathrm{u}}}, \quad \ddot{u} = A\underline{\ddot{\mathrm{u}}}, \tag{5.15}$$

to transform (5.10) to

$$\varepsilon^{3/2}\Big(\underline{\ddot{\mathrm{u}}}_1, 0, \dots, 0\Big) = \Phi(\underline{\mathrm{r}}, \underline{\mathrm{u}}, \underline{\dot{\mathrm{u}}}, \varepsilon), \tag{5.16}$$

with

$$\begin{aligned} \Phi(\underline{\mathrm{r}}, \underline{\mathrm{u}}, \underline{\dot{\mathrm{u}}}, \varepsilon) &= 6\pi e^{-4} A^t\Big(M^{\mathrm{reg}}(u, \dot{u}, \varepsilon) - G^{\mathrm{reg}}(r, u, \dot{u}, \varepsilon)\Big) \\ &= 6\pi e^{-4} A^t\Big(M^{\mathrm{reg}}(A\underline{\mathrm{u}}, A\underline{\dot{\mathrm{u}}}, \varepsilon) - G^{\mathrm{reg}}(A\underline{\mathrm{r}}, A\underline{\mathrm{u}}, A\underline{\dot{\mathrm{u}}}, \varepsilon)\Big). \end{aligned} \tag{5.17}$$

Writing $\Phi = (\Phi_\alpha)_{1\le\alpha\le N} \in (\mathbb{R}^3)^N$, our strategy is now to solve the $(N-1)$ equations

$$0 = \Phi_2(\underline{\mathrm{r}}, \underline{\mathrm{u}}, \underline{\dot{\mathrm{u}}}, \varepsilon), \quad \dots, \quad 0 = \Phi_N(\underline{\mathrm{r}}, \underline{\mathrm{u}}, \underline{\dot{\mathrm{u}}}, \varepsilon), \tag{5.18}$$

each in $\mathbb{R}^3$, for the $(N-1)$ variables $(\underline{\dot{\mathrm{u}}}_2, \dots, \underline{\dot{\mathrm{u}}}_N)$, also each in $\mathbb{R}^3$. This will yield a solution function

$$\underline{\mathrm{U}}_{2N} : \mathbb{R}^{3N} \times \mathbb{R}^{3N} \times \mathbb{R}^3 \times [0, \varepsilon_0] \to \mathbb{R}^{3N-3}, \quad (\underline{\dot{\mathrm{u}}}_2, \dots, \underline{\dot{\mathrm{u}}}_N) = \underline{\mathrm{U}}_{2N}(\underline{\mathrm{r}}, \underline{\mathrm{u}}, \underline{\dot{\mathrm{u}}}_1, \varepsilon)$$

i.e., we will have

$$0 = \Phi_2\Big(\underline{\mathrm{r}}, \underline{\mathrm{u}}, \underline{\dot{\mathrm{u}}}_1, \underline{\mathrm{U}}_{2N}(\underline{\mathrm{r}}, \underline{\mathrm{u}}, \underline{\dot{\mathrm{u}}}_1, \varepsilon), \varepsilon\Big), \quad \dots, \quad 0 = \Phi_N\Big(\underline{\mathrm{r}}, \underline{\mathrm{u}}, \underline{\dot{\mathrm{u}}}_1, \underline{\mathrm{U}}_{2N}(\underline{\mathrm{r}}, \underline{\mathrm{u}}, \underline{\dot{\mathrm{u}}}_1, \varepsilon), \varepsilon\Big) \tag{5.19}$$

for $(\underline{\mathrm{r}}, \underline{\mathrm{u}}, \underline{\dot{\mathrm{u}}}_1, \varepsilon) \in \mathbb{R}^{3N} \times \mathbb{R}^{3N} \times \mathbb{R}^3 \times [0, \varepsilon_0]$. Setting $\underline{\mathrm{x}} = (\underline{\mathrm{x}}_1, \underline{\mathrm{x}}_2) = (\underline{\mathrm{r}}, \underline{\mathrm{u}}) \in \mathbb{R}^{3N} \times \mathbb{R}^{3N}$ and $\underline{\mathrm{y}} = \underline{\dot{\mathrm{u}}}_1 \in \mathbb{R}^3$, in view of (5.16) we then need to solve

$$\begin{aligned} \underline{\dot{\mathrm{x}}} &= (\underline{\mathrm{u}}, \underline{\dot{\mathrm{u}}}) = \Big(\underline{\mathrm{u}}, \underline{\dot{\mathrm{u}}}_1, \underline{\mathrm{U}}_{2N}(\underline{\mathrm{r}}, \underline{\mathrm{u}}, \underline{\dot{\mathrm{u}}}_1, \varepsilon)\Big) = \Big(\underline{\mathrm{x}}_2, \underline{\mathrm{y}}, \underline{\mathrm{U}}_{2N}(\underline{\mathrm{x}}, \underline{\mathrm{y}}, \varepsilon)\Big) =: f_1(\underline{\mathrm{x}}, \underline{\mathrm{y}}, \varepsilon), && (5.20) \\ \varepsilon^{3/2}\underline{\dot{\mathrm{y}}} &= \Phi_1\Big(\underline{\mathrm{r}}, \underline{\mathrm{u}}, \underline{\dot{\mathrm{u}}}_1, \underline{\mathrm{U}}_{2N}(\underline{\mathrm{r}}, \underline{\mathrm{u}}, \underline{\dot{\mathrm{u}}}_1, \varepsilon), \varepsilon\Big) = \Phi_1\Big(\underline{\mathrm{x}}, \underline{\mathrm{y}}, \underline{\mathrm{U}}_{2N}(\underline{\mathrm{x}}, \underline{\mathrm{y}}, \varepsilon), \varepsilon\Big) =: g_1(\underline{\mathrm{x}}, \underline{\mathrm{y}}, \varepsilon), && (5.21) \end{aligned}$$

which turns out to be a standard singular perturbation problem, up to the factor of $\underline{\dot{\mathrm{y}}}$ which is $\varepsilon^{3/2}$ rather than ε. To achieve the latter, we transform

$$x(t) = \underline{\mathrm{x}}(\sqrt{\varepsilon}t), \quad y(t) = \underline{\mathrm{y}}(\sqrt{\varepsilon}t),$$

and arrive at the system

$$\dot{x} = \sqrt{\varepsilon} f_1(x, y, \varepsilon) =: f(x, y, \varepsilon), \quad \varepsilon \dot{y} = g_1(x, y, \varepsilon) =: g(x, y, \varepsilon), \tag{5.22}$$

which is of standard form and can be shown to allow for the existence of a (locally) invariant manifold if $\varepsilon > 0$ is small enough.

To carry out this program, we first have to investigate the solvability of (5.18).

Lemma 5.2 *There exist $\varepsilon_0 > 0$ and a smooth function $\underline{U}_{2N} : \mathrm{IR}^{3N} \times \mathrm{IR}^{3N} \times \mathrm{IR}^3 \times [0, \varepsilon_0] \to \mathrm{IR}^{3N-3}$, $(\underline{\dot{u}}_2, \ldots, \underline{\dot{u}}_N) = \underline{U}_{2N}(\underline{r}, \underline{u}, \underline{\dot{u}}_1, \varepsilon)$, such that (5.19) is satisfied.*

Proof: With $z = M^{\rm reg}(A\underline{u}, A\underline{\dot{u}}, \varepsilon) - G^{\rm reg}(A\underline{r}, A\underline{u}, A\underline{\dot{u}}, \varepsilon) \in \mathrm{IR}^{3N}$ we need to solve

$$0 = e_2 z_1 - e_1 z_2, \quad 0 = e_3 z_2 - e_2 z_3, \quad 0 = e_4 z_3 - e_3 z_4, \quad \ldots, \quad 0 = e_N z_{N-1} - e_{N-1} z_N, \tag{5.23}$$

for $(\underline{\dot{u}}_2, \ldots, \underline{\dot{u}}_N)$, cf. (5.14). According to (5.11) and (5.12) we have

$$z_\alpha = M_\alpha^{\rm reg}\Big((A\underline{u})_\alpha, \varepsilon\Big)(A\underline{\dot{u}})_\alpha - G_\alpha^{\rm reg}\Big(A\underline{r}, A\underline{u}, A\underline{\dot{u}}, \varepsilon\Big) \in \mathrm{IR}^3, \quad 1 \le \alpha \le N.$$

To decompose z_α appropriately, we introduce the notation $\underline{\dot{u}} = (\underline{\dot{u}}_1, \eta) \in \mathrm{IR}^3 \times \mathrm{IR}^{3N-3}$ with $\eta = (\underline{\dot{u}}_2, \ldots, \underline{\dot{u}}_N)$, and accordingly we split $A\underline{\dot{u}} = A_1\underline{\dot{u}}_1 + A_{2N}\eta$ for suitable linear $A_1 : \mathrm{IR}^3 \to \mathrm{IR}^{3N}$ and $A_{2N} : \mathrm{IR}^{3N-3} \to \mathrm{IR}^{3N}$, as given in (5.13). From the definition of $M_\alpha^{\rm reg}$ and $G_\alpha^{\rm reg}$, cf. (4.20), (4.19), and (5.8), we then obtain

$$\begin{aligned}
M_\alpha^{\rm reg}\Big((A\underline{u})_\alpha, \varepsilon\Big)(A\underline{\dot{u}})_\alpha &= m_\alpha (A\underline{\dot{u}})_\alpha + \varepsilon\bigg(\frac{1}{2} m_\alpha^*\Big((A\underline{u})_\alpha^{\rm reg}\Big)^2 + m_\alpha^*\Big((A\underline{u})_\alpha^{\rm reg}\cdot(A\underline{\dot{u}})_\alpha\Big)(A\underline{u})_\alpha^{\rm reg}\bigg) \\
&= m_\alpha(A_{2N}\eta)_\alpha + \bigg[m_\alpha(A_1\underline{\dot{u}}_1)_\alpha + \frac{\varepsilon}{2} m_\alpha^*\Big((A\underline{u})_\alpha^{\rm reg}\Big)^2 \\
&\quad + \varepsilon m_\alpha^*\Big((A\underline{u})_\alpha^{\rm reg}\cdot(A_1\underline{\dot{u}}_1)_\alpha\Big)(A\underline{u})_\alpha^{\rm reg}\bigg] + \varepsilon m_\alpha^*\Big((A\underline{u})_\alpha^{\rm reg}\cdot(A_{2N}\eta)_\alpha\Big)(A\underline{u})_\alpha^{\rm reg} \\
&=: m_\alpha(A_{2N}\eta)_\alpha + M_\alpha^{(1)}\Big((A\underline{u})_\alpha^{\rm reg}, (A_1\underline{\dot{u}}_1)_\alpha, \varepsilon\Big) + \varepsilon M_\alpha^{(2)}\Big((A\underline{u})_\alpha^{\rm reg}, (A_{2N}\eta)_\alpha\Big), \\
G_\alpha^{\rm reg}\Big(A\underline{r}, A\underline{u}, A\underline{\dot{u}}, \varepsilon\Big) &= \frac{e_\alpha}{4\pi}\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^N e_\beta \frac{\zeta_{\alpha\beta}}{|\zeta_{\alpha\beta}|^3} \\
&\quad + \varepsilon\frac{e_\alpha}{4\pi}\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^N e_\beta\bigg(-\frac{1}{2|\zeta_{\alpha\beta}|}(A\underline{\dot{u}})_\beta - \frac{((A\underline{\dot{u}})_\beta\cdot\zeta_{\alpha\beta})}{2|\zeta_{\alpha\beta}|^3}\zeta_{\alpha\beta} \\
&\qquad + \frac{\Big((A\underline{u})_\beta^{\rm reg}\Big)^2}{2|\zeta_{\alpha\beta}|^3}\zeta_{\alpha\beta} - \frac{3\Big((A\underline{u})_\beta^{\rm reg}\cdot\zeta_{\alpha\beta}\Big)^2}{2|\zeta_{\alpha\beta}|^5}\zeta_{\alpha\beta} \\
&\qquad - \frac{((A\underline{u})_\alpha^{\rm reg}\cdot(A\underline{u})_\beta^{\rm reg})}{|\zeta_{\alpha\beta}|^3}\zeta_{\alpha\beta} + \frac{((A\underline{u})_\alpha^{\rm reg}\cdot\zeta_{\alpha\beta})}{|\zeta_{\alpha\beta}|^3}(A\underline{u})_\beta^{\rm reg}\bigg) \\
&=: G_\alpha^{(0)}(A\underline{r}) + \varepsilon G_\alpha^{(2)}\Big(A\underline{r}, A_{2N}\eta\Big) + \varepsilon G_\alpha^{(1)}\Big(A\underline{r}, (A\underline{u})^{\rm reg}, A_1\underline{\dot{u}}_1\Big),
\end{aligned}$$

with

$$\zeta_{\alpha\beta} = \chi_2\Big(|(A\underline{r})_\alpha - (A\underline{r})_\beta|\Big)\Big[(A\underline{r})_\alpha - (A\underline{r})_\beta\Big], \tag{5.24}$$

cf. (5.8), and

$$\begin{aligned}
G^{(0)}_\alpha(A\underline{r}) &= \frac{e_\alpha}{4\pi}\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^N e_\beta\frac{\zeta_{\alpha\beta}}{|\zeta_{\alpha\beta}|^3},\\
G^{(2)}_\alpha\Big(A\underline{r}, A_{2N}\eta\Big) &= \frac{e_\alpha}{4\pi}\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^N e_\beta\bigg(-\frac{1}{2|\zeta_{\alpha\beta}|}\,(A_{2N}\eta)_\beta-\frac{((A_{2N}\eta)_\beta\cdot\zeta_{\alpha\beta})}{2|\zeta_{\alpha\beta}|^3}\,\zeta_{\alpha\beta}\bigg).
\end{aligned}$$

In view of (5.9) we note the bounds

$$\Big|M^{(2)}_\alpha\Big((A\underline{u})^{\rm reg}_\alpha, (A_{2N}\eta)_\alpha\Big)\Big|+\Big|G^{(2)}_\alpha\Big(A\underline{r}, A_{2N}\eta\Big)\Big|\le C|\eta|,\quad 1\le\alpha\le N, \tag{5.25}$$

which are valid for all $(\underline{r},\underline{u},\eta)\in\mathbb{R}^{3N}\times\mathbb{R}^{3N}\times\mathbb{R}^{3N-3}$. Omitting the arguments and recalling (5.13), the equations from (5.23) can be rewritten as

$$\begin{aligned}
0 &= \Big(e_2^2m_1+e_1^2m_2\Big)\underline{\dot u}_2-e_1e_3m_2\,\underline{\dot u}_3+\varepsilon\Big(e_2M^{(2)}_1-e_1M^{(2)}_2\Big)+\varepsilon\Big(e_1G^{(2)}_2-e_2G^{(2)}_1\Big)\\
&\quad+\Big(e_1G^{(0)}_2-e_2G^{(0)}_1\Big)+\Big(e_2M^{(1)}_1-e_1M^{(1)}_2\Big)+\varepsilon\Big(e_1G^{(1)}_2-e_2G^{(1)}_1\Big),\\
0 &= -e_1e_3m_2\,\underline{\dot u}_2+\Big(e_3^2m_2+e_2^2m_3\Big)\underline{\dot u}_3-e_2e_4m_3\,\underline{\dot u}_4+\varepsilon\Big(e_3M^{(2)}_2-e_2M^{(2)}_3\Big)+\varepsilon\Big(e_2G^{(2)}_3-e_3G^{(2)}_2\Big)\\
&\quad+\Big(e_2G^{(0)}_3-e_3G^{(0)}_2\Big)+\Big(e_3M^{(1)}_2-e_2M^{(1)}_3\Big)+\varepsilon\Big(e_2G^{(1)}_3-e_3G^{(1)}_2\Big),\\
0 &= -e_2e_4m_3\,\underline{\dot u}_3+\Big(e_4^2m_3+e_3^2m_4\Big)\underline{\dot u}_4-e_3e_5m_4\,\underline{\dot u}_5+\varepsilon\Big(e_4M^{(2)}_3-e_3M^{(2)}_4\Big)+\varepsilon\Big(e_3G^{(2)}_4-e_4G^{(2)}_3\Big)\\
&\quad+\Big(e_3G^{(0)}_4-e_4G^{(0)}_3\Big)+\Big(e_4M^{(1)}_3-e_3M^{(1)}_4\Big)+\varepsilon\Big(e_3G^{(1)}_4-e_4G^{(1)}_3\Big),\\
\vdots & \qquad\qquad\qquad\qquad \vdots\\
0 &= -e_{N-2}e_Nm_{N-1}\,\underline{\dot u}_{N-1}+\Big(e_N^2m_{N-1}+e_{N-1}^2m_N\Big)\underline{\dot u}_N\\
&\quad+\varepsilon\Big(e_NM^{(2)}_{N-1}-e_{N-1}M^{(2)}_N\Big)+\varepsilon\Big(e_{N-1}G^{(2)}_N-e_NG^{(2)}_{N-1}\Big)\\
&\quad+\Big(e_{N-1}G^{(0)}_N-e_NG^{(0)}_{N-1}\Big)+\Big(e_NM^{(1)}_{N-1}-e_{N-1}M^{(1)}_N\Big)+\varepsilon\Big(e_{N-1}G^{(1)}_N-e_NG^{(1)}_{N-1}\Big).
\end{aligned}$$

Note that this is a linear system for $\eta=(\underline{\dot u}_2,\ldots,\underline{\dot u}_N)$, since both $M^{(2)}_\alpha$ and $G^{(2)}_\alpha$ do depend on η linearly, whereas $M^{(1)}_\alpha$, $G^{(0)}_\alpha$, and $G^{(1)}_\alpha$ are independent of η. Accordingly, the system has the form

$$0=\mathcal{M}^{(0)}\eta+\varepsilon\mathcal{M}^{(2)}(\underline{r},\underline{u})\eta-\mathcal{R}(\underline{r},\underline{u},\underline{\dot u}_1,\varepsilon), \tag{5.26}$$

with the $(3N-3)\times(3N-3)$-matrix $\mathcal{M}^{(0)}=\mathcal{M}^{(0)}(e_1,\ldots,e_N,m_1,\ldots,m_N)$ being defined as

$$\begin{aligned}
\mathcal{M}^{(0)}\eta &= \Big(\Big(e_2^2m_1+e_1^2m_2\Big)\eta_2-e_1e_3m_2\eta_3,\ -e_1e_3m_2\eta_2+\Big(e_3^2m_2+e_2^2m_3\Big)\eta_3-e_2e_4m_3\eta_4,\\
&\qquad -e_2e_4m_3\eta_3+\Big(e_4^2m_3+e_3^2m_4\Big)\eta_4-e_3e_5m_4\eta_5\,,\ldots,\\
&\qquad -e_{N-2}e_Nm_{N-1}\eta_{N-1}+\Big(e_N^2m_{N-1}+e_{N-1}^2m_N\Big)\eta_N\Big),
\end{aligned} \tag{5.27}$$

and moreover

$$\begin{aligned}
\mathcal{M}^{(2)}(\underline{r},\underline{u})\eta &= \Big(e_\alpha M^{(2)}_{\alpha-1}-e_{\alpha-1}M^{(2)}_\alpha+e_{\alpha-1}G^{(2)}_\alpha-e_\alpha G^{(2)}_{\alpha-1}\Big)_{2\le\alpha\le N},\\
\mathcal{R}(\underline{r},\underline{u},\underline{\dot u}_1,\varepsilon) &= \Big(e_\alpha G^{(0)}_{\alpha-1}-e_{\alpha-1}G^{(0)}_\alpha+e_{\alpha-1}M^{(1)}_\alpha-e_\alpha M^{(1)}_{\alpha-1}+\varepsilon e_\alpha G^{(1)}_{\alpha-1}-\varepsilon e_{\alpha-1}G^{(1)}_\alpha\Big)_{2\le\alpha\le N}.
\end{aligned}$$

By (5.25) we have

$$\left|\mathcal{M}^{(2)}(\underline{\mathrm{r}},\underline{\mathrm{u}})\eta\right| \le C|\eta|, \quad (\underline{\mathrm{r}},\underline{\mathrm{u}},\eta) \in \mathrm{IR}^{3N} \times \mathrm{IR}^{3N} \times \mathrm{IR}^{3N-3},$$

i.e., $|\mathcal{M}^{(2)}(\underline{\mathrm{r}},\underline{\mathrm{u}})| \le C$ as a linear map, uniformly in $(\underline{\mathrm{r}},\underline{\mathrm{u}})$. Choosing $\varepsilon > 0$ small enough, hence (5.26) can be solved for η (even explicitly by means of a von Neumann-series), as soon as we know that $\mathcal{M}^{(0)}$ is invertible; this is verified in Lemma 5.3 below. The associated solution function $\underline{\mathrm{U}}_{2N}$ is smooth w.r.t. all variables since, due to regularizing, $\zeta_{\alpha\beta}$ is a smooth function of $\underline{\mathrm{r}}$. □

Lemma 5.3 *Denote $\mathcal{M}^{(0)} = \mathcal{M}_N^{(0)} \in \mathrm{IR}^{(3N-3)\times(3N-3)}$ the matrix defined by (5.27). Then*

$$\det \mathcal{M}^{(0)} = \Big[\Big(\prod_{j=2}^{N-1} e_j^2\Big)\Big(\sum_{j=1}^{N} e_j^2 \prod_{\substack{i=1\\ i\neq j}}^{N} m_i\Big)\Big]^3 > 0.$$

Proof: The first observation to make is that $\det \mathcal{M}_N^{(0)} = \big[\det \mathcal{A}_N\big]^3$, where

$$\mathcal{A}_N = \begin{pmatrix} e_2^2 m_1 + e_1^2 m_2 & -e_1e_3m_2 & 0 & \dots & 0 \\ -e_1e_3m_2 & e_3^2m_2 + e_2^2m_3 & -e_2e_4m_3 & \dots & 0 \\ 0 & -e_2e_4m_3 & e_4^2m_3 + e_3^2m_4 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & e_N^2 m_{N-1} + e_{N-1}^2 m_N \end{pmatrix} \in \mathrm{IR}^{N-1};$$

this can be verified by induction. Hence we need to prove that

$$\det \mathcal{A}_N = \Big(\prod_{j=2}^{N-1} e_j^2\Big)\Big(\sum_{j=1}^{N} e_j^2 \prod_{\substack{i=1\\ i\neq j}}^{N} m_i\Big). \tag{5.28}$$

For $N = 2$ we have $\det \mathcal{A}_2 = \mathcal{A}_2 = e_2^2 m_1 + e_1^2 m_2$, whereas for $N = 3$ we calculate $\det \mathcal{A}_3 = e_2^2(e_1^2 m_2 m_3 + e_2^2 m_1 m_3 + e_3^2 m_1 m_2)$, thus (5.28) is satisfied. If this is already known for some N, then $\det \mathcal{A}_{N+1} = (e_{N+1}^2 m_N + e_N^2 m_{N+1}) \det \mathcal{A}_N - e_{N-1}^2 e_{N+1}^2 m_N^2 \det \mathcal{A}_{N-1}$ and the induction hypothesis lead to

$$\begin{aligned}
\det \mathcal{A}_{N+1} &= (e_{N+1}^2 m_N + e_N^2 m_{N+1})\Big(\prod_{j=2}^{N-1} e_j^2\Big)\Big(\sum_{j=1}^{N} e_j^2 \prod_{\substack{i=1\\ i\neq j}}^{N} m_i\Big) \\
&\quad - e_{N-1}^2 e_{N+1}^2 m_N^2 \Big(\prod_{j=2}^{N-2} e_j^2\Big)\Big(\sum_{j=1}^{N-1} e_j^2 \prod_{\substack{i=1\\ i\neq j}}^{N-1} m_i\Big) \\
&= \Big(\prod_{j=2}^{N-1} e_j^2\Big)\Big\{e_{N+1}^2 m_N e_N^2 \Big(\prod_{\substack{i=1\\ i\neq N}}^{N} m_i\Big) + e_{N+1}^2 m_N \Big(\sum_{j=1}^{N-1} e_j^2 \prod_{\substack{i=1\\ i\neq j}}^{N} m_i\Big) \\
&\qquad + e_N^2 m_{N+1}\Big(\sum_{j=1}^{N} e_j^2 \prod_{\substack{i=1\\ i\neq j}}^{N} m_i\Big) - e_{N+1}^2 m_N^2 \Big(\sum_{j=1}^{N-1} e_j^2 \prod_{\substack{i=1\\ i\neq j}}^{N-1} m_i\Big)\Big\} \\
&= \Big(\prod_{j=2}^{N} e_j^2\Big)\Big\{e_{N+1}^2 m_N \Big(\prod_{\substack{i=1\\ i\neq N}}^{N} m_i\Big) + m_{N+1}\Big(\sum_{j=1}^{N} e_j^2 \prod_{\substack{i=1\\ i\neq j}}^{N} m_i\Big)\Big\} = \Big(\prod_{j=2}^{N} e_j^2\Big)\Big(\sum_{j=1}^{N+1} e_j^2 \prod_{\substack{i=1\\ i\neq j}}^{N+1} m_i\Big),
\end{aligned}$$

completing the proof of (5.28). □

The next step is to study the second equation in (5.22) for $\varepsilon = 0$ in greater detail.

Lemma 5.4 *For $\varepsilon = 0$ the equation $g(\underline{x}, \underline{y}, 0) = 0$ is solved for $\underline{y} = \underline{\dot{u}}_1$ by the function*

$$\underline{y} = \underline{h}_0(\underline{x}) = \underline{h}_0(\underline{r}, \underline{u}) = \underline{h}_0(\underline{r}) = \frac{1}{4\pi e^2} \sum_{1\le\alpha<\beta\le N} e_\alpha e_\beta \Big(\frac{e_\alpha}{m_\alpha} - \frac{e_\beta}{m_\beta}\Big) \frac{\zeta_{\alpha\beta}}{|\zeta_{\alpha\beta}|^3}, \quad \textit{with} \quad e^2 = \sum_{\alpha=1}^N e_\alpha^2,$$

cf. also (5.24). Moreover, the eigenvalues of $D_{\underline{y}}\, g(\underline{x}, \underline{y}, 0) \in \mathbb{R}^{3\times 3}$ are $\frac{6\pi}{e^4} \sum_{\alpha=1}^N e_\alpha^2 m_\alpha > 0$ (3 times), and hence bounded away from the imaginary axis.

Proof: The simplest way to find $\underline{y} = \underline{h}_0(\underline{x})$ is to use the original variables and consider (5.10) for $\varepsilon = 0$. Since

$$M^{\rm reg}(u, \dot{u}, 0) = \big(m_\alpha \dot{u}_\alpha\big)_{1\le\alpha\le N} \quad \text{and} \quad G^{\rm reg}(r, u, \dot{u}, 0) = \Big(\frac{e_\alpha}{4\pi} \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^N e_\beta \frac{\xi_{\alpha\beta}^{\rm reg}}{|\xi_{\alpha\beta}^{\rm reg}|^3}\Big)_{1\le\alpha\le N}, \tag{5.29}$$

cf. (4.20) and (4.19), eq. (5.10) reads as

$$m_\alpha \dot{u}_\alpha = \frac{e_\alpha}{4\pi} \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^N e_\beta \frac{\xi_{\alpha\beta}^{\rm reg}}{|\xi_{\alpha\beta}^{\rm reg}|^3}, \quad 1 \le \alpha \le N. \tag{5.30}$$

The relevant transformations are $r = A\underline{r}$ and $\dot{u} = A\underline{\dot{u}}$, whence $\underline{\dot{u}}_1 = (A^{-1}\dot{u})_1$. It may be verified that A^{-1} has the general form

$$A^{-1} z = e^{-2}\Big(\sum_{\alpha=1}^N e_\alpha z_\alpha, *, \ldots, *\Big), \quad z = (z_1, \ldots, z_N) \in (\mathbb{R}^3)^N, \tag{5.31}$$

and therefore (5.30) implies

$$\underline{y} = \underline{\dot{u}}_1 = \frac{1}{4\pi e^2} \sum_{\alpha=1}^N \frac{e_\alpha^2}{m_\alpha} \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^N e_\beta \frac{\zeta_{\alpha\beta}}{|\zeta_{\alpha\beta}|^3} = \frac{1}{4\pi e^2} \sum_{1\le\alpha<\beta\le N} e_\alpha e_\beta \Big(\frac{e_\alpha}{m_\alpha} - \frac{e_\beta}{m_\beta}\Big) \frac{\zeta_{\alpha\beta}}{|\zeta_{\alpha\beta}|^3},$$

the latter due to $\zeta_{\beta\alpha} = -\zeta_{\alpha\beta}$. For the eigenvalues of $D_{\underline{y}}\, g(\underline{x}, \underline{y}, 0)$, we note that due to (5.22), (5.21), (5.17), and (5.14)

$$\begin{aligned} g(\underline{x}, \underline{y}, 0) &= \Phi_1\big(\underline{r}, \underline{u}, \underline{\dot{u}}_1, \underline{U}_{2N}(\underline{r}, \underline{u}, \underline{\dot{u}}_1, 0), 0\big) \\ &= 6\pi e^{-4} \Big[A^t\Big(M^{\rm reg}(A\underline{u}, A\underline{\dot{u}}, 0) - G^{\rm reg}(A\underline{r}, A\underline{u}, A\underline{\dot{u}}, 0)\Big)\Big]_1 \\ &= 6\pi e^{-4} \sum_{\alpha=1}^N e_\alpha \Big([M^{\rm reg}(A\underline{u}, A\underline{\dot{u}}, 0)]_\alpha - [G^{\rm reg}(A\underline{r}, A\underline{u}, A\underline{\dot{u}}, 0)]_\alpha\Big) \\ &= 6\pi e^{-4} \sum_{\alpha=1}^N e_\alpha \Big(m_\alpha \dot{u}_\alpha - \frac{e_\alpha}{4\pi} \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^N e_\beta \frac{\xi_{\alpha\beta}^{\rm reg}}{|\xi_{\alpha\beta}^{\rm reg}|^3}\Big), \end{aligned}$$

where we have used (5.29) and passed to the original variables. Observing $\underline{\mathrm{y}} = \dot{\underline{\mathrm{u}}}_1$, $\dot{u}_\alpha = [A\dot{\underline{\mathrm{u}}}]_\alpha$, and (5.13), we hence obtain

$$D_{\underline{\mathrm{y}}}\, g(\underline{\mathrm{x}}, \underline{\mathrm{y}}, 0) = 6\pi e^{-4} \sum_{\alpha=1}^{N} e_\alpha m_\alpha \Big(\frac{\partial \dot{u}_\alpha}{\partial \dot{\underline{\mathrm{u}}}_1}\Big) = 6\pi e^{-4} \sum_{\alpha=1}^{N} e_\alpha m_\alpha e_\alpha \mathrm{id}_{\mathbb{R}^3} = \Big(6\pi e^{-4} \sum_{\alpha=1}^{N} e_\alpha^2 m_\alpha\Big)\, \mathrm{id}_{\mathbb{R}^3},$$

and this yields the claim. □

According to Lemma 5.4 we see that the assumptions $(H1)$-$(H3)$ of [9, Sect. 1.1 & 1.2] are satisfied, and therefore we find a (locally) invariant manifold for (5.22); cf. [9, Thm. 2]. Transferred back to (5.20) and (5.21) this result can be applied as follows. We define

$$\underline{\mathrm{K}}_0 = \Big\{\underline{\mathrm{x}} = (\underline{\mathrm{r}}, \underline{\mathrm{u}}) \in \mathbb{R}^{3N} \times \mathbb{R}^{3N} : |A\underline{\mathrm{r}}| \le 4C_q,\ |A\underline{\mathrm{u}}| \le 4C_v\Big\}$$

with C_q and C_v from (3.15) and (3.12), respectively, and we let

$$\underline{\mathrm{I}}_0 = \Big\{(\underline{\mathrm{x}}, \underline{\mathrm{y}}) : \underline{\mathrm{y}} = \underline{\mathrm{h}}_0(\underline{\mathrm{x}}),\ \underline{\mathrm{x}} \in \underline{\mathrm{K}}_0\Big\},$$

with $\underline{\mathrm{h}}_0(\underline{\mathrm{x}})$ from Lemma 5.4.

Lemma 5.5 *For every $k \in \mathbb{N}$ with $k \ge 4$ there exist $\varepsilon_1 > 0$ and a C^k-function $\underline{h} : [0, \varepsilon_1] \times \underline{K}_0 \to \mathbb{R}^3$ such that*

$$\underline{I}_\varepsilon = \Big\{(\underline{x}, \underline{y}) : \underline{y} = \underline{h}_\varepsilon(\underline{x}),\ \underline{x} \in \underline{K}_0\Big\} \tag{5.32}$$

is locally invariant w.r.t. (5.20) and (5.21), where $\underline{h}_\varepsilon(\underline{x}) = \underline{h}(\varepsilon, \underline{x})$. In particular this means the following. Consider $(\underline{x}(\tau_0), \underline{y}(\tau_0)) = (\underline{r}(\tau_0), \underline{u}(\tau_0), \dot{\underline{u}}_1(\tau_0)) \in \mathbb{R}^{3N} \times \mathbb{R}^{3N} \times \mathbb{R}^3$ such that $|A\underline{r}(\tau_0)| \le 2C_q$ as well as $|A\underline{u}(\tau_0)| \le 2C_v$ and

$$\dot{\underline{u}}_1(\tau_0) = \underline{h}_\varepsilon\Big(\underline{r}(\tau_0), \underline{u}(\tau_0)\Big)$$

hold. Then the corresponding solution $(\underline{x}(t), \underline{y}(t)) = (\underline{r}(t), \underline{u}(t), \dot{\underline{u}}_1(t))$ of (5.20) and (5.21) with this data at $t = \tau_0$ exists at least until $\tau_1 > \tau_0$ and satisfies

$$\dot{\underline{u}}_1(t) = \underline{h}_\varepsilon\Big(\underline{r}(t), \underline{u}(t)\Big), \quad t \in [\tau_0, \tau_1], \tag{5.33}$$

where $\tau_1 > \tau_0$ denotes the longest time such that $|A\underline{r}(t)| \le 3C_q$ and $|A\underline{u}(t)| \le 3C_v$ for $t \in [\tau_0, \tau_1]$. Moreover we have

$$|\underline{h}_\varepsilon - \underline{h}_0|_{C_b^1(\underline{K}_0)} \le C\varepsilon \tag{5.34}$$

for a constant $C > 0$ depending only on the input constants C_q and C_v. The dynamics on $\underline{I}_\varepsilon$ is governed by $\dot{\underline{x}} = f_1(\underline{x}, \underline{h}_\varepsilon(\underline{x}), \varepsilon)$, i.e.,

$$\dot{\underline{r}} = \underline{u}, \quad \dot{\underline{u}}_1 = \underline{h}_\varepsilon(\underline{r}, \underline{u}), \quad (\dot{\underline{u}}_2, \ldots, \dot{\underline{u}}_N) = \underline{U}_{2N}\Big(\underline{r}, \underline{u}, \underline{h}_\varepsilon(\underline{r}, \underline{u}), \varepsilon\Big), \tag{5.35}$$

cf. (5.20).

The C_b^1-estimate (5.34) is not given explicitly in [9], but may be validated along the lines of [18, 17] where the analogous statement is shown for C_b^0. With this preparation we can finally come to the

Proof of Theorem 5.1: We only have to undo the transformations to find $\hat{h}_\varepsilon$, and this way we arrive at

$$\hat{h}_\varepsilon(r,u) = A\Big(\underline{\mathrm{h}}_\varepsilon(A^{-1}r, A^{-1}u), \underline{\mathrm{U}}_{2N}(A^{-1}r, A^{-1}u, \underline{\mathrm{h}}_\varepsilon(A^{-1}r, A^{-1}u), \varepsilon)\Big), \quad (r,u) \in K_0.$$

To verify that then $\mathcal{I}_\varepsilon$ from (5.3) is locally invariant in the sense stated, we note that (5.5) and the definition of χ_1 and χ_2, cf. (5.8), imply that (5.1) and (5.7) do agree for $t \in [\tau_0, \tau_1]$. Hence (5.1) is equivalent to (5.20) and (5.21) via the transformation from (5.15) for $t \in [\tau_0, \tau_1]$, and therefore the local invariance of $\mathcal{I}_\varepsilon$ is a direct consequence of the local invariance of $\underline{\mathrm{I}}_\varepsilon$ from (5.32) in Lemma 5.5, for (5.4) cf. (5.33). Concerning (5.6), in view of (5.34) we can write $\underline{\mathrm{h}}_\varepsilon(\underline{\mathrm{r}}, \underline{\mathrm{u}}) = \underline{\mathrm{h}}_0(\underline{\mathrm{r}}) + \Delta_\varepsilon$ with $|\Delta_\varepsilon|_{C_b^1(\underline{K}_0)} \le C\varepsilon$. Hence (5.31), (5.35), and Lemma 5.4 imply

$$\begin{aligned} e^{-2}\sum_{\beta=1}^{N} e_\beta \dot{u}_\beta &= (A^{-1}\dot{u})_1 = \dot{\underline{\mathrm{u}}}_1 = \underline{\mathrm{h}}_\varepsilon(\underline{\mathrm{r}}, \underline{\mathrm{u}}) = \underline{\mathrm{h}}_0(\underline{\mathrm{r}}) + \Delta_\varepsilon \\ &= \frac{1}{4\pi e^2} \sum_{1\le\beta<\beta'\le N} e_\beta e_{\beta'} \Big(\frac{e_\beta}{m_\beta} - \frac{e_{\beta'}}{m_{\beta'}}\Big) \frac{\zeta_{\beta\beta'}}{|\zeta_{\beta\beta'}|^3} + \Delta_\varepsilon \\ &= \frac{1}{4\pi e^2} \sum_{1\le\beta<\beta'\le N} e_\beta e_{\beta'} \Big(\frac{e_\beta}{m_\beta} - \frac{e_{\beta'}}{m_{\beta'}}\Big) \frac{\xi_{\beta\beta'}}{|\xi_{\beta\beta'}|^3} + \Delta_\varepsilon, \end{aligned} \tag{5.36}$$

the latter since here $\zeta_{\beta\beta'} = (A\underline{r})_\beta - (A\underline{r})_{\beta'} = r_\beta - r_{\beta'} = \xi_{\beta\beta'}$ according to (5.5), cf. (5.24). From (5.36) we obtain (5.6) by differentiation and observing that $|\dot{\Delta}_\varepsilon|_{C_b^0(\underline{K}_0)} \le C\varepsilon$. $\Box$

6 Comparison of the full and the effective system

We prove Theorem 3.5 and assume that the stage is set as is described in the theorem. Then we define

$$h_\varepsilon(r,u) = \varepsilon^2 \hat{h}_\varepsilon\Big(\varepsilon r, \varepsilon^{-1/2}u\Big), \quad (r,u) \in K_\varepsilon, \tag{6.1}$$

with the function $\hat{h}$ from Theorem 5.1; note that $(r,u) \in K_\varepsilon$ is equivalent to $(\varepsilon r, \varepsilon^{-1/2}u) \in K_0$, cf. (5.2). We let $\tau_0 = \varepsilon^{3/2} t_0$ and

$$\bar{r}_\alpha(t) = \varepsilon r_\alpha(\varepsilon^{-3/2}t), \quad \bar{u}_\alpha(t) = \varepsilon^{-1/2} u_\alpha(\varepsilon^{-3/2}t), \quad 1 \le \alpha \le N. \tag{6.2}$$

Hence (3.20) implies that

$$\dot{\bar{u}}_\alpha(\tau_0) = \varepsilon^{-2}\dot{u}_\alpha(t_0) = \varepsilon^{-2} h_\varepsilon\Big(r_\alpha(t_0), u_\alpha(t_0)\Big) = \hat{h}_\varepsilon\Big(\varepsilon r_\alpha(t_0), \varepsilon^{-1/2}u_\alpha(t_0)\Big) = \hat{h}_\varepsilon\Big(\bar{r}_\alpha(\tau_0), \bar{u}_\alpha(\tau_0)\Big) \tag{6.3}$$

for $1 \le \alpha \le N$. In addition we deduce from (3.15), (3.12), (3.8), and (3.11) that

$$\begin{aligned} &|\bar{r}(\tau_0)| = \varepsilon|r(t_0)| = \varepsilon|q(t_0)| := \varepsilon \max_{1\le\alpha\le N} |q_\alpha(t_0)| \le C_q, \\ &|\bar{u}(\tau_0)| = \varepsilon^{-1/2}|u(t_0)| = \varepsilon^{-1/2}|v(t_0)| \le C_v, \quad \text{and} \\ &C_* \le \varepsilon|q_\alpha(t_0) - q_\beta(t_0)| = |\bar{r}_\alpha(\tau_0) - \bar{r}_\beta(\tau_0)| \le C^* \quad (\alpha \ne \beta), \end{aligned}$$

whence taking into account (6.3) we see that the assumptions of Theorem 5.1 are satisfied. We denote by $t_1 \in]t_0, T\varepsilon^{-3/2}]$ the longest time such that

$$|r(t)| \le 3C_q\varepsilon^{-1}, \quad |u(t)| \le 3C_v\sqrt{\varepsilon}, \quad \text{and} \quad (C_*/3)\varepsilon^{-1} \le |r_\alpha(t) - r_\beta(t)| \le 3C^*\varepsilon^{-1} \ \ (\alpha \ne \beta), \tag{6.4}$$

are valid simultaneously for $t \in [t_0, t_1]$, corresponding to the longest time $\tau_1 \in]\tau_0, T]$ such that

$$|\bar{r}(t)| \le 3C_q, \quad |\bar{u}(t)| \le 3C_v, \quad \text{and} \quad (C_*/3) \le |\bar{r}_\alpha(t) - \bar{r}_\beta(t)| \le 3C^* \ \ (\alpha \neq \beta),$$

are verified for $t \in [\tau_0, \tau_1]$, cf. (5.5). We infer from Theorem 5.1 that the solution $(r(t), u(t), \dot{u}(t))$ of the effective equation (1.8) with data given by (3.20) exists at least for $t \in [t_0, t_1]$ and satisfies

$$\dot{u}(t) = h_\varepsilon\Big(r(t), u(t)\Big), \quad t \in [t_0, t_1], \tag{6.5}$$

due to (5.4). Moreover,

$$\sum_{\beta=1}^N e_\beta \ddot{u}_\beta = \frac{1}{2} \sum_{\substack{\beta,\beta'=1\\ \beta\neq\beta'}}^N \frac{e_\beta e_{\beta'}}{4\pi}\Big(\frac{e_\beta}{m_\beta} - \frac{e_{\beta'}}{m_{\beta'}}\Big)\Big[\frac{1}{|\xi_{\beta\beta'}|^3}(u_\beta - u_{\beta'}) - \frac{3}{|\xi_{\beta\beta'}|^5}\,\xi_{\beta\beta'}\cdot(u_\beta - u_{\beta'})\,\xi_{\beta\beta'}\Big] + \mathcal{O}(\varepsilon^{9/2}) \tag{6.6}$$

for $t \in [t_0, t_1]$, by transforming (5.6) back utilizing (6.2); recall that due to the convention made prior to Lemma 4.7, in fact the microscopic variables in (5.6) are used, i.e., r_α, u_α, etc., have to be replaced by $\bar{r}_\alpha$, $\bar{u}_\alpha$, etc., and those are transformed back via (6.2).

We need to prove that $t_1 = T\varepsilon^{-3/2}$ holds, and for this purpose we compare the true solution and the solution of the effective equation (1.8) for times $t \in [t_0, t_1]$. We recall from (4.17) in Lemma 4.6 that

$$M_\alpha(v_\alpha)\dot{v}_\alpha = G_\alpha(q, v, \dot{v}) + \frac{e_\alpha}{6\pi}\sum_{\beta=1}^N e_\beta \ddot{v}_\beta + \mathcal{O}(\varepsilon^4) = G_\alpha(q, v, \dot{v}) + \mathcal{O}(\varepsilon^{7/2}), \quad \alpha = 1, \ldots, N, \tag{6.7}$$

for $t \in [t_0, T\varepsilon^{-3/2}]$, where in the last step we have used Lemma 3.3, noting that $t_0 \ge \tau_{**}$. In addition, direct calculation reveals that (1.8) may be reformulated as

$$M_\alpha(u_\alpha)\dot{u}_\alpha = G_\alpha(r, u, \dot{u}) + \frac{e_\alpha}{6\pi}\sum_{\beta=1}^N e_\beta \ddot{u}_\beta, \quad \alpha = 1, \ldots, N. \tag{6.8}$$

From (6.6) and (6.4) we deduce that

$$\Big|\sum_{\beta=1}^N e_\beta \ddot{u}_\beta\Big| \le C\varepsilon^{7/2}, \quad t \in [t_0, t_1],$$

with $C > 0$ depending only on the input constants, and accordingly we obtain from (6.8) that

$$M_\alpha(u_\alpha)\dot{u}_\alpha = G_\alpha(r, u, \dot{u}) + \mathcal{O}(\varepsilon^{7/2}), \quad \alpha = 1, \ldots, N, \quad t \in [t_0, t_1]. \tag{6.9}$$

Next we observe that (6.5) and (6.1) yield

$$|\dot{u}_\alpha(t)| \le C\varepsilon^2, \quad 1 \le \alpha \le N, \quad t \in [t_0, t_1], \tag{6.10}$$

since in particular $\hat{h}_\varepsilon : [0, \varepsilon_1] \times K_0 \to \mathbb{R}^{3N}$ is bounded. As we can use both the bounds from Lemma 3.2 and the bounds from (6.4) and (6.10) for $t \in [t_0, t_1]$, it is therefore possible to proceed exactly as in [15, p. 449/450] and to deduce, comparing (6.7) and (6.9), that

$$|q_\alpha(t) - r_\alpha(t)| \le C\sqrt{\varepsilon}, \quad |v_\alpha(t) - u_\alpha(t)| \le C\varepsilon^2, \quad 1 \le \alpha \le N, \quad t \in [t_0, t_1]; \tag{6.11}$$

the only property of t_1 which enters here is $t_1 \le C\varepsilon^{-3/2}$. However, we know from Lemma 3.2 that

$$|q(t)| \le C_q\,\varepsilon^{-1}, \quad |v(t)| \le C_v\sqrt{\varepsilon}, \quad \text{and} \quad C_*\varepsilon^{-1} \le |q_\alpha(t) - q_\beta(t)| \le C^*\varepsilon^{-1} \ \ (\alpha \ne \beta), \tag{6.12}$$

in particular for $t \in [t_0, T\varepsilon^{-3/2}]$. Since all constants thus far do depend only on the input constants, by choosing $\varepsilon > 0$ small enough and observing (6.12) and (6.11), we therefore arrive at

$$|r(t)| \le 2C_q\varepsilon^{-1}, \quad |u(t)| \le 2C_v\sqrt{\varepsilon}, \quad \text{and} \quad (C_*/2)\varepsilon^{-1} \le |r_\alpha(t) - r_\beta(t)| \le 2C^*\varepsilon^{-1} \ \ (\alpha \ne \beta),$$

being valid for $t \in [t_0, t_1]$. In view of the definition of t_1 this leads to a contradiction, unless $t_1 = T\varepsilon^{-3/2}$. Hence (6.11) shows that (3.21) holds as well, since we can use (6.7) and (6.9) to obtain from (6.11) the further estimate $|\dot v_\alpha(t) - \dot u_\alpha(t)| \le C\varepsilon^{7/2}$, cf. [15, (4.6)]. In addition, (3.22) is a consequence of (1.8) and (6.6). Finally to verify (3.23), it follows by means of direct calculation from Lemma 4.6 and from (3.12) that

$$\frac{d}{dt}\,\mathcal{H}_{\rm RR}(q(t), v(t), \dot v(t)) = -\frac{1}{6\pi}\Big(\sum_{\alpha=1}^N e_\alpha \dot v_\alpha(t)\Big)^2 + \mathcal{O}(\varepsilon^{9/2}), \quad t \in [t_0, T\varepsilon^{-3/2}]. \tag{6.13}$$

Therefore (3.21), (6.13), (3.19), (3.20), (3.13), and (6.10) yield

$$\begin{aligned}
&\mathcal{H}_{\rm D}(q(t), v(t)) - \mathcal{H}_{\rm D}(r(t), u(t))\\
&= \mathcal{H}_{\rm RR}(q(t), v(t), \dot v(t)) - \mathcal{H}_{\rm RR}(r(t), u(t), \dot u(t)) + \sum_{\alpha,\beta=1}^N \frac{e_\alpha e_\beta}{6\pi}\Big[v_\alpha(t)\cdot \dot v_\beta(t) - u_\alpha(t)\cdot \dot u_\beta(t)\Big]\\
&= \mathcal{H}_{\rm RR}(q(t), v(t), \dot v(t)) - \mathcal{H}_{\rm RR}(r(t), u(t), \dot u(t)) + \mathcal{O}(\varepsilon^4)\\
&= \int_{t_0}^t \frac{d}{dt'}\Big(\mathcal{H}_{\rm RR}(q, v, \dot v) - \mathcal{H}_{\rm RR}(r, u, \dot u)\Big)\,dt'\\
&\quad + \mathcal{H}_{\rm RR}(q(t_0), v(t_0), \dot v(t_0)) - \mathcal{H}_{\rm RR}(r(t_0), u(t_0), \dot u(t_0)) + \mathcal{O}(\varepsilon^4)\\
&= \int_{t_0}^t \Big[\frac{1}{6\pi}\Big(\sum_{\alpha=1}^N e_\alpha \dot u_\alpha\Big)^2 - \frac{1}{6\pi}\Big(\sum_{\alpha=1}^N e_\alpha \dot v_\alpha\Big)^2 + \mathcal{O}(\varepsilon^{9/2})\Big]\,dt'\\
&\quad + \sum_{\alpha,\beta=1}^N \frac{e_\alpha e_\beta}{6\pi}\Big[u_\alpha(t_0)\cdot \dot u_\beta(t_0) - v_\alpha(t_0)\cdot \dot v_\beta(t_0)\Big] + \mathcal{O}(\varepsilon^4)\\
&= \int_{t_0}^t \Big[\mathcal{O}(\varepsilon^{11/2}) + \mathcal{O}(\varepsilon^{9/2})\Big]\,dt' + \mathcal{O}(\varepsilon^4) = \mathcal{O}(\varepsilon^3),
\end{aligned}$$

for $t \le C\varepsilon^{-3/2}$. This completes the proof of Theorem 3.5. □

7 Appendix: Proof of Lemma 3.4

This section is devoted to the proof of Lemma 3.4. To calculate $\ddot v\,(t)$ from (3.3), we first note that $\frac{d}{dt}(m_{{\rm b}\alpha}\gamma_\alpha v_\alpha(t)) = m_{0\alpha}(v_\alpha(t))\dot v_\alpha(t)$, where the (3×3)-matrices $m_{0\alpha}(v_\alpha)$ are defined as

$$m_{0\alpha}(v_\alpha)(z) = m_{{\rm b}\alpha}(\gamma_\alpha z + \gamma_\alpha^3(v_\alpha\cdot z)v_\alpha), \quad z \in {\rm I\!R}^3. \tag{7.1}$$

Thus differentiating (3.3) once, it follows that for $\alpha = 1, \ldots, N$ we have

$$\begin{aligned}
\dot v_\alpha &= m_{0\alpha}(v_\alpha)^{-1}\int d^3x\,\rho_\alpha(x - q_\alpha)\Big([E(x) - E_{v_\alpha}(x - q_\alpha)] + v_\alpha \wedge [B(x) - B_{v_\alpha}(x - q_\alpha)]\Big)\\
&= m_{0\alpha}(v_\alpha)^{-1}\int d^3x\,\rho_\alpha(x)\Big(Z_1(x + q_\alpha, t) + v_\alpha \wedge Z_2(x + q_\alpha, t)\Big) + R_\alpha(t),
\end{aligned} \tag{7.2}$$

with

$$m_{0\alpha}(v_\alpha)^{-1}z = m_{\mathrm{b}\alpha}{}^{-1}\gamma_\alpha^{-1}(z - (v_\alpha \cdot z)v_\alpha), \quad z \in \mathbb{R}^3, \tag{7.3}$$

denoting the matrix inverse of $m_{0\alpha}(v_\alpha)$; here and henceforth we often omit the argument t of q_α, v_α, $\dot{v}_\alpha$, etc. Moreover,

$$R_\alpha(t) = m_{0\alpha}(v_\alpha)^{-1}\Big(\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N}\int d^3x\,\rho_\alpha(x-q_\alpha)\big[E_{v_\beta}(x-q_\beta) + v_\alpha \wedge B_{v_\beta}(x-q_\beta)\big]\Big), \tag{7.4}$$

as well as

$$Z(x,t) = \begin{pmatrix} Z_1(x,t) \\ Z_2(x,t)\end{pmatrix} = \begin{pmatrix} E(x,t) - \sum_{\beta=1}^N E_{v_\beta(t)}(x-q_\beta(t)) \\ B(x,t) - \sum_{\beta=1}^N B_{v_\beta(t)}(x-q_\beta(t))\end{pmatrix} \tag{7.5}$$

in (7.2). An important observation is that

$$\dot{Z}(t) = \mathcal{A}Z(t) - f(t)\,, \quad \text{with} \quad \mathcal{A} = \begin{pmatrix} 0 & \nabla\wedge \\ -\nabla\wedge & 0\end{pmatrix} \tag{7.6}$$

the Maxwell operator, and

$$f(x,t) = \begin{pmatrix} f_1(x,t) \\ f_2(x,t)\end{pmatrix} = \sum_{\beta=1}^{N}\begin{pmatrix} (\dot{v}_\beta(t)\cdot\nabla_v)E_{v_\beta(t)}(x-q_\beta(t)) \\ (\dot{v}_\beta(t)\cdot\nabla_v)B_{v_\beta(t)}(x-q_\beta(t))\end{pmatrix}; \tag{7.7}$$

see [15, Section 5.2]. We also note that $\nabla\cdot f_1 = 0 = \nabla\cdot f_2$, since $\nabla\cdot E_{v_\beta} = e_\beta\varphi$ and $\nabla\cdot B_{v_\beta} = 0$ are calculated from (2.3). Differentiating (7.2) once more, we obtain

$$\begin{aligned} \ddot{v}_\alpha &= \Big(\frac{d}{dt}m_{0\alpha}(v_\alpha)^{-1}\Big)m_{0\alpha}(v_\alpha)\big[\dot{v}_\alpha - R_\alpha(t)\big] + m_{0\alpha}(v_\alpha)^{-1}M_\alpha(t) \\ &\quad + m_{0\alpha}(v_\alpha)^{-1}\int d^3x\,\rho_\alpha(x)\big(\dot{v}_\alpha\wedge Z_2(x+q_\alpha,t)\big) + \dot{R}_\alpha(t), \end{aligned} \tag{7.8}$$

with the main term

$$M_\alpha(t) = \int d^3x\,\rho_\alpha(x)\big[(L_\alpha(t)Z_1)(x+q_\alpha(t),t) + v_\alpha(t)\wedge(L_\alpha(t)Z_2)(x+q_\alpha(t),t)\big], \tag{7.9}$$

where $L_\alpha(t)\phi = (v_\alpha(t)\cdot\nabla)\phi + \dot{\phi}$ for a general function $\phi = \phi(x,t)$. Finally, upon differentiating (7.8) it follows that

$$\begin{aligned} \dddot{v}_\alpha &= \Big(\frac{d^2}{dt^2}m_{0\alpha}(v_\alpha)^{-1}\Big)m_{0\alpha}(v_\alpha)\big[\dot{v}_\alpha - R_\alpha(t)\big] + \Big(\frac{d}{dt}m_{0\alpha}(v_\alpha)^{-1}\Big)\Big(\frac{d}{dt}m_{0\alpha}(v_\alpha)\Big)\big[\dot{v}_\alpha - R_\alpha(t)\big] \\ &\quad + \Big(\frac{d}{dt}m_{0\alpha}(v_\alpha)^{-1}\Big)m_{0\alpha}(v_\alpha)\big[\ddot{v}_\alpha - \dot{R}_\alpha(t)\big] + \Big(\frac{d}{dt}m_{0\alpha}(v_\alpha)^{-1}\Big)M_\alpha(t) + m_{0\alpha}(v_\alpha)^{-1}\dot{M}_\alpha(t) \\ &\quad + \ddot{R}_\alpha(t) + \Big(\frac{d}{dt}m_{0\alpha}(v_\alpha)^{-1}\Big)\int d^3x\,\rho_\alpha(x)\big(\dot{v}_\alpha\wedge Z_2(x+q_\alpha,t)\big) \\ &\quad + m_{0\alpha}(v_\alpha)^{-1}\int d^3x\,\rho_\alpha(x)\big[\ddot{v}_\alpha\wedge Z_2(x+q_\alpha,t) + \dot{v}_\alpha\wedge(L_\alpha(t)Z_2)(x+q_\alpha,t)\big]. \end{aligned} \tag{7.10}$$

Most of these terms are directly seen to be at least of the desired order $\mathcal{O}(\varepsilon^5)$. Indeed, using (7.1), (7.3), Lemma 3.2, and Lemma 3.3, one derives that

$$\big|m_{0\alpha}(v_\alpha)\big| + \big|m_{0\alpha}(v_\alpha)^{-1}\big| \le C, \quad \Big|\frac{d}{dt}m_{0\alpha}(v_\alpha)\Big| + \Big|\frac{d}{dt}m_{0\alpha}(v_\alpha)^{-1}\Big| \le C\varepsilon^{5/2}, \quad t\in[0,T\varepsilon^{-3/2}], \tag{7.11}$$

$$\Big|\frac{d^2}{dt^2}m_{0\alpha}(v_\alpha)^{-1}\Big| \le C\varepsilon^4, \quad t\in[\tau_{**},T\varepsilon^{-3/2}]. \tag{7.12}$$

Moreover, following [15, Section 5.2 & 5.3] we have

$$|R_\alpha(t)| + \left|\int d^3x\, \rho_\alpha(x) Z(x+q_\alpha(t),t)\right| \le C\varepsilon^2, \quad |\dot R_\alpha(t)| \le C\varepsilon^{7/2}, \quad t\in[0,T\varepsilon^{-3/2}], \tag{7.13}$$

and also

$$|M_\alpha(t)| + \left|\int d^3x\, \rho_\alpha(x)(L_\alpha(t)Z)(x+q_\alpha(t),t)\right| \le C\varepsilon^{7/2}, \quad t\in[\tau_{**},T\varepsilon^{-3/2}], \tag{7.14}$$

for $1\le\alpha\le N$. Thus we find from (7.10) and (7.11)–(7.14) that

$$|\dddot{v}_\alpha(t)| \le C|\dot M_\alpha(t)| + |\ddot R_\alpha(t)| + C\varepsilon^{11/2}, \quad \alpha=1,\ldots,N, \quad t\in[\tau_{**},T\varepsilon^{-3/2}]. \tag{7.15}$$

In Section 7.5 below we will show

$$|\ddot R_\alpha(t)| \le C\varepsilon^5, \quad \alpha=1,\ldots,N, \quad t\in[\tau_{**},T\varepsilon^{-3/2}], \tag{7.16}$$

cf. (7.106). Introducing

$$\mathcal{L}_\alpha(t)\phi = (\dot v_\alpha(t)\cdot\nabla)\phi + (v_\alpha(t)\cdot\nabla)^2\phi + 2(v_\alpha(t)\cdot\nabla)\dot\phi + \ddot\phi \tag{7.17}$$

for a general $\phi=\phi(x,t)$ and observing $\frac{d}{dt}[(L_\alpha(t)\phi)(x+q_\alpha(t))] = (\mathcal{L}_\alpha(t)\phi)(x+q_\alpha(t))$, we moreover deduce from (7.9) that

$$\begin{aligned}\dot M_\alpha(t) &= \int d^3x\,\rho_\alpha(x)\Big[(\mathcal{L}_\alpha(t)Z_1)(x+q_\alpha,t) + v_\alpha\wedge(\mathcal{L}_\alpha(t)Z_2)(x+q_\alpha,t)\Big]\\ &\quad + \int d^3x\,\rho_\alpha(x)\Big(\dot v_\alpha\wedge(L_\alpha(t)Z_2)(x+q_\alpha,t)\Big).\end{aligned}$$

In view of Lemma 3.2 and (7.14) we hence obtain

$$\begin{aligned}|\dot M_\alpha(t)| &\le \left|\int d^3x\,\rho_\alpha(x)\Big[(\mathcal{L}_\alpha(t)Z_1)(x+q_\alpha(t),t) + v_\alpha\wedge(\mathcal{L}_\alpha(t)Z_2)(x+q_\alpha(t),t)\Big]\right| + C\varepsilon^{11/2}\\ &\le C\left|\int d^3x\,\rho_\alpha(x)(\mathcal{L}_\alpha(t)Z)(x+q_\alpha(t),t)\right| + C\varepsilon^{11/2}, \quad t\in[\tau_{**},T\varepsilon^{-3/2}].\end{aligned}$$

Putting this together with (7.15) and (7.16), we have seen that

$$|\dddot{v}_\alpha(t)| \le C\varepsilon^5 + C\left|\int d^3x\,\rho_\alpha(x)(\mathcal{L}_\alpha(t)Z)(x+q_\alpha(t),t)\right|, \quad \alpha=1,\ldots,N, \quad t\in[\tau_{**},T\varepsilon^{-3/2}]. \tag{7.18}$$

In order to bound the main term on the right-hand side of (7.18), we calculate

$$\begin{aligned}\frac{d}{dt}(\mathcal{L}_\alpha(t)Z(\cdot,t)) &= \mathcal{A}(\mathcal{L}_\alpha(t)Z(\cdot,t)) - \mathcal{L}_\alpha(t)f(\cdot,t) + (\ddot v_\alpha\cdot\nabla)Z(\cdot,t)\\ &\quad + 2(v_\alpha\cdot\nabla)(\dot v_\alpha\cdot\nabla)Z(\cdot,t) + 2(\dot v_\alpha\cdot\nabla)\dot Z(\cdot,t),\end{aligned}$$

where we have used (7.6). Denoting $U(t)$, $t\in\mathbb{R}$, the group of isometries in $L^2(\mathbb{R}^3)^3\oplus L^2(\mathbb{R}^3)^3$ generated by the Maxwell operator $\mathcal{A}$ from (7.6), we therefore find for any $t_1\in[0,t]$

$$\begin{aligned}\int d^3x\,\rho_\alpha(x)(\mathcal{L}_\alpha(t)Z)(x+q_\alpha(t),t) &= \int d^3x\,\rho_\alpha(x)\Big[U(t-t_1)\Big(\mathcal{L}_\alpha(t_1)Z(\cdot,t_1)\Big)\Big](x+q_\alpha(t))\\ &\quad + \int d^3x\,\rho_\alpha(x)\int_{t_1}^t ds\,\Big[U(t-s)\Big(-\mathcal{L}_\alpha(s)f(\cdot,s)\\ &\qquad + (\ddot v_\alpha(s)\cdot\nabla)Z(\cdot,s)\\ &\qquad + 2(v_\alpha(s)\cdot\nabla)(\dot v_\alpha(s)\cdot\nabla)Z(\cdot,s)\\ &\qquad + 2(\dot v_\alpha(s)\cdot\nabla)\dot Z(\cdot,s)\Big)\Big](x+q_\alpha(t))\\ &=: T_{\mathrm{data}} + T_1 + T_2 + T_3 + T_4.\end{aligned} \tag{7.19}$$

This expression will be bounded term by term, for different choices of $t_1 \geq \tau_{**}$; note that we need to select $t_1 \geq \tau_{**}$ in order to use the bound on $\ddot{v}_\alpha(s)$ from Lemma 3.3 for $s \geq t_1$. First some estimates are derived below in Sections 7.1-7.5 which are then used in Section 7.6 to complete the proof of Lemma 3.4. Since our treatment of data terms in similar situations in the previous papers [15, 14, 13] was not completely accurate, we will give a more detailed account here. To obtain the necessary estimates, we will frequently use the following three technical lemmas.

Lemma 7.1 *For given $f = (f_1, f_2)$ with $\nabla \cdot f_1 = 0$ and $\nabla \cdot f_2 = 0$ we have for $W(t,s,x) = (W_1(t,s,x), W_2(t,s,x)) = [U(t-s)f(\cdot,s)](x)$ that*

$$\begin{aligned} W_1(t,s,x) &= \frac{1}{4\pi(t-s)^2}\int_{|y-x|=(t-s)} d^2y\,\Big[(t-s)\nabla\wedge f_2(y,s) + f_1(y,s) + ((y-x)\cdot\nabla)f_1(y,s)\Big],\\ W_2(t,s,x) &= \frac{1}{4\pi(t-s)^2}\int_{|y-x|=(t-s)} d^2y\,\Big[-(t-s)\nabla\wedge f_1(y,s) + f_2(y,s) + ((y-x)\cdot\nabla)f_2(y,s)\Big]. \end{aligned}$$

Proof: See [15, Lemma 5.1]. □

Lemma 7.2 *Defining*

$$\zeta_v(x) = \frac{1}{\left[(1-v^2)x^2 + (x\cdot v)^2\right]^{1/2}}, \quad \text{we have} \quad \hat{\zeta}_v(k) = \sqrt{\frac{2}{\pi}}\,\frac{1}{k^2-(k\cdot v)^2}, \quad |v|<1.$$

In addition, $|\nabla^j\zeta_v(x)| \leq C|x|^{-(1+j)}$ for $0\leq j\leq 4$, $x\in\mathbb{R}^3$, and $|v|\leq\bar{v}<1$. More generally, even $|\nabla_v^l\nabla^j\zeta_v(x)| \leq C|x|^{-(j+1)}$ for $0\leq l,j\leq 4$.

Proof: Through direct calculation. □

Lemma 7.3 *Assume $|x|,|y|\leq R_\varphi$ and $1\leq\alpha,\beta\leq N$, and recall*

$$\tau_{**} = (C_*/8)\varepsilon^{-1},$$

with the constant C_* from Lemma 3.2, cf. (3.16). Moreover, let $\tilde{x} = x - y + q_\alpha(t) - q_\beta(t-\tau) - z$ with some $0\leq\tau\leq t\leq T\varepsilon^{-3/2}$ and $z\in\mathbb{R}^3$. Then the following assertions hold for $\varepsilon>0$ small enough.

(a) If $\alpha\neq\beta$, $\tau\leq\tau_{**}$, and $|\tilde{x}|=\tau$, then $|z|\geq(C_*/4)\varepsilon^{-1}$.

(b) If $\alpha\neq\beta$, $\tau\geq\tau_{**}$, and $|\tilde{x}|=\tau$, then $|\tilde{x}|\geq(C_*/8)\varepsilon^{-1}$.

(c) If $\alpha=\beta$, $\tau\geq 8R_\varphi$, and $|\tilde{x}|=\tau$, then $|z|\geq\tau/4$.

Proof: (a) By (3.8) and (3.12) we have $|z| = |\tilde{x}-[x-y+q_\alpha(t)-q_\beta(t-\tau)]| \geq |q_\alpha(t)-q_\beta(t)|-|x|-|y|-|q_\beta(t)-q_\beta(t-\tau)|-|\tilde{x}| \geq C_*\varepsilon^{-1}-2R_\varphi-C_v\sqrt{\varepsilon}\tau-\tau \geq (C_*/2)\varepsilon^{-1}-2\tau \geq (C_*/4)\varepsilon^{-1}$, if $\varepsilon>0$ is chosen sufficiently small. (b) Obvious. (c) Similarly as in (a) we find $|z| = |\tilde{x}-[x-y+q_\alpha(t)-q_\alpha(t-\tau)]| \geq |\tilde{x}|-2R_\varphi-|q_\alpha(t)-q_\alpha(t-\tau)| \geq \tau-2R_\varphi-C_v\sqrt{\varepsilon}\tau \geq \tau/2-2R_\varphi \geq \tau/4$. □

7.1 Bounding T_1

To deal with

$$T_1(t,t_1) = -\int d^3x\,\rho_\alpha(x)\int_{t_1}^t ds\,\Big[U(t-s)\Big(\mathcal{L}_\alpha(s)f(\cdot,s)\Big)\Big](x+q_\alpha(t)) \tag{7.20}$$

from the right-hand side of (7.19), we introduce the notation

$$\Phi_v = \begin{pmatrix} E_v \\ B_v \end{pmatrix}, \quad \text{thus} \quad f(x,t) = \sum_{\beta=1}^N (\dot v_\beta(t)\cdot\nabla_v)\Phi_{v_\beta(t)}(x-q_\beta(t)). \tag{7.21}$$

Utilizing (7.17), a somewhat lengthy calculation reveals that

$$\begin{aligned}(\mathcal{L}_\alpha(s)f)(x,s) \;=\; \sum_{\beta=1}^N \Big\{&(\dddot v_\beta\cdot\nabla_v) + 3(\ddot v_\beta\cdot\nabla_v)(\dot v_\beta\cdot\nabla_v) + 2([v_\alpha - v_\beta]\cdot\nabla)(\ddot v_\beta\cdot\nabla_v) \\ &+ (\dot v_\beta\cdot\nabla_v)^3 + ([\dot v_\alpha - \dot v_\beta]\cdot\nabla)(\dot v_\beta\cdot\nabla_v) + 2([v_\alpha - v_\beta]\cdot\nabla)(\dot v_\beta\cdot\nabla_v)^2 \\ &+ ([v_\alpha - v_\beta]\cdot\nabla)^2(\dot v_\beta\cdot\nabla_v)\Big\}\,\Phi_{v_\beta}(x-q_\beta),\end{aligned} \tag{7.22}$$

where all $v_\alpha, v_\beta, \ldots$, etc., are evaluated at time s. We will not go through the estimate of all these terms when substituted back to (7.20), but only the first and the last one will be dealt with in some detail. Since the last term contains the maximal number two of ∇'s and the minimal a priori power ε^3 (cf. Lemma 3.2 and Lemma 3.3), and as the first term contains a third derivative $\dddot v_\beta$ which we are about to estimate, it is clear that all other constituents of (7.22) will be easier to handle; note that due to $T\varepsilon^{-3/2} \geq t \geq s \geq t_1 \geq \tau_{**}$ in the integral, we may as well use the a priori estimate on $\ddot v_\alpha$ from Lemma 3.3.

To begin with, we consider the last expression $([v_\alpha - v_\beta]\cdot\nabla)^2(\dot v_\beta\cdot\nabla_v)\Phi_{v_\beta}(x-q_\beta)$. Due to the difference, we can restrict to $\alpha \neq \beta$, as will be supposed in the sequel. Using this expression in (7.20), we see that it suffices to verify

$$\left|\int d^3x\,\rho_\alpha(x)\int_{t_1}^t ds\,\Big[U(t-s)\Big(([v_\alpha(s)-v_\beta(s)]\cdot\nabla)^2(\dot v_\beta(s)\cdot\nabla_v)\,\Phi_{v_\beta(s)}(\cdot - q_\beta(s))\Big)\Big](x+q_\alpha(t))\right| \leq C\varepsilon^5 \tag{7.23}$$

for $t \in [t_1, T\varepsilon^{-3/2}]$. By means of the solution formulas from Lemma 7.1 we can rewrite this in Fourier transformed form. Recalling $\Phi_v = (E_v, B_v)$ as well as $\rho_\alpha = e_\alpha\varphi$, we hence need to show

$$\begin{aligned}\Bigg| e_\alpha \int d^3k\,\overline{\hat\varphi(k)}\int_{t_1}^t ds\,e^{ik\cdot[q_\beta(s)-q_\alpha(t)]}\,([v_\alpha(s)-v_\beta(s)]\cdot k)^2(\dot v_\beta(s)\cdot\nabla_v) \\ \times\Big\{\frac{\sin|k|(t-s)}{|k|}\,\mathcal{F}(\nabla\wedge B_{v_\beta(s)})(k) + \cos|k|(t-s)\mathcal{F}(E_{v_\beta(s)})(k)\Big\}\Bigg| \leq C\varepsilon^5\end{aligned} \tag{7.24}$$

for $t \in [t_1, T\varepsilon^{-3/2}]$, with $\mathcal{F}$ denoting Fourier transform. For simplicity, we will concentrate only on the first expression containing $\frac{\sin|k|(t-s)}{|k|}$, the other term with $\cos|k|(t-s)$ can be bounded in a similar way. From (2.3) we deduce the relation $\mathcal{F}(\nabla\wedge B_v)(k) = [k^2 v - (v\cdot k)k]\hat\phi_v(k)$, and (2.4) yields

$$\dot v\cdot\nabla_v\hat\phi_v(k) = 2e\,\frac{(v\cdot k)(\dot v\cdot k)}{k^2-(v\cdot k)^2}\,\hat\phi_v(k),$$

where $e = e_\beta$ for $v = v_\beta(s)$. Moreover, Lemma 7.2 and (2.4) imply $\hat{\phi}_v(k) = \sqrt{\frac{\pi}{2}} e\,\hat{\varphi}(k)\hat{\zeta}_v(k)$. Thus calculating $(\dot{v}_\beta(s)\cdot\nabla_v)\mathcal{F}(\nabla\wedge B_{v_\beta(s)})(k)$ explicitly, it follows that it is enough to prove that

$$\Big| \int d^3k\,|\hat{\varphi}(k)|^2 \int_{t_1}^t ds\, e^{ik\cdot[q_\beta(s)-q_\alpha(t)]}\,([v_\alpha(s)-v_\beta(s)]\cdot k)^2\,\frac{\sin|k|(t-s)}{|k|}\,\hat{\zeta}_{v_\beta(s)}(k)$$
$$\times\Big[k^2\dot{v}_\beta(s) - (\dot{v}_\beta(s)\cdot k)k + 2k^2\frac{(v_\beta(s)\cdot k)(\dot{v}_\beta(s)\cdot k)}{k^2-(v_\beta(s)\cdot k)^2}\,v_\beta(s) - 2\,\frac{(v_\beta(s)\cdot k)^2(\dot{v}_\beta(s)\cdot k)}{k^2-(v_\beta(s)\cdot k)^2}\,k\Big]\Big| \le C\varepsilon^5 \tag{7.25}$$

for $t\in[t_1, T\varepsilon^{-3/2}]$. Counting the powers of ε in view of Lemma 3.2, we see that both last terms have an additional ε compared to the first two terms, the order in k being k^2 for all four terms. Therefore the last two terms are easier to handle, and thus dropped, since the same method can be used as will be explained for the first two terms. Hence we are going to show that $|A^{(1)}(t,t_1)| \le C\varepsilon^5$ for $t\in[t_1, T\varepsilon^{-3/2}]$, with

$$A^{(1)}(t,t_1) = \int d^3k\,|\hat{\varphi}(k)|^2 \int_0^{t-t_1} d\tau\, e^{ik\cdot[q_\beta(t-\tau)-q_\alpha(t)]}\,([v_\alpha(t-\tau)-v_\beta(t-\tau)]\cdot k)^2$$
$$\times\,\frac{\sin|k|\tau}{|k|}\,\hat{\zeta}_{v_\beta(t-\tau)}(k)\Big[k^2\dot{v}_\beta(t-\tau) - (\dot{v}_\beta(t-\tau)\cdot k)k\Big]. \tag{7.26}$$

Recalling the definition of $\tau_{**} = (C_*/8)\varepsilon^{-1}$ from Lemma 7.3, we split the integral

$$\int_0^{t-t_1} d\tau = \int_0^{\tau_{**}} d\tau + \int_{\tau_{**}}^{t-t_1} d\tau, \tag{7.27}$$

and accordingly we decompose $A^{(1)}(t,t_1) = A^{(1)}_{[0,\tau_{**}]}(t) + A^{(1)}_{[\tau_{**},t-t_1]}(t,t_1)$. In case that $t-t_1\le\tau_{**} = \mathcal{O}(\varepsilon^{-1})$, the whole term $A^{(1)}(t,t_1)$ can be bounded as is $A^{(1)}_{[0,\tau_{**}]}(t)$.

To begin with $A^{(1)}_{[0,\tau_{**}]}(t)$, we write this term as a double convolution in space variables as

$$\begin{aligned}\Big|A^{(1)}_{[0,\tau_{**}]}(t)\Big| &= C\Big|\int_0^{\tau_{**}} d\tau \int\!\!\int d^3x\,d^3y\,\varphi(x)\varphi(y)\\ &\times\int d^3z\,(\eta_{\alpha\beta}(\tau)\cdot\nabla)^2\big(\nabla\wedge(\dot{v}_\beta(t-\tau)\wedge\nabla)\big)\zeta_{v_\beta(t-\tau)}(z)\frac{1}{|\tilde{x}|}\,\delta(|\tilde{x}|-\tau)\Big|_{\tilde{x}=x-y+q_\alpha(t)-q_\beta(t-\tau)-z}\Big|,\end{aligned} \tag{7.28}$$

where we have set $\eta_{\alpha\beta}(\tau) = v_\alpha(t-\tau)-v_\beta(t-\tau)$ for brevity; recall that $\mathcal{F}(\frac{\sin|k|\tau}{|k|}) = \frac{1}{4\pi|x|}\delta(|x|-\tau)$, and note also $k\wedge(\dot{v}\wedge k) = k^2\dot{v} - (\dot{v}\cdot k)k$. Using Lemma 3.2, Lemma 7.2 with $j=4$, and Lemma 7.3(a), we deduce that for $t\in[0,T\varepsilon^{-3/2}]$

$$\begin{aligned}\Big|A^{(1)}_{[0,\tau_{**}]}(t)\Big| &\le C\varepsilon^3\int_0^{\tau_{**}} d\tau\int\!\!\int d^3x\,d^3y\,\varphi(x)\varphi(y)\int d^3z\,\frac{1}{|z|^5}\,\frac{1}{|\tilde{x}|}\,\delta(|\tilde{x}|-\tau)\\ &\le C\varepsilon^8\int_0^{\tau_{**}}\frac{d\tau}{\tau}\int\!\!\int d^3x\,d^3y\,\varphi(x)\varphi(y)\int_{|z-[x-y+q_\alpha(t)-q_\beta(t-\tau)]|=\tau} d^2z\\ &\le C\varepsilon^8\tau_{**}^2 \le C\varepsilon^6.\end{aligned} \tag{7.29}$$

Next we turn to bound $A^{(1)}_{[\tau_{**},t]}(t,t_1)$, corresponding to $\int_{\tau_{**}}^{t-t_1} d\tau(\ldots)$ in (7.26). Again we write this term as a double convolution in space variables, but this time as

$$\begin{aligned}\Big|A^{(1)}_{[\tau_{**},t]}(t,t_1)\Big| &= C\Big|\int_{\tau_{**}}^{t-t_1} d\tau\int\!\!\int d^3x\,d^3y\,\varphi(x)\varphi(y)\int d^3z\,\zeta_{v_\beta(t-\tau)}(z)\\ &\times(\eta_{\alpha\beta}(\tau)\cdot\nabla)^2\big(\nabla\wedge(\dot{v}_\beta(t-\tau)\wedge\nabla)\big)\frac{1}{|\tilde{x}|}\,\delta(|\tilde{x}|-\tau)\Big|_{\tilde{x}=x-y+q_\alpha(t)-q_\beta(t-\tau)-z}\Big|,\end{aligned}$$

where $\nabla = \nabla_{\tilde{x}}$. Hence it follows from Lemma 3.2, Lemma 7.2 with $j = 0$, and Lemma 7.3(b) that for $t \in [0, T\varepsilon^{-3/2}]$

$$\left|A^{(1)}_{[\tau_{**},t]}(t,t_1)\right| \leq C\varepsilon^3 \int\int d^3x\, d^3y\, \varphi(x)\varphi(y) \int_{\tau_{**}}^{t-t_1} d\tau \int d^3z\, \frac{1}{|z|}\, \frac{1}{|\tilde{x}|^5}\, \delta(|\tilde{x}| - \tau) \tag{7.30}$$

$$\leq C\varepsilon^8 \int\int d^3x\, d^3y\, \varphi(x)\varphi(y) \int_{\tau_{**}}^{t-t_1} d\tau \int_{|z-[x-y+q_\alpha(t)-q_\beta(t-\tau)]|=\tau} d^2z\, \frac{1}{|z|}. \tag{7.31}$$

Now observe that for $a \in \mathbb{R}^3$ and $\tau > 0$

$$\int_{|z-a|=\tau} d^2z\, \frac{1}{|z|} = \frac{2\pi\tau}{|a|}\Big(|a| + \tau - \big||a| - \tau\big|\Big) \leq 4\pi\tau, \tag{7.32}$$

in both cases $|a| \geq \tau$ and $|a| \leq \tau$. We thus deduce from (7.31) that

$$\left|A^{(1)}_{[\tau_{**},t]}(t,t_1)\right| \leq C\varepsilon^8 \int_{\tau_{**}}^{t-t_1} d\tau\tau \leq C\varepsilon^8 t^2 \leq C\varepsilon^5 \tag{7.33}$$

for $t \in [0, T\varepsilon^{-3/2}]$. Summarizing (7.29) and (7.33), we have proved that $|A^{(1)}(t,t_1)| \leq C\varepsilon^5$ for $t \in [t_1, T\varepsilon^{-3/2}]$, and bounding the remaining terms similarly, we conclude that (7.23) is satisfied.

Hence we can return from (7.22) to (7.20) and carry through this argument just elaborated for each of the terms besides the one containing $\ddot{v}_\beta$. In this way we arrive at

$$|T_1(t,t_1)| \leq \left|\sum_{\beta=1}^N \int d^3x\, \rho_\alpha(x) \int_{t_1}^t ds\, \Big[U(t-s)\Big((\ddot{v}_\beta\,(s)\cdot\nabla_v)\Phi_{v_\beta(s)}(\cdot - q_\beta(s))\Big)\Big](x+q_\alpha(t))\right| + C\varepsilon^5 \tag{7.34}$$

being satisfied for $t_1 \geq \tau_{**}$ and $t \in [t_1, T\varepsilon^{-3/2}]$. In order to deal with the $\ddot{v}_\beta$–term in (7.34), we will consider this expression under different circumstances, elaborated in the next three sections.

7.1.1 A standard estimate

We are going to show first that

$$\begin{aligned} &\left|\int d^3x\, \rho_\alpha(x) \int_{t_2}^t ds\, \Big[U(t-s)\Big((\ddot{v}_\beta\,(s)\cdot\nabla_v)\,\Phi_{v_\beta(s)}(\cdot - q_\beta(s))\Big)\Big](x+q_\alpha(t))\right| \\ &\leq C\Big(\max_{1\leq\kappa\leq N} |e_\kappa|^2\Big)\Big(\sup_{s\in[t_2,T\varepsilon^{-3/2}]}\ \max_{1\leq\kappa\leq N} |\,\ddot{v}_\kappa\,(s)|\Big) \end{aligned} \tag{7.35}$$

for $1 \leq \alpha, \beta \leq N$ and $t \in [t_2, T\varepsilon^{-3/2}]$, with $t_2 \in [0,t]$ remaining to be specified only later. The argument for $\alpha \neq \beta$ is similar to the one leading to the previous estimates. Once more we can employ the solution formulas from Lemma 7.1 and rewrite the term in question in Fourier transformed form. Dropping again the term with $\cos|k|(t-s)$, we may proceed as before and calculate $(\ddot{v}_\beta\,(s)\cdot\nabla_v)\mathcal{F}(\nabla\wedge B_{v_\beta(s)})(k)$ explicitly. This results in

$$\begin{aligned} &\Bigg|e_\alpha e_\beta \int d^3k\, |\hat{\varphi}(k)|^2 \int_{t_2}^t ds\, e^{ik\cdot[q_\beta(s)-q_\alpha(t)]}\, \frac{\sin|k|(t-s)}{|k|}\, \hat{\zeta}_{v_\beta(s)}(k)\, \Big[k^2\ \ddot{v}_\beta\,(s) - (\ddot{v}_\beta\,(s)\cdot k)k \\ &\qquad + 2k^2\frac{(v_\beta(s)\cdot k)(\ddot{v}_\beta\,(s)\cdot k)}{k^2 - (v_\beta(s)\cdot k)^2}\, v_\beta(s) - 2\,\frac{(v_\beta(s)\cdot k)^2(\ddot{v}_\beta\,(s)\cdot k)}{k^2 - (v_\beta(s)\cdot k)^2}\, k\Big]\Bigg| \\ &\leq C\Big(\max_{1\leq\kappa\leq N} |e_\kappa|^2\Big)\Big(\sup_{s\in[t_2,T\varepsilon^{-3/2}]}\ \max_{1\leq\kappa\leq N} |\,\ddot{v}_\kappa\,(s)|\Big) \end{aligned}$$

to be verified for $t \in [t_2, T\varepsilon^{-3/2}]$. Note that here we kept track of e_α and e_β, since they being small will be important at a later point. Again the last two terms on the left-hand side have an additional ε, and hence are omitted. Thus we need to show $|A^{(2)}(t,t_2)| \le C(\sup_{s\in[t_2,T\varepsilon^{-3/2}]}\max_{1\le\kappa\le N}|\ddot{v}_\kappa(s)|)$ for $t \in [t_2, T\varepsilon^{-3/2}]$, with

$$A^{(2)}(t,t_2) = \int d^3k\,|\hat{\varphi}(k)|^2 \int_0^{t-t_2} d\tau\, e^{ik\cdot[q_\beta(t-\tau)-q_\alpha(t)]}\,\frac{\sin|k|\tau}{|k|}\,\hat{\zeta}_{v_\beta(t-\tau)}(k) \times \Big[k^2\,\ddot{v}_\beta(t-\tau) - (\ddot{v}_\beta(t-\tau)\cdot k)k\Big]. \tag{7.36}$$

First we consider the case $\alpha \neq \beta$, and for this purpose we recall the definition of $\tau_{**} = (C_*/8)\varepsilon^{-1}$ from Lemma 7.3. Analogously to (7.27) we write $\int_0^{t-t_2} d\tau = \int_0^{\tau_{**}} d\tau + \int_{\tau_{**}}^{t-t_2} d\tau$, and correspondingly split $A^{(2)}(t,t_2) = A^{(2)}_{[0,\tau_{**}]}(t) + A^{(2)}_{[\tau_{**},t-t_2]}(t,t_2)$; again the case $t - t_2 \le \tau_{**}$ is simpler so that we are going to assume $t - t_2 \ge \tau_{**}$. To start with, from Lemma 7.2 and Lemma 7.3(a) we find

$$\begin{aligned}
\Big|A^{(2)}_{[0,\tau_{**}]}(t)\Big| &= C\Big|\int_0^{\tau_{**}} d\tau \int\!\!\int d^3x\,d^3y\,\varphi(x)\varphi(y)\int d^3z\,\Big(\nabla\wedge(\ddot{v}_\beta(t-\tau)\wedge\nabla)\Big)\zeta_{v_\beta(t-\tau)}(z) \\
&\qquad\times\frac{1}{|\tilde{x}|}\,\delta(|\tilde{x}|-\tau)\Big|_{\tilde{x}=x-y+q_\alpha(t)-q_\beta(t-\tau)-z}\Big| \\
&\le C\Big(\sup_{s\in[t-\tau_{**},t]}\max_{1\le\kappa\le N}|\ddot{v}_\kappa(s)|\Big)\int\!\!\int d^3x\,d^3y\,\varphi(x)\varphi(y)\int_0^{\tau_{**}} d\tau\int d^3z\,\frac{1}{|z|^3}\,\frac{1}{|\tilde{x}|}\,\delta(|\tilde{x}|-\tau) \\
&\le C\varepsilon^3\Big(\sup_{s\in[t-\tau_{**},t]}\max_{1\le\kappa\le N}|\ddot{v}_\kappa(s)|\Big)\int\!\!\int d^3x\,d^3y\,\varphi(x)\varphi(y)\int_0^{\tau_{**}}\frac{d\tau}{\tau}\int_{|z-[x-y+q_\alpha(t)-q_\beta(t-\tau)]|=\tau} d^2z \\
&\le C\varepsilon^3\tau_{**}^2\Big(\sup_{s\in[t-\tau_{**},t]}\max_{1\le\kappa\le N}|\ddot{v}_\kappa(s)|\Big) \le C\varepsilon\Big(\sup_{s\in[t_2,T\varepsilon^{-3/2}]}\max_{1\le\kappa\le N}|\ddot{v}_\kappa(s)|\Big).
\end{aligned}\tag{7.37}$$

Concerning $A^{(2)}_{[\tau_{**},t-t_2]}(t,t_2)$, in view of Lemma 7.2, Lemma 7.3(b), and (7.32) we can estimate

$$\begin{aligned}
\Big|A^{(2)}_{[\tau_{**},t-t_2]}(t,t_2)\Big| &= C\Big|\int_{\tau_{**}}^{t-t_2} d\tau\int\!\!\int d^3x\,d^3y\,\varphi(x)\varphi(y)\int d^3z\,\zeta_{v_\beta(t-\tau)}(z) \\
&\qquad\times\Big(\nabla\wedge(\ddot{v}_\beta(t-\tau)\wedge\nabla)\Big)\frac{1}{|\tilde{x}|}\,\delta(|\tilde{x}|-\tau)\Big|_{\tilde{x}=x-y+q_\alpha(t)-q_\beta(t-\tau)-z}\Big| \\
&\le C\Big(\sup_{s\in[t_2,t-\tau_{**}]}\max_{1\le\kappa\le N}|\ddot{v}_\kappa(s)|\Big)\int\!\!\int d^3x\,d^3y\,\varphi(x)\varphi(y) \\
&\qquad\times\int_{\tau_{**}}^{t-t_2} d\tau\int d^3z\,\frac{1}{|z|}\,\frac{1}{|\tilde{x}|^3}\,\delta(|\tilde{x}|-\tau) \\
&\le C\varepsilon^3\Big(\sup_{s\in[t_2,t-\tau_{**}]}\max_{1\le\kappa\le N}|\ddot{v}_\kappa(s)|\Big)\int\!\!\int d^3x\,d^3y\,\varphi(x)\varphi(y) \\
&\qquad\times\int_{\tau_{**}}^{t-t_2} d\tau\int_{|z-[x-y+q_\alpha(t)-q_\beta(t-\tau)]|=\tau}\frac{d^2z}{|z|} \\
&\le C\varepsilon^3t^2\Big(\sup_{s\in[t_2,t-\tau_{**}]}\max_{1\le\kappa\le N}|\ddot{v}_\kappa(s)|\Big) \le C\Big(\sup_{s\in[t_2,T\varepsilon^{-3/2}]}\max_{1\le\kappa\le N}|\ddot{v}_\kappa(s)|\Big)
\end{aligned}\tag{7.38}$$

for $t \le T\varepsilon^{-3/2}$. Summarizing (7.37) and (7.38), we have seen that

$$|A^{(2)}(t,t_2)| \le C\Big(\sup_{s\in[t_2,T\varepsilon^{-3/2}]}\max_{1\le\kappa\le N}|\ddot{v}_\kappa(s)|\Big),\quad t\in[t_2,T\varepsilon^{-3/2}],\quad \alpha\neq\beta. \tag{7.39}$$

To handle the case $\alpha = \beta$, we split $\int_0^t d\tau = \int_0^{8R_\varphi} d\tau + \int_{8R_\varphi}^{t-t_2} d\tau$ in (7.36), and accordingly $A^{(2)}(t,t_2) = A^{(2)}_{[0,8R_\varphi]}(t) + A^{(2)}_{[8R_\varphi,t-t_2]}(t,t_2)$; again w.l.o.g. we may assume that $t - t_2 \geq 8R_\varphi$. Then Lemma 7.2 and Lemma 7.3(c) imply

$$\begin{aligned}
\Big|A^{(2)}_{[8R_\varphi,t-t_2]}(t,t_2)\Big| &= C\Big|\int_{8R_\varphi}^{t-t_2} d\tau \int\int d^3x\, d^3y\, \varphi(x)\varphi(y) \int d^3z \Big(\nabla \wedge (\ddot{v}_\alpha\,(t-\tau) \wedge \nabla)\Big)\zeta_{v_\alpha(t-\tau)}(z) \\
&\qquad\qquad \times \frac{1}{|\tilde{x}|}\,\delta(|\tilde{x}|-\tau)\Big|_{\tilde{x}=x-y+q_\alpha(t)-q_\alpha(t-\tau)-z}\Big| \\
&\leq C\Big(\sup_{s\in[t_2,t-8R_\varphi]}\max_{1\leq\kappa\leq N}|\ddot{v}_\kappa\,(s)|\Big)\int\int d^3x\, d^3y\, \varphi(x)\varphi(y) \\
&\qquad\qquad \times \int_{8R_\varphi}^{t-t_2} d\tau \int d^3z\, \frac{1}{|z|^3}\,\frac{1}{|\tilde{x}|}\,\delta(|\tilde{x}|-\tau) \\
&\leq C\Big(\sup_{s\in[t_2,t-8R_\varphi]}\max_{1\leq\kappa\leq N}|\ddot{v}_\kappa\,(s)|\Big)\int\int d^3x\, d^3y\, \varphi(x)\varphi(y) \\
&\qquad\qquad \times \int_{8R_\varphi}^{t-t_2} \frac{d\tau}{\tau^4}\int_{|z-[x-y+q_\alpha(t)-q_\alpha(t-\tau)]|=\tau} d^2z \\
&\leq C\Big(\sup_{s\in[t_2,t-8R_\varphi]}\max_{1\leq\kappa\leq N}|\ddot{v}_\kappa\,(s)|\Big)\int\int d^3x\, d^3y\, \varphi(x)\varphi(y)\int_{8R_\varphi}^{t-t_2}\frac{d\tau}{\tau^2} \\
&\leq C\Big(\sup_{s\in[t_2,T\varepsilon^{-3/2}]}\max_{1\leq\kappa\leq N}|\ddot{v}_\kappa\,(s)|\Big).
\end{aligned} \tag{7.40}$$

On the other hand, we have the simple estimate

$$\begin{aligned}
\Big|A^{(2)}_{[0,8R_\varphi]}(t)\Big| &= C\Big|\int_0^{8R_\varphi} d\tau \int\int d^3x\, d^3y \Big(\nabla \wedge (\ddot{v}_\alpha\,(t-\tau)\wedge\nabla)\Big)\varphi(x)\varphi(y) \\
&\qquad \times \int d^3z\, \zeta_{v_\alpha(t-\tau)}(z)\frac{1}{|\tilde{x}|}\,\delta(|\tilde{x}|-\tau)\Big|_{\tilde{x}=x-y+q_\alpha(t)-q_\alpha(t-\tau)-z}\Big| \\
&\leq C\Big(\sup_{s\in[t-8R_\varphi,t]}\max_{1\leq\kappa\leq N}|\ddot{v}_\kappa\,(s)|\Big)\int\int d^3x\, d^3y\, |\nabla^2\varphi(x)|\varphi(y) \\
&\qquad\qquad \times \int_0^{8R_\varphi} d\tau \int d^3z\, \frac{1}{|z|}\,\frac{1}{|\tilde{x}|}\,\delta(|\tilde{x}|-\tau) \\
&\leq C\Big(\sup_{s\in[t-8R_\varphi,t]}\max_{1\leq\kappa\leq N}|\ddot{v}_\kappa\,(s)|\Big)\int\int d^3x\, d^3y\, |\nabla^2\varphi(x)|\varphi(y) \\
&\qquad\qquad \times \int_0^{8R_\varphi}\frac{d\tau}{\tau}\int_{|z-[x-y+q_\alpha(t)-q_\alpha(t-\tau)]|=\tau}\frac{d^2z}{|z|} \\
&\leq C\Big(\sup_{s\in[t_2,T\varepsilon^{-3/2}]}\max_{1\leq\kappa\leq N}|\ddot{v}_\kappa\,(s)|\Big),
\end{aligned} \tag{7.41}$$

cf. (7.32). In view of (7.39), (7.40), and (7.41), and bounding the remaining terms in the same manner, it is verified that (7.35) holds. We will further use this in the form

$$\begin{aligned}
&\Big|\sum_{\beta=1}^N \int d^3x\, \rho_\alpha(x)\int_{t_2}^t ds\,\Big[U(t-s)\Big((\ddot{v}_\beta\,(s)\cdot\nabla_v)\,\Phi_{v_\beta(s)}(\cdot - q_\beta(s))\Big)\Big](x+q_\alpha(t))\Big| \\
&\leq C\Big(\max_{1\leq\kappa\leq N}|e_\kappa|^2\Big)\Big(\sup_{s\in[t_2,T\varepsilon^{-3/2}]}\max_{1\leq\kappa\leq N}|\ddot{v}_\kappa\,(s)|\Big)
\end{aligned} \tag{7.42}$$

for $1 \le \alpha \le N$ and $t \in [t_2, T\varepsilon^{-3/2}]$, with $t_2 \in [0,t]$ still being free to be chosen; the constant $C > 0$ is independent of t_2.

7.1.2 An a priori estimate to bound $|\dddot{v}_\alpha(t)|$ by ε^4

The purpose of this section is to prove that

$$
\begin{aligned}
|T_5(t,t_2)| &:= \Big| \sum_{\beta=1}^{N} \int d^3x\, \rho_\alpha(x) \int_{t_2}^{t_2+\tau_{**}} ds \Big[U(t-s) \Big((\dddot{v}_\beta\ (s) \cdot \nabla_v)\, \Phi_{v_\beta(s)}(\cdot - q_\beta(s)) \Big) \Big](x + q_\alpha(t)) \Big| \\
&\le C\varepsilon^4 \qquad (7.43)
\end{aligned}
$$

for $1 \le \alpha \le N$ and

$$
t \in [t_2 + \tau_{**}, T\varepsilon^{-3/2}], \qquad (7.44)
$$

with τ_{**} from (3.16). This estimate will later play the key role in showing that at least $|\dddot{v}| \cong \varepsilon^4$.

Since $U(t)$ is the group with generator $\mathcal{A}$, we calculate

$$
\begin{aligned}
&\frac{d}{ds} \Big[U(t-s) \Big((\ddot{v}_\beta(s) \cdot \nabla_v)\, \Phi_{v_\beta(s)}(\cdot - q_\beta(s)) \Big) \Big] \\
&= -U(t-s) \Big((\ddot{v}_\beta(s) \cdot \nabla_v)\, \mathcal{A}\Phi_{v_\beta(s)}(\cdot - q_\beta(s)) \Big) + U(t-s) \Big((\dddot{v}_\beta\ (s) \cdot \nabla_v)\, \Phi_{v_\beta(s)}(\cdot - q_\beta(s)) \Big) \\
&\quad + U(t-s) \Big((\ddot{v}_\beta(s) \cdot \nabla_v)(\dot{v}_\beta(s) \cdot \nabla_v)\, \Phi_{v_\beta(s)}(\cdot - q_\beta(s)) \Big) \\
&\quad - U(t-s) \Big((\ddot{v}_\beta(s) \cdot \nabla_v)(v_\beta(s) \cdot \nabla)\, \Phi_{v_\beta(s)}(\cdot - q_\beta(s)) \Big). \qquad (7.45)
\end{aligned}
$$

Next, calculating

$$
\mathcal{A}\Phi_v = (\nabla \wedge B_v, -\nabla \wedge E_v) = \Big((v \cdot \nabla)\nabla\phi_v - v\Delta\phi_v, (v \cdot \nabla)(v \wedge \nabla\phi_v) \Big) \qquad (7.46)
$$

explicitly from (2.3), we see that taking $\mathcal{A}\Phi_v$ has a similar effect as taking $(v \cdot \nabla)\Phi_v$, since both operations result in an additional v and an additional ∇-derivative. For simplicity we hence drop the term with $\mathcal{A}\Phi_v$ in (7.45). Substituting the remainder of (7.45) back in the definition of $T_5(t,t_2)$ then shows that

$$
\begin{aligned}
&|T_5(t,t_2)| \\
&\le \sum_{\beta=1}^{N} \Big| \int d^3x\, \rho_\alpha(x) \Big\{ U(t - [t_2 + \tau_{**}]) \Big((\ddot{v}_\beta(t_2 + \tau_{**}) \cdot \nabla_v)\, \Phi_{v_\beta(t_2+\tau_{**})}(\cdot - q_\beta(t_2 + \tau_{**})) \Big) \Big\} \\
&\qquad\qquad (x + q_\alpha(t)) \Big| \\
&\quad + \sum_{\beta=1}^{N} \Big| \int d^3x\, \rho_\alpha(x) \Big\{ U(t - t_2) \Big((\ddot{v}_\beta(t_2) \cdot \nabla_v)\, \Phi_{v_\beta(t_2)}(\cdot - q_\beta(t_2)) \Big) \Big\} (x + q_\alpha(t)) \Big| \\
&\quad + \sum_{\beta=1}^{N} \Big| \int d^3x\, \rho_\alpha(x) \int_{t_2}^{t_2+\tau_{**}} ds \Big\{ U(t-s) \Big((\ddot{v}_\beta(s) \cdot \nabla_v)(\dot{v}_\beta(s) \cdot \nabla_v)\, \Phi_{v_\beta(s)}(\cdot - q_\beta(s)) \Big) \Big\} \\
&\qquad\qquad (x + q_\alpha(t)) \Big| \\
&\quad + \sum_{\beta=1}^{N} \Big| \int d^3x\, \rho_\alpha(x) \int_{t_2}^{t_2+\tau_{**}} ds \Big\{ U(t-s) \Big((\ddot{v}_\beta(s) \cdot \nabla_v)(v_\beta(s) \cdot \nabla)\, \Phi_{v_\beta(s)}(\cdot - q_\beta(s)) \Big) \Big\}
\end{aligned}
$$

$$(x+q_\alpha(t))\Big|$$

$$=: \sum_{\beta=1}^{N} \Big(|T^\beta_{5,1}(t,t_2)| + |T^\beta_{5,2}(t,t_2)| + |T^\beta_{5,3}(t,t_2)| + |T^\beta_{5,4}(t,t_2)| \Big). \tag{7.47}$$

To bound these terms individually, we will make frequent use of the method developed before in Sections 7.1 and 7.1.1, without expanding all the details again. The recipe is always the same: (1.) Substitute the solution formulas from Lemma 7.1 and pass to Fourier transformed form in the $\int d^3x\, \rho_\alpha(x)(\ldots)$; (2.) Drop the term with factor $\cos|k|(t-s)$, since it can be handled the same way; (3.) Evaluate explicitly the ∇_v-derivatives and drop the easier terms which thereby have gained additional v's; (4.) If there is an $\int ds(\ldots)$, then change this through $t-s=\tau$ to an $\int d\tau(\ldots)$. Afterwards split the latter into two parts, by inserting τ_{**} from Lemma 7.3 if $\alpha\neq\beta$, and by inserting $8R_\varphi$ for $\alpha=\beta$; (5.) Finally rewrite the whole expression from Fourier transformed form to double convolution form $\int\int d^3x\, d^3y\, \varphi(x)\varphi(y)(\ldots)$. If $\alpha\neq\beta$ and $\tau\le\tau_{**}$, then place the k's as derivatives on the $\zeta_{v_\beta(t-\tau)}(z)$ and use Lemma 7.3(a), but for $\tau\ge\tau_{**}$ place the k's as derivatives on the $\frac{1}{|\tilde{x}|}$ and use Lemma 7.3(b). In case that $\alpha=\beta$, some extra care has to be taken, as will be explained case by case later, nevertheless it should be kept in mind that then Lemma 7.3(c) applies for $\tau\ge 8R_\varphi$.

Part A: Bounding $|T^\beta_{5,3}(t,t_2)| + |T^\beta_{5,4}(t,t_2)|$. Let us start with the case $\alpha\neq\beta$. Noting that here (and everywhere else) we have an additional ∇-derivative resulting from the $(\nabla\wedge B_v)$-term of the solution formula from Lemma 7.1, cf. the beginning of Section 7.1, and recalling that $\Phi_v\cong\nabla\phi_v$, we find that here

$$\begin{aligned}
&|T^\beta_{5,3}(t,t_2)| + |T^\beta_{5,4}(t,t_2)| \\
&\le C\int_{t_2}^{t_2+\tau_{**}} ds \,\Big|\int\int d^3x\, d^3y\, \varphi(x)\varphi(y) \\
&\qquad\qquad \times [\varepsilon^{11/2}\,\nabla^2 + \varepsilon^4\,\nabla^3] \int d^3z\, \zeta_{v_\beta(s)}(z)\frac{1}{|\tilde{x}|}\,\delta(|\tilde{x}|-(t-s))\Big|_{\tilde{x}=x-y+q_\alpha(t)-q_\beta(s)-z}\Big|,
\end{aligned}$$

where we have also used the bounds from Lemma 3.2 and Lemma 3.3, which is possible due to $s\ge t_2\ge\tau_{**}$. Transforming $t-s=\tau$, the integration interval becomes $\tau\in[t-(t_2+\tau_{**}), t-t_2]$. By hypothesis, $\tau_{**}\le t-t_2$. We may as well assume that $t-(t_2+\tau_{**})\le\tau_{**}$, since otherwise the first of the integrals below simply drops out. Following the above recipe we hence deduce from Lemma 7.2, Lemma 7.3(a), (b), and (7.32) that

$$\begin{aligned}
&|T^\beta_{5,3}(t,t_2)| + |T^\beta_{5,4}(t,t_2)| \\
&\le C\int\int d^3x\, d^3y\, \varphi(x)\varphi(y)\int_{t-(t_2+\tau_{**})}^{\tau_{**}} d\tau\,\Big(\varepsilon^{11/2}\,\varepsilon^3\,\frac{1}{\tau}\int_{|z-[x-y+q_\alpha(t)-q_\beta(t-\tau)]|=\tau} d^2z \\
&\qquad\qquad + \varepsilon^4\,\varepsilon^4\,\frac{1}{\tau}\int_{|z-[x-y+q_\alpha(t)-q_\beta(t-\tau)]|=\tau} d^2z\Big) \\
&\quad + C\int\int d^3x\, d^3y\, \varphi(x)\varphi(y)\int_{\tau_{**}}^{t-t_2} d\tau\,\Big(\varepsilon^{11/2}\,\varepsilon^3\int_{|z-[x-y+q_\alpha(t)-q_\beta(t-\tau)]|=\tau} d^2z\,\frac{1}{|z|} \\
&\qquad\qquad + \varepsilon^4\,\varepsilon^4\int_{|z-[x-y+q_\alpha(t)-q_\beta(t-\tau)]|=\tau} d^2z\,\frac{1}{|z|}\Big) \\
&\le C\Big(\varepsilon^{11/2}\,\varepsilon^3+\varepsilon^4\,\varepsilon^4\Big)\tau_{**}^2 + C\Big(\varepsilon^{11/2}\,\varepsilon^3+\varepsilon^4\,\varepsilon^4\Big)t^2 \le C\varepsilon^5,
\end{aligned} \tag{7.48}$$

according to $\tau_{**}=\mathcal{O}(\varepsilon^{-1})$ and $t\le T\varepsilon^{-3/2}$.

Next we consider the case $\alpha = \beta$. Here we have $t - t_2 \geq \tau_{**} = \mathcal{O}(\varepsilon^{-1}) > 8R_\varphi$, and again we can suppose w.l.o.g. that $t - (t_2 + \tau_{**}) \leq 8R_\varphi$. Partitioning the $d\tau$-integral this way, it follows from Lemma 7.2, Lemma 7.3(c), and (7.32) that

$$
\begin{aligned}
&|T^\alpha_{5,3}(t,t_2)| + |T^\alpha_{5,4}(t,t_2)| \\
&\leq C \int_{t-(t_2+\tau_{**})}^{8R_\varphi} d\tau \left| \int\int d^3x\, d^3y \left([\varepsilon^{11/2}\, \nabla^2 + \varepsilon^4\, \nabla^3] \varphi(x) \right) \varphi(y) \right. \\
&\qquad\qquad \left. \times \int d^3z\, \zeta_{v_\alpha(t-\tau)}(z) \frac{1}{|\tilde{x}|}\, \delta(|\tilde{x}| - \tau) \Big|_{\tilde{x} = x - y + q_\alpha(t) - q_\alpha(t-\tau) - z} \right| \\
&\quad + C \int_{8R_\varphi}^{t-t_2} d\tau \left| \int\int d^3x\, d^3y\, \varphi(x)\varphi(y) \right. \\
&\qquad\qquad \left. \times \int d^3z \left([\varepsilon^{11/2}\, \nabla^2 + \varepsilon^4\, \nabla^3]\, \zeta_{v_\alpha(t-\tau)}(z) \right) \frac{1}{|\tilde{x}|}\, \delta(|\tilde{x}| - \tau) \Big|_{\tilde{x} = x - y + q_\alpha(t) - q_\alpha(t-\tau) - z} \right| \\
&\leq C \int\int d^3x\, d^3y \left(\varepsilon^{11/2}\, |\nabla^2 \varphi(x)| + \varepsilon^4\, |\nabla^3 \varphi(x)| \right) \varphi(y) \int_{t-(t_2+\tau_{**})}^{8R_\varphi} d\tau\, \frac{1}{\tau} \\
&\qquad\qquad\qquad \times \int_{|z - [x - y + q_\alpha(t) - q_\alpha(t-\tau)]| = \tau} d^2z\, \frac{1}{|z|} \\
&\quad + C \int\int d^3x\, d^3y\, \varphi(x)\varphi(y) \int_{8R_\varphi}^{t-t_2} d\tau \left(\varepsilon^{11/2} \frac{1}{\tau^3} + \varepsilon^4 \frac{1}{\tau^4} \right) \frac{1}{\tau} \int_{|z - [x - y + q_\alpha(t) - q_\alpha(t-\tau)]| = \tau} d^2z \\
&\leq C\varepsilon^4 \int\int d^3x\, d^3y \left(|\nabla^2 \varphi(x)| + |\nabla^3 \varphi(x)| \right) \varphi(y) \int_0^{8R_\varphi} d\tau \\
&\quad + C \int\int d^3x\, d^3y\, \varphi(x)\varphi(y) \int_{8R_\varphi}^{t-t_2} d\tau \left(\varepsilon^{11/2} \frac{1}{\tau^2} + \varepsilon^4 \frac{1}{\tau^3} \right) \leq C\varepsilon^4.
\end{aligned} \tag{7.49}
$$

Summarizing (7.48) and (7.49), we have verified

$$
|T^\beta_{5,3}(t,t_2)| + |T^\beta_{5,4}(t,t_2)| \leq C\varepsilon^4 \tag{7.50}
$$

for all $1 \leq \beta \leq N$ and all t, t_2 satisfying the assumptions (7.44) of this section.

Part B: Bounding $|T^\beta_{5,1}(t,t_2)|$. We go back to (7.47) and deal with the contribution of the first term on the right-hand side. Again we begin with the case $\alpha \neq \beta$. Following the method used so far, we obtain from Lemma 3.3 that

$$
\begin{aligned}
|T^\beta_{5,1}(t,t_2)| \;\leq\; & C\varepsilon^{7/2} \left| \int\int d^3x\, d^3y\, \varphi(x)\varphi(y) \right. \\
& \left. \times \nabla^2 \int d^3z\, \zeta_{v_\beta(t_2+\tau_{**})}(z) \frac{1}{|\tilde{x}|}\, \delta\Big(|\tilde{x}| - (t - [t_2 + \tau_{**}])\Big) \Big|_{\tilde{x} = x - y + q_\alpha(t) - q_\beta(t_2+\tau_{**}) - z} \right|.
\end{aligned} \tag{7.51}
$$

First we assume that $\tau := t - [t_2 + \tau_{**}] \leq \tau_{**}$. Then we pass the derivatives to $\zeta_{v_\beta(t_2+\tau_{**})}(z)$ and invoke Lemma 7.3(a) and (7.32) to deduce

$$
\begin{aligned}
|T^\beta_{5,1}(t,t_2)| \;\leq\; & C\varepsilon^{7/2}\varepsilon^3 \int\int d^3x\, d^3y\, \varphi(x)\varphi(y) \\
& \times \int_{|z - [x - y + q_\alpha(t) - q_\beta(t_2+\tau_{**})]| = \tau} d^2z\, \frac{1}{|x - y + q_\alpha(t) - q_\beta(t_2+\tau_{**}) - z|} \\
\leq\; & C\varepsilon^{7/2}\varepsilon^3 \tau \leq C\varepsilon^{7/2}\varepsilon^3 \tau_{**} \leq C\varepsilon^{11/2}.
\end{aligned}
$$

On the other hand, if $\tau \geq \tau_{**}$ holds, then $\frac{1}{|\tilde{x}|}$ in (7.51) gets the ∇^2, and Lemma 7.3(b) together with (7.32) implies

$$\begin{aligned} |T^{\beta}_{5,1}(t,t_2)| &\leq C\varepsilon^{7/2}\varepsilon^3 \int\int d^3x\, d^3y\, \varphi(x)\varphi(y) \int_{|z-[x-y+q_\alpha(t)-q_\beta(t_2+\tau_{**})]|=\tau} d^2z\, \frac{1}{|z|} \\ &\leq C\varepsilon^{7/2}\varepsilon^3\tau \leq C\varepsilon^{7/2}\varepsilon^3 t \leq C\varepsilon^5 \end{aligned}$$

for $t \leq T\varepsilon^{-3/2}$. Thus we have found

$$|T^{\beta}_{5,1}(t,t_2)| \leq C\varepsilon^5, \quad \alpha \neq \beta. \tag{7.52}$$

The most difficult term to estimate in this section is

$$\begin{aligned} &T^{\alpha}_{5,1}(t,t_2) \\ &\quad = \int d^3x\, \rho_\alpha(x)\Big\{U(t-[t_2+\tau_{**}])\Big((\ddot{v}_\alpha(t_2+\tau_{**})\cdot\nabla_v)\,\Phi_{v_\alpha(t_2+\tau_{**})}(\cdot - q_\alpha(t_2+\tau_{**}))\Big)\Big\}(x+q_\alpha(t)), \end{aligned} \tag{7.53}$$

since for $\alpha = \beta$ there is no way to gain ε's through the particles being far apart, and as here $t = t_2 + \tau_{**}$ is possible, thus preventing us from using the decay induced by the Maxwell group $U(t)$ close to $t = t_2 + \tau_{**}$. What saves the argument is the observation that the integrand vanishes at $t = t_2 + \tau_{**}$, as is most easily seen by changing to Fourier transformed form. Indeed, e.g.

$$\int d^3x\, \rho_\alpha(x)\Big\{E_v(\cdot - q_\alpha(t))\Big\}(x+q_\alpha(t)) = \int d^3x\, \rho_\alpha(x) E_v(x) = C\int d^3k\, |\hat{\varphi}(k)|^2\, i[k-(v\cdot k)v] = 0 \tag{7.54}$$

due to the rotational symmetry of φ, recall (C). To exploit this, we calculate

$$\begin{aligned} &\frac{d}{ds}\Big[U(s-[t_2+\tau_{**}])\,\Phi_{v_\alpha(t_2+\tau_{**})}(\cdot + q_\alpha(s) - q_\alpha(t_2+\tau_{**}))\Big] \\ &\quad = U(s-[t_2+\tau_{**}])\,\mathcal{A}\Phi_{v_\alpha(t_2+\tau_{**})}(\cdot + q_\alpha(s) - q_\alpha(t_2+\tau_{**})) \\ &\qquad + U(s-[t_2+\tau_{**}])(v_\alpha(s)\cdot\nabla)\,\Phi_{v_\alpha(t_2+\tau_{**})}(\cdot + q_\alpha(s) - q_\alpha(t_2+\tau_{**})). \end{aligned} \tag{7.55}$$

Again we may rely on (7.46) to see that $\mathcal{A}\Phi_v$ is, from the point of view of estimates, exactly as $(v\cdot\nabla)\Phi_v$, so we will again drop the term containing $\mathcal{A}\Phi_v$. Integrating (7.55) over $s \in [t_2+\tau_{**}, t]$, substituting the remainder back into (7.53), and cancelling the zero term at $t = t_2 + \tau_{**}$, we then obtain

$$\begin{aligned} |T^{\alpha}_{5,1}(t,t_2)| \leq C\Big|\int d^3x\, \rho_\alpha(x) \int_{t_2+\tau_{**}}^{t} ds\, \Big\{U(s-[t_2+\tau_{**}])\Big(\\ (\ddot{v}_\alpha(t_2+\tau_{**})\cdot\nabla_v)(v_\alpha(s)\cdot\nabla)\,\Phi_{v_\alpha(t_2+\tau_{**})}(\cdot - q_\alpha(t_2+\tau_{**}))\Big)\Big\}(x+q_\alpha(s))\Big|. \end{aligned}$$

Thus we have gained one $v_\alpha(s) \cong \sqrt{\varepsilon}$ and one ∇-derivative, at the expense that now $\int_{t_2+\tau_{**}}^{t} ds$ appears. Applying our standard method yields in view of Lemma 3.3 and Lemma 3.2

$$\begin{aligned} |T^{\alpha}_{5,1}(t,t_2)| &\leq C\varepsilon^4\Big|\int_0^{t-[t_2+\tau_{**}]} d\tau \int\int d^3x\, d^3y\, \varphi(x)\varphi(y) \\ &\qquad \times \nabla^3 \int d^3z\, \zeta_{v_\alpha(t_2+\tau_{**})}(z)\frac{1}{|\tilde{x}|}\,\delta(|\tilde{x}|-\tau)\Big|_{\tilde{x}=x-y+q_\alpha(\tau+[t_2+\tau_{**}])-q_\alpha(t_2+\tau_{**})-z}\Big|. \end{aligned}$$

We split the integral at $\tau = 8R_\varphi$. If $t - [t_2 + \tau_{**}] \le 8R_\varphi$, then the second term can be omitted and the estimate is simpler. Passing the derivatives to $\zeta_{v_\alpha(t_2+\tau_{**})}(z)$ in $\int_{8R_\varphi}^{t-[t_2+\tau_{**}]} d\tau(\ldots)$, and then utilizing Lemma 7.3(c) to find $|z| \ge C\tau$ there, it follows that

$$\begin{aligned}
|T^\alpha_{5,1}(t,t_2)| \;\le\;& C\varepsilon^4 \int_0^{8R_\varphi} d\tau \int\int d^3x\, d^3y\, |\nabla^3\varphi(x)|\varphi(y) \\
&\qquad \times \frac{1}{\tau}\int_{|z-[x-y+q_\alpha(\tau+[t_2+\tau_{**}])-q_\alpha(t_2+\tau_{**})]|=\tau} d^2z\, \frac{1}{|z|} \\
&+ C\varepsilon^4 \int_{8R_\varphi}^{t-[t_2+\tau_{**}]} d\tau \int\int d^3x\, d^3y\, \varphi(x)\varphi(y) \\
&\qquad \times \frac{1}{\tau^5}\int_{|z-[x-y+q_\alpha(\tau+[t_2+\tau_{**}])-q_\alpha(t_2+\tau_{**})]|=\tau} d^2z \\
\le\;& C\varepsilon^4.
\end{aligned}$$

Therefore (7.52) allows us to summarize our findings as

$$|T^\beta_{5,1}(t,t_2)| \le C\varepsilon^4 \tag{7.56}$$

for all $1 \le \beta \le N$ and all t, t_2 obeying the assumptions (7.44) of this section.

Part C: Bounding $|T^\beta_{5,2}(t,t_2)|$. Again we go back to (7.47) and investigate the second term on the right-hand side. This expression is easier to handle, due to $t - t_2 \ge \tau_{**} = \mathcal{O}(\varepsilon^{-1})$. Here the standard method yields, no matter whether $\alpha \ne \beta$ or $\alpha = \beta$, that

$$\begin{aligned}
|T^\beta_{5,2}(t,t_2)| \;\le\;& C\varepsilon^{7/2}\Big|\int\int d^3x\, d^3y\, \varphi(x)\varphi(y) \\
&\qquad \times \int d^3z\, \zeta_{v_\beta(t_2)}(z)\Big(\nabla^2\frac{1}{|\tilde{x}|}\Big)\,\delta(|\tilde{x}|-(t-t_2))\Big|_{\tilde{x}=x-y+q_\alpha(t)-q_\beta(t_2)-z}\Big| \\
\le\;& C\varepsilon^{7/2}(t-t_2)^{-3}\int\int d^3x\, d^3y\, \varphi(x)\varphi(y)\int_{|z-[x-y+q_\alpha(t)-q_\beta(t_2)]|=(t-t_2)} d^2z\,\frac{1}{|z|} \\
\le\;& C\varepsilon^{7/2}(t-t_2)^{-2} \le C\varepsilon^{11/2}.
\end{aligned}$$

Together with (7.50) and (7.56) this proves that indeed (7.43) is verified, under the assumptions (7.44).

7.1.3 Improving $\ddot{v}$-estimates by $\varepsilon^{1/4}$

Here we are going to show that

$$\begin{aligned}
|T_6(t,t_3)| \;:=\;& \Big|\sum_{\beta=1}^N \int d^3x\, \rho_\alpha(x)\int_{t_3}^{t_3+\tau_{**}} ds\, \Big[U(t-s)\Big((\ddot{v}_\beta\,(s)\cdot\nabla_v)\,\Phi_{v_\beta(s)}(\cdot - q_\beta(s))\Big)\Big](x+q_\alpha(t))\Big| \\
\le\;& C\Big(\sup_{s\in[t_3,t_3+\tau_{**}]}\max_{1\le\kappa\le N}|\,\ddot{v}_\kappa\,(s)|\Big)\varepsilon^{1/4}
\end{aligned} \tag{7.57}$$

for $1 \le \alpha \le N$ and

$$t \in [t_3+\tau_{**}, T\varepsilon^{-3/2}]. \tag{7.58}$$

To verify (7.57), we define $T_6(t,t_3) = \sum_{\beta=1}^N T_6^\beta(t,t_3)$. In case that $\alpha \neq \beta$, our standard method implies

$$
\begin{aligned}
|T_6^\beta(t,t_3)| \;&\le\; C\Big(\sup_{s\in[t_3,t_3+\tau_{**}]}\max_{1\le\kappa\le N}|\ddot{v}_\kappa(s)|\Big)\,\Big|\int_{t-[t_3+\tau_{**}]}^{t-t_3} d\tau \int\!\!\int d^3x\,d^3y\,\varphi(x)\varphi(y) \\
&\qquad \times \nabla^2 \int d^3z\,\zeta_{v_\beta(t-\tau)}(z)\frac{1}{|\tilde{x}|}\,\delta(|\tilde{x}|-\tau)\Big|_{\tilde{x}=x-y+q_\alpha(t)-q_\beta(t-\tau)-z}\Big|.
\end{aligned}
$$

Again we split the $d\tau$-integral by introducing $\tau_{**}\le t-t_3$, cf. (7.58), from Lemma 7.3. W.l.o.g. we can suppose that $\tau_{**}\ge t-[t_3+\tau_{**}]$, as otherwise simply the first integral below drops out. In this manner we find from Lemma 7.3(a) and (7.32) that

$$
\begin{aligned}
|T_6^\beta(t,t_3)| \;&\le\; C\Big(\sup_{s\in[t_3,t_3+\tau_{**}]}\max_{1\le\kappa\le N}|\ddot{v}_\kappa(s)|\Big)\Big(\Big|\int_{t-[t_3+\tau_{**}]}^{\tau_{**}} d\tau\int\!\!\int d^3x\,d^3y\,\varphi(x)\varphi(y) \\
&\qquad \times \int d^3z\,\big(\nabla^2\zeta_{v_\beta(t-\tau)}(z)\big)\frac{1}{|\tilde{x}|}\,\delta(|\tilde{x}|-\tau)\Big|_{\tilde{x}=x-y+q_\alpha(t)-q_\beta(t-\tau)-z}\Big| \\
&\quad + \int_{\tau_{**}}^{t-t_3} d\tau\int\!\!\int d^3x\,d^3y\,\varphi(x)\varphi(y) \\
&\qquad \times \int d^3z\,\zeta_{v_\beta(t-\tau)}(z)\Big(\nabla^2\frac{1}{|\tilde{x}|}\Big)\,\delta(|\tilde{x}|-\tau)\Big|_{\tilde{x}=x-y+q_\alpha(t)-q_\beta(t-\tau)-z}\Big|\Big) \\
&\le\; C\Big(\sup_{s\in[t_3,t_3+\tau_{**}]}\max_{1\le\kappa\le N}|\ddot{v}_\kappa(s)|\Big)\Big(\int\!\!\int d^3x\,d^3y\,\varphi(x)\varphi(y) \\
&\qquad \times \varepsilon^3\int_{t-[t_3+\tau_{**}]}^{\tau_{**}} d\tau\,\frac{1}{\tau}\int_{|z-[x-y+q_\alpha(t)-q_\beta(t-\tau)]|=\tau} d^2z \\
&\quad + \int\!\!\int d^3x\,d^3y\,\varphi(x)\varphi(y)\int_{\tau_{**}}^{t-t_3} d\tau\,\frac{1}{\tau^3}\int_{|z-[x-y+q_\alpha(t)-q_\beta(t-\tau)]|=\tau} d^2z\,\frac{1}{|z|}\Big) \\
&\le\; C\Big(\sup_{s\in[t_3,t_3+\tau_{**}]}\max_{1\le\kappa\le N}|\ddot{v}_\kappa(s)|\Big)\Big(\varepsilon^3\tau_{**}^2+\int_{\tau_{**}}^{t-t_3} d\tau\,\frac{1}{\tau^2}\Big).
\end{aligned}
$$

Since $\tau_{**}=\mathcal{O}(\varepsilon^{-1})$ and $\int_{\tau_{**}}^{t-t_3}\frac{d\tau}{\tau^2}\le\int_{\tau_{**}}^{\infty}\frac{d\tau}{\tau^2}=\tau_{**}^{-1}$, we deduce that even

$$
|T_6^\beta(t,t_3)|\le C\Big(\sup_{s\in[t_3,t_3+\tau_{**}]}\max_{1\le\kappa\le N}|\ddot{v}_\kappa(s)|\Big)\varepsilon,\quad \alpha\neq\beta. \tag{7.59}
$$

As in the previous Section 7.1.2 the term where $\alpha=\beta$, i.e.,

$$
T_6^\alpha(t,t_3)=\int d^3x\,\rho_\alpha(x)\int_{t_3}^{t_3+\tau_{**}} ds\,\Big[U(t-s)\Big((\ddot{v}_\alpha(s)\cdot\nabla_v)\,\Phi_{v_\alpha(s)}(\cdot-q_\alpha(s))\Big)\Big](x+q_\alpha(t)),
$$

is critical. To derive the desired bound, we introduce

$$
\hat{\tau}=t_3+\tau_{**}-\varepsilon^{-1/4}. \tag{7.60}
$$

Since $\tau_{**}=\mathcal{O}(\varepsilon^{-1})$, we have $\hat{\tau}\ge t_3$. According to

$$
\int_{t_3}^{t_3+\tau_{**}} ds=\int_{t_3}^{\hat{\tau}} ds+\int_{\hat{\tau}}^{t_3+\tau_{**}} ds,
$$

we split

$$
T_6^\alpha(t,t_3)=T_{6,1}^\alpha(t,t_3)+T_{6,2}^\alpha(t,t_3). \tag{7.61}
$$

Following our standard method, we first see that in view of (7.32)

$$
\begin{aligned}
|T^{\alpha}_{6,1}(t,t_3)| &\le C\Big(\sup_{s\in[t_3,t_3+\tau_{**}]}\max_{1\le\kappa\le N}|\ddot{v}_\kappa(s)|\Big)\Big|\int_{t_3}^{\hat{\tau}} ds \int\!\!\int d^3x\, d^3y\, \varphi(x)\varphi(y) \\
&\qquad \times \int d^3z\, \zeta_{v_\alpha(s)}(z)\Big(\nabla^2\frac{1}{|\tilde{x}|}\Big)\,\delta(|\tilde{x}|-(t-s))\Big|_{\tilde{x}=x-y+q_\alpha(t)-q_\alpha(s)-z}\Big| \\
&\le C\Big(\sup_{s\in[t_3,t_3+\tau_{**}]}\max_{1\le\kappa\le N}|\ddot{v}_\kappa(s)|\Big)\int\!\!\int d^3x\, d^3y\, \varphi(x)\varphi(y) \\
&\qquad \times \int_{t_3}^{\hat{\tau}} ds\,(t-s)^{-3}\int_{|z-[x-y+q_\alpha(t)-q_\alpha(s)]|=(t-s)} d^2z\,\frac{1}{|z|} \\
&\le C\Big(\sup_{s\in[t_3,t_3+\tau_{**}]}\max_{1\le\kappa\le N}|\ddot{v}_\kappa(s)|\Big)\int_{t_3}^{\hat{\tau}} ds\,(t-s)^{-2} \\
&= C\Big(\sup_{s\in[t_3,t_3+\tau_{**}]}\max_{1\le\kappa\le N}|\ddot{v}_\kappa(s)|\Big)\frac{\hat{\tau}-t_3}{(t-\hat{\tau})(t-t_3)}.
\end{aligned}
$$

Now (7.58) and (7.60) yield

$$
\frac{\hat{\tau}-t_3}{(t-\hat{\tau})(t-t_3)} = \frac{\tau_{**}-\varepsilon^{-1/4}}{(t-\hat{\tau})(t-t_3)} \le \frac{1}{t-\hat{\tau}} \le \varepsilon^{1/4},
$$

whence

$$
|T^{\alpha}_{6,1}(t,t_3)| \le C\Big(\sup_{s\in[t_3,t_3+\tau_{**}]}\max_{1\le\kappa\le N}|\ddot{v}_\kappa(s)|\Big)\varepsilon^{1/4}. \tag{7.62}
$$

Consequently it remains to derive the same bound for

$$
T^{\alpha}_{6,2}(t,t_3) = \int_{\hat{\tau}}^{t_3+\tau_{**}} ds\,(\ddot{v}_\alpha(s)\cdot\nabla_v)\int d^3x\,\rho_\alpha(x)\,\Big[U(t-s)\Big(\Phi_{v_\alpha(s)}(\cdot-q_\alpha(s))\Big)\Big](x+q_\alpha(t)).
$$

Again the main observation is that $\int d^3x\,\rho_\alpha(x)\,[U(t-s)(\Phi_{v_\alpha(s)}(\cdot-q_\alpha(s)))](x+q_\alpha(t))=0$ for $s=t$, cf. (7.54). Similar to (7.55) we calculate

$$
\begin{aligned}
&\frac{d}{d\bar{s}}\Big[U(t-\bar{s})\,\Phi_{v_\alpha(\bar{s})}(\cdot+q_\alpha(t)-q_\alpha(\bar{s}))\Big] \\
&\quad = -U(t-\bar{s})\,\mathcal{A}\Phi_{v_\alpha(\bar{s})}(\cdot+q_\alpha(s)-q_\alpha(\bar{s})) \\
&\qquad + U(t-\bar{s})(\dot{v}_\alpha(\bar{s})\cdot\nabla_v)\,\Phi_{v_\alpha(\bar{s})}(\cdot+q_\alpha(s)-q_\alpha(\bar{s})) \\
&\qquad - U(t-\bar{s})(v_\alpha(\bar{s})\cdot\nabla)\,\Phi_{v_\alpha(\bar{s})}(\cdot+q_\alpha(s)-q_\alpha(\bar{s})).
\end{aligned} \tag{7.63}
$$

Using $\mathcal{A}\Phi_v \cong (v\cdot\nabla)\Phi_v$, cf. (7.46), we notice that the term containing $\mathcal{A}\Phi_v$ can be handled similarly to the last one on the right-hand side of (7.63), and hence it is dropped. Integrating the remainder over $\bar{s}\in[s,t]$, we get

$$
\begin{aligned}
|T^{\alpha}_{6,2}(t,t_3)| \le \Big|\int_{\hat{\tau}}^{t_3+\tau_{**}} ds \int_s^t d\bar{s}\,(\ddot{v}_\alpha(s)\cdot\nabla_v)\int d^3x\,\rho_\alpha(x)\,\Big[U(t-\bar{s})\Big((v_\alpha(\bar{s})\cdot\nabla)\Phi_{v_\alpha(\bar{s})}(\cdot-q_\alpha(\bar{s}))\Big) \\
- U(t-\bar{s})(\dot{v}_\alpha(\bar{s})\cdot\nabla)\Phi_{v_\alpha(\bar{s})}(\cdot-q_\alpha(\bar{s}))\Big)\Big](x+q_\alpha(t))\Big|.
\end{aligned} \tag{7.64}
$$

Let

$$
I_1 = \{s\in[\hat{\tau},t_3+\tau_{**}] : t-s\ge 1\},\quad I_2 = [\hat{\tau},t_3+\tau_{**}]\setminus I_1. \tag{7.65}
$$

Then

$$\int_{I_1} ds \int_s^t d\bar{s} = \int_{I_1} ds \Big(\int_s^{t-1} d\bar{s} + \int_{t-1}^t d\bar{s} \Big). \tag{7.66}$$

For the first part we infer from the standard method and (7.32) that

$$\begin{aligned}
\Big| \int_{I_1} ds \int_s^{t-1} d\bar{s} \ldots \Big| \ &\leq \ C\Big(\sup_{s\in[t_3,t_3+\tau_{**}]} \max_{1\leq\kappa\leq N} |\ \ddot{v}_\kappa\ (s)| \Big) \int_{t-[t_3+\tau_{**}]}^{t-\hat{\tau}} d\tau \int_1^\tau d\bar{\tau} \int\int d^3x\, d^3y\, \varphi(x)\varphi(y) \\
&\quad \times \Big| \int d^3z\, \zeta_{v_\alpha(\bar{s})}(z) \Big([\sqrt{\varepsilon}\,\nabla^3 + \varepsilon^2\nabla^2] \frac{1}{|\tilde{x}|} \Big)\, \delta(|\tilde{x}| - \bar{\tau}) \Big|_{\tilde{x}=x-y+q_\alpha(t)-q_\alpha(t-\bar{\tau})-z} \Big| \\
&\leq \ C\Big(\sup_{s\in[t_3,t_3+\tau_{**}]} \max_{1\leq\kappa\leq N} |\ \ddot{v}_\kappa\ (s)| \Big) \int_{t-[t_3+\tau_{**}]}^{t-\hat{\tau}} d\tau \int_1^\tau d\bar{\tau} \int\int d^3x\, d^3y\, \varphi(x)\varphi(y) \\
&\qquad \times \Big[\sqrt{\varepsilon}\, \frac{1}{\bar{\tau}^4} + \varepsilon^2 \frac{1}{\bar{\tau}^3} \Big] \int_{|z-[x-y+q_\alpha(t)-q_\alpha(t-\bar{\tau})]|=\bar{\tau}} d^2z\, \frac{1}{|z|} \\
&\leq \ C\Big(\sup_{s\in[t_3,t_3+\tau_{**}]} \max_{1\leq\kappa\leq N} |\ \ddot{v}_\kappa\ (s)| \Big) \int_{t-[t_3+\tau_{**}]}^{t-\hat{\tau}} d\tau \int_1^\tau d\bar{\tau} \Big[\sqrt{\varepsilon}\, \frac{1}{\bar{\tau}^3} + \varepsilon^2 \frac{1}{\bar{\tau}^2} \Big]. \\
&\leq \ C\Big(\sup_{s\in[t_3,t_3+\tau_{**}]} \max_{1\leq\kappa\leq N} |\ \ddot{v}_\kappa\ (s)| \Big) \int_{t-[t_3+\tau_{**}]}^{t-\hat{\tau}} d\tau\, [\sqrt{\varepsilon} + \varepsilon^2] \\
&\leq \ C\Big(\sup_{s\in[t_3,t_3+\tau_{**}]} \max_{1\leq\kappa\leq N} |\ \ddot{v}_\kappa\ (s)| \Big) \varepsilon^{1/4},
\end{aligned} \tag{7.67}$$

the latter since $-\hat{\tau} + [t_3 + \tau_{**}] = \varepsilon^{-1/4}$ due to (7.60). For the second part of (7.66), we once more apply the standard method, and passing all derivatives to φ, it follows that

$$\begin{aligned}
\Big| \int_{I_1} ds \int_{t-1}^t d\bar{s} \ldots \Big| \ &\leq \ C\Big(\sup_{s\in[t_3,t_3+\tau_{**}]} \max_{1\leq\kappa\leq N} |\ \ddot{v}_\kappa\ (s)| \Big) \int_{t-[t_3+\tau_{**}]}^{t-\hat{\tau}} d\tau \int_0^1 d\bar{\tau} \\
&\qquad \times \int\int d^3x\, d^3y\, \Big[\sqrt{\varepsilon}\, |\nabla^3\varphi(x)| + \varepsilon^2 |\nabla^2\varphi(x)| \Big] \varphi(y) \\
&\qquad \times \int d^3z\, |\zeta_{v_\alpha(\bar{s})}(z)|\, \frac{1}{|\tilde{x}|}\, \delta(|\tilde{x}| - \bar{\tau}) \Big|_{\tilde{x}=x-y+q_\alpha(t)-q_\alpha(t-\bar{\tau})-z} \\
&\leq \ C\Big(\sup_{s\in[t_3,t_3+\tau_{**}]} \max_{1\leq\kappa\leq N} |\ \ddot{v}_\kappa\ (s)| \Big) \sqrt{\varepsilon} \int_{t-[t_3+\tau_{**}]}^{t-\hat{\tau}} d\tau \\
&\leq \ C\Big(\sup_{s\in[t_3,t_3+\tau_{**}]} \max_{1\leq\kappa\leq N} |\ \ddot{v}_\kappa\ (s)| \Big) \varepsilon^{1/4}.
\end{aligned} \tag{7.68}$$

According to (7.65), it thus remains to bound the contribution of

$$\Big| \int_{I_2} ds \int_s^t d\bar{s} \ldots \Big|$$

to the right-hand side of (7.64). However, since for $s \in I_2$ we have $t - s \leq 1$, the length of the $d\bar{s}$-integration interval is bounded by 1. Hence we may repeat the argument leading to (7.68), as we gain an $\varepsilon^{1/2}$ from $v_\alpha(\bar{s})$, but we loose an $\varepsilon^{-1/4}$ due to $|I_2| \leq t_3 + \tau_{**} - \hat{\tau} = \varepsilon^{-1/4}$. Summarizing this observation to (7.59), (7.61), (7.62), (7.64), (7.67), and (7.68), we have completed the proof of (7.57).

7.2 Bounding T_{data}

The data contribution to (7.19) is

$$T_{\mathrm{data}}(t, t_1) = \int d^3x\, \rho_\alpha(x) \Big[U(t - t_1) \Big(\mathcal{L}_\alpha(t_1) Z(\cdot, t_1) \Big) \Big] (x + q_\alpha(t)). \tag{7.69}$$

Our aim here is to prove that

$$|T_{\rm data}(t,t_1)| \le C\varepsilon^{11/2}, \quad t_1\in[\tau_{**},t], \quad t\ge t_1+\tau_{**}. \tag{7.70}$$

First we recall from (7.17) that

$$\mathcal{L}_\alpha(t_1)Z(\cdot,t_1) = (\dot v_\alpha(t_1)\cdot\nabla)Z(\cdot,t_1) + (v_\alpha(t_1)\cdot\nabla)^2 Z(\cdot,t_1) + 2(v_\alpha(t_1)\cdot\nabla)\dot Z(\cdot,t_1) + \ddot Z(\cdot,t_1).$$

In view of $\dot Z = \mathcal{A}Z - f$, cf. (7.6), this can be rewritten as

$$\begin{aligned}\mathcal{L}_\alpha(t_1)Z(\cdot,t_1) \;&=\; (\dot v_\alpha(t_1)\cdot\nabla)Z(\cdot,t_1) + (v_\alpha(t_1)\cdot\nabla)^2 Z(\cdot,t_1) + 2(v_\alpha(t_1)\cdot\nabla)[\mathcal{A}Z(\cdot,t_1) - f(\cdot,t_1)]\\ &\quad + \mathcal{A}^2 Z(\cdot,t_1) - \mathcal{A}f(\cdot,t_1) - \dot f(\cdot,t_1).\end{aligned} \tag{7.71}$$

According to (7.5) and (2.5) we have $Z(x,0)\equiv 0$, whence (7.6) yields

$$\begin{aligned}Z(x,t_1) \;&=\; -\int_0^{t_1} ds\,\Big[U(t_1-s)f(\cdot,s)\Big](x)\\ &=\; -\sum_{\beta=1}^N \int_0^{t_1} ds\,\Big[U(t_1-s)\Big((\dot v_\beta(s)\cdot\nabla_v)\Phi_{v_\beta(s)}(\cdot - q_\beta(s))\Big)\Big](x).\end{aligned} \tag{7.72}$$

Since $\mathcal{A}\Phi_v \cong (v\cdot\nabla)\Phi_v$, cf. (7.46), and $\mathcal{A}U(t) = U(t)\mathcal{A}$, we take the liberty to treat $\mathcal{A}Z(\cdot,t_1)$ as $(v_\alpha(t_1)\cdot\nabla)Z(\cdot,t_1)$, $\mathcal{A}f(\cdot,t_1)$ as $(v_\alpha(t_1)\cdot\nabla)f(\cdot,t_1)$, and $\mathcal{A}^2Z(\cdot,t_1)$ as $(v_\alpha(t_1)\cdot\nabla)^2Z(\cdot,t_1)$, although in fact $v_\beta(s)$ do appear, but both $|v_\alpha(s)|$ and $|v_\beta(s)|$ will only be estimated by $\sqrt{\varepsilon}$ below, so this makes no difference. Hence from the point of view of estimates (7.71) becomes

$$\mathcal{L}_\alpha(t_1)Z(\cdot,t_1) \cong (\dot v_\alpha(t_1)\cdot\nabla)Z(\cdot,t_1) + (v_\alpha(t_1)\cdot\nabla)^2 Z(\cdot,t_1) + (v_\alpha(t_1)\cdot\nabla)f(\cdot,t_1) + \dot f(\cdot,t_1), \tag{7.73}$$

where we also dropped factors and "$-$"-signs. Since

$$\begin{aligned}\dot f(x,t_1) \;&=\; \sum_{\beta=1}^N \Big\{(\ddot v_\beta(t_1)\cdot\nabla_v)\Phi_{v_\beta(t_1)}(x-q_\beta(t_1)) + (\dot v_\beta(t_1)\cdot\nabla_v)^2\Phi_{v_\beta(t_1)}(x-q_\beta(t_1))\\ &\qquad - (v_\beta(t_1)\cdot\nabla)(\dot v_\beta(t_1)\cdot\nabla_v)\Phi_{v_\beta(t_1)}(x-q_\beta(t_1))\Big\},\end{aligned} \tag{7.74}$$

we see that in fact also

$$\mathcal{L}_\alpha(t_1)Z(\cdot,t_1) \cong (\dot v_\alpha(t_1)\cdot\nabla)Z(\cdot,t_1) + (v_\alpha(t_1)\cdot\nabla)^2 Z(\cdot,t_1) + \dot f(\cdot,t_1). \tag{7.75}$$

Substituting (7.75) back into (7.69), and taking into account (7.72) and (7.74), we arrive at

$$\begin{aligned}|T_{\rm data}(t,t_1)| \;&\le\; C\sum_{\beta=1}^N \Big|\int d^3x\,\rho_\alpha(x)\int_0^{t_1} ds\,\Big[U(t-s)\Big((\dot v_\alpha(t_1)\cdot\nabla)(\dot v_\beta(s)\cdot\nabla_v)\\ &\qquad + (v_\alpha(t_1)\cdot\nabla)^2(\dot v_\beta(s)\cdot\nabla_v)\Big)\Phi_{v_\beta(s)}(\cdot - q_\beta(s))\Big](x+q_\alpha(t))\Big|\\ &\quad + C\sum_{\beta=1}^N \Big|\int d^3x\,\rho_\alpha(x)\,\Big[U(t-t_1)\Big((\ddot v_\beta(t_1)\cdot\nabla_v) + (\dot v_\beta(t_1)\cdot\nabla_v)^2\\ &\qquad + (v_\beta(t_1)\cdot\nabla)(\dot v_\beta(t_1)\cdot\nabla_v)\Big)\Phi_{v_\beta(t_1)}(\cdot - q_\beta(t_1))\Big](x+q_\alpha(t))\Big|\\ &=:\; C\sum_{\beta=1}^N \Big(|T^\beta_{{\rm data},1}(t,t_1)| + |T^\beta_{{\rm data},2}(t,t_1)|\Big).\end{aligned} \tag{7.76}$$

To bound these terms we apply the standard method from Section 7.1, and utilizing Lemma 3.2 and Lemma 3.3 we find

$$
\begin{aligned}
&|T^\beta_{\mathrm{data},1}(t,t_1)|+|T^\beta_{\mathrm{data},2}(t,t_1)|\\
&\quad\le C\Big|\int_{t-t_1}^{t}d\tau\int\!\!\int d^3x\,d^3y\,\varphi(x)\varphi(y)\\
&\qquad\qquad\times[\varepsilon^4\nabla^3+\varepsilon^3\nabla^4]\int d^3z\,\zeta_{v_\beta(t-\tau)}(z)\frac{1}{|\tilde x|}\,\delta(|\tilde x|-\tau)\Big|_{\tilde x=x-y+q_\alpha(t)-q_\beta(t-\tau)-z}\Big|\\
&\quad+C\Big|\int\!\!\int d^3x\,d^3y\,\varphi(x)\varphi(y)\\
&\qquad\qquad\times[\varepsilon^{7/2}\nabla^2+\varepsilon^4\nabla^2+\varepsilon^{5/2}\nabla^3]\int d^3z\,\zeta_{v_\beta(t_1)}(z)\frac{1}{|\tilde x|}\,\delta(|\tilde x|-(t-t_1))\Big|_{\tilde x=x-y+q_\alpha(t)-q_\beta(t_1)-z}\Big|.
\end{aligned}
$$

Here all the terms can be handled in the same manner. Passing the derivatives to $\frac{1}{|\tilde x|}$ and observing $\tau\ge t-t_1\ge\tau_{**}=\mathcal{O}(\varepsilon^{-1})$, cf. (7.70), we deduce with (7.32) that

$$
\begin{aligned}
&|T^\beta_{\mathrm{data},1}(t,t_1)|+|T^\beta_{\mathrm{data},2}(t,t_1)|\\
&\le\; C\int\!\!\int d^3x\,d^3y\,\varphi(x)\varphi(y)\int_{t-t_1}^{t}d\tau\,\Big[\varepsilon^4\frac{1}{\tau^4}+\varepsilon^3\frac{1}{\tau^5}\Big]\int_{|z-[x-y+q_\alpha(t)-q_\beta(t-\tau)]|=\tau}d^2z\,\frac{1}{|z|}\\
&\quad+C\int\!\!\int d^3x\,d^3y\,\varphi(x)\varphi(y)\Big[\varepsilon^{7/2}(t-t_1)^{-3}+\varepsilon^{5/2}(t-t_1)^{-4}\Big]\int_{|z-[x-y+q_\alpha(t)-q_\beta(t_1)]|=(t-t_1)}d^2z\,\frac{1}{|z|}\\
&\le\; C\int_{t-t_1}^{\infty}d\tau\,\Big[\varepsilon^4\frac{1}{\tau^3}+\varepsilon^3\frac{1}{\tau^4}\Big]+C\Big[\varepsilon^{7/2}(t-t_1)^{-2}+\varepsilon^{5/2}(t-t_1)^{-3}\Big]\le C\varepsilon^{11/2}.
\end{aligned}
$$

This completes the proof of (7.70).

7.3 Bounding T_2

The contribution T_2 to (7.19) is

$$
T_2(t,t_1)=\int d^3x\,\rho_\alpha(x)\int_{t_1}^{t}d\tau\,\Big[U(t-\tau)\Big((\ddot v_\alpha(\tau)\cdot\nabla)Z(\cdot,\tau)\Big)\Big](x+q_\alpha(t)). \tag{7.77}
$$

We are going to show that

$$
|T_2(t,t_1)|\le C\varepsilon^5,\quad t_1\in[\tau_{**},t],\quad t\ge\tau_{**}. \tag{7.78}
$$

In view of (7.6) we have $\frac{d}{dt}(\nabla Z)=\mathcal{A}(\nabla Z)-\nabla f$. Since $Z(x,0)\equiv 0$ by (2.5), also $\nabla Z(x,0)\equiv 0$, whence

$$
\nabla Z(\cdot,\tau)=-\int_0^\tau ds\,U(\tau-s)\nabla f(\cdot,s). \tag{7.79}
$$

Utilizing this and (7.21) in (7.77), we obtain

$$
T_2(t,t_1)=-\sum_{\beta=1}^{N}\int d^3x\,\rho_\alpha(x)\int_{t_1}^{t}d\tau\int_0^\tau ds\,\Big[U(t-s)\Big((\ddot v_\alpha(\tau)\cdot\nabla)(\dot v_\beta(s)\cdot\nabla_v)\,\Phi_{v_\beta(s)}(\cdot-q_\beta(s))\Big)\Big](x+q_\alpha(t)).
$$

Defining the respective terms in the sum on the right-hand side as $T_2^\beta(t,t_1)$, we first note that for $\alpha\ne\beta$ the standard method applies. By assumption $t\ge\tau_{**}$, and for $\bar\tau\in[0,t-t_1]$ it plays no role whether $\tau_{**} \geq \bar\tau$ or $\tau_{**} \leq \bar\tau$, since e.g. in the former case it follows from Lemma 7.3(a), (b) that

$$\begin{aligned}
&\Big| \int_{\bar\tau}^{t} d\bar s \int\!\!\int d^3x\, d^3y\, \varphi(x)\varphi(y) \nabla^3 \int d^3z\, \zeta_{v_\beta(t-\bar s)}(z) \frac{1}{|\tilde x|}\, \delta(|\tilde x| - \bar s)\Big|_{\tilde x = x-y+q_\alpha(t)-q_\beta(t-\bar s)-z} \Big| \\
&\quad \leq C \int\!\!\int d^3x\, d^3y\, \varphi(x)\varphi(y) \Big(\int_{\bar\tau}^{\tau_{**}} d\bar s\, \varepsilon^4 \frac{1}{\bar s} \int_{|z-[x-y+q_\alpha(t)-q_\beta(t-\bar s)]|=\bar s} d^2z \\
&\qquad\qquad + \int_{\tau_{**}}^{t} d\bar s\, \varepsilon^4 \int_{|z-[x-y+q_\alpha(t)-q_\beta(t-\bar s)]|=\bar s} d^2z\, \frac{1}{|z|} \Big) \\
&\quad \leq C\Big(\varepsilon^4 \tau_{**}^2 + \varepsilon^4 t^2\Big) \leq C\varepsilon
\end{aligned}$$

for $t \leq T\varepsilon^{-3/2}$, and in case that $\tau_{**} \leq \bar\tau$ the same result is obtained. Hence from Lemma 3.2, Lemma 3.3, and by means of the standard method we infer that

$$\begin{aligned}
|T_2^\beta(t,t_1)| &\leq C\varepsilon^{11/2} \Big| \int_0^{t-t_1} d\bar\tau \int_{\bar\tau}^{t} d\bar s \int\!\!\int d^3x\, d^3y\, \varphi(x)\varphi(y) \\
&\qquad \times \nabla^3 \int d^3z\, \zeta_{v_\beta(t-\bar s)}(z) \frac{1}{|\tilde x|}\, \delta(|\tilde x| - \bar s)\Big|_{\tilde x = x-y+q_\alpha(t)-q_\beta(t-\bar s)-z} \Big| \\
&\leq C\varepsilon^{11/2} t\varepsilon \leq C\varepsilon^5, \quad \alpha \neq \beta.
\end{aligned} \tag{7.80}$$

Hence it remains to investigate the case $\alpha = \beta$. Then for e.g. $\bar\tau \geq 1$ we have

$$\begin{aligned}
&\Big| \int_{\bar\tau}^{t} d\bar s \int\!\!\int d^3x\, d^3y\, \varphi(x)\varphi(y) \int d^3z\, \zeta_{v_\alpha(t-\bar s)}(z) \Big(\nabla^3 \frac{1}{|\tilde x|}\Big)\, \delta(|\tilde x| - \bar s)\Big|_{\tilde x = x-y+q_\alpha(t)-q_\alpha(t-\bar s)-z} \Big| \\
&\quad \leq C \int\!\!\int d^3x\, d^3y\, \varphi(x)\varphi(y) \int_{\bar\tau}^{\infty} d\bar s\, \frac{1}{\bar s^4} \int_{|z-[x-y+q_\alpha(t)-q_\alpha(t-\bar s)]|=\bar s} d^2z\, \frac{1}{|z|} \\
&\quad \leq C \int_{\bar\tau}^{\infty} d\bar s\, \frac{1}{\bar s^3} \leq C \frac{1}{\bar\tau^2}.
\end{aligned}$$

On the other hand, simply

$$\Big| \int_0^1 d\bar s \int\!\!\int d^3x\, d^3y\, (\nabla^3\varphi(x))\varphi(y) \int d^3z\, \zeta_{v_\alpha(t-\bar s)}(z) \frac{1}{|\tilde x|}\, \delta(|\tilde x| - \bar s)\Big|_{\tilde x = x-y+q_\alpha(t)-q_\alpha(t-\bar s)-z} \Big| \leq C$$

holds. Consequently by the standard method

$$\begin{aligned}
|T_2^\alpha(t,t_1)| &\leq C\varepsilon^{11/2} \Big| \int_0^1 d\bar\tau \Big(\int_{\bar\tau}^1 + \int_1^t \Big) d\bar s \int\!\!\int d^3x\, d^3y\, \varphi(x)\varphi(y) \\
&\qquad \times \nabla^3 \int d^3z\, \zeta_{v_\alpha(t-\bar s)}(z) \frac{1}{|\tilde x|}\, \delta(|\tilde x| - \bar s)\Big|_{\tilde x = x-y+q_\alpha(t)-q_\alpha(t-\bar s)-z} \Big| \\
&\quad + C\varepsilon^{11/2} \Big| \int_1^{t-t_1} d\bar\tau \int_{\bar\tau}^{t} d\bar s \int\!\!\int d^3x\, d^3y\, \varphi(x)\varphi(y) \\
&\qquad \times \nabla^3 \int d^3z\, \zeta_{v_\alpha(t-\bar s)}(z) \frac{1}{|\tilde x|}\, \delta(|\tilde x| - \bar s)\Big|_{\tilde x = x-y+q_\alpha(t)-q_\alpha(t-\bar s)-z} \Big| \\
&\leq C\varepsilon^{11/2} + C\varepsilon^{11/2} \int_1^{t-t_1} d\bar\tau\, \frac{1}{\bar\tau^2} \leq C\varepsilon^{11/2}.
\end{aligned} \tag{7.81}$$

Summarizing (7.80) and (7.81), we see that (7.78) holds.

7.4 Bounding $T_3 + T_4$

In this section we will be dealing with

$$T_3(t,t_1) + T_4(t,t_1) = 2\int d^3x\,\rho_\alpha(x)\int_{t_1}^{t} d\tau\,\Big[U(t-\tau)\Big((v_\alpha(\tau)\cdot\nabla)(\dot v_\alpha(\tau)\cdot\nabla)Z(\cdot,\tau) + (\dot v_\alpha(\tau)\cdot\nabla)\dot Z(\cdot,\tau)\Big)\Big](x+q_\alpha(t)), \quad (7.82)$$

and it will be verified that

$$|T_3(t,t_1) + T_4(t,t_1)| \le C\varepsilon^5, \quad t_1 \ge \tau_{**}, \quad t\in[t_1+\tau_{**}, T\varepsilon^{-3/2}]. \quad (7.83)$$

Introducing

$$P_\alpha(t)\phi = (v_\alpha(t)\cdot\nabla)\nabla\phi + \nabla\dot\phi \quad (7.84)$$

for a general function $\phi = \phi(x,t)$, (7.82) can be rewritten as

$$T_3(t,t_1) + T_4(t,t_1) = 2\int d^3x\,\rho_\alpha(x)\int_{t_1}^{t} d\tau\,\Big[U(t-\tau)\Big(\dot v_\alpha(\tau)\cdot P_\alpha(\tau)Z(\cdot,\tau)\Big)\Big](x+q_\alpha(t)). \quad (7.85)$$

Then

$$\frac{d}{dt}\Big(P_\alpha(t)\phi\Big) = (\dot v_\alpha(t)\cdot\nabla)\nabla\phi + P_\alpha(t)\dot\phi$$

implies in view of (7.6) that

$$\frac{d}{dt}\Big(P_\alpha(t)Z\Big) = (\dot v_\alpha(t)\cdot\nabla)\nabla Z + P_\alpha(t)[\mathcal{A}Z - f] = \mathcal{A}\Big(P_\alpha(t)Z\Big) + (\dot v_\alpha(t)\cdot\nabla)\nabla Z - P_\alpha(t)f,$$

and this leads to

$$P_\alpha(\tau)Z(\cdot,\tau) = U(\tau-t_1)[P_\alpha(t_1)Z(\cdot,t_1)] + \int_{t_1}^{\tau} ds\,U(\tau-s)\Big\{(\dot v_\alpha(s)\cdot\nabla)\nabla Z(\cdot,s) - P_\alpha(s)f(\cdot,s)\Big\}. \quad (7.86)$$

Resubstituting (7.86) into (7.85) we get two terms, one from the data, and one main term.

7.4.1 Bounding the data term

The data term part of (7.85) is

$$\begin{aligned} T_{3+4,\mathrm{data}}(t,t_1) &= 2\int d^3x\,\rho_\alpha(x)\int_{t_1}^{t} d\tau\,\Big[U(t-t_1)\Big(\dot v_\alpha(\tau)\cdot P_\alpha(t_1)Z(\cdot,t_1)\Big)\Big](x+q_\alpha(t)) \\ &= 2\int d^3x\,\rho_\alpha(x)\Big[U(t-t_1)\Big([v_\alpha(t)-v_\alpha(t_1)]\cdot P_\alpha(t_1)Z(\cdot,t_1)\Big)\Big](x+q_\alpha(t)). \end{aligned}$$

Now (7.84) and (7.6) imply

$$\begin{aligned} P_\alpha(t_1)Z(\cdot,t_1) &= (v_\alpha(t_1)\cdot\nabla)\nabla Z(\cdot,t_1) + \nabla\dot Z(\cdot,t_1) \\ &= (v_\alpha(t_1)\cdot\nabla)\nabla Z(\cdot,t_1) + \mathcal{A}\nabla Z(\cdot,t_1) - \nabla f(\cdot,t_1) \\ &\cong (v_\alpha(t_1)\cdot\nabla)\nabla Z(\cdot,t_1) + \nabla f(\cdot,t_1), \end{aligned}$$

again from the point of view of estimates, since $\mathcal{A}\Phi_v \cong (v \cdot \nabla)\Phi_v$, cf. the remarks before (7.73). Using (7.79) and (7.21) we thus find

$$\begin{aligned}
|T_{3+4,\mathrm{data}}(t,t_1)| \;\leq\;& C\sum_{\beta=1}^{N}\Big|\int d^3x\,\rho_\alpha(x)\int_0^{t_1} ds\,\Big[U(t-s)\Big(([v_\alpha(t)-v_\alpha(t_1)]\cdot\nabla)(v_\alpha(t_1)\cdot\nabla) \\
&\qquad\times(\dot v_\beta(s)\cdot\nabla_v)\Phi_{v_\beta(s)}(\cdot-q_\beta(s))\Big)\Big](x+q_\alpha(t))\Big| \\
&+C\sum_{\beta=1}^{N}\Big|\int d^3x\,\rho_\alpha(x)\Big[U(t-t_1)\Big(([v_\alpha(t)-v_\alpha(t_1)]\cdot\nabla) \\
&\qquad\times(\dot v_\beta(t_1)\cdot\nabla_v)\Phi_{v_\beta(t_1)}(\cdot-q_\beta(t_1))\Big)\Big](x+q_\alpha(t))\Big|.
\end{aligned}$$

Concerning the right-hand side, counting powers of ε and ∇-derivatives we see that we have an $\varepsilon^3\nabla^4$ for the first term and an $\varepsilon^{5/2}\nabla^3$ for the second term (with $U(t)$ and Φ_v counting one ∇ each). Since $t-t_1 \geq \tau_{**} = \mathcal{O}(\varepsilon^{-1})$ we hence find

$$|T_{3+4,\mathrm{data}}(t,t_1)| \leq C\varepsilon^{11/2}, \tag{7.87}$$

exactly as the estimates on $T^\beta_{\mathrm{data},1}(t,t_1)$ and $T^\beta_{\mathrm{data},2}(t,t_1)$ from (7.76) have been derived in Section 7.2. Note that here no $\ddot v$-term appears, whence we do not need to assume $t_1 \geq \tau_{**}$ at this point.

7.4.2 Bounding the main term

By this we mean the contribution

$$\begin{aligned}
T_{3+4,\mathrm{main}}(t,t_1) \;:=\;& 2\int d^3x\,\rho_\alpha(x)\int_{t_1}^{t} d\tau\int_{t_1}^{\tau} ds\,\Big[U(t-s)\Big(\dot v_\alpha(\tau)\cdot\Big\{(\dot v_\alpha(s)\cdot\nabla)\nabla Z(\cdot,s) \\
&\qquad - P_\alpha(s)f(\cdot,s)\Big\}\Big)\Big](x+q_\alpha(t)) \\
=:\;& T_{3+4,\mathrm{main},1}(t,t_1) - T_{3+4,\mathrm{main},2}(t,t_1)
\end{aligned} \tag{7.88}$$

to (7.85).

To begin with $T_{3+4,\mathrm{main},1}(t,t_1)$, we can use (7.79) to rewrite this expression as

$$\begin{aligned}
&T_{3+4,\mathrm{main},1}(t,t_1) \\
&\quad = -2\int d^3x\,\rho_\alpha(x)\int_{t_1}^{t} d\tau\int_{t_1}^{\tau} ds\int_0^s d\sigma\,\Big[U(t-\sigma)\Big((\dot v_\alpha(\tau)\cdot\nabla)(\dot v_\alpha(s)\cdot\nabla)f(\cdot,\sigma)\Big)\Big](x+q_\alpha(t)).
\end{aligned}$$

Substituting (7.21) for f and observing that no $\ddot v$-terms are to be estimated, we hence arrive at

$$\begin{aligned}
&|T_{3+4,\mathrm{main},1}(t,t_1)| \\
&\quad\leq\; C\sum_{\beta=1}^{N}\int_0^t d\tau\int_0^\tau ds\int_0^s d\sigma\,\Big|\int d^3x\,\rho_\alpha(x)\,\Big[U(t-\sigma)\Big((\dot v_\alpha(\tau)\cdot\nabla)(\dot v_\alpha(s)\cdot\nabla) \\
&\qquad\qquad\times(\dot v_\beta(\sigma)\cdot\nabla_v)\,\Phi_{v_\beta(\sigma)}(\cdot-q_\beta(\sigma))\Big)\Big](x+q_\alpha(t))\Big|.
\end{aligned}$$

Observing $\int_0^t d\tau\int_0^\tau ds\int_0^s d\sigma = \int_0^t d\sigma\int_\sigma^t ds\int_s^t d\tau$, transforming $\tilde\sigma = t-\sigma$, $\tilde s = t-s$, and $\tilde\tau = t-\tau$, and then omitting the tilde, we see that

$$|T_{3+4,\mathrm{main},1}(t,t_1)|$$

$$\leq \; C\sum_{\beta=1}^{N}\int_0^t d\sigma\int_0^\sigma ds\int_0^s d\tau\,\Big|\int d^3x\,\rho_\alpha(x)\,\Big[U(\sigma)\Big((\dot v_\alpha(t-\tau)\cdot\nabla)(\dot v_\alpha(t-s)\cdot\nabla)$$
$$\times(\dot v_\beta(t-\sigma)\cdot\nabla_v)\,\Phi_{v_\beta(t-\sigma)}(\cdot-q_\beta(t-\sigma))\Big)\Big](x+q_\alpha(t))\Big|.$$

Invoking Lemma 3.2 and the standard method, it follows that

$$\begin{aligned}
|T_{3+4,\mathrm{main},1}(t,t_1)| \;&\leq\; C\varepsilon^6\sum_{\beta=1}^{N}\int_0^t d\sigma\int_0^\sigma ds\int_0^s d\tau\,\Big|\int\int d^3x\,d^3y\,\varphi(x)\varphi(y)\\
&\qquad\times\nabla^4\int d^3z\,\zeta_{v_\beta(t-\sigma)}(z)\frac{1}{|\tilde x|}\,\delta(|\tilde x|-\sigma)\Big|_{\tilde x=x-y+q_\alpha(t)-q_\beta(t-\sigma)-z}\Big|\\
&\leq\; C\varepsilon^6\sum_{\beta=1}^{N}\int_0^t d\sigma\,\sigma^2\,\Big|\int\int d^3x\,d^3y\,\varphi(x)\varphi(y)\\
&\qquad\times\nabla^4\int d^3z\,\zeta_{v_\beta(t-\sigma)}(z)\frac{1}{|\tilde x|}\,\delta(|\tilde x|-\sigma)\Big|_{\tilde x=x-y+q_\alpha(t)-q_\beta(t-\sigma)-z}\Big|.
\end{aligned}\tag{7.89}$$

First we consider $\alpha\neq\beta$. We may assume that $t\geq\tau_{**}$, with τ_{**} from Lemma 7.3, the case $t\leq\tau_{**}$ being simpler. Estimating $\sigma^2\leq t^2\leq C\varepsilon^{-3}$ for $t\leq T\varepsilon^{-3/2}$ and invoking Lemma 7.3(a), (b), as well as (7.32), we deduce

$$\begin{aligned}
&\varepsilon^6\Big(\int_0^{\tau_{**}}+\int_{\tau_{**}}^t\Big)d\sigma\,\sigma^2\,\Big|\int\int d^3x\,d^3y\,\varphi(x)\varphi(y)\\
&\qquad\times\nabla^4\int d^3z\,\zeta_{v_\beta(t-\sigma)}(z)\frac{1}{|\tilde x|}\,\delta(|\tilde x|-\sigma)\Big|_{\tilde x=x-y+q_\alpha(t)-q_\beta(t-\sigma)-z}\Big|\\
&\leq C\varepsilon^3\int_0^{\tau_{**}}d\sigma\,\int\int d^3x\,d^3y\,\varphi(x)\varphi(y)\,\varepsilon^5\,\frac{1}{\sigma}\int_{|z-[x-y+q_\alpha(t)-q_\beta(t-\sigma)]|=\sigma}d^2z\\
&\quad+C\varepsilon^3\int_{\tau_{**}}^t d\sigma\,\int\int d^3x\,d^3y\,\varphi(x)\varphi(y)\,\varepsilon^5\int_{|z-[x-y+q_\alpha(t)-q_\beta(t-\sigma)]|=\sigma}d^2z\,\frac{1}{|z|}\\
&\leq C\varepsilon^3\Big(\varepsilon^5\tau_{**}^2+\varepsilon^5t^2\Big)\leq C\varepsilon^5
\end{aligned}\tag{7.90}$$

for $t\leq T\varepsilon^{-3/2}$. On the other hand, for $\alpha=\beta$ we split at $\sigma=8R_\varphi$. Then Lemma 7.3(c) yields

$$\begin{aligned}
&\varepsilon^6\Big(\int_0^{8R_\varphi}+\int_{8R_\varphi}^t\Big)d\sigma\,\sigma^2\,\Big|\int\int d^3x\,d^3y\,\varphi(x)\varphi(y)\\
&\qquad\times\nabla^4\int d^3z\,\zeta_{v_\alpha(t-\sigma)}(z)\frac{1}{|\tilde x|}\,\delta(|\tilde x|-\sigma)\Big|_{\tilde x=x-y+q_\alpha(t)-q_\alpha(t-\sigma)-z}\Big|\\
&\leq C\varepsilon^6\int_0^{8R_\varphi}d\sigma\,\sigma^2\int\int d^3x\,d^3y\,|\nabla^4\varphi(x)|\varphi(y)\,\frac{1}{\sigma}\int_{|z-[x-y+q_\alpha(t)-q_\alpha(t-\sigma)]|=\sigma}d^2z\,\frac{1}{|z|}\\
&\quad+C\varepsilon^6\int_{8R_\varphi}^t d\sigma\,\sigma^2\int\int d^3x\,d^3y\,\varphi(x)\varphi(y)\,\frac{1}{\sigma^6}\int_{|z-[x-y+q_\alpha(t)-q_\alpha(t-\sigma)]|=\sigma}d^2z\\
&\leq C\varepsilon^6.
\end{aligned}\tag{7.91}$$

Summarizing (7.90) and (7.91), and going back to (7.89), we have shown that

$$|T_{3+4,\mathrm{main},1}(t,t_1)|\leq C\varepsilon^5\tag{7.92}$$

for the t, t_1 in question.

Finally we return to (7.88) and bound

$$T_{3+4,\mathrm{main},2}(t,t_1) = 2\int d^3x\,\rho_\alpha(x)\int_{t_1}^t d\tau\int_{t_1}^\tau ds\,\Big[U(t-s)\Big(\dot v_\alpha(\tau)\cdot P_\alpha(s)f(\cdot,s)\Big)\Big](x+q_\alpha(t)). \tag{7.93}$$

By means of (7.84) and (7.21) it is calculated that

$$P_\alpha(s)f(\cdot,s) = \nabla\sum_{\beta=1}^N\Big\{([v_\alpha-v_\beta]\cdot\nabla)(\dot v_\beta\cdot\nabla_v)+(\ddot v_\beta\cdot\nabla_v)+(\dot v_\beta\cdot\nabla_v)^2\Big\}\,\Phi_{v_\beta}(\cdot-q_\beta), \tag{7.94}$$

where all $v_\alpha, v_\beta, \ldots$, etc., are evaluated at time s. Since $|\dot v_\beta(s)|^2\cong\varepsilon^4$ but only $|\ddot v_\beta(s)|\cong\varepsilon^{7/2}$, the last term in (7.94) is better than the one next to the last, and hence it is dropped. Using the remainder in (7.93), we find that

$$\begin{aligned}
&|T_{3+4,\mathrm{main},2}(t,t_1)|\\
&\le C\sum_{\beta=1}^N\int_{t_1}^t d\tau\int_{t_1}^\tau ds\,\Big|\int d^3x\,\rho_\alpha(x)\Big[U(t-s)\Big((\dot v_\alpha(\tau)\cdot\nabla)([v_\alpha(s)-v_\beta(s)]\cdot\nabla)(\dot v_\beta(s)\cdot\nabla_v)\\
&\qquad\qquad +(\dot v_\alpha(\tau)\cdot\nabla)(\ddot v_\beta(s)\cdot\nabla_v)\Big)\,\Phi_{v_\beta(s)}(\cdot-q_\beta(s))\Big](x+q_\alpha(t))\Big|.
\end{aligned}$$

Since $s\ge t_1\ge\tau_{**}$, cf. (7.83), we have $|\ddot v_\beta(s)|\le C\varepsilon^{7/2}$ due to Lemma 3.3. Applying the standard method, we thus conclude from Lemma 3.2 that

$$\begin{aligned}
|T_{3+4,\mathrm{main},2}(t,t_1)| \;\le\;& C\sum_{\substack{\beta=1\\ \beta\ne\alpha}}^N\int_{t_1}^t d\tau\int_{t_1}^\tau ds\,\varepsilon^{9/2}\Big|\int\!\!\int d^3x\,d^3y\,\varphi(x)\varphi(y)\\
&\times\nabla^4\int d^3z\,\zeta_{v_\beta(s)}(z)\frac{1}{|\tilde x|}\,\delta(|\tilde x|-(t-s))\Big|_{\tilde x=x-y+q_\alpha(t)-q_\beta(s)-z}\Big|\\
&+C\sum_{\beta=1}^N\int_{t_1}^t d\tau\int_{t_1}^\tau ds\,\varepsilon^{11/2}\Big|\int\!\!\int d^3x\,d^3y\,\varphi(x)\varphi(y)\\
&\times\nabla^3\int d^3z\,\zeta_{v_\beta(s)}(z)\frac{1}{|\tilde x|}\,\delta(|\tilde x|-(t-s))\Big|_{\tilde x=x-y+q_\alpha(t)-q_\beta(s)-z}\Big|.
\end{aligned}$$

Using $\int_{t_1}^t d\tau\int_{t_1}^\tau ds\le\int_0^t d\tau\int_0^\tau ds$ and observing $\int_0^t d\tau\int_0^\tau ds=\int_0^t ds\int_s^t d\tau$, we then introduce the change of variables $\tilde s=t-s$, $\tilde\tau=t-\tau$, and omitting the tilde again we arrive at

$$\begin{aligned}
|T_{3+4,\mathrm{main},2}(t,t_1)| \;\le\;& C\varepsilon^{9/2}\sum_{\substack{\beta=1\\ \beta\ne\alpha}}^N\int_0^t ds\int_0^s d\tau\,\Big|\int\!\!\int d^3x\,d^3y\,\varphi(x)\varphi(y)\\
&\times\nabla^4\int d^3z\,\zeta_{v_\beta(t-s)}(z)\frac{1}{|\tilde x|}\,\delta(|\tilde x|-s)\Big|_{\tilde x=x-y+q_\alpha(t)-q_\beta(t-s)-z}\Big|\\
&+C\varepsilon^{11/2}\sum_{\beta=1}^N\int_0^t ds\int_0^s d\tau\,\Big|\int\!\!\int d^3x\,d^3y\,\varphi(x)\varphi(y)\\
&\times\nabla^3\int d^3z\,\zeta_{v_\beta(t-s)}(z)\frac{1}{|\tilde x|}\,\delta(|\tilde x|-s)\Big|_{\tilde x=x-y+q_\alpha(t)-q_\beta(t-s)-z}\Big|
\end{aligned}$$

$$
\begin{aligned}
&\le \; C\varepsilon^{9/2} \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N} \int_0^t ds\, s \,\Big| \int\!\!\int d^3x\, d^3y\, \varphi(x)\varphi(y) \\
&\qquad\qquad\qquad \times \nabla^4 \int d^3z\, \zeta_{v_\beta(t-s)}(z) \frac{1}{|\tilde{x}|}\, \delta(|\tilde{x}|-s)\Big|_{\tilde{x}=x-y+q_\alpha(t)-q_\beta(t-s)-z}\Big| \\
&\quad + C\varepsilon^{11/2} \sum_{\beta=1}^{N} \int_0^t ds\, s \,\Big| \int\!\!\int d^3x\, d^3y\, \varphi(x)\varphi(y) \\
&\qquad\qquad\qquad \times \nabla^3 \int d^3z\, \zeta_{v_\beta(t-s)}(z) \frac{1}{|\tilde{x}|}\, \delta(|\tilde{x}|-s)\Big|_{\tilde{x}=x-y+q_\alpha(t)-q_\beta(t-s)-z}\Big| .
\end{aligned}
$$

In the first term we have $\alpha \neq \beta$, and we estimate $s \le t \le C\varepsilon^{-3/2}$ to be left with an "$\varepsilon^3\nabla^4$". Exactly as in (7.90) we see that this part is bounded by $C\varepsilon^5$, whence

$$
\begin{aligned}
|T_{3+4,\mathrm{main},2}(t,t_1)| \;&\le\; C\varepsilon^5 + C\varepsilon^{11/2} \sum_{\beta=1}^{N} \int_0^t ds\, s \,\Big| \int\!\!\int d^3x\, d^3y\, \varphi(x)\varphi(y) \\
&\qquad\qquad \times \nabla^3 \int d^3z\, \zeta_{v_\beta(t-s)}(z) \frac{1}{|\tilde{x}|}\, \delta(|\tilde{x}|-s)\Big|_{\tilde{x}=x-y+q_\alpha(t)-q_\beta(t-s)-z}\Big| .
\end{aligned} \tag{7.95}
$$

For $\alpha \neq \beta$ we use $s \le t \le C\varepsilon^{-3/2}$, and by means of Lemma 7.3(a), (b), and (7.32) we see that

$$
\begin{aligned}
&\varepsilon^{11/2}\Big(\int_0^{\tau_{**}} + \int_{\tau_{**}}^t\Big) ds\, s \,\Big| \int\!\!\int d^3x\, d^3y\, \varphi(x)\varphi(y) \\
&\qquad\qquad \times \nabla^3 \int d^3z\, \zeta_{v_\beta(t-s)}(z) \frac{1}{|\tilde{x}|}\, \delta(|\tilde{x}|-s)\Big|_{\tilde{x}=x-y+q_\alpha(t)-q_\beta(t-s)-z}\Big| \\
&\le C\varepsilon^4 \int_0^{\tau_{**}} ds \int\!\!\int d^3x\, d^3y\, \varphi(x)\varphi(y)\, \varepsilon^4 \frac{1}{s} \int_{|z-[x-y+q_\alpha(t)-q_\beta(t-s)]|=s} d^2z \\
&\quad + C\varepsilon^4 \int_{\tau_{**}}^t ds \int\!\!\int d^3x\, d^3y\, \varphi(x)\varphi(y)\, \varepsilon^4 \int_{|z-[x-y+q_\alpha(t)-q_\beta(t-s)]|=s} d^2z\, \frac{1}{|z|} \\
&\le C\varepsilon^4\Big(\varepsilon^4\tau_{**}^2 + \varepsilon^4 t^2\Big) \le C\varepsilon^5
\end{aligned} \tag{7.96}
$$

for $t \le T\varepsilon^{-3/2}$. Finally for $\alpha = \beta$ we can invoke Lemma 7.3(c) to find

$$
\begin{aligned}
&\varepsilon^{11/2}\Big(\int_0^{8R_\varphi} + \int_{8R_\varphi}^t\Big) ds\, s \,\Big| \int\!\!\int d^3x\, d^3y\, \varphi(x)\varphi(y) \\
&\qquad\qquad \times \nabla^3 \int d^3z\, \zeta_{v_\alpha(t-s)}(z) \frac{1}{|\tilde{x}|}\, \delta(|\tilde{x}|-s)\Big|_{\tilde{x}=x-y+q_\alpha(t)-q_\alpha(t-s)-z}\Big| \\
&\le C\varepsilon^{11/2} \int_0^{8R_\varphi} ds\, s \int\!\!\int d^3x\, d^3y\, |\nabla^3\varphi(x)|\varphi(y)\, \frac{1}{s} \int_{|z-[x-y+q_\alpha(t)-q_\alpha(t-s)]|=s} d^2z\, \frac{1}{|z|} \\
&\quad + C\varepsilon^{11/2} \int_{8R_\varphi}^t ds\, s \int\!\!\int d^3x\, d^3y\, \varphi(x)\varphi(y)\, \frac{1}{s^5} \int_{|z-[x-y+q_\alpha(t)-q_\alpha(t-s)]|=s} d^2z \\
&\le C\varepsilon^{11/2}.
\end{aligned} \tag{7.97}
$$

In view of (7.95), (7.96), and (7.97) we infer that $|T_{3+4,\mathrm{main},2}(t,t_1)| \le C\varepsilon^5$, and together with (7.87) and (7.92) we conclude that indeed (7.83) holds.

7.5 Bounding $|\ddot{R}_\alpha(t)|$

We are going to verify the estimate (7.16). A direct calculation starting from (7.4) reveals that

$$\begin{aligned}\ddot{R}_\alpha(t) &= \Big(\frac{d^2}{dt^2}\, m_{0\alpha}(v_\alpha)^{-1}\Big) m_{0\alpha}(v_\alpha) R_\alpha(t) + \Big(\frac{d}{dt}\, m_{0\alpha}(v_\alpha)^{-1}\Big)\Big(\frac{d}{dt}\, m_{0\alpha}(v_\alpha)\Big) R_\alpha(t) \\ &\quad + \Big(\frac{d}{dt}\, m_{0\alpha}(v_\alpha)^{-1}\Big) m_{0\alpha}(v_\alpha)\dot{R}_\alpha(t) \\ &\quad + \Big(\frac{d}{dt}\, m_{0\alpha}(v_\alpha)^{-1}\Big) m_{0\alpha}(v_\alpha)\Big[\dot{R}_\alpha(t) - \Big(\frac{d}{dt}\, m_{0\alpha}(v_\alpha)^{-1}\Big) m_{0\alpha}(v_\alpha) R_\alpha(t)\Big] \\ &\quad + m_{0\alpha}(v_\alpha)^{-1} \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N} S_{\alpha\beta}(t), \end{aligned} \tag{7.98}$$

with the terms

$$\begin{aligned} S_{\alpha\beta}(t) &= \int d^3x\, \rho_\alpha(x-q_\alpha)\Big\{(\ddot{v}_\beta\cdot\nabla_v)E_{v_\beta} + (\dot{v}_\beta\cdot\nabla_v)^2 E_{v_\beta} + 2(\dot{v}_\beta\cdot\nabla_v)([v_\alpha - v_\beta]\cdot\nabla)E_{v_\beta} \\ &\quad + ([\dot{v}_\alpha - \dot{v}_\beta]\cdot\nabla)E_{v_\beta} + ([v_\alpha - v_\beta]\cdot\nabla)^2 E_{v_\beta} + \ddot{v}_\alpha\wedge B_{v_\beta} \\ &\quad + 2\dot{v}_\alpha\wedge(\dot{v}_\beta\cdot\nabla_v)B_{v_\beta} + 2\dot{v}_\alpha\wedge([v_\alpha - v_\beta]\cdot\nabla)B_{v_\beta} + v_\alpha\wedge(\ddot{v}_\beta\cdot\nabla_v)B_{v_\beta} \\ &\quad + v_\alpha\wedge(\dot{v}_\beta\cdot\nabla_v)^2 B_{v_\beta} + v_\alpha\wedge(\dot{v}_\beta\cdot\nabla_v)([v_\alpha - v_\beta]\cdot\nabla)B_{v_\beta} \\ &\quad + v_\alpha\wedge([\dot{v}_\alpha - \dot{v}_\beta]\cdot\nabla)B_{v_\beta} + v_\alpha\wedge(\dot{v}_\beta\cdot\nabla_v)([v_\alpha - v_\beta]\cdot\nabla)B_{v_\beta} \\ &\quad + v_\alpha\wedge([v_\alpha - v_\beta]\cdot\nabla)^2 B_{v_\beta}\Big\}(x-q_\beta), \end{aligned} \tag{7.99}$$

where all q_α, q_β, etc., are evaluated at time t. Invoking the bounds (7.11)–(7.13), (7.98) yields

$$|\ddot{R}_\alpha(t)| \le C\varepsilon^6 + C\max_{\beta\neq\alpha}|S_{\alpha\beta}(t)|, \quad t\in[\tau_{**}, T\varepsilon^{-3/2}]. \tag{7.100}$$

To estimate $S_{\alpha\beta}(t)$, we introduce for $\alpha\neq\beta$ the interaction terms

$$\begin{aligned}\nabla\Psi_{\alpha\beta}(t) &:= \int d^3x\, \rho_\alpha(x-q_\alpha(t))\nabla\phi_{v_\beta(t)}(x-q_\beta(t)) \\ &= (-i)\, e_\alpha e_\beta \int d^3k\, k \frac{|\hat{\varphi}(k)|^2}{k^2 - (k\cdot v_\beta(t))^2}\, e^{ik\cdot[q_\beta(t)-q_\alpha(t)]} \\ &= \frac{e_\alpha e_\beta}{4\pi}\int\!\!\int d^3x d^3y\, \varphi(x-q_\alpha(t))\varphi(y-q_\beta(t))\nabla\zeta_{v_\beta(t)}(x-y), \end{aligned} \tag{7.101}$$

with $\zeta_v(x)$ from Lemma 7.2, and for $l,j\ge 0$ we will also need the derivatives

$$\begin{aligned}\nabla_v^l\nabla^j\Psi_{\alpha\beta}(t) &:= \int d^3x\, \rho_\alpha(x-q_\alpha(t))\nabla_v^l\nabla^j\phi_{v_\beta(t)}(x-q_\beta(t)) \\ &= \frac{e_\alpha e_\beta}{4\pi}\int\!\!\int d^3x d^3y\, \varphi(x-q_\alpha(t))\varphi(y-q_\beta(t))\nabla_v^l\nabla^j\zeta_{v_\beta(t)}(x-y). \end{aligned} \tag{7.102}$$

To illustrate the method for bounding $S_{\alpha\beta}(t)$, let us first consider

$$S_{\alpha\beta,1}(t) = \int d^3x\, \rho_\alpha(x-q_\alpha(t))(\ddot{v}_\beta\cdot\nabla_v)E_{v_\beta(t)}(x-q_\beta(t)),$$

which comprises the first term in (7.99). Recalling $E_v(x) = -\nabla\phi_v(x) + (v\cdot\nabla\phi_v(x))v$ from (2.3) and calculating $(\ddot{v}\cdot\nabla_v)E_v(x)$ explicitly, by means of (7.101) and (7.102) we may rewrite $S_{\alpha\beta,1}(t)$ as

$$\begin{aligned} S_{\alpha\beta,1}(t) &= -(\ddot{v}_\beta(t)\cdot\nabla_v)\nabla\Psi_{\alpha\beta}(t) + v_\beta(t)\,(\ddot{v}_\beta(t)\cdot\nabla)\Psi_{\alpha\beta}(t) \\ &\quad + v_\beta(t)\,(\ddot{v}_\beta(t)\cdot\nabla_v)(v_\beta(t)\cdot\nabla)\Psi_{\alpha\beta}(t) + \ddot{v}_\beta(t)\,(v_\beta(t)\cdot\nabla)\Psi_{\alpha\beta}(t). \end{aligned} \tag{7.103}$$

From Lemma 7.2 we obtain

$$\begin{aligned} \left|\nabla_v^l\nabla^j\Psi_{\alpha\beta}(t)\right| &\le C\int\!\!\int d^3x d^3y\,\varphi(x-q_\alpha(t))\varphi(y-q_\beta(t))|x-y|^{-(j+1)} \\ &= C\int\!\!\int d^3x d^3y\,\varphi(x)\varphi(y)|x-y+q_\alpha(t)-q_\beta(t)|^{-(j+1)}. \end{aligned} \tag{7.104}$$

Since $\alpha\neq\beta$ we have $|x-y+q_\alpha(t)-q_\beta(t)| \ge |q_\alpha(t)-q_\beta(t)| - 2R\varphi \ge C_*\varepsilon^{-1} - 2R_\varphi \ge (C_*/2)\varepsilon^{-1}$ for $|x|,|y|\le R_\varphi$ and $t\in[0,T\varepsilon^{-3/2}]$, due to Lemma 3.2. Therefore (7.104) yields

$$\left|\nabla_v^l\nabla^j\Psi_{\alpha\beta}(t)\right| \le C\varepsilon^{j+1}, \quad l,j\ge 0, \quad \alpha\neq\beta, \quad t\in[0,T\varepsilon^{-3/2}], \tag{7.105}$$

and applied to (7.103) we find in view of Lemma 3.2 and Lemma 3.3 that

$$|S_{\alpha\beta,1}(t)| \le C\varepsilon^{7/2}\varepsilon^2 + C\sqrt{\varepsilon}\varepsilon^{7/2}\varepsilon^2 + C\sqrt{\varepsilon}\varepsilon^{7/2}\sqrt{\varepsilon}\varepsilon^2 + C\varepsilon^{7/2}\sqrt{\varepsilon}\varepsilon^2 \le C\varepsilon^{11/2}, \quad t\in[\tau_{**},T\varepsilon^{-3/2}].$$

In principle, all other terms in (7.99) may be handled in the same manner, the rule of thumb being that one first counts the powers of ε due to Lemma 3.2, then one counts the ∇-derivatives, with $E_v, B_v \cong \nabla\phi_v$, and a ∇^j gives an additional ε^{j+1}, whereas the ∇_v-derivatives do not hurt the estimate. This way term by term can be bounded, the worst ones being

$$\begin{aligned} \bigg|\int d^3x\,\rho_\alpha(x-q_\alpha(t))\Big\{&([\dot{v}_\alpha(t)-\dot{v}_\beta(t)]\cdot\nabla)E_{v_\beta(t)}(x-q_\beta(t)) \\ &+ ([v_\alpha(t)-v_\beta(t)]\cdot\nabla)^2E_{v_\beta(t)}(x-q_\beta(t))\bigg| \le C\varepsilon^5. \end{aligned}$$

Consequently, (7.100) shows that also

$$|\ddot{R}_\alpha(t)| \le C\varepsilon^5, \quad \alpha = 1,\dots,N, \quad t\in[\tau_{**},T\varepsilon^{-3/2}], \tag{7.106}$$

holds.

7.6 Conclusion of the proof of Lemma 3.4

We recall from (3.16) that $\tau_{**} = (C_*/8)\varepsilon^{-1}$, cf. also Lemma 7.3. To begin with, we fix some $\alpha\in\{1,\dots,N\}$. From (7.18) and (7.19) we know that for $t\in[\tau_{**},T\varepsilon^{-3/2}]$ we have

$$\begin{aligned} |\ddot{v}_\alpha(t)| &\le C\varepsilon^5 + C\left|\int d^3x\,\rho_\alpha(x)(\mathcal{L}_\alpha(t)Z)(x+q_\alpha(t),t)\right| \\ &\le C\varepsilon^5 + |T_{\mathrm{data}}(t,t_1)| + |T_1(t,t_1)| + |T_2(t,t_1)| + |T_3(t,t_1)+T_4(t,t_1)|, \end{aligned}$$

with t_1 still to be selected. If below we can moreover ensure that

$$t_1\in[\tau_{**},t], \quad t\in[t_1+\tau_{**},T\varepsilon^{-3/2}], \quad t\ge\tau_{**}, \tag{7.107}$$

then (7.70), (7.78), and (7.83) imply

$$|\dddot{v}_\alpha(t)| \le C\varepsilon^5 + |T_1(t,t_1)|.$$

Concerning $T(t,t_1)$, we rely on the bounds obtained in Section 7.1. Assuming (7.107) it is found from (7.34) that for $t \in [\tau_{**}, T\varepsilon^{-3/2}]$

$$|\dddot{v}_\alpha(t)| \le C\varepsilon^5 + \Big|\sum_{\beta=1}^N \int d^3x\, \rho_\alpha(x) \int_{t_1}^t ds\, \Big[U(t-s)\Big((\dddot{v}_\beta(s)\cdot\nabla_v)\Phi_{v_\beta(s)}(\cdot - q_\beta(s))\Big)\Big](x+q_\alpha(t))\Big|. \quad (7.108)$$

Now we fix $t \in [2\tau_{**}, T\varepsilon^{-3/2}]$ and set $t_1 = \tau_{**}$. Then (7.107) holds, and (7.108) implies

$$\begin{aligned} |\dddot{v}_\alpha(t)| \le\ & C\varepsilon^5 \\ & + \Big|\sum_{\beta=1}^N \int d^3x\, \rho_\alpha(x) \int_{\tau_{**}}^{2\tau_{**}} ds\, \Big[U(t-s)\Big((\dddot{v}_\beta(s)\cdot\nabla_v)\Phi_{v_\beta(s)}(\cdot - q_\beta(s))\Big)\Big](x+q_\alpha(t))\Big| \\ & + \Big|\sum_{\beta=1}^N \int d^3x\, \rho_\alpha(x) \int_{2\tau_{**}}^{t} ds\, \Big[U(t-s)\Big((\dddot{v}_\beta(s)\cdot\nabla_v)\Phi_{v_\beta(s)}(\cdot - q_\beta(s))\Big)\Big](x+q_\alpha(t))\Big|. \end{aligned}$$

Utilizing (7.43) with $t_2 = \tau_{**}$ and (7.42) with $t_2 = 2\tau_{**}$, it follows that

$$|\dddot{v}_\alpha(t)| \le C\varepsilon^4 + C\Big(\max_{1\le\kappa\le N}|e_\kappa|^2\Big)\Big(\sup_{s\in[2\tau_{**},T\varepsilon^{-3/2}]}\max_{1\le\kappa\le N}|\dddot{v}_\kappa(s)|\Big)$$

for $t \in [2\tau_{**}, T\varepsilon^{-3/2}]$, whence

$$\sup_{t\in[2\tau_{**},T\varepsilon^{-3/2}]}\max_{1\le\kappa\le N}|\dddot{v}_\kappa(t)| \le C\varepsilon^4, \quad (7.109)$$

if the $|e_\kappa|$ are chosen small enough. Next we fix $t \in [3\tau_{**}, T\varepsilon^{-3/2}]$ and set $t_1 = 2\tau_{**}$. Then again (7.107) is satisfied, therefore by (7.108)

$$\begin{aligned} |\dddot{v}_\alpha(t)| \le\ & C\varepsilon^5 \\ & + \Big|\sum_{\beta=1}^N \int d^3x\, \rho_\alpha(x) \int_{2\tau_{**}}^{3\tau_{**}} ds\, \Big[U(t-s)\Big((\dddot{v}_\beta(s)\cdot\nabla_v)\Phi_{v_\beta(s)}(\cdot - q_\beta(s))\Big)\Big](x+q_\alpha(t))\Big| \\ & + \Big|\sum_{\beta=1}^N \int d^3x\, \rho_\alpha(x) \int_{3\tau_{**}}^{t} ds\, \Big[U(t-s)\Big((\dddot{v}_\beta(s)\cdot\nabla_v)\Phi_{v_\beta(s)}(\cdot - q_\beta(s))\Big)\Big](x+q_\alpha(t))\Big|. \end{aligned}$$

For the first part (7.57) applies with $t_3 = 2\tau_{**}$, whereas for the second part we can use (7.42) with $t_2 = 3\tau_{**}$. Accordingly we infer

$$\begin{aligned} |\dddot{v}_\alpha(t)| &\le C\varepsilon^5 + C\Big(\sup_{s\in[2\tau_{**},3\tau_{**}]}\max_{1\le\kappa\le N}|\dddot{v}_\kappa(s)|\Big)\varepsilon^{1/4} \\ &\quad + C\Big(\max_{1\le\kappa\le N}|e_\kappa|^2\Big)\Big(\sup_{s\in[3\tau_{**},T\varepsilon^{-3/2}]}\max_{1\le\kappa\le N}|\dddot{v}_\kappa(s)|\Big) \\ &\le C\varepsilon^{17/4} + C\Big(\max_{1\le\kappa\le N}|e_\kappa|^2\Big)\Big(\sup_{s\in[3\tau_{**},T\varepsilon^{-3/2}]}\max_{1\le\kappa\le N}|\dddot{v}_\kappa(s)|\Big), \end{aligned}$$

where we have used (7.109). As this hold for all $t \in [3\tau_{**}, T\varepsilon^{-3/2}]$, by choosing the $|e_\kappa|$ small enough we can ensure

$$\sup_{t\in[3\tau_{**},T\varepsilon^{-3/2}]} \max_{1\le\kappa\le N} |\dddot{v}_\kappa(t)| \le C\varepsilon^{17/4}. \tag{7.110}$$

Now it is clear how this procedure is iterated to gain factors $\varepsilon^{1/4}$. From (7.110) we obtain the bound

$$\sup_{t\in[4\tau_{**},T\varepsilon^{-3/2}]} \max_{1\le\kappa\le N} |\dddot{v}_\kappa(t)| \le C\varepsilon^{9/2},$$

and so on, until the power ε^5 is reached. Then no further improvement is possible, since there are other error terms of order $\mathcal{O}(\varepsilon^5)$ in (7.108). This way we arrive at

$$\sup_{t\in[6\tau_{**},T\varepsilon^{-3/2}]} \max_{1\le\kappa\le N} |\dddot{v}_\kappa(t)| \le C\varepsilon^5,$$

and this completes the proof of Lemma 3.4. □

8 Appendix: Proof of Lemma 4.3

The proof follows the lines of the proof of [15, Lemma 3.2], although some care has to be taken since the key estimate on $\dddot{v}_\alpha(t)$ from Lemma 3.4 does not hold for $t \in [0, T\varepsilon^{-3/2}]$, but only for $t \in [6\tau_{**}, T\varepsilon^{-3/2}]$. We will consider only assertion (b) of Lemma 4.3, the other parts being verified similarly. Recalling $\xi_{\alpha\beta} = q_\alpha(t) - q_\beta(t)$ and $\alpha \ne \beta$, we first introduce

$$\begin{aligned}
D_1(t) &= i\int_0^t d\tau \int d^3k\, |\hat{\varphi}(k)|^2 e^{-ik\cdot\xi_{\alpha\beta}} \Big\{ e^{-ik\cdot[q_\beta(t)-q_\beta(t-\tau)]} - e^{-ik\cdot[\tau v_\beta - \frac{1}{2}\tau^2\dot{v}_\beta + \frac{1}{6}\tau^3\ddot{v}_\beta]} \Big\} \frac{\sin|k|\tau}{|k|} k \\
&= -\nabla_\xi \int_0^t d\tau \int d^3k\, |\hat{\varphi}(k)|^2 e^{-ik\cdot\xi_{\alpha\beta}} \Big\{ e^{-ik\cdot[q_\beta(t)-q_\beta(t-\tau)]} - e^{-ik\cdot[\tau v_\beta - \frac{1}{2}\tau^2\dot{v}_\beta + \frac{1}{6}\tau^3\ddot{v}_\beta]} \Big\} \frac{\sin|k|\tau}{|k|} \\
&= -\nabla_\xi \int\!\!\int d^3x\, d^3y\, \varphi(x)\varphi(y) \int_0^t d\tau \Big\{ \psi_\tau\Big(x - y + \xi_{\alpha\beta} + q_\beta(t) - q_\beta(t-\tau)\Big) \\
&\qquad - \psi_\tau\Big(x - y + \xi_{\alpha\beta} + \tau v_\beta - \frac{1}{2}\tau^2\dot{v}_\beta + \frac{1}{6}\tau^3\ddot{v}_\beta\Big)\Big\},
\end{aligned} \tag{8.1}$$

where $\psi_\tau(x) = (4\pi|x|)^{-1}\delta(|x| - \tau)$. To proceed further, we need two technical lemmas.

Lemma 8.1 *For $|x|, |y| \le R_\varphi$ and $t \in [t_0, T\varepsilon^{-3/2}]$ fixed, where $t_0 = 4(R_\varphi + C^*\varepsilon^{-1})$, cf. (3.14), we define the function $\theta = \theta(\tau) = \theta(\tau; x - y, t, \alpha, \beta)$ through*

$$\theta(\tau) = \tau - |x - y + q_\alpha(t) - q_\beta(t-\tau)|, \quad \tau \in [0, t].$$

Then $2 \ge \theta'(\tau) \ge 3/4$, and there exists a unique $\tau_0 = \tau_0(x - y, t, \alpha, \beta) \in [0, t_0]$ such that $\theta(\tau_0) = 0$. More precisely, the estimate

$$(C_*/2)\varepsilon^{-1} \le \tau_0 \le 2C^*\varepsilon^{-1} \tag{8.2}$$

holds.

Proof: We have $\theta'(\tau) = 1 - \frac{x-y+q_\alpha(t)-q_\beta(t-\tau)}{|x-y+q_\alpha(t)-q_\beta(t-\tau)|} \cdot v_\beta(t-\tau)$, whence $2 \ge \theta'(\tau) \ge 3/4$ in view of Lemma 3.2 for ε small enough. For the other claims, we first note that $0 \ge \theta(0) = -|x-y+q_\alpha(t)-q_\beta(t)| \ge$

$-(2R_\varphi + C^*\varepsilon^{-1}) \geq -2C^*\varepsilon^{-1}$ and therefore also $\theta(t_0) = \theta(0) + \int_0^{t_0} \theta'(\tau)\, d\tau \geq -2C^*\varepsilon^{-1} + 3t_0/4 = 3R_\varphi + C^*\varepsilon^{-1} > 0$, whence θ has a unique zero τ_0 satisfying $\tau_0 \in [0, t_0]$. To verify (8.2), we estimate

$$\begin{aligned} |x - y + q_\alpha(t) - q_\beta(t-\tau)| &\geq |q_\alpha(t) - q_\beta(t)| - |x| - |y| - |q_\beta(t) - q_\beta(t-\tau)| \\ &\geq C_*\varepsilon^{-1} - 2R_\varphi - C\sqrt{\varepsilon}\tau \end{aligned}$$

by (3.8) and (3.12). Since $\tau_0 \in [0, t_0]$ we have $\sqrt{\varepsilon}\tau_0 \leq C\varepsilon^{-1/2}$, thus

$$\tau_0 = |x - y + q_\alpha(t) - q_\beta(t-\tau_0)| \geq (C_*/2)\varepsilon^{-1}$$

for ε sufficiently small. On the other hand, in view of (3.11) and (3.12) also

$$\begin{aligned} \tau_0 = |x - y + q_\alpha(t) - q_\beta(t-\tau_0)| &\leq |q_\alpha(t) - q_\beta(t)| + |x| + |y| + |q_\beta(t) - q_\beta(t-\tau_0)| \\ &\leq C^*\varepsilon^{-1} + 2R_\varphi + C\sqrt{\varepsilon}\tau_0, \end{aligned}$$

and therefore $\tau_0 \leq 2C^*\varepsilon^{-1}$ for ε small enough. □

Lemma 8.2 *In the setting of Lemma 8.1 we now define* $\bar\theta = \bar\theta(\tau) = \bar\theta(\tau; x-y, t, \alpha, \beta)$ *by*

$$\bar\theta(\tau) = \tau - \left| x - y + \xi_{\alpha\beta}(t) + \tau v_\beta(t) - \frac{1}{2}\tau^2\dot v_\beta(t) + \frac{1}{6}\tau^3\ddot v_\beta(t)\right|, \quad \tau \in [0, t].$$

Then $2 \geq \bar\theta'(\tau) \geq 3/4$, *and there exists a unique* $\tau_1 = \tau_1(x-y, t, \alpha, \beta) \in [0, t_0]$ *such that* $\bar\theta(\tau_1) = 0$. *Again we can arrange for*

$$(C_*/2)\varepsilon^{-1} \leq \tau_1 \leq 2C^*\varepsilon^{-1} \tag{8.3}$$

to be satisfied.

Proof: Here we have $\bar\theta'(\tau) = 1 - \frac{(\ldots)}{|\ldots|} \cdot \Big(v_\beta(t) - \tau\dot v_\beta(t) + \frac{1}{2}\tau^2\ddot v_\beta(t)\Big)$. Due to $t \geq t_0 \geq \tau_{**} = (C_*/8)\varepsilon^{-1}$ we obtain $\tau^2|\ddot v_\beta(t)| \leq Ct^2\varepsilon^{7/2} \leq C\sqrt{\varepsilon}$ by Lemma 3.3, and also $|v_\beta(t)| + \tau|\dot v_\beta(t)| \leq C\sqrt{\varepsilon}$ in view of Lemma 3.2. Since $\bar\theta(0) = \theta(0)$ we can proceed as before in the proof of Lemma 8.1. □

Returning to (8.1) and using Lemmas 8.1 and 8.2, we may thus simply write

$$D_1(t) = -\frac{1}{4\pi}\int\int d^3x\, d^3y\, \varphi(x)\varphi(y)\, \nabla_\xi\Big(\tau_0^{-1} - \tau_1^{-1}\Big). \tag{8.4}$$

Calculating $\nabla_\xi\tau_0^{-1}$ and $\nabla_\xi\tau_1^{-1}$ from the defining properties $\theta(\tau_0) = 0$ and $\bar\theta(\tau_1) = 0$, we arrive at

$$\begin{aligned} \nabla_\xi\tau_0^{-1} &= -\tau_0^{-3}\Big\{\Big(x - y + \xi_{\alpha\beta} + q_\beta(t) - q_\beta(t-\tau_0)\Big) \\ &\qquad + \Big(x - y + \xi_{\alpha\beta} + q_\beta(t) - q_\beta(t-\tau_0)\Big) \cdot v_\beta(t-\tau_0)\nabla_\xi\tau_0\Big\}, \\ \nabla_\xi\tau_1^{-1} &= -\tau_1^{-3}\Big\{\Big(x - y + \xi_{\alpha\beta} + \tau_1 v_\beta - \frac{1}{2}\tau_1^2\dot v_\beta + \frac{1}{6}\tau_1^3\ddot v_\beta\Big) \\ &\qquad + \Big(x - y + \xi_{\alpha\beta} + \tau_1 v_\beta - \frac{1}{2}\tau_1^2\dot v_\beta + \frac{1}{6}\tau_1^3\ddot v_\beta\Big) \cdot \Big(v_\beta - \tau_1\dot v_\beta + \frac{1}{2}\tau_1^2\ddot v_\beta\Big)\nabla_\xi\tau_1\Big\}. \end{aligned}$$

Therefore

$$\begin{aligned}
\left|\nabla_\xi \tau_0^{-1} - \nabla_\xi \tau_1^{-1}\right| \;\le\; & \left|\tau_0^{-3} - \tau_1^{-3}\right| \left|x - y + \xi_{\alpha\beta} + \tau_1 v_\beta - \frac{1}{2}\tau_1^2 \dot v_\beta + \frac{1}{6}\tau_1^3 \ddot v_\beta\right| \\
& \quad \times \Big[1 + \Big(|v_\beta| + \tau_1 |\dot v_\beta| + \frac{1}{2}\tau_1^2 |\ddot v_\beta|\Big) |\nabla_\xi \tau_1|\Big] \\
& + \tau_0^{-3} |q_\beta(t-\tau_0) - q_\beta(t-\tau_1)| \\
& + \tau_0^{-3} \left|q_\beta(t) - q_\beta(t-\tau_1) - \tau_1 v_\beta + \frac{1}{2}\tau_1^2 \dot v_\beta - \frac{1}{6}\tau_1^3 \ddot v_\beta\right| \\
& + \tau_0^{-3} \Big|x - y + \xi_{\alpha\beta} + q_\beta(t) - q_\beta(t-\tau_0)\Big| |v_\beta(t-\tau_0)| |\nabla_\xi(\tau_0 - \tau_1)| \\
& + \tau_0^{-3} \Big|x - y + \xi_{\alpha\beta} + q_\beta(t) - q_\beta(t-\tau_0)\Big| |v_\beta(t-\tau_0) - v_\beta(t-\tau_1)| |\nabla_\xi \tau_1| \\
& + \tau_0^{-3} \Big|x - y + \xi_{\alpha\beta} + q_\beta(t) - q_\beta(t-\tau_0)\Big| \\
& \quad \times \left|v_\beta(t-\tau_1) - v_\beta + \tau_1 \dot v_\beta - \frac{1}{2}\tau_1^2 \ddot v_\beta\right| |\nabla_\xi \tau_1| \\
& + \tau_0^{-3} |q_\beta(t-\tau_0) - q_\beta(t-\tau_1)| |v_\beta(t-\tau_1)| |\nabla_\xi \tau_1| \\
& + \tau_0^{-3} \left|q_\beta(t) - q_\beta(t-\tau_1) - \tau_1 v_\beta + \frac{1}{2}\tau_1^2 \dot v_\beta - \frac{1}{6}\tau_1^3 \ddot v_\beta\right| |v_\beta(t-\tau_1)| |\nabla_\xi \tau_1|.
\end{aligned}$$

Hence Lemma 3.2, Lemma 3.3, (8.2), and (8.3) yield

$$\begin{aligned}
\left|\nabla_\xi \tau_0^{-1} - \nabla_\xi \tau_1^{-1}\right| \;\le\; & C\varepsilon^{-1} \left|\tau_0^{-3} - \tau_1^{-3}\right| \Big[1 + \sqrt{\varepsilon} |\nabla_\xi \tau_1|\Big] + C\varepsilon^{7/2} |\tau_0 - \tau_1| \\
& + C\varepsilon^3 \left|q_\beta(t) - q_\beta(t-\tau_1) - \tau_1 v_\beta + \frac{1}{2}\tau_1^2 \dot v_\beta - \frac{1}{6}\tau_1^3 \ddot v_\beta\right| + C\varepsilon^{5/2} |\nabla_\xi(\tau_0 - \tau_1)| \\
& + C\varepsilon^4 |\tau_0 - \tau_1| |\nabla_\xi \tau_1| + C\varepsilon^2 \left|v_\beta(t-\tau_1) - v_\beta + \tau_1 \dot v_\beta - \frac{1}{2}\tau_1^2 \ddot v_\beta\right| |\nabla_\xi \tau_1| \\
& + C\varepsilon^{7/2} \left|q_\beta(t) - q_\beta(t-\tau_1) - \tau_1 v_\beta + \frac{1}{2}\tau_1^2 \dot v_\beta - \frac{1}{6}\tau_1^3 \ddot v_\beta\right| |\nabla_\xi \tau_1|. \qquad (8.5)
\end{aligned}$$

To bound the right-hand side further we note that $t - \tau_1 \ge t_0 - 2C^*\varepsilon^{-1} = 4R_\varphi + 2C^*\varepsilon^{-1} \ge (3C_*/4)\varepsilon^{-1} = 6\tau_{**}$ according to (8.3), recall (3.16). Thus $|\dddot v_\beta(s)| \le C\varepsilon^5$ for all $s \in [t-\tau_1, t]$ by Lemma 3.4, and this implies that

$$\begin{aligned}
q_\beta(t) &= q_\beta(t-\tau_1) + \tau_1 v_\beta - \frac{1}{2}\tau_1^2 \dot v_\beta + \frac{1}{6}\tau_1^3 \ddot v_\beta + \mathcal{O}(\varepsilon^5 \tau_1^4), \\
v_\beta(t-\tau_1) &= v_\beta - \tau_1 \dot v_\beta + \frac{1}{2}\tau_1^2 \ddot v_\beta + \mathcal{O}(\varepsilon^5 \tau_1^3).
\end{aligned}$$

Utilizing this observation and the definition of τ_0 and τ_1, we moreover obtain

$$\begin{aligned}
|\tau_0 - \tau_1| &\le \left|q_\beta(t) - q_\beta(t-\tau_0) - \tau_1 v_\beta + \frac{1}{2}\tau_1^2 \dot v_\beta - \frac{1}{6}\tau_1^3 \ddot v_\beta\right| \le C\sqrt{\varepsilon}|\tau_0 - \tau_1| + C\varepsilon^5 \tau_1^4 \\
&\le C\sqrt{\varepsilon}|\tau_0 - \tau_1| + C\varepsilon,
\end{aligned}$$

whence $|\tau_0 - \tau_1| \le C\varepsilon$ and consequently also $|\tau_0^{-3} - \tau_1^{-3}| \le C\varepsilon^5$. Next it is verified that $|\nabla_\xi \tau_1| \le C$, and with some more effort also that $|\nabla_\xi(\tau_0 - \tau_1)| \le C\varepsilon^{3/2}$. Invoking all these estimates on the right-hand side of (8.5) it follows that $|\nabla_\xi \tau_0^{-1} - \nabla_\xi \tau_1^{-1}| \le C\varepsilon^4$, and recalling (8.4), we finally obtain

$$\sup_{t \in [t_0, T\varepsilon^{-3/2}]} |D_1(t)| \le C\varepsilon^4. \qquad (8.6)$$

The next step is to consider

$$
\begin{aligned}
D_2(t) &= i\int_0^t d\tau \int d^3k\, |\hat\varphi(k)|^2 e^{-ik\cdot\xi_{\alpha\beta}} \Big\{ e^{-ik\cdot[\tau v_\beta - \frac{1}{2}\tau^2\dot v_\beta + \frac{1}{6}\tau^3\ddot v_\beta]} - \Big(1 - ik\cdot\Big[\tau v_\beta - \frac{1}{2}\tau^2\dot v_\beta + \frac{1}{6}\tau^3\ddot v_\beta\Big] \\
&\qquad -\frac{1}{2}\Big[\tau^2(k\cdot v_\beta)^2 - \tau^3(k\cdot v_\beta)(k\cdot\dot v_\beta)\Big] + \frac{i}{6}\tau^3(k\cdot v_\beta)^3\Big)\Big\}\, \frac{\sin|k|\tau}{|k|}k.
\end{aligned}
$$

With $\psi_\tau(x) = (4\pi|x|)^{-1}\delta(|x|-\tau)$ this may be rewritten as

$$
\begin{aligned}
D_2(t) &= -\nabla_\xi \int\!\!\int d^3x\, d^3y\, \varphi(x)\varphi(y) \int_0^t d\tau\, \Big\{ \psi_\tau\Big(x-y+\xi_{\alpha\beta} + \tau v_\beta - \frac{1}{2}\tau^2\dot v_\beta + \frac{1}{6}\tau^3\ddot v_\beta\Big) \\
&\qquad - \psi_\tau\Big(x-y+\xi_{\alpha\beta}\Big) - \nabla\psi_\tau\Big(x-y+\xi_{\alpha\beta}\Big)\cdot\Big[\tau v_\beta - \frac{1}{2}\tau^2\dot v_\beta + \frac{1}{6}\tau^3\ddot v_\beta\Big] \\
&\qquad - \Big[\frac{1}{2}\tau^2(v_\beta\cdot\nabla)^2 - \frac{1}{2}\tau^3(v_\beta\cdot\nabla)(\dot v_\beta\cdot\nabla) + \frac{1}{6}\tau^3(v_\beta\cdot\nabla)^3\Big]\psi_\tau\Big(x-y+\xi_{\alpha\beta}\Big)\Big\}.
\end{aligned}
$$

By expanding $\psi_\tau(\cdot)$ about $x-y+\xi_{\alpha\beta}$ and using a similar technique as for $D_1(t)$ it can be verified that also

$$
\sup_{t\in[t_0,\,T\varepsilon^{-3/2}]} |D_2(t)| \le C\varepsilon^4. \tag{8.7}
$$

Finally we introduce

$$
\begin{aligned}
D_3(t) &= i\int_t^\infty d\tau \int d^3k\, |\hat\varphi(k)|^2 e^{-ik\cdot\xi_{\alpha\beta}} \Big(1 - ik\cdot\Big[\tau v_\beta - \frac{1}{2}\tau^2\dot v_\beta + \frac{1}{6}\tau^3\ddot v_\beta\Big] \\
&\qquad -\frac{1}{2}\Big[\tau^2(k\cdot v_\beta)^2 - \tau^3(k\cdot v_\beta)(k\cdot\dot v_\beta)\Big] + \frac{i}{6}\tau^3(k\cdot v_\beta)^3\Big)\, \frac{\sin|k|\tau}{|k|}k,
\end{aligned}
$$

and it remains to notice that as in [15, p. 466/467] we obtain $D_3(t) = 0$ for $t\in[t_0, T\varepsilon^{-3/2}]$. Taking into account (8.6) and (8.7), we see that the assertion of Lemma 4.3(b) is satisfied. □

9 Appendix: Proof of Lemma 3.1

First it will be convenient to transform $(\bar r, \bar u)$ to the time scale of (q, v). To this purpose we introduce

$$
r_\alpha(t) = \varepsilon^{-1}\bar r_\alpha(\varepsilon^{3/2}t), \quad u_\alpha(t) = \sqrt{\varepsilon}\,\bar u_\alpha(\varepsilon^{3/2}t), \tag{9.1}
$$

where the $(\bar r_\alpha(t), \bar u_\alpha(t))$ are the solution to the system induced by (3.6) with data $(\bar r_\alpha^0, \bar u_\alpha^0)$ from (3.7). Then

$$
r_\alpha(0) = q_\alpha^0, \quad u_\alpha(0) = v_\alpha^0, \tag{9.2}
$$

by (3.7), and the corresponding equations are

$$
m_\alpha \dot u_\alpha = \sum_{\substack{\beta=1\\ \beta\ne\alpha}}^N \frac{e_\alpha e_\beta}{4\pi}\,\frac{r_\alpha - r_\beta}{|r_\alpha - r_\beta|^3}, \quad \alpha = 1,\dots,N, \tag{9.3}
$$

valid for $t\in[0, (\tau_{\mathrm{C}} - \delta_0)\varepsilon^{-3/2}]$.

Lemma 9.1 *We have*

$$C_0\varepsilon^{-1} \le \inf_{t\in[0,(\tau_{\mathrm{C}}-\delta_0)\varepsilon^{-3/2}]} |r_\alpha(t)-r_\beta(t)|, \qquad \sup_{t\in[0,(\tau_{\mathrm{C}}-\delta_0)\varepsilon^{-3/2}]} |r_\alpha(t)-r_\beta(t)| \le C^0\varepsilon^{-1} \quad (\alpha\ne\beta), \tag{9.4}$$

and

$$\sup_{t\in[0,(\tau_{\mathrm{C}}-\delta_0)\varepsilon^{-3/2}]} |u_\alpha(t)| \le C\sqrt{\varepsilon}, \tag{9.5}$$

with constants C_0, C^0, and $C>0$, depending only on τ_{C}, δ_0, and the data.

Proof: The bounds in (9.4) follow from (9.1) and the definition of τ_{C}. For (9.5), note that the system (9.3) is Hamiltonian with conserved energy

$$\mathcal{H}_{\mathrm{C}}(r,u) = \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha u_\alpha^2 + \frac{1}{2}\sum_{\substack{\alpha,\beta=1\\ \alpha\ne\beta}}^{N} \frac{e_\alpha e_\beta}{4\pi|r_\alpha - r_\beta|},$$

whence

$$\frac{1}{2} m_\alpha u_\alpha^2(t) \le \mathcal{H}_{\mathrm{C}}(r(0),u(0)) - \frac{1}{2}\sum_{\substack{\alpha,\beta=1\\ \alpha\ne\beta}}^{N} \frac{e_\alpha e_\beta}{4\pi|r_\alpha(t) - r_\beta(t)|} \le C\varepsilon,$$

the latter in view of (9.4), (9.2), and (3.5). □

To prove Lemma 3.1 we set

$$C_* = \min\left\{\frac{C_1}{2}, \frac{C_0}{2}\right\},$$

with C_1 from (3.4) and C_0 from (9.4), and we introduce

$$\hat{t} = \sup\left\{t_1 \in [0, \min\{\tau_{\mathrm{C}}-\delta_0, T_0\}\varepsilon^{-3/2}] : C_*\varepsilon^{-1} \le \inf_{t\in[0,t_1]} |q_\alpha(t)-q_\beta(t)| \ \text{ for } \ \alpha\ne\beta\right\}.$$

Hence we need to show that in fact $\hat{t} = \min\{\tau_{\mathrm{C}}-\delta_0, T_0\}\varepsilon^{-3/2}$. Note that $\hat{t}>0$ according to (3.4), and also

$$C_*\varepsilon^{-1} \le \inf_{t\in[0,\hat{t}\,]} |q_\alpha(t)-q_\beta(t)|, \quad \alpha\ne\beta. \tag{9.6}$$

Utilizing the method of [15, Lemma 2.1], cf. also Lemma 3.2, we know that a lower bound of type (9.6) leads to the further bounds

$$\sup_{t\in[0,\hat{t}\,]} |q_\alpha(t)-q_\beta(t)| \le C_4\varepsilon^{-1} \quad (\alpha\ne\beta), \qquad \sup_{t\in[0,\hat{t}\,]} |v_\alpha(t)| \le C_4\sqrt{\varepsilon}, \tag{9.7}$$

with some constant $C_4>0$ depending only on C_1, C_2, C_3, and C_*, but not on $\hat{t}\le C\varepsilon^{-3/2}$. Observe that these estimates hold from $t=0$, and not only from $t=\mathcal{O}(\varepsilon^{-1})=t_0$, since we only need to wait for this time span in case that we have to deal with expressions resulting from data terms. This happens only when in the course of proof some time derivative of the difference function $Z(x,t)$, cf. (7.5), is to be estimated. However, for (9.7) such terms do not appear, cf. [15, Sect. 5.2], they first come up when we estimate $|\ddot{v}_\alpha(t)|$. In addition, (9.7) is obtained without the assumption that the $|e_\alpha|$ be small, cf. [15, Sect. 5.1].

These bounds can be used to derive the appropriate lower-order effective equation for the true solution.

Lemma 9.2 *For $t \in [0, \hat{t}\,]$ we have*

$$m_\alpha \dot{v}_\alpha = \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{N} \frac{e_\alpha e_\beta}{4\pi} \frac{q_\alpha - q_\beta}{|q_\alpha - q_\beta|^3} + \mathcal{O}(\varepsilon^{5/2}), \quad \alpha = 1, \ldots, N. \tag{9.8}$$

Proof: In principle we can follow the argument of Section 4 and expand the Lorentz force $F_\alpha(t)$ from (4.1), with the difference being that we need to get a bound right from $t = 0$, and not only from $t = t_0 = \mathcal{O}(\varepsilon^{-1})$. This means that Lemma 4.1 cannot be used as is to be expected since for small times of order $\mathcal{O}(\varepsilon^{-1})$ (before interaction) the effective Coulomb force in (9.8) will be due to the initial force $F_\alpha^{(0)}(t)$. It is only at times after $t = \mathcal{O}(\varepsilon^{-1})$ that the retarded part of the field $F_\alpha^{(r)}(t)$ makes its influence felt. However, from the viewpoint of a proof there is no sharp time $t_* = \mathcal{O}(\varepsilon^{-1})$ at which this transition does manifest itself, whence a separation of the force as $F_\alpha(t) = F_\alpha^{(0)}(t) + F_\alpha^{(r)}(t)$ will not lead to the optimal bound. Instead of this we write $F_\alpha(t)$ as a single integral as follows. We first continue the particle trajectories and velocities to $t = -\infty$ and define

$$\tilde{q}_\alpha(t) = \left\{ \begin{array}{ccl} q_\alpha(t) & : & t \in [0, \infty[\\ q_\alpha^0 + t v_\alpha^0 & : & t \in]-\infty, 0] \end{array} \right. , \quad \tilde{v}_\alpha(t) = \left\{ \begin{array}{ccl} v_\alpha(t) & : & t \in [0, \infty[\\ v_\alpha^0 & : & t \in]-\infty, 0] \end{array} \right. ; \tag{9.9}$$

for simplicity the tilde is omitted henceforth. Using the relation

$$\int_{-\infty}^{0} ds\, e^{ik\cdot q_\beta(s)} \frac{\sin|k|(t-s)}{|k|} = \frac{e^{ik\cdot q_\beta^0}}{k^2 - (k\cdot v_\beta^0)^2} \Big(\cos|k|t + i(k\cdot v_\beta^0)\, \frac{\sin|k|t}{|k|} \Big)$$

and (2.5), a straightforward calculation shows that we have the representation

$$\begin{aligned} F_\alpha^{(0)}(t) \;=\; & \sum_{\beta=1}^{N} e_\alpha e_\beta \int_{-\infty}^{0} ds \int d^3k\, |\hat{\varphi}(k)|^2 e^{-ik\cdot[q_\alpha(t) - q_\beta(s)]} \Big\{ -\cos|k|(t-s)\, v_\beta(s) + i\, \frac{\sin|k|(t-s)}{|k|}\, k \\ & - i\, \frac{\sin|k|(t-s)}{|k|}\, v_\alpha(t) \wedge (k \wedge v_\beta(s)) \Big\}, \end{aligned}$$

hence also

$$\begin{aligned} F_\alpha(t) \;=\; & \sum_{\beta=1}^{N} e_\alpha e_\beta \int_{-\infty}^{t} ds \int d^3k\, |\hat{\varphi}(k)|^2 e^{-ik\cdot[q_\alpha(t) - q_\beta(s)]} \Big\{ -\cos|k|(t-s)\, v_\beta(s) + i\, \frac{\sin|k|(t-s)}{|k|}\, k \\ & - i\, \frac{\sin|k|(t-s)}{|k|}\, v_\alpha(t) \wedge (k \wedge v_\beta(s)) \Big\}, \end{aligned}$$

in view of (4.3) and (4.4). This form of $F_\alpha(t)$ is appropriate to deduce (9.8). The argument proceeds entirely along the lines of Section 4, but its realization is much easier, since we only have to take into account the contributions of order $\mathcal{O}(\varepsilon^2)$. To illustrate how (9.9) enters, we pick out a single term, e.g.

$$A(t) = i \int_{-\infty}^{t} ds \int d^3k\, |\hat{\varphi}(k)|^2 e^{-ik\cdot[q_\alpha(t) - q_\beta(s)]}\, \frac{\sin|k|(t-s)}{|k|}\, k$$

for $\alpha \neq \beta$, in order to draw a parallel to Lemma 4.3(b), cf. Section 8. We are going to show that

$$A(t) = i \int_{-\infty}^{t} ds \int d^3k\, |\hat{\varphi}(k)|^2\, \frac{\sin|k|(t-s)}{|k|}\, k + \mathcal{O}(\varepsilon^{5/2}), \quad t \in [0, \hat{t}\,]. \tag{9.10}$$

To verify this, we introduce the difference

$$\begin{aligned}
D(t) &= i\int_{-\infty}^{t} ds \int d^3k\, |\hat\varphi(k)|^2 \Big(e^{-ik\cdot[q_\alpha(t)-q_\beta(s)]}-1\Big)\,\frac{\sin|k|(t-s)}{|k|}\,k \\
&= i\int_{0}^{\infty} d\tau \int d^3k\, |\hat\varphi(k)|^2 \Big(e^{-ik\cdot[q_\alpha(t)-q_\beta(t-\tau)]}-1\Big)\,\frac{\sin|k|\tau}{|k|}\,k \\
&= -\nabla_\xi \int\!\!\int d^3x\, d^3y\, \varphi(x)\varphi(y) \\
&\qquad \times \int_0^\infty d\tau\, \Big\{\psi_\tau\Big(x-y+\xi_{\alpha\beta}+q_\beta(t)-q_\beta(t-\tau)\Big)-\psi_\tau\Big(x-y+\xi_{\alpha\beta}\Big)\Big\}, \qquad (9.11)
\end{aligned}$$

with $\psi_\tau(x)=(4\pi|x|)^{-1}\delta(|x|-\tau)$ and $\xi_{\alpha\beta}=q_\alpha(t)-q_\beta(t)$, cf. (8.1). Next note that $q_\alpha\in C^1(\mathbb{R})$ and moreover

$$|v_\alpha(t)|\le C\sqrt{\varepsilon},\quad t\in]-\infty,\hat t\,],\quad 1\le\alpha\le N, \qquad (9.12)$$

by (9.9), (9.7), and (3.5). For fixed $|x|,|y|\le R_\varphi$ and $t\in[0,\hat t\,]$ we define the function

$$\theta:[0,\infty[\to\mathbb{R},\qquad \theta(\tau)=\tau-|x-y+q_\alpha(t)-q_\beta(t-\tau)|.$$

Then $\theta(0)=-|x-y+q_\alpha(t)-q_\beta(t)|=-\mathcal{O}(\varepsilon^{-1})$ by (9.6) and (9.7), and in addition $\theta'(\tau)=1-\frac{x-y+q_\alpha(t)-q_\beta(t-\tau)}{|x-y+q_\alpha(t)-q_\beta(t-\tau)|}\cdot v_\beta(t-\tau)=\mathcal{O}(1)$ due to (9.12), as $t-\tau\in]-\infty,t]\subset]-\infty,\hat t\,]$. Hence $\theta(\cdot)$ has a unique zero $\tau_0=\tau_0(x-y,t,\alpha,\beta)=\mathcal{O}(\varepsilon^{-1})$ in $[0,\infty[$. Setting $\tau_1=|x-y+\xi_{\alpha\beta}(t)|=\mathcal{O}(\varepsilon^{-1})$, we see that (9.11) can be rewritten as

$$4\pi D(t)=-\int\!\!\int d^3x\,d^3y\,\varphi(x)\varphi(y)\,\nabla_\xi(\tau_0^{-1}-\tau_1^{-1}). \qquad (9.13)$$

From the definitions we find that

$$\begin{aligned}
\nabla_\xi\tau_0^{-1} &= -\tau_0^{-3}\Big\{\Big(x-y+\xi_{\alpha\beta}+q_\beta(t)-q_\beta(t-\tau_0)\Big) \\
&\qquad +\Big(x-y+\xi_{\alpha\beta}+q_\beta(t)-q_\beta(t-\tau_0)\Big)\cdot v_\beta(t-\tau_0)\nabla_\xi\tau_0\Big\}, \\
\nabla_\xi\tau_1^{-1} &= -\tau_1^{-3}(x-y+\xi_{\alpha\beta}).
\end{aligned}$$

Since $|\nabla_\xi\tau_0|\le C$, it follows by means of (9.12), and recalling $\tau_0^{-1}=\mathcal{O}(\varepsilon)$, that

$$\nabla_\xi\tau_0^{-1}=-\tau_0^{-3}\Big(x-y+\xi_{\alpha\beta}+q_\beta(t)-q_\beta(t-\tau_0)\Big)+\mathcal{O}(\varepsilon^{5/2}).$$

Next observe that $|q_\beta(t)-q_\beta(t-\tau_0)|\le C\sqrt{\varepsilon}\tau_0\le C\varepsilon^{-1/2}$ in case that $t-\tau_0\ge 0$, due to (9.7). However, if $t-\tau_0\le 0$, then $|q_\beta(t)-q_\beta(t-\tau_0)|=|q_\beta(t)-q_\beta^0|\le C\sqrt{\varepsilon}t\le C\sqrt{\varepsilon}\tau_0\le C\varepsilon^{-1/2}$. Thus

$$\nabla_\xi\tau_0^{-1}=-\tau_0^{-3}(x-y+\xi_{\alpha\beta})+\mathcal{O}(\varepsilon^{5/2})$$

in all cases, and also

$$|\tau_0-\tau_1|\le|q_\beta(t)-q_\beta(t-\tau_0)|\le C\varepsilon^{-1/2}.$$

Therefore

$$|\nabla_\xi(\tau_0^{-1}-\tau_1^{-1})|\le C\varepsilon^{-1}|\tau_0^{-3}-\tau_1^{-3}|+C\varepsilon^{5/2}\le C\varepsilon^5\max\{\tau_0^2,\tau_1^2\}|\tau_0-\tau_1|+C\varepsilon^{5/2}\le C\varepsilon^{5/2},$$

and in view of (9.13) this completes the proof of (9.10). Since it can be verified that all remaining terms can be handled in an analogously manner, we deduce that (9.8) holds. □

Using (9.8) it is possible to complete the proof of Lemma 3.1. In view of (9.8), (9.3), (9.4), and (9.6), we have

$$\begin{aligned} m_\alpha |\dot v_\alpha - \dot u_\alpha| &\le \Big| \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^N \frac{e_\alpha e_\beta}{4\pi} \frac{q_\alpha - q_\beta}{|q_\alpha - q_\beta|^3} - \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^N \frac{e_\alpha e_\beta}{4\pi} \frac{r_\alpha - r_\beta}{|r_\alpha - r_\beta|^3} \Big| + C\varepsilon^{5/2} \\ &\le C\varepsilon^3 \sum_{\beta=1}^N |q_\beta - r_\beta| + C\varepsilon^{5/2} \end{aligned}$$

for $t \in [0, \hat t\,]$. By the argument given in [15, p. 449/450] this yields

$$|q_\alpha(t) - r_\alpha(t)| \le C\varepsilon^{5/2-3} = C\varepsilon^{-1/2}, \quad |v_\alpha(t) - u_\alpha(t)| \le C\varepsilon^{5/2-3/2} = C\varepsilon, \quad t \in [0, \hat t\,], \tag{9.14}$$

for $\alpha = 1, \ldots, N$. Consequently, by (9.4) and (9.14) we obtain for $t \in [0, \hat t\,]$

$$\begin{aligned} |q_\alpha(t) - q_\beta(t)| &\ge |r_\alpha(t) - r_\beta(t)| - |q_\alpha(t) - r_\alpha(t)| - |q_\beta(t) - r_\beta(t)| \\ &\ge C_0\varepsilon^{-1} - C\varepsilon^{-1/2} \ge (3C_0/4)\varepsilon^{-1}, \end{aligned}$$

the latter if $\varepsilon > 0$ is chosen small enough. Since $3C_0/4 > C_*$, this leads to a contradiction to the definition of $\hat t$ in case that $\hat t < \min\{\tau_{\rm C} - \delta_0, T_0\}\varepsilon^{-3/2}$, whence we must have $\hat t = \min\{\tau_{\rm C} - \delta_0, T_0\}\varepsilon^{-3/2}$ as was to be shown. □

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