Slow Motion of Charges Interacting with the Maxwell Field

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Abstract. We study the Abraham model for N charges interacting with the Maxwell field. On the scale of the charge diameter, R_φ, the charges are a distance $\varepsilon^{-1}R_\varphi$ apart and have a velocity $\sqrt{\varepsilon}c$ with ε a small dimensionless parameter. We follow the motion of the charges over times of the order $\varepsilon^{-3/2}R_\varphi/c$ and prove that on this time scale their motion is well approximated by the Darwin Lagrangian. The mass is renormalized. The interaction is dominated by the instantaneous Coulomb forces, which are of the order ε^2. The magnetic fields and first order retardation generate the Darwin correction of the order ε^3. Radiation damping would be of the order $\varepsilon^{7/2}$.

1. The Abraham model

In electrodynamics it is customary to consider either the dynamics of charges in prescribed electromagnetic fields or the dynamics of fields at prescribed motion of charges. For most applications such a separation suffices. But more fundamentally one would like to have a coupled dynamical system jointly for the charges and the Maxwell field. Such a model was proposed early 1900 by M. Abraham. To avoid short-distance singularities, he assumes that the charge is spread out over a distance R_φ, which physically is of the order of the classical electron radius. Thus charge α has a charge distribution ρ_α which for simplicity we take to be

$$\rho_\alpha(x) = e_\alpha\varphi(x), \quad x \in \mathbb{R}^3,$$

where the common form factor φ is smooth, radial, normalized, and of compact support,

$$0 \le \varphi \in C_0^\infty(\mathbb{R}^3), \quad \varphi(x) = \varphi_r(|x|), \quad \int d^3x\varphi(x) = 1, \quad \varphi(x) = 0 \text{ for } |x| \ge R_\varphi.$$

The position of charge α at time t, $t \in \mathbb{R}$, is denoted by $q_\alpha(t)$ and its velocity by $v_\alpha(t)$, $\alpha = 1, \dots, N$. The charges determine then the charge distribution ρ and the current j as

$$\rho(x,t) = \sum_{\alpha=1}^N \rho_\alpha(x - q_\alpha(t)) \quad \text{and} \quad j(x,t) = \sum_{\alpha=1}^N \rho_\alpha(x - q_\alpha(t))v_\alpha(t), \tag{1.1}$$

which satisfy charge conservation by fiat. The Maxwell field, consisting of the electric field E and the magnetic field B, evolves according to

$$c^{-1}\frac{\partial}{\partial t}B(x,t) = -\nabla \wedge E(x,t), \quad c^{-1}\frac{\partial}{\partial t}E(x,t) = \nabla \wedge B(x,t) - c^{-1}j(x,t) \tag{1.2}$$

with the constraints

$$\nabla \cdot E(x,t) = \rho(x,t), \quad \nabla \cdot B(x,t) = 0, \tag{1.3}$$

c denoting the velocity of light. The charges generate the electromagnetic field which in turn determines the forces on the charges through the Lorentz force equation

$$\frac{d}{dt}\Big(m_{\mathrm{b}\alpha}\gamma_\alpha v_\alpha(t)\Big) = \int d^3x\, \rho_\alpha(x - q_\alpha(t))\Big[E(x,t) + c^{-1}v_\alpha(t) \wedge B(x,t)\Big] \tag{1.4}$$

for $\alpha = 1, \ldots, N$. Here $m_{\mathrm{b}\alpha}$ is the bare mass of charge α and γ_α the relativistic factor $\gamma_\alpha = (1 - v_\alpha^2/c^2)^{-1/2}$, which ensures $|v_\alpha| < c$. Note that there are no direct forces acting between the particles. Eqns. (1.1)–(1.4) are known as Abraham model for N (nonrotating) charges.

We define the energy function by

$$\mathcal{H}(E, B, \vec{q}, \vec{v}) = \sum_{\alpha=1}^{N} m_{\mathrm{b}\alpha}\gamma_\alpha + \frac{1}{2}\int d^3x\, [E^2(x) + B^2(x)], \tag{1.5}$$

with $\vec{q} = (q_1, \ldots, q_N)$ and $\vec{v} = (v_1, \ldots, v_N)$. It then may be seen that the initial value problem corresponding to (1.1)–(1.4) has a unique weak solution of finite energy and that $\mathcal{H}$ is conserved by this solution, compare with [**1**] for the case of a single particle.

The Abraham model is semi-relativistic, since the charge distribution is rigid in a prescribed Lorentz frame and remains Lorentz uncontracted independent of its velocity. To have a relativistic model with extended charges requires more effort. There has been many attempts over the years, but a physically satisfactory theory was first developed by Nodvik [**2**]. In contrast to the charge his mass distribution is concentrated in the center, which is physically somewhat artificial and makes the theory singular from a dynamical point of view. For a general mass and charge distribution Appel and Kiessling [**3**] formulate the relativistic model through a covariant action and study some of its properties, in particular the zero bare mass limit. Despite this progress our understanding of the relativistic dynamics is rudimentary compared to the results on the Abraham model.

For a single charge we have studied a variety of rather obvious dynamical questions, in particular the long time behavior, including scattering theory and the convergence to travelling wave solutions [**4, 1**], and the effective dynamics under slowly varying external potentials (adiabatic limit), covering the approximation of the true motion by the center manifold dynamics of the Lorentz-Dirac equation [**5, 6, 7, 8**]. For many particles results are sparse. The goal of the present contribution is to understand the effective dynamics in the limit of slow motion.

2. Slow motion

As one learns in every course on electromagnetism slowly moving charges interact statically through the Coulomb force. Thus, given N charges with positions r_α, velocities u_α, charges e_α, and masses m_α, $\alpha = 1, \ldots, N$, their motion is governed by the Lagrangian

$$\mathcal{L}_{\mathrm{C}} = \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha u_\alpha^2 - \frac{1}{2} \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{N} \frac{e_\alpha e_\beta}{4\pi|r_\alpha - r_\beta|}. \tag{2.1}$$

The first term is the kinetic energy and the second term is the Coulomb potential. On purpose we use different symbols for position and velocity in order to distinguish from the 'true' position and velocity as governed by the Abraham model. Our task is to derive (2.1) from the Abraham model.

The textbook argument interprets slow motion as the velocity of light $c \to \infty$ in (1.1)–(1.4). At first sight this prescription seems to give the correct result, but in fact it fails in a subtle way. Firstly, instead of the Coulomb potential in (2.1) one obtains the Coulomb potential smeared by φ. But more importantly one concludes $m_\alpha = m_{\mathrm{b}\alpha}$ which physically cannot be correct, since even if the charge is moving slowly, its mass is renormalized and m_α differs from $m_{\mathrm{b}\alpha}$ by the contribution from the self-field. Thus we have to discuss more precisely what means "slow motion" in terms of *dimensionless* quantities.

We assume that initially the particles are very far apart on the scale set by R_φ. Thus we require, for $\alpha \neq \beta$, that

$$|q_\alpha(0) - q_\beta(0)| \cong \varepsilon^{-1} R_\varphi \tag{2.2}$$

with $\varepsilon > 0$ small. If particles would come together as close as R_φ, our equations of motion are not trustworthy anyhow. In addition, we require that the initial velocities be small compared to the speed of light,

$$|v_\alpha(0)| \cong \sqrt{\varepsilon} c. \tag{2.3}$$

The initial electromagnetic field is chosen, in essence, such as to minimize the energy function $\mathcal{H}$ from (1.5) subject to the constraints (2.2),(2.3). With these initial conditions, it takes a time of order $\varepsilon^{-1} R_\varphi / c$ for a signal to travel between the particles. This means that on any longer time scale, retardation effects are small. If particles interact through Coulomb forces, as will have to be proved, the strength of the forces is of order ε^2 since the distance is of order $\varepsilon^{-1} R_\varphi$. Followed over a time span $\varepsilon^{-3/2} R_\varphi / c$, this yields a change in position of order $\varepsilon^{-1} R_\varphi$ and in velocity of order $\sqrt{\varepsilon} c$. On this basis we expect the orders of magnitude (2.2) and (2.3) to remain valid over times of order $\varepsilon^{-3/2} R_\varphi / c$. Thus physically we consider slow particles that are far apart, and we want to follow their motion over long times.

Granted the validity of the Coulomb dynamics, more ambitiously we may ask for corrections. For example, in a strongly ionized plasma under standard conditions the velocities cannot be considered as small compared to the velocity of light, cf. [**9**, §65], and one adds to (2.1) the Darwin term with Lagrangian

$$\begin{aligned} \mathcal{L}_{\mathrm{D}} = \mathcal{L}_{\mathrm{C}} &+ \sum_{\alpha=1}^{N} \frac{1}{8c^2} m_\alpha^* u_\alpha^4 \\ &+ \frac{1}{4c^2} \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{N} \frac{e_\alpha e_\beta}{4\pi |r_\alpha - r_\beta|} \Big(u_\alpha \cdot u_\beta + |r_\alpha - r_\beta|^{-2} (u_\alpha \cdot [r_\alpha - r_\beta])(u_\beta \cdot [r_\alpha - r_\beta]) \Big), \end{aligned} \tag{2.4}$$

where $m_\alpha^* u_\alpha^4$ is a correction to the kinetic energy, m_α^* depending on the precise model ($m_\alpha^* = m_\alpha$ for a relativistic particle), and the Coulomb potential is improved to the velocity dependent Darwin potential.

In theoretical physics it is common practice to count the post-Coulombian corrections in orders of v/c relative to the motion through pure Coulomb forces. Thus the Darwin term is the first correction and of order $(v/c)^2$. The next correction is of order $(v/c)^3$ and accounts for damping through radiation, which reads

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}_\mathrm{D}}{\partial u_\alpha}\right) = \frac{\partial \mathcal{L}_\mathrm{D}}{\partial r_\alpha} + (e_\alpha/\, 6\pi c^3) \sum_{\beta=1}^{N} e_\beta \ddot{v}_\beta, \tag{2.5}$$

$\alpha = 1, \dots, N$. (2.5) has a repulsive center manifold. Any initial condition away from it rapidly converges to c. Such runaway solutions are an 'artifact' of the expansion in ε and are not seen by the Abraham model whose solution is well approximated by the solution of (2.5) on its center manifold. To obtain the effective second order equation, at the present level of precision it suffices to substitute the Hamiltonian dynamics to lowest order which yields

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial \mathcal{L}_\mathrm{D}}{\partial u_\alpha}\right) &= \frac{\partial \mathcal{L}_\mathrm{D}}{\partial r_\alpha} + \frac{e_\alpha}{6\pi c^3}\frac{1}{2}\sum_{\substack{\beta,\beta'=1\\ \beta\neq\beta'}}^{N} \left(\frac{e_\beta}{m_\beta} - \frac{e_{\beta'}}{m_{\beta'}}\right)\frac{e_\beta e_{\beta'}}{4\pi|r_\beta - r_{\beta'}|^3} \\ &\Big((u_\beta - u_{\beta'}) - 3\frac{(r_\beta - r_{\beta'})\cdot(u_\beta - u_{\beta'})}{|r_\beta - r_{\beta'}|^2}(r_\beta - r_{\beta'})\Big).\end{aligned} \tag{2.6}$$

Note that if the ratio e_α/m_α does not depend on α, then the radiation reaction vanishes and the system does not emit dipole radiation. The next order correction is $(v/c)^4$ and of Lagrangian form. It is discussed in [**9, 10**].

In general relativity, there is a huge effort to obtain corrections to the Newtonian orbits, which as a problem is similar to the one discussed here. The most famous example is the Hulse-Taylor binary pulsar, where two highly compact neutron stars of roughly solar mass revolve around each other with a period of 7.8 h [**11**]. In this case $(v/c) = 10^{-3}$. For gravitational systems there is only quadrupole radiation which is of order $(v/c)^5$. To this order the theory agrees with the observed radio signals within 0,3%. In newly designed experiments one expects highly improved precision which will require corrections up to order $(v/c)^{11}$.

3. Comparison dynamics

We set

$$m_\alpha = m_{\mathrm{b}\alpha} + \frac{4}{3}e_\alpha^2 m_e \quad \text{and} \quad m_\alpha^* = m_{\mathrm{b}\alpha} + \frac{16}{15}e_\alpha^2 m_e$$

with the electromagnetic mass $m_e = \frac{1}{2}\int d^3k\, |\hat{\varphi}(k)|^2 k^{-2}$ and the Darwin Lagrangian

$$\begin{aligned}\mathcal{L}_\mathrm{D}(\vec{r},\vec{u}) &= \sum_{\alpha=1}^{N}\Big(\frac{1}{2}m_\alpha u_\alpha^2 + \frac{\varepsilon}{8}m_\alpha^* u_\alpha^4\Big) - \frac{1}{2}\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{N}\frac{e_\alpha e_\beta}{4\pi|r_\alpha - r_\beta|} \\ &+ \frac{\varepsilon}{4}\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{N}\frac{e_\alpha e_\beta}{4\pi|r_\alpha - r_\beta|}\Big(u_\alpha\cdot u_\beta + |r_\alpha - r_\beta|^{-2}(u_\alpha\cdot[r_\alpha - r_\beta])(u_\beta\cdot[r_\alpha - r_\beta])\Big)\end{aligned}$$

for $\vec{r} = (r_1, \dots, r_N)$ and $\vec{u} = (u_1, \dots, u_N)$. The comparison dynamics is then

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}_\mathrm{D}}{\partial u_\alpha}\right) = \frac{\partial \mathcal{L}_\mathrm{D}}{\partial r_\alpha}, \quad \alpha = 1, \dots, N. \tag{3.1}$$

It conserves the energy

$$\mathcal{H}_{\mathrm{D}}(\vec{r},\vec{u}) = \sum_{\alpha=1}^{N}\Big(\frac{1}{2}m_\alpha u_\alpha^2 + \varepsilon\frac{3}{8}m_\alpha^* u_\alpha^4\Big) + \frac{1}{2}\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{N}\frac{e_\alpha e_\beta}{4\pi|r_\alpha - r_\beta|}\,. \tag{3.2}$$

Because of the Coulomb singularity, in general the solutions to (3.1) will exist only locally in time, the only exception being when all charges have the same sign, in which case energy conservation yields global existence. In the corresponding gravitational problem, for a set of positive phase space measure, mass can be transported to infinity in a finite time [**12**]. We do not know whether this can happen also for the Coulomb problem. We set

$$q_\alpha^0 = \varepsilon^{-1} r_\alpha^0 \quad \text{and} \quad v_\alpha^0 = \sqrt{\varepsilon}u_\alpha^0, \quad \alpha = 1,\dots,N, \tag{3.3}$$

with $r_\alpha^0 \neq r_\beta^0$ for $\alpha \neq \beta$. Then (2.2) and (2.3) are satisfied. During the initial time slip of order ε^{-1} the fields build up the forces between particles and adjust to their motion. Thus during that period the dynamics of the particles is not well approximated by the Darwin Lagrangian and we correct the initial data of the comparison dynamics to the true positions and velocities only at the end of the initial time slip. To take into account that the comparison dynamics will have no global solutions in time, in general, we define $\tau \in]0,\infty]$ to be the first time when either $\lim_{t\to\tau^-}|r_\alpha(t) - r_\beta(t)| = 0$ for some $\alpha\neq\beta$ or $\lim_{t\to\tau^-}|r_\alpha(t)| = \infty$ for some α holds for the comparison dynamics (3.1).

As our main approximation result we state

THEOREM 3.1. *Let $T > 0$ be fixed. Define $\tau \in]0,\infty]$ as above and fix some $\delta_0 \in]0,\tau[$. For the Abraham model let the initial data be given by (3.3) and the initial fields such that the corresponding field energy is minimal. Furthermore we require $|e_\alpha| \le \bar{e}$, with $\bar{e}$ depending only on the initial data. Let $t_0 = 4(R_\varphi + C^*\varepsilon^{-1})$. We adjust the initial data of the comparison dynamics such that $q_\alpha(t_0) = \varepsilon^{-1}r_\alpha(\varepsilon^{3/2}t_0)$ and $v_\alpha(t_0) = \sqrt{\varepsilon}u_\alpha(\varepsilon^{3/2}t_0)$, $\alpha = 1,\dots,N$.*

Then there exists a constant $C > 0$ such that for all $t \in [t_0, \min\{\tau-\delta_0, T\}\,\varepsilon^{-3/2}]$ we have

$$|q_\alpha(t) - \varepsilon^{-1}r_\alpha(\varepsilon^{3/2}t)| \le C\sqrt{\varepsilon}, \quad |v_\alpha(t) - \sqrt{\varepsilon}u_\alpha(\varepsilon^{3/2}t)| \le C\varepsilon^2, \quad \alpha = 1,\dots,N. \tag{3.4}$$

The complete proof has appeared in [**13**] and we only add some general comments.

(i) If one is satisfied with the precision from the pure Coulomb dynamics, then in (3.4) we lose one power in ε. In this case, we can adjust the initial data of the comparison dynamics at time $t = 0$, and (3.4) holds for all $t \in [0, \min\{\tau-\delta_0, T\}\,\varepsilon^{-3/2}]$.

(ii) The major step in our proof is to obtain bounds on the positions $q_\alpha(t)$, including their first, second, and third time derivatives, which are of the right order in ε. To achieve this we use a contraction argument on the solution of Maxwell equations with the approximate charge soliton solution subtracted. Unfortunately, at this point the restriction to a sufficiently small $\bar{e}$ enters. We believe that it is an artifact of our proof and it would be of interest to find an alternative which avoids this assumption.

(iii) We are currently working to improve the approximation such that radiation damping is included. For a single particle in an external potential this problem is treated in [**7**]. With several particles the same scheme works. However one has to also control the retarded interaction between particles at the precision level of radiation damping which requires some new techniques. Even for a single particle the approximation theorem is less perfect than stated in Theorem 3.1 and the many particle case will share that fate. Because the comparison dynamics is unstable, the unevitable errors are magnified. Only the difference in the energy of the true solution and the energy of the comparison solution can shown to be of the right order in ε. The closeness of trajectories is not guaranteed beyond the estimates of Theorem 3.1. What really happens remains unexplored. The physical hope is that the closeness of trajectories as order ε over times of order $\varepsilon^{-3/2}$ can be achieved through properly adjusting the initial data of the comparison dynamics.

(iv) The starting point of reference [**10**] is already an approximate Lagrangian and not the Abraham model. Even on a formal level, preferably in the context of the relatistivic model of [**3**], it would be of interest to understand whether to this order the approximate Lagrangian and the full model yield the same answer.

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