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On the Rigouros Applicability of Oseledets' Ergodic Theorem to Obtain Lyapunov Exponents for Non-Smooth Dynamical Systems

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Abstract

We investigate how Oseledets' theorem can be applied to define Lyapunov exponents also for dynamical systems with discontinuous right-hand sides.

Key words: Lyapunov exponents, Oseledets' theorem, non-smooth dynamical systems, oscillator with friction

1 Introduction

In the theory of smooth dynamical systems, Lyapunov exponents are thought of to be a well-established tool for the qualitative analysis of the system, since they describe the exponential convergence or divergence of nearby trajectories. To explain this, consider $\dot{x}=f(x)$ with $f\in C^1(\mathbb{R}^d,\mathbb{R}^d)$ and corresponding semiflow $\varphi:[0,\infty[\times\mathbb{R}^d\to\mathbb{R}^d$, i.e. $x(\cdot)=\varphi(\cdot,x_0)$ solves the differential equation and has $x(0)=x_0$. Let x and y be two fixed solutions with $x(0)=x_0$ and $y(0)=y_0$. To study how small changes $|y_0-x_0|$ of the initial values affect the difference of the solutions at a later time $t>0$, we write

$$y(t)-x(t)=\varphi(t,y_0)-\varphi(t,x_0)\approx\partial_x\varphi(t,x_0)\cdot(y_0-x_0)=:\partial_x\varphi(t,x_0)\cdot v_0=:v(t)\,.$$

Hence the quantity

$$\chi(x_0,v_0):=\limsup_{t\to\infty}\frac{1}{t}\ln\frac{|\partial_x\varphi(t,x_0)\cdot v_0|}{|v_0|}=\limsup_{t\to\infty}\frac{1}{t}\ln\frac{|v(t)|}{|v_0|}\tag{1}$$

somehow describes the evolution of small pertubation of the initial values, and is called a Lyapunov exponent.

In order to relate $\chi(x_0,v_0)$ really to the long-time bevaviour or the stability of the system under consideration, of course the $\limsup_{t\to\infty}$ in (1) has to exist as a $\lim_{t\to\infty}$. Therefore one tries to obtain information in which cases this $\limsup_{t\to\infty}$ in fact may be replaced by a $\lim_{t\to\infty}$, and for, a powerful tool is provided by Oseledets' multiplicative ergodic theorem Oseledets (1968), which we state here in a version for semiflows from Ruelle (1979, Thm. B3), applied to $M=\mathbb{R}^d$. We remark that the added ergodic part follows just like in the discrete case, cf. e.g. Ledrappier (1984, Thm. 3.1).

The following notations are used in this theorem: $(\varphi^t)_{t\geq 0}:\mathbb{R}^d\to\mathbb{R}^d$ is called a measurable semiflow iff $(t,x)\mapsto\varphi^t(x)=\varphi(t,x)$ is (Borel-) measurable, $\varphi^0=\mathrm{id}_{\mathbb{R}^d}$ and $\varphi^{t+s}=\varphi^t\circ\varphi^s$ for all $t,s\in[0,\infty[$. Moreover, $(\varphi^t)_{t\geq 0}$ leaves invariant some measure μ iff $\varphi^t(\mu)=\mu$ for all $t\in[0,\infty[$. Finally $T:[0,\infty[\times\mathbb{R}^d\to L(\mathbb{R}^d)$ (where $L(\mathbb{R}^d)$ are the real $d\times d$-matrices) is called a cocycle for $(\varphi^t)_{t\geq 0}$, iff T is measurable and $T(t+s,x)=T(t,\varphi^s(x))T(s,x)$ for $t,s\geq 0$ and $x\in\mathbb{R}^d$.

Theorem *Let μ be a probability measure on $(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d))$ and suppose that $(\varphi^t)_{t\geq 0}:\mathbb{R}^d\to\mathbb{R}^d$ is a measurable semiflow which leaves μ invariant. Let $T:[0,\infty[\times\mathbb{R}^d\to L(\mathbb{R}^d)$ be a cocycle for $(\varphi^t)_{t\geq 0}$. Then, if the integrability conditions*

$$\sup_{\theta\in[0,1]}\ln^+|T(\theta,\cdot)|\in L^1(\mathbb{R}^d,\mu)\quad\textit{and}\quad\sup_{\theta\in[0,1]}\ln^+|T(1-\theta,\varphi^\theta(\cdot))|\in L^1(\mathbb{R}^d,\mu)\tag{2}$$

with $\ln^+(x)=\max\{\ln(x),0\}$ are satisfied, there exists $\Gamma\subset\mathbb{R}^d$ with $\mu(\Gamma)=1$ such that $\varphi^t(\Gamma)\subset\Gamma$ for all $t\geq 0$, and the following properties hold if $x\in\Gamma$:

(a) $\lim_{t\to\infty}\Big(T(t,x)^*T(t,x)\Big)^{1/2t}=\Lambda_x$ *exists.*

(b) Let $\exp\lambda_x^{(1)}<\ldots<\exp\lambda_x^{(k)}$ be the eigenvalues of Λ_x (where $k=k(x)$, the $\lambda_x^{(i)}$ are real and $\lambda_x^{(1)}$ may be $-\infty$), and $U_x^{(1)},\ldots,U_x^{(k)}$ the corresponding eigenspaces. Let $m_x^{(i)}=\dim U_x^{(i)}$. Then the functions $x\mapsto\lambda_x^{(i)},m_x^{(i)}$ are (φ^t)-invariant. With $V_x^{(0)}=\{0\}$ and $V_x^{(i)}=U_x^{(1)}+\ldots+U_x^{(i)}$ we have for $i=1,\ldots,k$

$$\lim_{t\to\infty}\frac{1}{t}\ln|T(t,x)v|=\lambda_x^{(i)}\quad\textit{when}\quad v\in V_x^{(i)}\setminus V_x^{(i-1)}.\tag{3}$$

In addition, if μ is ergodic, then $x\mapsto k(x)$ and $x\mapsto\lambda_x^{(i)}$ are constant on Γ.

Now note that in the case of a smooth dynamical system, $T(t,x)=\partial_x\varphi^t(x)$ defines a cocycle for $(\varphi^t)_{t\geq 0}$, since by the chain rule

$$T(t+s,x)=\partial_x\varphi^{t+s}(x)=\partial_x(\varphi^t(\varphi^s(x)))=\partial_x\varphi^t(\varphi^s(x))\,\partial_x\varphi^s(x)=T(t,\varphi^s(x))T(s,x).$$

Thus one can apply Oseledets' theorem in particular to obtain a $\lim_{t\to\infty}$ in (1) instead of a $\limsup_{t\to\infty}$ for all $v_0\in\mathbb{R}^d$ and for all $x_0\in\mathbb{R}^d$ in a set of μ-measure equal to one, at least if one assumes (or proves) the existence of a corresponding measure μ. Of course the applicability of Oseledets' theorem mainly relies on the fact that φ is C^1 w.r.t. $x\in\mathbb{R}^d$, and hence on $f\in C^1(\mathbb{R}^d,\mathbb{R}^d)$.

In this paper, we intend to show how Oseledets' theorem may be used to obtain Lyapunov exponents also for some dynamical systems with non-smooth right-hand sides. As an example, we consider the equation of a friction-oscillator

$$\ddot{x}(t)+x(t)+\alpha\Big[c(v_0)+\mathrm{sgn}(\dot{x}(t)-1)\,c(v_0(\dot{x}(t)-1))\Big]=w(t)\tag{4}$$

which, already suitably normalized, describes the slipping or sticking of a periodically forced mass on a belt turning around with constant velocity $v=1$, cf. Popp/Stelter (1990:a), (1990:b), Stelter (1990), and Figure 1 (to be found at the end of this paper). Here $w(t)=\gamma\cos(\eta t)$ is the excitation, $\gamma\geq 0$ and $\eta,\alpha,v_0>0$ are parameters, and the friction characteristic $c:\mathrm{I\!R}\to\mathrm{I\!R}$ is choosen to be

$$c(v)=\frac{\mu_0-\mu_1}{1+\lambda_0|v|}+\mu_1 \tag{5}$$

with $\lambda_0\geq 0$ and $\mu_0>\mu_1>0$, i.e. the static friction coefficient exceeds the kinetic friction coefficient. We remark that (4) always will be considered as a three-dimensional autonomous system by letting $(x_1,x_2,x_3)=(x,\dot{x},t)$. Since by (4) it is only clear which right-hand-sides to choose in $\mathcal{D}^-=\{x\in\mathrm{I\!R}^3:x_2<1\}$ or in $\mathcal{D}^+=\{x\in\mathrm{I\!R}^3:x_2>1\}$, (4) has to be made meaningful also for $\dot{x}(t)=1$. For that, the equation has to be replaced by the corresponding differential inclusion

$$\ddot{x}(t)+x(t)\in-\alpha\Big[c(v_0)+\mathrm{Sgn}(v_0(\dot{x}(t)-1))\,c(v_0(\dot{x}(t)-1))\Big]+w(t) \tag{6}$$

with $\mathrm{Sgn}(v)=v/|v|$ for $v\neq 0$ and $\mathrm{Sgn}(0)=[-1,1]$. To determine the behaviour of the system at the discontinuity surface $\mathcal{D}=\{x\in\mathrm{I\!R}^3:x_2=1\}$, suppose that $\dot{x}(t)=1$ for $t\in]t_0,t_1[$. Then (6) implies $b_+(t)\leq x(t)\leq b_-(t)$ for $t\in]t_0,t_1[$ with

$$b_\pm(x_3)=-\alpha[c(v_0)\pm c(0)]+w(x_3). \tag{7}$$

Therefore, a trajectory which arrives at $\mathcal{D}$ outside of the sliding-motion-region $\mathcal{S}=\{x\in\mathrm{I\!R}^3: x_2=1,b_+(x_3)\leq x_1\leq b_-(x_3)\}$ cannot rest in $\mathcal{D}$ for a while, but has to pass from $\mathcal{D}^-$ to $\mathcal{D}^+$ or vice versa. On the other hand, by calculating the changes of a solution in direction of the normal $(0,1,0)$ to $\mathcal{D}$ (cf. Filippov (1988, p.55/56)), it may be seen that a solution arriving at $\mathcal{D}$ in $\mathcal{S}_0=\{x\in\mathrm{I\!R}^3:x_2=1,b_+(x_3)<x_1<b_-(x_3)\}$ cannot leave the surface until it reaches the right boundary of $\mathcal{S}_0$ described by $x_1=b_-(x_3)$, since in that case the vector-field is directed towards $\mathcal{S}$ from both sides. A qualitative sketch of the vector-field is given in Figure 2.

To sum up, the appropriate choice of the right-hand side corresponding to the autonomous version of (4) is

$$f_\pm(x)=\begin{pmatrix} x_2 \\ -x_1-\alpha\Big[c(v_0)\pm c(v_0(x_2-1))\Big]+w(x_3) \\ 1 \end{pmatrix} \quad\text{and}\quad f_0(x)=\begin{pmatrix}1\\0\\1\end{pmatrix} \tag{8}$$

in

$$M_+=\mathcal{D}^+\cup\{x\in\mathrm{I\!R}^3:x_2=1,x_1<b_+(x_3)\},\ M_0=\{x\in\mathrm{I\!R}^3:x_2=1,b_+(x_3)\leq x_1<b_-(x_3)\},$$
$$\text{and}\quad M_-=\mathcal{D}^-\cup\{x\in\mathrm{I\!R}^3:x_2=1,\,b_-(x_3)\leq x_1\}, \tag{9}$$

respectively. Note that we choose f_+ on $\{x\in\mathrm{I\!R}^3:x_2=1,x_1<b_+(x_3)\}$ and f_- on $\{x\in\mathrm{I\!R}^3: x_2=1,\,b_-(x_3)\leq x_1\}$, and that in the above argueing we also implicitely made the assumption that a solution cannot hit $\mathcal{D}$ infinitely often in a finite time, since for both of these restrictions there is at least some numerical evidence, cf. e.g. Kunze/Küpper (1997).

Because we have to deal with different right-hand sides now, it is therefore no longer possible to apply Oseledets' theorem directly. To construct a canonical cocycle also for such non-smooth systems, we will proceed as follows: For every $x\in\mathrm{I\!R}^3$ there will be a sequence $(t_i(x))_{i\in\mathrm{I\!N}_0}$ such that $0=t_0(x)<t_1(x)<t_2(x)\ldots$, and such that $t_i(x)$ is the i's switching time

from one right-hand side of the differential equation to the next for a trajectory starting in x. Hence in $]t_i(x), t_{i+1}(x)[$ the corresponding semiflow φ should be differentiable w.r.t. x like in the smooth case, but there might be a jump of $\partial_x \varphi^t(x)$ at the next switching time $t_{i+1}(x)$. Therefore one has to derive a transformation $L_{i+1}(x)$, which decribes the transition from the derivative from the left to the one from the right at $t_{i+1}(x)$, i.e. with

$$\lim_{t \to t_{i+1}(x)^+} \partial_x \varphi^t(x) = L_{i+1}(x) \left(\lim_{t \to t_{i+1}(x)^-} \partial_x \varphi^t(x) \right).$$

By means of the $L_{i+1}(x)$ then a cocycle can be constructed.

To carry out this program rigorously, we will start by defining a semiflow φ from some general assumptions in Section 2. Note that e.g. in (4) above solutions are not unique to the left, since they may stay in the discontinuity surface. Therefore φ will only be a semiflow in general. In Section 2 we shall also introduce a suitable forward invariant $G \subset \mathbb{R}^3$ on which $\partial_x \varphi^t(x)$ may be expected to exist. To see that some points have to be excluded, consider Figures 3a and 3b below. One problem might arise due to the fact that a trajectory reaches the discontinuity surface tangentially at some time t, since the time for passing through the loop is not needed any longer, as it was the case at time $t - \varepsilon$. A second nuisance is generated by trajectories which hit $\{x \in \mathbb{R}^3 : x_2 = 1, x_1 = b_+(x_3)\}$, cf. Figure 3b, because in an arbitrary small neighbourhood of the corresponding initial value x_0 there can be points which are mapped to M_0, but also points which are mapped to M_+ by the semiflow, and the last ones will first have to pass some time through M_+ until the next change of right-hand side. Thus both described phenomena are likely to result in a discontinuity of some $t_i(\cdot)$, and since the $t_i(x)$ clearly have to enter the definition of φ, in particular $\partial_x \varphi^t(x)$ may not exists for such points.

After having derived the appropriate transition matrices $L_{i+1}(x)$ in Section 3, we will give a definition of the cocycle in Section 4 and show how Lyapunov exponents can be obtained by means of this cocycle for the example of the friction-oscillator (4) considered as a motivation above.

We finally remark that an extensive study of ergodic properties of billiard systems (which are also non-smooth) can be found in Katok/Strelcyn (1986), cf. also the related Lamba (1993) for the case of an impacting oscillator (without friction).

2 Preliminaries

We start by describing non-smooth systems in a formal way. To do this, the following assumptions will be made furtheron:

We are given a partition $\mathbb{R}^d = \bigcup_{k=1}^n M_k$ of $\mathbb{R}^d$ in mutually disjoint and connected C^1-manifolds M_k. In addition, we are also given functions $f_1, \ldots, f_n \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ and $h_1, \ldots, h_n \in C^1(\mathbb{R}^d, \mathbb{R})$, and for every $x \in \mathbb{R}^d$ a sequence $(t_i(x))_{i \in \mathbb{N}_0}$ without accumulation point such that $0 = t_0(x) < t_1(x) < t_2(x) < t_3(x) \ldots$.

Let

$$f(\cdot, x_0) = f_k \quad \text{and} \quad h(\cdot, x_0) = h_k \quad \text{iff} \quad x_0 \in M_k \quad \text{for} \quad k \in I = \{1, \ldots, n\}. \tag{10}$$

It will be helpful to keep the following picture in mind: $f(\cdot, x_0) = f_k$ should be thought of as the right-hand side of $\dot{x} = f_k(x)$ if one starts in $x_0 \in M_k$, and this right-hand side of the differential equation remains unchanged as long as the trajectory does not leave M_k. The $t_i(x_0)$ are the consecutive switching-times from one right-hand side to the next, and $h(\cdot,x_0)$ stands for the next condition of type $h_k=0$ to be satisfied by a trajectory starting in $x_0 \in M_k$. In case of the friction-oscillator, we have

$$h_\pm(x)=x_2-1 \quad \text{and} \quad h_0(x)=x_1-b_-(x_3) \quad \text{for} \quad x\in\mathbb{R}^3, \tag{11}$$

where we used indices $\pm$ and 0 instead of $k=1,2,3$.

To define a semiflow by means of the f_k, we will need the following notation. If $g\in C^1(\mathbb{R}^d,\mathbb{R}^d)$, then Φ_g will denote the (semi-) flow induced by g, i.e. $x=\Phi_g(\cdot,x_0)$ uniquely solves $\dot{x}=g(x)$, $x(0)=x_0$.

Lemma 1 *Define* $\varphi:[0,\infty[\times\mathbb{R}^d\to\mathbb{R}^d$ *through* $\varphi^t(x)=\Phi_{f(\cdot,x)}(t,x)$ *for* $t\in[0,t_1(x)]$ *and*

$$\varphi^t(x)=\Phi_{f(\cdot,\varphi^{t_i(x)}(x))}\Big(t-t_i(x),\varphi^{t_i(x)}(x)\Big) \quad \textit{for} \quad i\in\mathbb{N}, \quad t\in]t_i(x),t_{i+1}(x)].$$

Then $(\varphi^t)_{t\geq 0}:\mathbb{R}^d\to\mathbb{R}^d$ *is a semiflow, if the follwing two consistency conditions are satisfied:*

(A1) $i\in\mathbb{N}_0,\, x\in\mathbb{R}^d,\, t\in[t_i(x),t_{i+1}(x)[\quad\Longrightarrow\quad t_j(\varphi^t(x))=t_{i+j}(x)-t$ *for all* $j\in\mathbb{N}$.
and

(A2) For all $i\in\mathbb{N}_0$ *and* $x\in\mathbb{R}^d$ *there exists* $k\in I$ *such that* $\varphi([t_i(x),t_{i+1}(x)[,x)\subset M_k$. *Moreover,* $\varphi(\cdot,x)$ *is continuous on* $[0,\infty)$ *for* $x\in\mathbb{R}^d$.

Proof: For $x\in M_k$ we have $\varphi^0(x)=\Phi_{f_k}(0,x)=x$ and hence $\varphi^0=\mathrm{id}_{\mathbb{R}^d}$. To obtain the semigroup-property, we will show that $\varphi^{t+s}(x)=\varphi^t(\varphi^s(x))$ for all $x\in\mathbb{R}^d$, $s\in[t_i(x),t_{i+1}(x)[$ and $t+s\in]t_{i+j}(x),t_{i+j+1}(x)]$ with $i\in\mathbb{N}_0$ and $j\in\mathbb{N}_0$. For that, consider first the case $j=0$. Then $f(\cdot,\varphi^{t_i(x)}(x))=f_k$ for some $k\in I$, and consequently by $(A2)$ also $f(\cdot,\varphi^s(x))=f_k$. Moreover, with $y=\varphi^s(x)$ we have $t\in[0,t_1(y)]$ since $t_1(y)=t_{i+1}(x)-s$ by $(A1)$. Thus the fact that Φ_{f_k} is a flow implies

$$\begin{aligned}\varphi^{t+s}(x) &= \Phi_{f_k}\Big(t+s-t_i(x),\varphi^{t_i(x)}(x)\Big)=\Phi_{f_k}\Big(t,\Phi_{f_k}(s-t_i(x),\varphi^{t_i(x)}(x))\Big)=\Phi_{f_k}(t,\varphi^s(x))\\ &= \Phi_{f_k}(t,y)=\varphi^t(y)=\varphi^t(\varphi^s(x)).\end{aligned}$$

For the induction step fix $s\in[t_i(x),t_{i+1}(x)[$ and $t+s\in]t_{i+j+1}(x),t_{i+j+2}(x)]$. Then the assumption may be applied to $\tilde{t}=t_{i+j+1}(x)-s$ and $\tilde{s}=s$ to give by $(A1)$, again with $y=\varphi^s(x)$,

$$\varphi^{t_{j+1}(y)}(y)=\varphi^{t_{i+j+1}(x)-s}(\varphi^s(x))=\varphi^{\tilde{t}}(\varphi^{\tilde{s}}(x))=\varphi^{\tilde{t}+\tilde{s}}(x)=\varphi^{t_{i+j+1}(x)}(x),$$

and hence $t\in]t_{i+j+1}(x)-s,t_{i+j+2}(x)-s]=]t_{j+1}(y),t_{j+2}(y)]$ implies

$$\begin{aligned}\varphi^{t+s}(x) &= \Phi_{f(\cdot,\varphi^{t_{i+j+1}(x)}(x))}\Big(t+s-t_{i+j+1}(x),\varphi^{t_{i+j+1}(x)}(x)\Big)\\ &= \Phi_{f(\cdot,\varphi^{t_{j+1}(y)}(y))}\Big(t-t_{j+1}(y),\varphi^{t_{j+1}(y)}(y)\Big)=\varphi^t(y)=\varphi^t(\varphi^s(x)).\end{aligned}$$

Since the continuity of $\varphi(\cdot,x)$ is immediate because of $\Phi_{f_k}(0,y)=y$ for all $k\in I$ and $y\in\mathbb{R}^d$, the proof is complete. $\square$

Condition $(A1)$ means that the t_i have to be defined from the concrete problem in such a way that the j's hitting time after the i's is just the $(i+j)$'s in total, and $(A2)$ says that a trajectory may not leave a manifold between consecutive switching times.

The next simple lemma is stated for further reference.

Lemma 2 *Let (A1) and (A2) be satisfied, and* $\xi_i(x)=\varphi^{t_i(x)}(x)$ *for* $i\in\mathbb{N}_0$ *and* $x\in\mathbb{R}^d$. *Then*

$$\xi_i(\varphi^t(x))=\xi_{i+j}(x) \quad \text{for all} \quad x\in\mathbb{R}^d,\, i,j\in\mathbb{N}_0 \quad \text{and} \quad t\in[t_j(x),t_{j+1}(x)[.$$

In particular, $\xi_i(\xi_j(x))=\xi_{i+j}(x)$ *for* $x\in\mathbb{R}^d$ *and* $i,j\in\mathbb{N}_0$.

Proof: This follows from (A1) and Lemma 1, since $y=\varphi^t(x)$ with $t\in[t_j(x),t_{j+1}(x)[$ has $t_i(y)=t_{i+j}(x)-t$ and hence $\xi_i(y)=\varphi^{t_i(y)}(y)=\varphi^{t_{i+j}(x)-t}(\varphi^t(x))=\xi_{i+j}(x)$. □

The following notations will be used in the sequel: If g_1,g_2 are $\mathbb{R}^d$-valued functions, then $\langle g_1,g_2\rangle(z)=\langle g_1(z),g_2(z)\rangle$, where $\langle u,v\rangle$ denotes the inner product in $\mathbb{R}^d$. Moreover, for $r>0$ and $x\in\mathbb{R}^d$ we let $B_r(x)=\{y\in\mathbb{R}^d : |y-x|<r\}$. For h from (10) we will always write $\nabla h(x,y)$ instead of $(\partial_x h(\cdot,y))(x)$, if the first component of h is denoted by x. Analogously, $f'(x,y)=(\partial_x f(\cdot,y))(x)\in L(\mathbb{R}^d)$.

The next lemma introduces suitable sets, on which the semiflow can be expected to be differentiable w.r.t. x, cf. the introduction and Figures 3a and 3b.

Lemma 3 *Let* $\Omega_0=\Delta_0=\mathbb{R}^d$ *and*

$$\Omega_{i+1}=\Omega_i \cap \{x_0\in\mathbb{R}^d : \langle\nabla h(\cdot,\xi_i(x_0)), f(\cdot,\xi_i(x_0))\rangle(\xi_{i+1}(x_0))\neq 0\}$$

as well as

$$\begin{aligned}\Delta_{i+1} = \Delta_i \cap \{x_0\in\mathbb{R}^d : \xi_i(x_0)\in M_k \Rightarrow \exists l\in I\ \forall t\in[t_i(x_0),t_{i+1}(x_0)[\ \exists\varepsilon>0 :\\ \xi_1(B_\varepsilon(\varphi^t(x_0))\cap M_k)\subset M_l\}\end{aligned}$$

for $i\in\mathbb{N}_0$. *Then* $G_1=\bigcap_{i=0}^\infty\Omega_i$ *and* $G_2=\bigcap_{i=0}^\infty\Delta_i$ *are forward invariant, i.e.* $\varphi^t(G_p)\subset G_p$ *for all* $t\in[0,\infty[$ *and* $p=1,2$. *In particular,* $G=G_1\cap G_2$ *is forward invariant.*

Proof: This follows directly from (A2) and Lemma 2. □

Now we can show the regularity of the t_i and ξ_i which will be needed to define a cocycle. To relate the t_i to h, a condition (A3) will be needed, which states that after the start of a trajectory at some initial $x\in\mathbb{R}^d$, $h(\cdot,x)$ should be exactly the one h_k such that the condition $h_k=0$ is satisfied at the next hitting time. Like (A1) and (A2), this condition should also be satisfied for semiflows constructed from "reasonable" non-smooth systems.

Lemma 4 *Suppose that besides (A1) and (A2) also*

(A3) $h(\Phi_{f(\cdot,x)}(t,x),x)=0 \iff t=t_1(x)$ *for* $x\in\mathbb{R}^d$ *and* $t\in[0,\infty[$.

Then for $i\in\mathbb{N}_0$ *and* $k\in I$, *in case that* $M_k\cap\Omega_i\cap\Delta_i\neq\emptyset$ *with* Ω_i *and* Δ_i *from Lemma 3,*

$$t_i\in C^1(M_k\cap\Omega_i\cap\Delta_i,\mathbb{R}) \quad \text{and} \quad \xi_i\in C^1(M_k\cap\Omega_i\cap\Delta_i,\mathbb{R}^d).$$

Proof: The claim will be proven by induction. For that, we shall also show in every step that

$$\begin{aligned}&i\in\mathbb{N}_0,\, x_0\in M_k\cap\Omega_i\cap\Delta_i,\, \xi_i(x_0)\in M_l,\, \eta>0\\ &\Longrightarrow \exists\rho>0 : \xi_i(B_\rho(x_0)\cap M_k\cap\Omega_i\cap\Delta_i)\subset B_\eta(\xi_i(x_0))\cap M_l \qquad (12)\end{aligned}$$

is valid, and we will use the abbreviation $\Sigma_{k,i} = M_k \cap \Omega_i \cap \Delta_i$. Since $t_0(x)=0$ and $\xi_0(x)=x$ in $\mathbb{R}^d$, the case $i=0$ is obvious. For the induction step we assume that the claim has already been proven for some $i\in\mathbb{N}$ and that $x_0\in\Sigma_{k,i+1}\subset\Sigma_{k,i}$ for some $k\in I$. By definition of $C^1(\cdot)$ and by assumption we hence find $\delta_i>0$ as well as functions $\tau_i\in C^1(B_{\delta_i}(x_0),\mathbb{R})$ and $\phi_i\in C^1(B_{\delta_i}(x_0),\mathbb{R}^d)$ such that $\tau_i(x)=t_i(x)$ and $\phi_i(x)=\xi_i(x)$ for $x\in B_{\delta_i}(x_0)\cap\Sigma_{k,i}$. Choose the $l\in I$ with $y_0=\xi_i(x_0)\in M_l$, i.e. $f(\cdot,y_0)=f_l$ and $h(\cdot,y_0)=h_l$. Also $t_1(y_0)=t_{i+1}(x_0)-t_i(x_0)$ by $(A1)$, and in particular

$$\xi_i(B_{\rho_i}(x_0)\cap\Sigma_{k,i})\subset M_l \tag{13}$$

for some $\rho_i>0$ by (12). Let $g_{i+1}(t,x)=h_l(\Phi_{f_l}(t-\tau_i(x),\phi_i(x)))$ for $(t,x)\in\mathbb{R}\times B_{\delta_i}(x_0)$. Then by definition of the semiflow and by $(A3)$ applied to y_0

$$g_{i+1}(t_{i+1}(x_0),x_0)=h_l(\Phi_{f_l}(t_{i+1}(x_0)-t_i(x_0),y_0))=h(\Phi_{f(\cdot,y_0)}(t_1(y_0),y_0),y_0)=0.$$

In addition we obtain $\partial_t g_{i+1}(t,x)=\langle\nabla h_l,f_l\rangle(\Phi_{f_l}(t-\tau_i(x),\phi_i(x)))$, and therefore, because of $\Phi_{f_l}(t_{i+1}(x_0)-t_i(x_0),\xi_i(x_0))=\xi_{i+1}(x_0)$, we arrive at

$$\partial_t g_{i+1}(t_{i+1}(x_0),x_0)=\langle\nabla h_l,f_l\rangle(\xi_{i+1}(x_0))=\langle\nabla h(\cdot,\xi_i(x_0)),f(\cdot,\xi_i(x_0))\rangle(\xi_{i+1}(x_0))\neq 0 \tag{14}$$

by definition of Ω_{i+1}. Thus we may apply the implicit function theorem to find $\gamma>0$, $\sigma>0$, and a unique C^1-function $\tau_{i+1}:B_\gamma(x_0)\to]t_{i+1}(x_0)-\sigma,t_{i+1}(x_0)+\sigma[$ such that $\tau_{i+1}(x_0)=t_{i+1}(x_0)$ and $g_{i+1}(\tau_{i+1}(x),x)=0$ in $B_\gamma(x_0)$. We define $\delta_{i+1}=\delta_i\wedge\rho_i\wedge\gamma$ and hence have $\tau_{i+1}\in C^1(B_{\delta_{i+1}}(x_0),\mathbb{R})$. To show $t_{i+1}\in C^1(\Sigma_{k,i+1},\mathbb{R})$ it is therefore enough to prove that $t_{i+1}=\tau_{i+1}$ in $B_{\delta_{i+1}}(x_0)\cap\Sigma_{k,i+1}$. For that, fix x in this set and let $y=\xi_i(x)$. Thus $f(\cdot,y)=f_l$ and $h(\cdot,y)=h_l$ by (13). Therefore

$$0=g_{i+1}(\tau_{i+1}(x),x)=h_l(\Phi_{f_l}(\tau_{i+1}(x)-t_i(x),\xi_i(x)))=h(\Phi_{f(\cdot,y)}(\tau_{i+1}(x)-t_i(x),y),y)$$

yields $t_{i+1}(x)=t_1(y)+t_i(x)=\tau_{i+1}(x)-t_i(x)+t_i(x)=\tau_{i+1}(x)$ by $(A1)$ and $(A3)$. Moreover, we can define $\phi_{i+1}(x)=\Phi_{f_l}(\tau_{i+1}(x)-\tau_i(x),\phi_i(x))$ to obtain $\phi_{i+1}\in C^1(B_{\delta_{i+1}}(x_0),\mathbb{R}^d)$ as well as $\phi_{i+1}(x)=\Phi_{f_l}(t_{i+1}(x)-t_i(x),\xi_i(x))=\xi_{i+1}(x)$ for $x\in B_{\delta_{i+1}}(x_0)\cap\Sigma_{k,i+1}$, so that we also get $\xi_{i+1}\in C^1(\Sigma_{k,i+1},\mathbb{R}^d)$.

Finally we have to prove (12) for $i+1$. For that, fix $x_0\in\Sigma_{k,i+1}$ and $\eta>0$, and assume that $\xi_{i+1}(x_0)\in M_l$ as well as $\xi_i(x_0)\in M_p$ for $l,p\in I$. Choose $\delta>0$ and $\phi_{i+1}\in C^1(B_\delta(x_0),\mathbb{R}^d)$ such that $\phi_{i+1}=\xi_{i+1}$ in $B_\delta(x_0)\cap\Sigma_{k,i+1}$. Since $\xi_1(\xi_i(x_0))=\xi_{i+1}(x_0)\in M_l$ and since $x_0\in\Delta_{i+1}$, by definition of Δ_{i+1} applied for $t=t_i(x_0)$, and by Lemma 2, we find $\varepsilon>0$ such that $\xi_1(B_\varepsilon(\xi_i(x_0))\cap M_p)\subset M_l$. Moreover, because of $\Omega_{i+1}\cap\Delta_{i+1}\subset\Omega_i\cap\Delta_i$, by (12) for i there exists $\rho\in]0,\delta]$ with $\xi_i(B_\rho(x_0)\cap\Sigma_{k,i})\subset B_\varepsilon(\xi_i(x_0))\cap M_p$. Consequently, by Lemma 2

$$\xi_{i+1}(B_\rho(x_0)\cap\Sigma_{k,i+1})\subset\xi_{i+1}(B_\rho(x_0)\cap\Sigma_{k,i})=\xi_1(\xi_i(\ldots))\subset\xi_1(B_\varepsilon(\xi_i(x_0))\cap M_p)\subset M_l. \tag{15}$$

We also may assume that $\rho>0$ is small enough to ensure $\phi_{i+1}(B_\rho(x_0))\subset B_\eta(\phi_{i+1}(x_0))=B_\eta(\xi_{i+1}(x_0))$, and thus

$$\xi_{i+1}(B_\rho(x_0)\cap\Sigma_{k,i+1})=\phi_{i+1}(B_\rho(x_0)\cap\Sigma_{k,i+1})\subset B_\eta(\xi_{i+1}(x_0)). \tag{16}$$

Hence (15) and (16) show that (12) also holds for $i+1$. □

Corollary 1 *Suppose that (A1)-(A3) are satisfied. Then for* $i \in \mathbb{N}_0$, $x_0 \in M_k \cap \Omega_{i+1} \cap \Delta_{i+1}$ *and* $t \in]t_i(x_0), t_{i+1}(x_0)[$ *there exists* $r > 0$ *such that* $t \in]t_i(x), t_{i+1}(x)[$ *for* $x \in B_r(x_0) \cap M_k \cap \Omega_{i+1} \cap \Delta_{i+1}$ *and* $\varphi^t(\cdot) \in C^1(B_r(x_0) \cap M_k \cap \Omega_{i+1} \cap \Delta_{i+1}, \mathbb{R}^d)$.

Proof: By Lemma 4 we find $\delta > 0$, $\tau_i, \tau_{i+1} \in C^1(B_\delta(x_0), \mathbb{R})$ and $\phi_i \in C^1(B_\delta(x_0), \mathbb{R}^d)$ such that $\tau_i = t_i$ and $\tau_{i+1} = t_{i+1}$ in $B_\delta(x_0) \cap \Sigma_{k,i+1}$ as well as $\phi_i = \xi_i$ in $B_\delta(x_0) \cap \Sigma_{k,i}$. If $y_0 = \xi_i(x_0) \in M_l$ we also may assume that $\xi_i(B_\delta(x_0) \cap \Sigma_{k,i}) \subset M_l$ by (12) and since $\Omega_{i+1} \cap \Delta_{i+1} \subset \Omega_i \cap \Delta_i$. Then by continuity of τ_i and τ_{i+1} there exists $r \in]0, \delta]$ with $|\tau_i(x) - t_i(x_0)| \leq (t - t_i(x_0))/2$ and $|\tau_{i+1}(x) - t_{i+1}(x_0)| \leq (t_{i+1}(x_0) - t)/2$ for $x \in B_r(x_0)$. Therefore $t \in]t_i(x), t_{i+1}(x)[$ for $x \in B_r(x_0) \cap \Sigma_{k,i+1}$, and if we define $\theta(x) = \Phi_{f_l}(t - \tau_i(x), \phi_i(x))$, then we obtain $\theta \in C^1(B_r(x_0), \mathbb{R}^d)$ as well as

$$\varphi^t(x) = \Phi_{f(\cdot,\xi_i(x))}(t - t_i(x), \xi_i(x)) = \Phi_{f_l}(t - \tau_i(x), \phi_i(x)) = \theta(x)$$

for $x \in B_r(x_0) \cap \Sigma_{k,i+1} \subset B_\delta(x_0) \cap \Sigma_{k,i}$. $\square$

3 Transition of the linearization at switching times

Our next objective is to derive the transition matrices at times $t_{i+1}(x)$. We remark that the analogous formula also was obtained before in Müller (1995) by formal investigation of nearby trajectories, and by a full linearization of the functions corresponding to the h_k. Note also that in particular in the case of a smooth dynamical systems, where all f_k are the same, one has $L_{i+1}(x) = \mathrm{id}_{\mathbb{R}^d}$ below, i.e. the transition at switching times is smooth.

Lemma 5 *Suppose that (A1)-(A3) hold. For* $i \in \mathbb{N}_0$ *and* $x_0 \in \Omega_{i+2} \cap \Delta_{i+2}$ *define* $L_{i+1}(x_0) \in L(\mathbb{R}^d)$ *through*

$$L_{i+1}(x_0) = \frac{[f(\xi_{i+1}(x_0), \xi_{i+1}(x_0)) - f(\xi_{i+1}(x_0), \xi_i(x_0))]}{\langle \nabla h(\cdot, \xi_i(x_0)), f(\cdot, \xi_i(x_0)) \rangle (\xi_{i+1}(x_0))} \langle \nabla h(\xi_{i+1}(x_0), \xi_i(x_0)), \cdot \rangle + \mathrm{id}_{\mathbb{R}^d}.$$

Then

$$\lim_{t \to t_{i+1}(x_0)^+} \partial_x \varphi^t(x_0) = L_{i+1}(x_0) \left(\lim_{t \to t_{i+1}(x_0)^-} \partial_x \varphi^t(x_0) \right).$$

Proof: Choose $k \in I$ such that $x_0 \in M_k$ and $l, p \in I$ in accordance with (12) such that $x_i = \xi_i(x_0) \in M_l$ and $x_{i+1} = \xi_{i+1}(x_0) \in M_p$. This is possible, since in particular $x_0 \in \Omega_{i+1} \cap \Delta_{i+1} \subset \Omega_i \cap \Delta_i$, and the following formal calculations are justified by Lemma 4 and Corollary 1. By definition of the semiflow we obtain for $t \in]t_{i+1}(x_0), t_{i+2}(x_0)[$

$$\begin{aligned} \partial_x \varphi^t(x_0) &= \partial_t \Phi_{f_p}(t - t_{i+1}(x_0), \xi_{i+1}(x_0)) (-t'_{i+1}(x_0)) + \partial_x \Phi_{f_p}(t - t_{i+1}(x_0), \xi_{i+1}(x_0)) \xi'_{i+1}(x_0) \\ &= -f_p(\Phi_{f_p}(t - t_{i+1}(x_0), x_{i+1})) t'_{i+1}(x_0) + \partial_x \Phi_{f_p}(t - t_{i+1}(x_0), x_{i+1}) \xi'_{i+1}(x_0) \end{aligned} \tag{17}$$

and thus, since $\Phi_{f_p}(0, y) = y$ and $\partial_x \Phi_{f_p}(0, y) = \mathrm{id}_{\mathbb{R}^d}$ for all $y \in \mathbb{R}^d$

$$\lim_{t \to t_{i+1}(x_0)^+} \partial_x \varphi^t(x_0) = -f_p(x_{i+1}) t'_{i+1}(x_0) + \xi'_{i+1}(x_0) = -f_p(x_{i+1}) \langle t'_{i+1}(x_0), \cdot \rangle + \xi'_{i+1}(x_0) \in L(\mathbb{R}^d). \tag{18}$$

Let

$$\Gamma_l^i(t, x) = \Phi_{f_l}(t - t_i(x), \xi_i(x)) \quad \text{for} \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d. \tag{19}$$

Then $\partial_t \Gamma_l^i(t,x) = f_l(\Gamma_l^i(t,x))$, and for $t \in]t_i(x_0), t_{i+1}(x_0)[$ and x in a suitable neighbourhood of x_0 we have $\varphi^t(x) = \Gamma_l^i(t,x)$ as well as

$$\begin{aligned}\xi_{i+1}(x) &= \varphi^{t_{i+1}(x)}(x) = \Phi_{f(\cdot,\varphi^{t_i(x)}(x))}(t_{i+1}(x) - t_i(x), \varphi^{t_i(x)}(x)) \\ &= \Phi_{f_l}(t_{i+1}(x) - t_i(x), \xi_i(x)) = \Gamma_l^i(t_{i+1}(x), x).\end{aligned}$$

Therefore

$$\begin{aligned}\xi'_{i+1}(x_0) &= \partial_t \Gamma_l^i(t_{i+1}(x_0), x_0)\, t'_{i+1}(x_0) + \partial_x \Gamma_l^i(t_{i+1}(x_0), x_0) = f_l(x_{i+1})\, t'_{i+1}(x_0) + \partial_x \Gamma_l^i(t_{i+1}(x_0), x_0) \\ &= f_l(x_{i+1}) \langle t'_{i+1}(x_0), \cdot \rangle + \partial_x \Gamma_l^i(t_{i+1}(x_0), x_0) \in L(\mathbb{R}^d). \qquad (20)\end{aligned}$$

Retaining the notation from Lemma 4 we also have formally $g_{i+1}(t,x) = h_l(\Gamma_l^i(t,x))$, and consequently $\partial_x g_{i+1}(t_{i+1}(x_0), x_0) = \nabla h_l(x_{i+1})\, \partial_x \Gamma_l^i(t_{i+1}(x_0), x_0)$. Thus by (14)

$$t'_{i+1}(x_0) = -\frac{\partial_x g_{i+1}(t_{i+1}(x_0), x_0)}{\partial_t g_{i+1}(t_{i+1}(x_0), x_0)} = -\frac{1}{\langle \nabla h_l, f_l \rangle (x_{i+1})} \langle \nabla h_l(x_{i+1}), \partial_x \Gamma_l^i(t_{i+1}(x_0), x_0) \cdot \rangle. \qquad (21)$$

Hence we can insert (20) and (21) into (18) to obtain

$$\begin{aligned}\lim_{t \to t_{i+1}(x_0)^+} \partial_x \varphi^t(x_0) &= [f_l(x_{i+1}) - f_p(x_{i+1})] \langle t'_{i+1}(x_0), \cdot \rangle + \partial_x \Gamma_l^i(t_{i+1}(x_0), x_0) \\ &= \frac{[f_p(x_{i+1}) - f_l(x_{i+1})]}{\langle \nabla h_l, f_l \rangle (x_{i+1})} \langle \nabla h_l(x_{i+1}), \partial_x \Gamma_l^i(t_{i+1}(x_0), x_0) \cdot \rangle + \partial_x \Gamma_l^i(t_{i+1}(x_0), x_0).\end{aligned}$$

Therefore $f_l = f(\cdot, \xi_i(x_0))$, $f_p = f(\cdot, \xi_{i+1}(x_0))$ and $h_l = h(\cdot, \xi_i(x_0))$ together with

$$\partial_x \varphi^t(x_0) = \partial_x \Gamma_l^i(t, x_0) \to \partial_x \Gamma_l^i(t_{i+1}(x_0), x_0) \quad \text{as} \quad t \to t_{i+1}(x_0)^- \qquad (22)$$

yields the claimed form of the transition matrix $L_{i+1}(x_0)$. □

4 Definition of the cocycle and applications

By means of Lemma 5 we now can construct a canonical cocycle.

Theorem 1 *Suppose that (A1)-(A3) are satisfied, and let the forward invariant set $G \subset \mathbb{R}^d$ be defined according to Lemma 3. Define $T : [0,\infty[\times \mathbb{R}^d \to L(\mathbb{R}^d)$ for $x \in G$ through $T(t,x) = \partial_x \varphi^t(x)$ if $t \neq t_i(x)$ for all $i \in \mathbb{N}$ as well as*

$$T(t_{i+1}(x), x) = L_{i+1}(x)(\partial_x \Gamma_l^i(t_{i+1}(x), x))$$

for $i \in \mathbb{N}_0$, if $\xi_i(x) \in M_l$ with Γ_l^i from (19). Moreover, let $T(t,x) = 0$ for $x \in \mathbb{R}^d \setminus G$. Then T is a cocycle for $(\varphi^t)_{t \geq 0}$.

Proof: We have to show that

$$T(t+s, x) = T(t, \varphi^s(x))\, T(s, x) \quad \text{for} \quad t, s \in [0, \infty[\quad \text{and} \quad x \in \mathbb{R}^d, \qquad (23)$$

and we continue to argue formally, which again may be justified by Lemma 4 and Corollary 1. Firstly, we remark that (23) is fulfilled for $x \in \mathbb{R}^d \setminus G$. Note also that (23) holds if $t = 0$

or if $s=0$, since G is forward invariant and $T(0,y)=\mathrm{id}_{\mathbb{R}^d}$ for all $y\in G$. So fix $t,s>0$ and $x\in G$, and assume first that $s\neq t_k(x)$ and $t+s\neq t_k(x)$ for all $k\in\mathbb{N}$, thus $s\in]t_i(x),t_{i+1}(x)[$ and $t+s\in]t_j(x),t_{j+1}(x)[$ for some $i,j\in\mathbb{N}$. Then also $t\neq t_k(\varphi^s(x))$ for all $k\in\mathbb{N}$ by $(A1)$, and hence by the invariance of G and by Corollary 1 we may differentiate $\varphi^{t+s}(x)=\varphi^t(\varphi^s(x))$ from Lemma 1 w.r.t. x to obtain by definition of T (23) for such s and $t+s$.

Now if $s=t_i(x)$ or $t+s=t_j(x)$, then in all cases $s+\tau\in]t_i(x),t_{i+1}(x)[$ and $t+(s+\tau)\in]t_j(x),t_{j+1}(x)[$ for $\tau>0$ sufficiently small. Consequently $T(t+s+\tau,x)=T(t,\varphi^{s+\tau}(x))T(s+\tau,x)$ by the first part. Since $T(\cdot,x)$ is right-continuous by definition of T and by Lemma 5, cf. (22), it is hence sufficient to prove that for fixed $t>0$ and $y_0\in G$ the function $T(t,\cdot)$ is continuous at y_0 on the set $D_{y_0}=\{y\in G: t_k(y)\leq t_k(y_0)\,\text{for all}\,k\in\mathbb{N}\}$. Indeed, in this case we can take, due to the invariance of G, $y_0=\varphi^s(x)\in G$ and $y=\varphi^{s+\tau}(x)\in G$, because then $t_k(y)=t_k(\varphi^{s+\tau}(x))=t_{i+k}(x)-(s+\tau)\leq t_{i+k}(x)-s=t_k(\varphi^s(x))=t_k(y_0)$ for $k\in\mathbb{N}$ by $(A1)$. Thus we may use the continuity of $\varphi(\cdot,x)$, cf. Lemma 1, to let $\tau\to 0^+$ and to obtain (23) in the limit.

To prove the continuity of $T(t,\cdot)$ at $y_0\in G$ on D_{y_0}, we may assume that $t=t_{i+1}(y_0)$ for some $i\in\mathbb{N}_0$, since otherwise Corollary 1 shows that $T(t,\cdot)=\partial_x\varphi^t(\cdot)$ is continuous even in a whole G-neighbourhoud U (not restricted to D_{y_0}) of y_0. So let $t=t_{i+1}(y_0)$ and choose $l,p\in I$ such that $\xi_i(y_0)\in M_l$ and $\xi_{i+1}(y_0)\in M_p$. Hence by definition and by (18)

$$T(t,y_0)=L_{i+1}(y_0)(\partial_x\Gamma_l^i(t_{i+1}(y_0),y_0))=-f_p(\xi_{i+1}(y_0))\langle t'_{i+1}(y_0),\cdot\rangle+\xi'_{i+1}(y_0). \tag{24}$$

Because of $t_i(y_0)<t=t_{i+1}(y_0)<t_{i+2}(y_0)$, we have $t\in[t_{i+1}(y),t_{i+2}(y)[$ for $y\in D_{y_0}$ sufficiently near to y_0 by definition of D_{y_0} and by Lemma 4. For $y\in G$ near to y_0 we also have $\xi_i(y)\in M_l$ and $\xi_{i+1}(y)\in M_p$ by (12). Since thus either

$$T(t,y)=L_{i+1}(y)(\partial_x\Gamma_l^i(t_{i+1}(y),y))$$

if $t=t_{i+1}(y)$ or, cf. (17),

$$\begin{aligned} T(t,y) &= \partial_x\varphi^t(y)=-f_p(\Phi_{f_p}(t_{i+1}(y_0)-t_{i+1}(y),\xi_{i+1}(y)))\langle t'_{i+1}(y),\cdot\rangle \\ &\quad +\partial_x\Phi_{f_p}(t_{i+1}(y_0)-t_{i+1}(y),\xi_{i+1}(y))\,\xi'_{i+1}(y) \end{aligned}$$

if $t\in]t_{i+1}(y),t_{i+2}(y)[$, the claim hence follows from (24), the regularity of the functions involved, from $\Phi_{f_p}(0,\xi_{i+1}(y_0))=\xi_{i+1}(y_0)$ and $\partial_x\Phi_{f_p}(0,\xi_{i+1}(y_0))=\mathrm{id}_{\mathbb{R}^d}$. □

In the next corollary to Theorem 1 we show that T has to satisfy a certain integral equation.

Corollary 2 *In the situation of Theorem 1, if $x\in G$ and $t\in[t_i(x),t_{i+1}(x)[$, we have*

$$T(t,x)=P_i(0,x)+\int_0^t P_i(s,x)\,F(s,x)\,T(s,x)\,ds, \tag{25}$$

where $P_i(s,x)=L_i(x)\cdot\ldots\cdot L_{j+2}(x)\cdot L_{j+1}(x)\in L(\mathbb{R}^d)$ for $s\in[t_j(x),t_{j+1}(x)[$, $P_i(s,x)=\mathrm{id}_{\mathbb{R}^d}$ in case that $j\geq i$, and $F(s,x)=f'(\varphi^s(x),\xi_j(x))$ for $s\in[t_j(x),t_{j+1}(x)[$.

Proof: We first consider the case $i=0$, i.e. $t\in[0,t_1(x)[$ for some fixed $x\in G$. Then $P_0(s,x)=\mathrm{id}_{\mathbb{R}^d}$ and $F(s,x)=f'(\varphi^s(x),x)$ for $s\in[0,t_1(x)[$. If we let $f_k=f(\cdot,x)$, we obtain by definition $\varphi^s(x)=\Phi_{f_k}(s,x)=\Gamma_k^0(s,x)$ for $s\in[0,t_1(x)[$, and thus, cf. (19),

$$\begin{aligned} \partial_t T(s,x) &= \partial_t(\partial_x\Gamma_k^0(s,x))=\partial_x(\partial_t\Gamma_k^0(s,x))=\partial_x(f_k(\Gamma_k^0(s,x))) \\ &= f'_k(\Gamma_k^0(s,x))\,\partial_x\Gamma_k^0(s,x)=F(s,x)T(s,x). \end{aligned} \tag{26}$$

Since $T(0,x)=\mathrm{id}_{\mathbb{R}^d}$ this may be integrated to give (25) for $i=0$. If (25) already holds for some $i\in\mathbb{N}$, then in particular by definition of $T(t_i(x),x)$ and by assumption

$$T(t_i(x),x)=L_i(x)\Big(\lim_{t\to t_i(x)^-} T(t,x)\Big)=L_i(x)\,P_{i-1}(0,x)+\int_0^{t_i(x)} L_i(x)\,P_{i-1}(s,x)\,F(s,x)\,T(s,x)\,ds. \tag{27}$$

Moreover, if $f_l=f(\cdot,\xi_i(x))$, then $\varphi^s(x)=\Gamma_l^i(s,x)$ for $s\in[t_i(x),t_{i+1}(x)[$, and hence the differential equation for T from (26) is also valid for $s\in[t_i(x),t_{i+1}(x)[$. Integrating this from $t_i(x)$ to t and using (27) we thus obtain for $t\in[t_i(x),t_{i+1}(x)[$

$$\begin{aligned}T(t,x) &= L_i(x)\,P_{i-1}(0,x)+\int_0^{t_i(x)} L_i(x)\,P_{i-1}(s,x)\,F(s,x)\,T(s,x)\,ds+\int_{t_i(x)}^t F(s,x)\,T(s,x)\,ds\\ &= P_i(0,x)+\int_0^t P_i(s,x)\,F(s,x)\,T(s,x)\,ds,\end{aligned}$$

by definition of P_i and F. □

We remark that (25) is similiar to formulas for solutions of certain implusive equations, studied e.g. in Bainov/Zabreiko/Kostadinov (1988).

Since T from Theorem 1 defines a cocycle, for an application of Oseledets' theorem one has to make sure that the integrabilty conditions (2) are satisfied. An easy criterion for this (which nevertheless can be verified for the friction-oscillator) is given in

Corollary 3 *Suppose that (A1)-(A3) are satisfied and that* $|L_i(x)|\le 1$ *for* $i\in\mathbb{N}$ *and* $x\in G$. *If in addition* $\max_{k=1,\dots,n}|f_k'|_\infty\le C$ *for some constant* $C>0$, *then for every* $t_*>0$ *there exists* $C_{t_*}>0$ *such that*

$$\sup\Big\{|T(t,x)|: t\in[0,t_*],\, x\in\mathbb{R}^d\Big\}\le C_{t_*}.$$

In particular, (2) holds.

Proof: Since by assumption and by definition of P_i and F we have $|P_i(s,x)|\le 1$ and $|F(s,x)|\le C$ for all $i\in\mathbb{N}$, $s\in[0,\infty[$ and $x\in G$, the claim follows from (25) and Gronwall's lemma, because we only have to care about the $x\in G$, by definition of T. □

Finally we want to show now that Oseledets' theorem may be used to imply the existence of Lyapunov exponents in the example of the friction-oscillator.

So we assume that there can be constructed a corresponding semiflow which satisfies $(A1)$-$(A3)$ (as it was made plausible above), and that there exists an ergodic probability measure μ on $(\mathbb{R}^3,\mathcal{B}(\mathbb{R}^3))$, which is left invariant by $(\varphi^t)_{t\ge 0}$. To verify the boundedness conditions from Corollary 3 with M_+, M_-, M_0 from (9), with f_+, f_-, f_0 from (8), and with h_+, h_- and h_0 from (11), note first that f_+', f_-' and f_0' are uniformly bounded in $\mathbb{R}^3$ by definition of the friction characteristic c, cf. (5). To check $|L_i(x)|\le 1$, note that according to the direction of the vector field (cf. Figure 2) there are only five possible transitions, namely $M_+\to M_-$, $M_-\to M_+$, $M_+\to M_0$, $M_-\to M_0$, and $M_0\to M_-$ for a trajectory. We investigate as an example $M_+\to M_-$, since the other cases are similiar. So we suppose that for some $x_0\in G$ and $i\in\mathbb{N}_0$ we have $f(\cdot,\xi_i(x_0))=f_+$, $f(\cdot,\xi_{i+1}(x_0))=f_-$ and $h(\cdot,\xi_i(x_0))=h_+$, i.e. $\nabla h_+(y)=(0,1,0)$ for all $y\in\mathbb{R}^3$. In addition, for $y\in\mathbb{R}^3$ we obtain $f_-(y)-f_+(y)=(0,2\alpha c(0),0)$, and therefore by

Definition, cf. Lemma 5, since $y=\xi_{i+1}(x_0)$ has $y_2=1$,

$$L_{i+1}(x_0)=\frac{(0,2\alpha c(0),0)}{-y_1-\alpha[c(v_0)+c(0)]+w(y_3)}\left\langle\begin{pmatrix}0\\1\\0\end{pmatrix},\cdot\right\rangle+\mathrm{id}_{\mathbb{R}^3}=\begin{pmatrix}1&0&0\\0&\theta(y)&0\\0&0&1\end{pmatrix}$$

with

$$\theta(y)=1-\frac{2\alpha c(0)}{y_1+\alpha[c(v_0)+c(0)]-w(y_3)}=1-\frac{2\alpha c(0)}{y_1-b_-(y_3)+2\alpha c(0)}.$$

Again by the direction of the vector field, a transition $M_+\to M_-$ is only possible at y if $y_1\geq b_-(y_3)$, and thus $\theta(y)\in[0,1[$, hence $|L_{i+1}(x_0)|=1$.

5 Concluding Remarks

1.) It would be desirable in the example of the friction-oscillator to prove the existence not only of an ergodic probability measure μ which leaves the semiflow invariant, but of such a μ with sufficiently large support. A corresponding "canonical" μ should be supported in a neighbourhood of an attractor of the system. In addition, in order that the Lyapunov exponents obtained for the system are in accordance with the real observed behaviour of the system, also $\mathbb{R}^d\setminus G$ should be neglectable w.r.t. μ. Therefore it needs further clarification whether it is indeed a somehow generic property of a trajectory not to arrive tangentially at the surface and not to meet $\{x\in\mathbb{R}^3: x_2=1, x_1=b_-(x_3)\}$.

Moreover, our choice of the friction characteristic was (5) to obtain $|c'|_\infty$ bounded. Again it depends on the support of μ whether other characteristics behaving like v^2 as $v\to\infty$ may also be used.

2.) We made the implicit assumption in case of the friction oscillator that no trajectory hits the discontinuity surface inifinitely often in a finite time to have $t_i(x)\to\infty$ as $i\to\infty$. Since there are interesting examples of problems where this assumption definitively does not hold (like in the case of a rotor which may touch a rigid boundary, cf. Szczygielski (1986)), also generalizations in this direction should be aimed at.

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