# The singular perturbation limit of a variational problem from nonlinear fiber optics

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**Abstract**

Among the fiber optical models the dispersion managed (DM) systems have attracted interest due to their favorable properties compared to the more traditional approaches which rely on classical solitons. Numerical and physical experiments have shown that in the limit of strong dispersion management, there exist stable nearly periodic pulses. It was later found that these breathing pulses correspond to the ground states solutions of a certain averaged DM problem with positive and even zero residual dispersion. It was not clear, however, how DM solitons in the zero residual dispersion regime are related to those ones in the positive residual dispersion regime. In this paper, DM solitons in the mean-zero case are obtained as DM solitons in the limit of vanishing positive residual dispersion. This corresponds to studying the singular perturbation limit of energy functionals of the form $\varphi^{(\varepsilon)}(u) = \frac{\varepsilon}{2}\int_{\mathbb{R}} |u'|^2\,dx - F(u)$ with a highly non-local operator $F$.

## 1 Introduction and main results

Over the last two decades models for optical fibers which are based on dispersion management have become increasingly popular [3, 12, 16]. One obvious reason for the interest in the subject is due to the fact that in practise such fiber lines support pulses whose tails decay very rapidly (like Gaussians $\sim e^{-x^2}$, in contrast to the decay $\sim e^{-|x|}$ of the traditional $\operatorname{sech}^2(x)$-solitons; see [8, 4]). Since accordingly the mutual interference of the tails is much weaker, better results for the transmission of optical signals at high bit rates can be obtained [11, 18]. A second advantage of DM-based models is the stability of the pulses over long periods of time [5, 23]. Thirdly, it has been discussed numerically [13, 14] that DM solitons are also stable in the regime of zero residual dispersion, that is, pulses at small positive residual dispersion appear to persist as the residual dispersion is taken to zero; a mathematical proof of this fact is the subject of the present paper. Since another important reason for considering the zero residual dispersion case is that this regime plays a role for the destabilizing Gordon-Haus jitter (see [19]), it is desirable also from a practical point of view to understand the limit of vanishing residual dispersion.

The model frequently considered in the context of DM optical fibers is described by

$$iu_z + d(z)u_{tt} + |u|^2 u = 0, \tag{1.1}$$

where $u$ is the envelope of the optical pulse (electromagnetic wave), the evolution of the latter being governed by Maxwell's equations; see [6, 15, 22]. In (1.1), $z$ is the distance along the fiber whereas $t$ denotes (retarded) time. The group velocity dispersion function $d(z) = \tilde{d}(z) + \langle d\rangle$ is periodic with residual dispersion $\langle d\rangle$, whereas $\tilde{d}(z)$ has mean value zero over one compensation period. It is by now well-known that (1.1), in the strong DM regime, can be averaged over the compensation period (which is taken as the unit interval) to yield an effective equation in the limit of vanishing residual dispersion $\langle d\rangle \simeq \varepsilon \to 0^+$. Several papers discuss in detail the averaged equations and their derivation, their relation to (1.1), and the associated scales; see e.g. [1, 5, 20, 21, 23] and the references therein. For our purposes we follow [5, 7, 23] and only note that, after some lengthy manipulation and suitable choice of the constants, an effective equation turns out to be governed by the Hamiltonian

$$\mathcal{H}^{(\varepsilon)}(u) = \frac{\varepsilon}{2}\int_{\mathbb{R}} |u_t|^2\,dt - \int_0^1\int_{\mathbb{R}} |T(z)u|^4\,dt\,dz.$$

The corresponding Euler-Lagrange equation is

$$iu_z + \frac{\varepsilon}{2}u_{tt} + Q(u) = 0, \tag{1.2}$$

where

$$Q(u) = \frac{1}{4}\int_0^1 T(z)^{-1}(|T(z)u|^2\,T(z)u)\,dz$$

denotes the averaged nonlinearity. An implicit assumption inherent to the possible reduction of (1.1) to (1.2) is that $d(z)$ is a one-periodic step function; here $u(t,z) = T(z)u(t,0)$ denotes the solution to $iu_z + d(z)u_{tt} = 0$. In order to find ground state solutions of (1.2) one therefore discusses, in the case $\varepsilon > 0$, the constrained variational problems

$$\mathcal{H}_E^{(\varepsilon)} = \inf\Big\{\mathcal{H}^{(\varepsilon)}(u) : u \in H^1, \int_{\mathbb{R}} |u|^2\,dt = E\Big\},$$

where $H^1 = H^1(\mathbb{R};\mathbb{C})$ is the usual Sobolev space of square integrable functions $u$ such that $\int_{\mathbb{R}} |u_t|^2\,dt$ is finite. It has been shown in [23] that for every $\varepsilon > 0$ and $E > 0$ the infimum $\mathcal{H}_E^{(\varepsilon)}$ is attained, cf. also [9]; we remark that the numerical results in [7] indicate that minimizers are non-unique. In the case $\varepsilon = 0$ the corresponding constrained variational problems are

$$\mathcal{H}_E = \inf\Big\{\mathcal{H}(u) : u \in L^2, \int_{\mathbb{R}} |u|^2\,dt = E\Big\}$$

with

$$\mathcal{H}(u) = -\int_0^1\int_{\mathbb{R}} |T(z)u|^4\,dt\,dz,$$

and the existence of a minimizer has been proved in [10]. The general problem which had to be overcome for $\varepsilon = 0$ is the lack of compactness for minimizing sequences, due to the absence of a priori bounds in spaces different from $L^2$.

Here our main concern is to study the singular perturbation limit $\varepsilon \to 0$ of the ground state energies $\mathcal{H}_E^{(\varepsilon)}$ and of the minimizers. We are going to show that $\mathcal{H}_E^{(\varepsilon)} \to \mathcal{H}_E$, and that for the respective minimizers we have $u^{(\varepsilon)} \to u$ in $L^2$, up to selection of a subsequence, rotation, and translation of the original sequence (see Theorem 1.2 below). Hence we indeed find that pulses at small positive residual dispersion do persist in the limit of vanishing residual dispersion.

To simplify matters from the outset, we will assume that the dispersion function $d(z)$ is constant ($\equiv 1$) on the interval $[0,1]$; in general one could split the original period interval $[0,1]$ of $d(z)$ into the pieces on which $d(z)$ is constant and consider each piece separately. Moreover, we will denote more intuitively $t$ what is $z$ above (since it plays the role of a time variable), and accordingly we will write $x$ instead of $t$. For example, $e^{it\partial_x^2}u_0 = T(t)u_0$ denotes the evolution operator of the free Schrödinger equation in one space dimension, i.e., $u(t,x) = (T(t)u_0)(x)$ solves

$$iu_t + u_{xx} = 0, \quad u(0,x) = u_0(x). \tag{1.3}$$

Thus our aim is to investigate the limit $\varepsilon \to 0$ of the energies

$$P_\lambda^{(\varepsilon)} = \inf\Big\{\varphi^{(\varepsilon)}(u) : u \in H^1,\ \int_{\mathbb{R}} |u|^2\,dx = \lambda\Big\} \tag{1.4}$$

and of the corresponding minimizers, where the functionals are given by

$$\varphi^{(\varepsilon)}(u) = \frac{\varepsilon}{2}\int_{\mathbb{R}} |u'(x)|^2\,dx - \int_0^1\int_{\mathbb{R}} |(T(t)u)(x)|^4\,dx\,dt. \tag{1.5}$$

The limiting energy is

$$P_\lambda = \inf\Big\{\varphi(u) : u \in L^2,\ \int_{\mathbb{R}} |u|^2\,dx = \lambda\Big\}, \tag{1.6}$$

with associated functional

$$\varphi(u) = -\int_0^1\int_{\mathbb{R}} |(T(t)u)(x)|^4\,dx\,dt. \tag{1.7}$$

We note that due to the presence of the term $|(T(t)u)(x)|^4$ both the $\varphi^{(\varepsilon)}$ and $\varphi$ are non-local. Together with the fact that the underlying spatial domain $\mathbb{R}$ is unbounded, this leads to mathematically interesting questions. Of course the unbounded domain excludes concentration effects at the boundary, which are typically found in the limit $\varepsilon \to 0$ for positive solutions of $-\varepsilon^2\Delta u = f(u)$ in a bounded domain $\Omega \subset \mathbb{R}^n$, under suitable assumptions on $f$; cf. [17] and the references therein. To proceed further, we recall the result from [10].

**Lemma 1.1** *Let $(u_j) \subset L^2$ be a minimizing sequence for $P_\lambda$, i.e., we have $\int_{\mathbb{R}} |u_j|^2\,dx = \lambda$ as well as $\varphi(u_j) \to P_\lambda$. Then there exists a subsequence (not relabelled) as well as sequences $(\xi_j) \subset \mathbb{R}$ and $(x_j) \subset \mathbb{R}$ such that $e^{-i\xi_j(x_j+\cdot)}u_j(x_j+\cdot) \to u$ in $L^2$, and $u$ is a minimizer for $P_\lambda$.*

The main result of the present paper is summarized in the following theorem.

**Theorem 1.2** *For every $\lambda > 0$ we have $P_\lambda^{(\varepsilon)} \to P_\lambda$ as $\varepsilon \to 0$. In addition, if $(u_j) \subset H^1$ is a sequence of minimizers for $P_\lambda^{(\varepsilon_j)}$ for some sequence $\varepsilon_j \to 0$, then there exist a subsequence (not relabelled) as well as sequences $(\xi_j) \subset \mathbb{R}$ and $(x_j) \subset \mathbb{R}$ such that $e^{-i\xi_j(x_j+\cdot)}u_j(x_j+\cdot) \to u$ in $L^2$, and $u$ is a minimizer for $P_\lambda$.*

The proof of this theorem is elaborated in the next Section 2. Concerning notation, we write $L^p = L^p(\mathbb{R};\mathbb{C})$ and $H^s = H^s(\mathbb{R};\mathbb{C})$, with corresponding norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{H^s}$, respectively. The inner product on $L^2$ is denoted $(u,v)_{L^2} = \int_{\mathbb{R}} u\bar{v}\,dx$, whereas the Fourier transform of $u \in L^2$ is $\hat{u}(\xi) = (2\pi)^{-1/2}\int_{\mathbb{R}} e^{-ix\xi}u(x)\,dx$.

## 2 Proof of Theorem 1.2

We start with some basic properties of $P_\lambda^{(\varepsilon)}$ and $P_\lambda$ from (1.4) and (1.6), respectively.

**Lemma 2.1** *There exist constants $C_1, C_2 > 0$ such that $-C_1\lambda^2 \le P_\lambda^{(\varepsilon)} \le -C_2\lambda^2$ and $-C_1\lambda^2 \le P_\lambda \le -C_2\lambda^2$ are satisfied for all $\lambda > 0$ and $\varepsilon \in ]0, \lambda/2]$. Moreover,*

$$P_\lambda^{(\varepsilon)} \searrow P_\lambda^* \ge P_\lambda, \quad \varepsilon \to 0, \tag{2.1}$$

*for $P_\lambda^* := \inf_{\varepsilon>0} P_\lambda^{(\varepsilon)}$. We also have $P_\lambda = \lambda^2 P_1$ for $\lambda > 0$, and*

$$\varphi(u) \ge P_{\|u\|_{L^2}^2} = \|u\|_{L^2}^4 P_1, \quad u \in L^2. \tag{2.2}$$

**Proof:** For the first part, we follow [23, Thm. B.1], cf. also [10, Lemma 2.5]. From the elementary inequality $a^4 \le \frac{1}{2\eta}a^2 + \frac{\eta}{2}a^6$ with $\eta = \|u\|_{L^2}^{-2}$, and using Strichartz' inequality

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |T(t)u|^6 \, dxdt \le C\|u\|_{L^2}^6,$$

cf. [2, Thm. 3.2.5(i)], we see that

$$\begin{aligned} |\varphi(u)| &\le \int_0^1 \int_{\mathbb{R}} \Big(\frac{1}{2\eta}|T(t)u|^2 + \frac{\eta}{2}|T(t)u|^6\Big)\, dxdt \\ &\le \frac{1}{2\eta}\,\|u\|_{L^2}^2 + C\eta\|u\|_{L^2}^6 \le C_1\|u\|_{L^2}^4 \end{aligned}$$

for $u \in L^2$. Thus for $u \in H^1$ with $\int_{\mathbb{R}} |u|^2\, dx = \lambda$ we obtain the lower bound

$$\varphi^{(\varepsilon)}(u) \ge \varphi(u) \ge -C_1\|u\|_{L^2}^4 = -C_1\lambda^2,$$

hence $P_\lambda \ge -C_1\lambda^2$. On the other hand, for $v(x) = Ae^{-x^2/2}$, where $A^2 = \pi^{-1/2}\lambda$, it follows that $\|v\|_{L^2}^2 = \lambda$ and $(T(t)v)(x) = \frac{A}{\sqrt{1+2it}}\, e^{-x^2/(2+4it)}$. Thus

$$P_\lambda^{(\varepsilon)} \le \varphi^{(\varepsilon)}(v) = \frac{\varepsilon}{4}\lambda - 2^{-3/2}\pi^{-1/2}\arctan(2)\lambda^2 \le -C_2\lambda^2,$$

with the explicit constant $C_2 = 2^{-3/2}\pi^{-1/2}\arctan(2) - 1/8 > 0$. To verify $-C_1\lambda^2 \le P_\lambda \le -C_2\lambda^2$ one may proceed analogously, cf. [10, Lemma 2.5]. Concerning (2.1), we note that obviously $\varphi^{(\varepsilon_2)}(u) \ge \varphi^{(\varepsilon_1)}(u)$ for $\varepsilon_2 \ge \varepsilon_1 > 0$ and $u \in H^1$, whence $P_\lambda^{(\varepsilon_2)} \ge P_\lambda^{(\varepsilon_1)}$ shows that the function $\varepsilon \mapsto P_\lambda^{(\varepsilon)}$ is increasing, and thus tends to its infimum as $\varepsilon \to 0$. If $\varepsilon > 0$, $\lambda > 0$, and $u \in H^1$ is such that $\int_{\mathbb{R}} |u|^2\, dx = \lambda$, then $\varphi^{(\varepsilon)}(u) \ge \varphi(u) \ge P_\lambda$, and therefore $P_\lambda^{(\varepsilon)} \ge P_\lambda$ leads to $P_\lambda^* \ge P_\lambda$ in the limit $\varepsilon \to 0$. The relation $P_\lambda = \lambda^2 P_1$ is obtained from a straightforward scaling argument, and this in turn implies (2.2), cf. also [10, Lemma 2.5]. $\square$

The following lemma states a useful property of $\varphi$.

**Lemma 2.2** *There exists a constant $C > 0$ such that*

$$|\varphi(u) - \varphi(v)| \le C\Big(\|u\|_{L^2} + \|v\|_{L^2}\Big)^3 \|u - v\|_{L^2}, \quad u, v \in L^2. \tag{2.3}$$

**Proof:** Cf. [10, Lemma 2.2]. $\square$

The vanishing of the singularly perturbed term from (1.5) in the limit $\varepsilon \to 0$ is guaranteed by the next result.

**Lemma 2.3** *Let $(u_j) \subset H^1$ be a sequence of minimizers for $P_\lambda^{(\varepsilon_j)}$, for some sequence $\varepsilon_j \to 0$. Then*

$$\varepsilon_j \int_{\mathbb{R}} |u_j'|^2 \, dx \to 0, \quad j \to \infty.$$

**Proof:** By considering suitable subsequences we may assume that $\varepsilon_{j+1} \le \varepsilon_j/2$ for $j \in \mathbb{N}$. Then $u_j \in H^1$ and $\int_{\mathbb{R}} |u_j|^2 \, dx = \lambda$ imply

$$\begin{aligned} P_\lambda^{(\varepsilon_{j+1})} &\le \varphi^{(\varepsilon_{j+1})}(u_j) \\ &= \frac{\varepsilon_{j+1} - \varepsilon_j}{2} \int_{\mathbb{R}} |u_j'|^2 \, dx + \varphi^{(\varepsilon_j)}(u_j) \\ &= \frac{\varepsilon_{j+1} - \varepsilon_j}{2} \int_{\mathbb{R}} |u_j'|^2 \, dx + P_\lambda^{(\varepsilon_j)}, \end{aligned}$$

and consequently

$$\begin{aligned} \frac{\varepsilon_j}{4} \int_{\mathbb{R}} |u_j'|^2 \, dx &\le \frac{\varepsilon_j - \varepsilon_{j+1}}{2} \int_{\mathbb{R}} |u_j'|^2 \, dx \\ &\le P_\lambda^{(\varepsilon_j)} - P_\lambda^{(\varepsilon_{j+1})} \to P_\lambda^* - P_\lambda^* = 0, \quad j \to \infty, \end{aligned}$$

by (2.1). $\square$

To prove Theorem 1.2, we start by verifying that the $P_\lambda^{(\varepsilon)}$ converge to $P_\lambda$.

**Lemma 2.4** *As $\varepsilon \to 0$ we have $P_\lambda^{(\varepsilon)} \to P_\lambda$.*

**Proof:** Due to Lemma 2.1, we only have to show that $P_\lambda^* \le P_\lambda$, where

$$P_\lambda^* = \lim_{\varepsilon \to 0} P_\lambda^{(\varepsilon)} = \inf_{\varepsilon > 0} P_\lambda^{(\varepsilon)}.$$

To see this, we note that according to [10] (cf. Lemma 1.1) we have $P_\lambda = \varphi(v)$ for a suitable minimizing function $v \in L^2$ with $\int_{\mathbb{R}} |v|^2 \, dx = \lambda$. We fix $\phi \in C_0^\infty(\mathbb{R})$ with values in $[0, 1]$ such that $\phi(\xi) = 1$ for $|\xi| \le 1$ and $\phi(\xi) = 0$ for $|\xi| \ge 2$, and for $R > 0$ we write $\phi_R(\xi) = \phi(\xi/R)$. We then define $v_R = (\phi_R \hat{v})\check{} = \check{\phi}_R * v$ as well as $v^R = v - v_R$ to obtain $v = v_R + v^R$,

$$\|v^R\|_{L^2}^2 = \int_{|\xi| \ge R} |(1 - \phi_R)\hat{v}|^2 \, d\xi \le \int_{|\xi| \ge R} |\hat{v}|^2 \, d\xi =: \sigma(R) \tag{2.4}$$

and

$$\begin{aligned} \|v_R\|_{L^2}^2 &= \int_{\mathbb{R}} |\phi_R \hat{v}|^2 \, d\xi \\ &= \lambda - \int_{|\xi| \ge R} (1 - \phi_R^2)|\hat{v}|^2 \, d\xi \\ &\ge \lambda - \int_{|\xi| \ge R} |\hat{v}|^2 \, d\xi = \lambda - \sigma(R), \end{aligned}$$

with $\sigma(R) \to 0$ as $R \to \infty$. In addition,

$$\|v_R\|_{\dot{H}^1}^2 = \int_{\mathbb{R}} |\xi|^2 |\phi_R \hat{v}|^2 \, d\xi = \int_{|\xi| \le 2R} |\xi|^2 |\phi_R \hat{v}|^2 \, d\xi \le CR^2 \|\hat{v}\|_{L^2}^2 = CR^2. \tag{2.5}$$

Hence, setting $u_R = \lambda^{1/2} \|v_R\|_{L^2}^{-1} v_R$, we have $u_R \in H^1$ and $\|u_R\|_{L^2}^2 = \lambda$. By definition of $P_\lambda^{(\varepsilon)}$ this together with (2.3) yields the estimate

$$\begin{aligned}
P_\lambda^{(\varepsilon)} &\le \varphi^{(\varepsilon)}(u_R) \\
&= \frac{\varepsilon}{2} \int_{\mathbb{R}} |u_R'|^2 \, dx + \varphi(u_R) \\
&= \frac{\varepsilon}{2} \lambda \|v_R\|_{L^2}^{-2} \int_{\mathbb{R}} |v_R'|^2 \, dx + \lambda^2 \|v_R\|_{L^2}^{-4} \varphi(v_R) \\
&\le C\varepsilon(\lambda - \sigma(R))^{-1} R^2 + \lambda^2 (\lambda - \sigma(R))^{-2} \Big( |\varphi(v) - \varphi(v_R)| + \varphi(v) \Big) \\
&\le C\varepsilon R^2 + C\Big( \|v\|_{L^2} + \|v_R\|_{L^2} \Big)^3 \|v^R\|_{L^2} + \lambda^2 (\lambda - \sigma(R))^{-2} P_\lambda \\
&\le C\varepsilon R^2 + C\sigma(R)^{1/2} + (1 + C\sigma(R))^2 P_\lambda \,,
\end{aligned}$$

for $R$ sufficiently large (depending only on the fixed $\lambda$). Taking the limits $\varepsilon \to 0$ first and then $R \to \infty$, this leads to $P_\lambda^* \le P_\lambda$ as desired. □

To validate the remaining assertion of Theorem 1.2 we fix a sequence $(u_j) \subset H^1 \subset L^2$ of minimizers for $P_\lambda^{(\varepsilon_j)}$ corresponding to a sequence $\varepsilon_j \to 0$, i.e., we have $\int_{\mathbb{R}} |u_j|^2 \, dx = \lambda$ and $\varphi^{(\varepsilon_j)}(u_j) = P_\lambda^{(\varepsilon_j)}$ for $j \in \mathbb{N}$. Then

$$r_j := \varepsilon_j \int_{\mathbb{R}} |u_j'|^2 \, dx \to 0, \quad j \to \infty, \tag{2.6}$$

by Lemma 2.3. In view of Lemma 1.1 it suffices to verify that $(u_j)$ is a minimizing sequence for $P_\lambda$. To see this, we note that by definition of $\varphi$ and $\varphi^{(\varepsilon_j)}$, respectively,

$$|\varphi(u_j) - P_\lambda| = \left| \varphi^{(\varepsilon_j)}(u_j) - \frac{\varepsilon_j}{2} \int_{\mathbb{R}} |u_j'|^2 \, dx - P_\lambda \right| \le |P_\lambda^{(\varepsilon_j)} - P_\lambda| + \frac{1}{2} r_j \to 0, \quad j \to \infty,$$

due to Lemma 2.4 and (2.6). Hence we have $\lim_{j\to\infty} \varphi(u_j) = P_\lambda$, which in turn implies that $(u_j)$ is a minimizing sequence for $P_\lambda$. This completes the proof of Theorem 1.2. □

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