A Nonlinear Characterization of Hilbert Spaces

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Abstract

We show that a Banach space is Hilbert if and only if its duality map maps line segments to convex sets.

Properties of the duality map of a Banach space X often had been used to characterize geometric properties of X; cf. e.g. [1] or [3]. In this note we shall prove the following characterization of Hilbert spaces:

Theorem *Let $(X, |\cdot|)$ be a real Banach space. Then X is Hilbert if and only if the duality map $\mathcal{F}$ of X maps line segments to convex sets.*

Recall that the (in general multivalued) duality map $\mathcal{F}: X \to 2^X \setminus \{\emptyset\}$ of X is

$$\mathcal{F}(x) = \{x^* \in X^* : |x^*| = |x|, x^*(x) = |x|^2\} .$$

We shall also need the semi-inner products $(\cdot,\cdot)_\pm : X \times X \to \mathbb{R}$, definied as

$$(x,y)_+ = \max\{y^*(x) : y^* \in \mathcal{F}(y)\} \text{ and } (x,y)_- = \min\{y^*(x) : y^* \in \mathcal{F}(y)\}$$

for $x, y \in X$. All important properties of $\mathcal{F}$ and $(\cdot,\cdot)_\pm$ are collected in Propositions 12.3. and 13.1. of [2] or Proposition 32.22 of [4].

If $x, y \in X$, then $[x,y] := \{ty + (1-t)x : t \in [0,1]\}$ will be the line segment joining x and y, and if a line segment $[x,y]$ is given, we let $x_t := ty + (1-t)x$ for $t \in \mathbb{R}$; an analogous notation will be used for $x^*, y^* \in X^*$.

Before going on to the proof of the theorem, let us state some simple lemmas which will be needed in the sequel.

Lemma 1 *If* $|x|=|y|=1$ *and* $|x_{t_0}|=1$ *for some* $t_0\in(0,1)$*, then* $|x_t|=1$ *for all* $t\in[0,1]$.

Proof. You may draw a picture to see that e.g. $0<|x_t|<1$ for some $t\in(t_0,1)$ is impossible, since then $z=s\cdot x_{t_0}\in\mathbb{R}_0^+\cdot x_{t_0}\cap[x,|x_t|^{-1}x_t]$ for $s=[t-(1-|x_t|)\,t_0]^{-1}t>1$ must have $|z|\leq 1$, a contradiction. $\square$

Lemma 2 *If* $\dim X=2$, $f:X\to X^*$ *is one-to-one and satisfies* $f(\alpha x)=\alpha f(x)$ *and* $f([x,y])\subset[f(x),f(y)]$ *for* $\alpha\in\mathbb{R}$ *and* $x,y\in X$*, then* f *is linear.*

Proof. Suppose that $x,y\in X$ are linearly independent and $f(x_{1/2})=t_0\,f(y)+(1-t_0)\,f(x)$ for some $1/2\neq t_0\in[0,1]$. Then $x_{t_0}\in[x,y]\cap[2(1-t_0)x,2t_0y]$ implies $f(x_{t_0})\in[f(x),f(y)]\cap[f(2(1-t_0)x),f(2t_0y)]=\{f(x_{1/2})\}$, hence the contradiction $t_0=1/2$. Therefore $f(x+y)=f(x)+f(y)$. $\square$

Lemma 3 *X is Hilbert if and only if* $\mathcal{F}(x)+\mathcal{F}(y)\subset\mathcal{F}(x+y)$ *for all* $x,y\in X$*, and therefore in particular, if* $\mathcal{F}$ *is single-valued and linear.*

Proof. $|x\pm y|^2=(x^*\pm y^*)(x\pm y)$ for $x^*\in\mathcal{F}(x)$ and $y^*\in\mathcal{F}(y)$ implies the parallelogramm identity for $|\cdot|$. $\square$

Now we turn to the

Proof of the theorem: 1.) W.l.o.g. we may assume that $\dim X=2$, since in the general case the duality map $\mathcal{F}_0$ of $X_0=\mathbb{R}x\oplus\mathbb{R}y$ (with linearly independent $x,y\in X$) also maps line segments to convex sets. This clearly is implied by the following two facts, valid for $z\in X_0$: (i.) if $z^*\in\mathcal{F}_0(z)$, then $x^*\in\mathcal{F}(z)$ for every extension $x^*\in X^*$ of $z^*\in X_0^*$ with same norm as z^* (ii.) if $x^*\in\mathcal{F}(z)$, then $x^*|_{X_0}\in\mathcal{F}_0(z)$.

2.) If X were not strictly convex, then $|x|=1=|y|$ and $|x+y|=2$ for some linearly independent $x,y\in X$. Choose $x^*\in X^*$ and $y^*\in Y^*$ with $x^*(y)=(y,x)_-$ and $y^*(x)=(x,y)_-$. Then $x^*(y)=1=y^*(x)$ is impossible, since this would yield $x^*=y^*$, and hence $0\in[-y^*,y^*]\subset\mathcal{F}([-x,y])$, a contradiction. Moreover, $[x^*,y^*]\subset\mathcal{F}([x,y])$ implies that for every $s\in[0,1]$ there is a $t\in[0,1]$ with $x_s^*\in\mathcal{F}(x_t)$, and consequently

$$t\varphi(s)=s(1-y^*(x))\,,\ \text{where}\quad \varphi(s)=s(1-y^*(x))+(1-s)(x^*(y)-1)\qquad(1)$$

by Lemma 1. Now $x^*(y)<1$ and $y^*(x)<1$ is also impossible by (1), because $\varphi(0)<0<\varphi(1)$ and $\varphi'(s)>0$ in $[0,1]$ would give a zero $s_0\in(0,1)$ of φ in this case. Therefore we may assume w.l.o.g. that $x^*(y)<1$ and $y^*(x)=1$, hence $[x^*,y^*]\subset\mathcal{F}(x)$. For $z^*\in\mathcal{F}(x)$ we have $z^*=x^*_s$, where $s=(1-x^*(y))^{-1}(z^*-x^*)(y)\in[0,1]$, and consequently $[x^*,y^*]=\mathcal{F}(x)$. This implies

$$\frac{1}{2}(x^*-y^*)\in[x^*,-y^*]\subset\mathcal{F}([-x,x])=[-1,1]\cdot[x^*,y^*],$$

a contradiction to the fact that x^* and y^* are linearly independent.

3.) By using 2.) it may be shown in similiar way that X^* is strictly convex.

4.) By 2.) and 3.), $\mathcal{F}:X\to X^*$ is single-valued, one-to-one and onto with inverse $\mathcal{F}^*$, the duality map of X^*. Hence we only have to show that $\mathcal{F}([x,y])\subset[\mathcal{F}(x),\mathcal{F}(y)]$ for linearly independent $x,y\in X$ satisfying $|y|\le|x|$, by Lemma 2 and Lemma 3. If we let $x^*=\mathcal{F}(x)$ and $y^*=\mathcal{F}(y)$, then $[x^*,y^*]\subset\mathcal{F}([x,y])$ by assumption. This yields $x^*_s=\mathcal{F}(x_{\phi(s)})$ for a well-defined function $\phi:[0,1]\to[0,1]$ with $\phi(0)=0$ and $\phi(1)=1$. Since it is sufficient to prove that ϕ is onto, we shall check the continuity of ϕ. The definition of $\mathcal{F}$ implies

$$\phi(s)[s(|y|^2-y^*(x))+(1-s)(x^*(y)-|x|^2)]=|x^*_s|^2-sy^*(x)-(1-s)|x|^2$$

for $s\in[0,1]$. Therefore the interesting points are the zeros of $\varphi(s)$, the $[\dots]$ on the left-hand side. Because of $\varphi(0)\le 0$ and $\varphi'(s)=\varphi'(0)\ge 0$ it may be seen by distinguishing cases, that the most interesting one is $\varphi(0)<0$, $\varphi'(0)>0$ and $\varphi(1)\ge 0$. Then the unique zero of φ is $s_0=(\varphi'(0))^{-1}(|x|^2-x^*(y))\in(0,1]$, and $\lim_{s\to s_0}(|x^*_s|^2-sy^*(x)-(1-s)|x|^2)/\varphi(s)$ does exist, by the L'Hospital theorem, since $(d/ds)|x^*_s|^2=2(y^*-x^*,x^*_s)_-\to 2(y^*-x^*,x^*_{s_0})_-$ for $s\to s_0$. As a consequence, ϕ is onto, and this completes the proof. □

Let us finally note in passing that the symmetry of $(\cdot,\cdot)_+$ or $(\cdot,\cdot)_-$ is also sufficient for X to be Hilbert. Indeed, if one of the semi-inner products is symmetric, then both are, since $(x,-y)_\pm=(-x,y)_\pm=-(x,y)_\mp$. Moreover, for $x,y\in X$ and $t>0$ we have

$$\begin{aligned}
|x|\,(x-ty,y)_+ &= |x|\,(y,x-ty)_+\\
&= |x|\,|x-ty|\inf_{\lambda>0}\lambda^{-1}(|(x-ty)+\lambda y|-|x-ty|)\\
&\le |x-ty|\,|x|\,t^{-1}(|x|-|x-ty|)\\
&\le |x-ty|\,|x|\sup_{\lambda>0}\lambda^{-1}(|x|-|x-\lambda y|)=|x-ty|\,(y,x)_-\,.
\end{aligned}$$

Therefore the continuity of $(\cdot, y)_+$ implies $(\cdot,\cdot)_+ = (\cdot,\cdot)_-$. Hence $(\cdot,\cdot)_+$ is linear in the first, and due to the symmetry, also in the second argument. This clearly yields the parallelogramm identity for $|\cdot|$.

References

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