Exchange of stability and finite-dimensional dynamics in a bifurcation problem with marginally stable continuous spectrum

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April 21, 2003

Abstract

We consider solutions bifurcating from a spatially homogeneous equilibrium under the assumption that the associated linearization possesses continuous spectrum up to the imaginary axis, for all values of the bifurcation parameter, and that a real eigenvalue crosses the imaginary axis. For a model we investigate the nonlinear stability of the trivial solution with respect to spatially localized perturbations, prove the occurrence of a pitchfork bifurcation of equilibria and the nonlinear stability of the bifurcating equilibria, again with respect to spatially localized perturbations.

1 Introduction

The Lyapunov-Schmidt method and the centre manifold theorem can be applied if a ground state loses stability and allow us then to reduce a priori infinite-dimensional bifurcation problems to finite-dimensional ones. Both of these methods typically are useful if the spectrum of the linearization separates into a finite-dimensional part close to the imaginary axis and an infinite-dimensional part which lies strictly in the left half-plane of the complex plane. In the present paper we are interested in a situation where these spectral assumptions from classical bifurcation theory are not satisfied, in particular in the case when there are finitely many eigenvalues close to the imaginary axis, but the continuous spectrum of the linearization about the trivial solution touches the imaginary axis for all values of the bifurcation parameter. Such a situation is encountered in many applications. A typical example is the flow around a body in the Navier-Stokes equation [1, 2]. In these problems marginally stable solutions get unstable as finitely many eigenvalues are crossing the imaginary axis. Then those unstable systems show structures which are spatially localized, so that a connection could be expected to the finitely many unstable eigenfunctions that are as well spatially localized. Hence the question occurs if a finite dimensional description of the infinite dimensional nonlinear dynamics is possible.

As a first step towards an understanding of this question we consider here

$$\partial_t U = LU + N(U), \quad U|_{t=0} = U_0, \tag{1}$$

for $t \geq 0$, $x \in \mathbb{R}$, and $U(x,t) \in \mathbb{R}$ as an example of such a system, where

$$\begin{aligned} LU &= \partial_x^2 U + c\partial_x U + \alpha r U, \\ N(U) &= \partial_x(U^3), \\ r(x) &= \operatorname{sech}^2(x), \end{aligned}$$

for parameters $\alpha, c > 0$. This model seems to be quite special, but it already incorporates most aspects also shared by the real world examples which we have in mind. For all values of $\alpha, c > 0$ the linearization L possesses continuous spectrum up to the imaginary axis and finitely many eigenvalues. For simplicity, henceforth we take $\alpha = 2$ ensuring that for $c < 2$ there is only one eigenvalue of L at

$$\lambda_c = 1 - \frac{c^2}{4} \tag{2}$$

with an explicitly known eigenfunction $\varphi_{0,c}(x) = e^{-cx/2}\operatorname{sech}(x)$.

Remark 1.1 For later reference, we note that this follows by introducing $\tilde{U}(x) = e^{cx/2}U(x)$ and observing that $\tilde{L}\tilde{U} = \partial_x^2\tilde{U} + \alpha r(x)\tilde{U} = (\lambda + \frac{c^2}{4})\tilde{U}$ in case that $LU = \lambda U$; it is well-known that $\tilde{L}$ has a single eigenvalue $\tilde{\lambda} = 1$, with corresponding eigenfunction $\tilde{U}(x) = \operatorname{sech}(x)$; see [7, Sect. 4.3].

Although for L there is continuous spectrum up to the imaginary axis, this special system (1) allows for a *finite-dimensional reduction*, and it exhibits an *exchange of stability* from the trivial solution to the bifurcating equilibria as in classical bifurcation theory, when there is no spectrum close to the imaginary axis except for finitely many eigenvalues.

In contrast to classical bifurcation theory the stability of the trivial solution $U = 0$ is by no means obvious for $c > 2$, due to continuous spectrum up to the imaginary axis. Therefore as a first step in Section 2 we prove the nonlinear stability of the trivial solution $U = 0$ for $c > 2$ with respect to sufficiently small and spatially localized perturbations. In Section 3 we verify that a finite-dimensional reduction is possible, i.e., we show that close to $c = 2$ the Lyapunov-Schmidt method can be utilized to establish the occurrence of a pitchfork bifurcation of stationary solutions. In Section 4 we prove the exchange of stability, i.e., nonlinear stability of the bifurcating equilibria, again with respect to sufficiently small and spatially localized perturbations.

The basic ideas are now explained in the following remarks.

Remark 1.2 (a) The dynamics of the linearized system is as follows. The term αrU amplifies perturbations of $U = 0$ close to the point $x = 0$. By the term $c\partial_x U$, perturbations, initially concentrated at $x = 0$, are transported away from $x = 0$, whence the amplification loses its influence as time goes by. Hence after a certain time the diffusion term will play the major role and the perturbations will vanish with polynomial decay rates.
(b) The nonlinearity $\partial_x(U^3)$ is chosen in such a way that it is asymptotically irrelevant with respect to diffusion, i.e., the nonlinear system

$$\partial_t U = \partial_x^2 U + \partial_x(U^3), \quad U|_{t=0} = U_0, \tag{3}$$

shows the same asymptotic behaviour as the linear diffusion equation for sufficiently small and spatially localized data U_0. In three extended space directions all nonlinear terms are irrelevant with respect to diffusion, which hopefully will allow us to apply our techniques to more realistic examples.
(c) The nonlinearity $\partial_x(U^3)$ will play a role close to the point $x = 0$, where the spatial amplification αrU is located. It will lead to a nonlinear bound on the size of the bifurcating solutions.
(d) Note that $U = 0$ is a priori not an isolated equilibrium. Every constant $U = const.$ is a solution of (3), i.e., every C_b^0-neighbourhood of $U = 0$ contains infinitely many equilibria. However, $U = 0$ is isolated in a space of functions which decay to zero as $x \to \infty$.
(e) In order to recover the diffusive behaviour behind the amplification we use some exponential weights in space. See for instance [4] or [8] and the references cited therein.

Investigations of bifurcation scenarios including continuous spectrum can also be found in a number of other papers, see for instance [11, 5, 12, 10]. It is the aim of further research to apply the present ideas also to more complicated models.

Notation: Throughout the paper constants are generally just denoted by C. We use the space L^1 of Lebesgue-integrable functions equipped with the norm $\|u\|_{L^1} = \int_{\mathbb{R}} |u(x)|dx$, and the space C_b^0 of uniformly continuous and bounded functions equipped with the norm $\|u\|_{L^\infty} = \sup_{x\in\mathbb{R}} |u(x)|$. As usual, the norm on $L^1 \cap L^\infty$ is $\|u\|_{L^1\cap L^\infty} = \|u\|_{L^1} + \|u\|_{L^\infty}$. Sometimes we denote functions $x \mapsto r(x)$ by $r(x)$ instead of r, in particular $x \mapsto e^{\beta x}$ by $e^{\beta x}$.

Acknowledgements: This paper is a cooperation within the DFG-Schwerpunkt "Analysis, Numerik und Ergodentheorie dynamischer Systeme", supported by the grants Ku 587/5-3 and Mi 459/2-3.

2 Nonlinear stability of the trivial solution for $c > 2$

In this section we prove that the trivial solution $u = 0$ is nonlinearly stable for $c > 2$ with respect to sufficiently small spatially localized perturbations. This is by no means obvious, due to fact that the linearization possesses continuous spectrum up to the imaginary axis. The result will be established by simultaneously considering the solution U of (1) in three different topologies, and the idea is as follows.

To exploit the diffusive behaviour behind the amplification, we need estimates both in the $\|\cdot\|_{L^1}$– and the $\|\cdot\|_{L^\infty}$–norm; we make this evident in our notation by denoting the same solution U by $u = U$, if $u \in C_b^0$, and by $v = U$, if $v \in L^1$. In order to control the influence of the amplification, we use some exponentially weighted norm in space which guarantees that the initial perturbations are not too big in front of the amplification, i.e., we introduce $w(x,t) = U(x,t)e^{\beta x}$ with $\beta > 0$ to be chosen later appropriately.

A short calculation reveals that then u, v, and w are solutions of the system

$$\begin{aligned} \partial_t u &= \partial_x^2 u + c\partial_x u + \alpha r(x)e^{-\beta x}w + \partial_x(u^3), \\ \partial_t v &= \partial_x^2 v + c\partial_x v + \alpha r(x)e^{-\beta x}w + \partial_x(u^2 v), \\ \partial_t w &= \partial_x^2 w + (c-2\beta)\partial_x w + (\beta^2 - c\beta + \alpha r(x))w + (\partial_x - \beta)(u^2 w). \end{aligned} \tag{4}$$

In the equations for u and v, we expressed the spatially localized amplification term $\alpha r(x)U$ in terms of the weighted variable w. This is possible since $\sup_{x\in\mathbb{R}} |r(x)e^{-\beta x}| < \infty$, as long as we select $|\beta| < 2$. As in Remark 1.1, it can be shown that the spectrum of the linear operator

$$\Lambda w = \partial_x^2 w + (c-2\beta)\partial_x w + (\beta^2 - c\beta + \alpha r(x))w$$

consists of a single eigenvalue at $1 - c^2/4$ and of continuous spectrum contained in $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \leq \beta^2 - c\beta\}$. Hence for $c > 2$ and $\beta^2 - c\beta < 0$ the variable w will be exponentially damped. Thus, the diffusive variables u and v satisfy (with $\xi = x - ct$) the system

$$\partial_t u = \partial_\xi^2 u + \partial_\xi(u^3) + \ \ldots\ , \tag{5}$$

$$\partial_t v = \partial_\xi^2 v + \partial_\xi(u^2 v) + \ \ldots\ , \tag{6}$$

the dots indicating terms that vanish at an exponential rate. Note that this system is coupled by the initial conditions and that it is equivalent to (3), if we choose $u|_{t=0} = v|_{t=0} = U_0$ and $w|_{t=0} = U_0 e^{\beta x}$. For system (5), (6), it is well known that the diffusive behaviour of the linearized system can be used to control the nonlinear terms and to prove diffusive behaviour also for the nonlinear system. Thus, for sufficiently small spatially localized perturbations the u–variable will decay as $\mathcal{O}(t^{-1/2})$, the v–variable will remain bounded, and w will decay at an exponential rate.

These arguments are made rigorous in the following theorem.

Theorem 2.1 *Let $c > 2$, choose any $\beta \in]0, 2[$, and define*

$$\mu = -\frac{1}{3} \max\{1 - c^2/4,\ \beta^2 - c\beta\} > 0. \tag{7}$$

Then for all $\delta_1 > 0$ there exists $\delta_2 > 0$ such that for the solutions (u, v, w) of (4)

$$\sup_{t \in [0,\infty[} \Big((1+t)^{1/2} \|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{L^1} + e^{\mu t} \|w(\cdot,t)\|_{L^1 \cap L^\infty} \Big) \leq \delta_1,$$

provided the initial data satisfy $u(\cdot,0) = v(\cdot,0)$ and

$$\|u(\cdot,0)\|_{L^\infty} + \|u(\cdot,0)\|_{L^1} + \|w(\cdot,0)\|_{L^1 \cap L^\infty} \leq \delta_2.$$

Proof. We define

$$Bu = \partial_x^2 u + c\partial_x u \tag{8}$$

and

$$\Lambda w = \partial_x^2 w + (c - 2\beta)\partial_x w + (\beta^2 - c\beta + \alpha r(x))w. \tag{9}$$

Moreover, we let

$$\begin{aligned}
a(t) &= \sup_{s \in [0,t]} \Big((1+s)^{1/2} \|u(\cdot,s)\|_{L^\infty} \Big), \\
b(t) &= \sup_{s \in [0,t]} \|v(\cdot,s)\|_{L^1}, \\
c(t) &= \sup_{s \in [0,t]} \Big(e^{\mu s} \|w(\cdot,s)\|_{L^1 \cap L^\infty} \Big);
\end{aligned}$$

we henceforth omit the dot and just write $u(s)$, etc. Applying the variation of constants formula to (4) gives

$$\begin{aligned}
u(t) &= e^{Bt} u_0 + \int_0^t e^{B(t-s)} \Big[\alpha r e^{-\beta x} w + \partial_x(u^3) \Big](s)\, ds, \\
v(t) &= e^{Bt} u_0 + \int_0^t e^{B(t-s)} \Big[\alpha r e^{-\beta x} w + \partial_x(u^2 v) \Big](s)\, ds, \\
w(t) &= e^{\Lambda t} w_0 + \int_0^t e^{\Lambda(t-s)} \Big[(\partial_x - \beta)(u^2 w) \Big](s)\, ds.
\end{aligned}$$

We note that

$$\begin{aligned}
&\|e^{Bt}\|_{L^1 \to L^1} \leq C, \quad \|e^{Bt}\|_{L^1 \to L^\infty} \leq Ct^{-1/2}, \quad \|e^{Bt}\|_{L^\infty \to L^\infty} \leq C, \\
&\|e^{Bt}\partial_x\|_{L^1 \to L^1} \leq Ct^{-1/2}, \quad \|e^{Bt}\partial_x\|_{L^\infty \to L^\infty} \leq Ct^{-1/2}, \\
&\|e^{\Lambda t}\|_{L^1 \cap L^\infty \to L^1 \cap L^\infty} \leq Ce^{-2\mu t}, \quad \|e^{\Lambda t}\partial_x\|_{L^1 \cap L^\infty \to L^1 \cap L^\infty} \leq Ce^{-2\mu t} t^{-1/2},
\end{aligned}$$

for positive constants C. The first five estimates follow from the fact that the solution of $\partial_t u = \partial_x^2 u + c\partial_x u$, $u|_{t=0} = u_0$, is given by $u(x,t) = u_{\text{diffus}}(x + ct, t)$, where

$$u_{\text{diffus}}(x,t) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} u_0(y) dy.$$

Concerning the estimates on $e^{\Lambda t}$, these are a consequence of the spectrum of the sectorial operator Λ being contained in $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \leq -3\mu\}$; see [9, Thm. 1.3.4].

So we first get

$$\begin{aligned} e^{\mu t}\|w(t)\|_{L^1\cap L^\infty} &\leq e^{\mu t}\|e^{\Lambda t}w_0\|_{L^1\cap L^\infty} \\ &\quad + e^{\mu t}\int_0^t \|e^{\Lambda(t-s)}(\partial_x - \beta)\|_{L^1\cap L^\infty\to L^1\cap L^\infty}\|u^2(s)w(s)\|_{L^1\cap L^\infty}\,ds. \end{aligned}$$

Due to $\|u^2(s)w(s)\|_{L^1\cap L^\infty} \leq \|u(s)\|^2_{L^\infty}\|w(s)\|_{L^1\cap L^\infty}$ we can continue

$$\begin{aligned} e^{\mu t}\|w(t)\|_{L^1\cap L^\infty} &\leq Cc(0) + Ca(t)^2c(t)\,e^{\mu t}\int_0^t e^{-2\mu(t-s)}(t-s)^{-1/2}(1+s)^{-1}e^{-\mu s}\,ds \\ &\leq C\Big(c(0) + a(t)^2c(t)\Big), \end{aligned}$$

the latter since the integral is a bounded function of t. Taking the supremum over time yields

$$c(t) \leq C\Big(c(0) + a(t)^2c(t)\Big) \tag{10}$$

for some constant $C > 0$.

Next we have

$$\begin{aligned} \|v(t)\|_{L^1} &\leq \|e^{Bt}\|_{L^1\to L^1}\|u_0\|_{L^1} + \int_0^t \|e^{B(t-s)}\partial_x\|_{L^1\to L^1}\|u^2(s)v(s)\|_{L^1}\,ds \\ &\quad + \int_0^t \|e^{B(t-s)}\|_{L^1\to L^1}\|\alpha r e^{-\beta x}w(s)\|_{L^1}\,ds \\ &\leq Cb(0) + Ca(t)^2b(t)\Big(\int_0^t (t-s)^{-1/2}(1+s)^{-1}\,ds\Big) \\ &\quad + C\int_0^t e^{-\mu s}\Big(e^{\mu s}\|w(s)\|_{L^1\cap L^\infty}\Big)\,ds, \end{aligned}$$

where we used that $\sup_{x\in\mathbb{R}}|r(x)e^{-\beta x}| < \infty$ for $|\beta| < 2$. The first integral even decays as $\mathcal{O}(t^{-1/2}\ln t)$ whereas $\int_0^t e^{-\mu s}\,ds$ is bounded. Hence we obtain

$$b(t) \leq C\Big(b(0) + a(t)^2b(t) + c(t)\Big). \tag{11}$$

Finally,

$$\begin{aligned} &(1+t)^{1/2}\|u(t)\|_{L^\infty} \\ &\quad\leq (1+t)^{1/2}\min\Big\{\|e^{Bt}\|_{L^1\to L^\infty}\|u_0\|_{L^1}, \|e^{Bt}\|_{L^\infty\to L^\infty}\|u_0\|_{L^\infty}\Big\} \\ &\qquad + (1+t)^{1/2}\int_0^t \|e^{B(t-s)}\partial_x\|_{L^\infty\to L^\infty}\|u(s)\|^3_{L^\infty}\,ds \\ &\qquad + (1+t)^{1/2}\int_0^t \|e^{B(t-s)}\|_{L^1\to L^\infty}\|\alpha r e^{-\beta x}w(s)\|_{L^1}\,ds \\ &\quad\leq C\Big(a(0) + b(0)\Big) + Ca(t)^3\Big((1+t)^{1/2}\int_0^t (t-s)^{-1/2}(1+s)^{-3/2}\,ds\Big) \\ &\qquad + Cc(t)\Big((1+t)^{1/2}\int_0^t (t-s)^{-1/2}e^{-\mu s}ds\Big), \end{aligned}$$

The first integral may be evaluated explicitly as $2\frac{\sqrt{t}}{1+t}$, whence it is bounded when multiplied by $(1+t)^{1/2}$. Also the term after $c(t)$ is bounded, as follows e.g. splitting $\int_0^t ds = \int_0^{t/2} ds + \int_{t/2}^t ds$. Thus we obtain

$$a(t) \leq C\Big(a(0) + b(0) + a(t)^3 + c(t)\Big). \tag{12}$$

Summarizing estimates (12), (11) and (10), we have shown that for $t \in [0, \infty[$

$$\begin{aligned} a(t) &\leq C_1\Big(a(0) + b(0) + a(t)^3 + c(t)\Big), \\ b(t) &\leq C_1\Big(b(0) + a(t)^2 b(t) + c(t)\Big), \\ c(t) &\leq C_1\Big(c(0) + a^2(t)c(t)\Big), \end{aligned}$$

for a constant $C_1 > 0$. For given $\delta_1 > 0$, w.l.o.g. $\frac{C_1}{9}\delta_1^2 \leq \frac{1}{2}$, we hence can select

$$\delta_2 < \min\left\{\frac{1}{3}, \frac{1}{6C_1(1+2C_1)}\right\}\delta_1,$$

as may be seen by considering the longest time interval $[0, t_1]$ on which $a(t) \leq \frac{1}{3}\delta_1$, $b(t) \leq \frac{1}{3}\delta_1$, and $c(t) \leq \frac{1}{3}\delta_1$ are satisfied simultaneously. Then the estimate for c implies $c(t) \leq 2C_1 c(0) \leq 2C_1\delta_2$ on $[0, t_1]$, and this can be used in the remaining estimates on a and b to see that in fact $t_1 = \infty$. $\square$

For the original system (1) we have shown:

Corollary 2.2 *Let $c > 2$, choose any $\beta \in]0, 2[$, and define*

$$\mu = -\frac{1}{3}\max\{1 - c^2/4,\ \beta^2 - c\beta\} > 0.$$

Then for all $\delta_1 > 0$ there exists $\delta_2 > 0$ such that for the solution U of (1)

$$\sup_{t\in[0,\infty[}\Big((1+t)^{1/2}\|U(\cdot,t)\|_{L^\infty} + \|U(\cdot,t)\|_{L^1} + e^{\mu t}\|U(\cdot,t)e^{\beta x}\|_{L^1\cap L^\infty}\Big) \leq \delta_1,$$

provided the data satisfy

$$\|U(\cdot,0)\|_{L^\infty} + \|U(\cdot,0)\|_{L^1} + \|U(\cdot,0)e^{\beta x}\|_{L^1\cap L^\infty} \leq \delta_2.$$

Remark 2.3 (a) It is not necessary to assume $u(\cdot,0) = v(\cdot,0)$ in Theorem 2.1, if an additional estimate on $\|u(\cdot,t)\|_{L^1}$ is included. More precisely, we have similarly

$$\sup_{t\in[0,\infty[}\Big(\|u(\cdot,t)\|_{L^1} + (1+t)^{1/2}\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{L^1} + e^{\mu t}\|w(\cdot,t)\|_{L^1\cap L^\infty}\Big) \leq \delta_1,$$

if

$$\|u(\cdot,0)\|_{L^1} + \|u(\cdot,0)\|_{L^\infty} + \|v(\cdot,0)\|_{L^1} + \|w(\cdot,0)\|_{L^1\cap L^\infty} \leq \delta_2$$

holds for the data.

(b) With the help of the maximum principle, for this special nonlinearity we can prove in the same way the following result.

Let $c > 2$, choose any $\beta \in]0, 2[$, and define μ by (7). Then for all $\delta_1 > 0$ there exists $\delta_2 > 0$ such that

$$\sup_{t\in[0,\infty[} \Big(\|u(\cdot,t)\|_{L^\infty} + e^{\mu t}\|w(\cdot,t)\|_{L^\infty} \Big) \leq \delta_1,$$

if the data satisfy

$$\|u(\cdot,0)\|_{L^\infty} + \|w(\cdot,0)\|_{L^\infty} \leq \delta_2.$$

3 The reduction and occurrence of a pitchfork bifurcation

In this section we consider (1) for $0 < 1-\frac{c^2}{4} =: \varepsilon^2 \ll 1$. Since now there is one positive eigenvalue $\lambda_c = \varepsilon^2 > 0$, cf. (2), the trivial solution $U = 0$ is unstable. Here we are interested in the bifurcation scenario for $\varepsilon > 0$ close to zero, and due to the continuous spectrum of L not lying strictly in the left half-plane, classical methods of bifurcation theory (as the Lyapunov-Schmidt method and the centre manifold reduction) a priori do not apply.
However, according to the form of the spectrum for $\varepsilon > 0$ there exists a one-dimensional unstable manifold tangential to the subspace spanned by the eigenfunction $\varphi_{0,c}(x) = e^{-cx/2}\mathrm{sech}(x)$ associated to the eigenvalue λ_c. Because of the subsequent analysis we expect this manifold to end in the two equilibria which we show to bifurcate from the trivial solution.

In order to prove this result, we again introduce a weight in space which forces the functions to vanish exponentially for $x \to \infty$. Then the equilibrium $U = 0$ will be isolated, since all other equilibria closeby in C_b^0 are excluded. The special form of the equation will allow us to apply a Lyapunov-Schmidt reduction.

As before we consider $u = U \in C_b^0$ and $w \in C_b^0$, where $w(x,t) = U(x,t)e^{\beta x}$; we do not need v any longer. Here $\beta \in]0, 2[$ is fixed with $\beta < c < 2$, independently of $\varepsilon > 0$ small enough. We are looking for stationary solutions of (4), and they satisfy

$$\begin{aligned} (13)\qquad 0 &= \partial_x^2 u + c\partial_x u + \alpha r(x)e^{-\beta x}w + \partial_x(u^3), \\ (14)\qquad 0 &= \partial_x^2 w + (c-2\beta)\partial_x w + (\beta^2 - c\beta + \alpha r(x))w + (\partial_x - \beta)(u^2 w), \end{aligned}$$

which we write in an abstract way as

$$\begin{aligned} (15)\qquad 0 &= Bu + Mw + \partial_x(u^3), \\ (16)\qquad 0 &= \Lambda w + (\partial_x - \beta)(u^2 w), \end{aligned}$$

with $(Mw)(x) = \alpha r(x)e^{-\beta x}w(x) = 2\mathrm{sech}^2(x)e^{-\beta x}w(x)$, and B and Λ from (8) and (9), respectively.

The linear operator B has continuous spectrum $\{-k^2 + cik \mid k \in \mathbb{R}\}$ and no eigenvalues, whereas the linear operator Λ possesses continuous spectrum $\{-k^2 + (c - 2\beta)ik + \beta^2 - c\beta \mid k \in \mathbb{R}\}$ and the eigenvalue $\lambda_c = \varepsilon^2$, as may be verified by the method mentioned in Remark 1.1.

Therefore Λ has a spectral gap of order $\mathcal{O}(|\beta^2 - c\beta|) = \mathcal{O}(1)$, and so we can introduce the $\mathcal{O}(1)$-bounded and Λ-invariant projection on the space spanned by the associated eigenfunction $\varphi_{\beta,c}(x) = e^{\beta x} e^{-cx/2} \mathrm{sech}(x)$. This projection can be defined by means of a Dunford integral $P_{\beta,c}(w) = \frac{1}{2\pi i} \int_\Gamma (\Lambda - \lambda \mathrm{Id})^{-1} w \, d\lambda$ in C_b^0, where $\Gamma \subset \mathbb{C}$ is a curve in the resolvent set of Λ surrounding the eigenvalue λ_c. For brevity we set $E_c = P_{\beta,c}$, $E_s = \mathrm{Id} - E_c$, and we define the linear and bounded $e_c : C_b^0 \to \mathbb{R}$ implicitly by $e_c(u)\varphi_{\beta,c} = E_c(u)$. Since we will need an explicit formula for E_c as well for $c \neq 2\beta$, we remark that

$$E_c(w) = \frac{1}{2} \Big(e^{-\beta x} e^{cx/2} w, \mathrm{sech}(\cdot) \Big)_{L^2} \varphi_{\beta,c} \, , \tag{17}$$

at least for functions w for which this expression makes sense. This may be shown by switching from w to $\tilde{w}(x) = e^{-\beta x} e^{cx/2} w(x)$ and noting that then $\Lambda w = \lambda w + g$ reads as $\partial_x^2 \tilde{w} + \alpha r(x) \tilde{w} = (\lambda + \frac{c^2}{4}) \tilde{w} + e^{-\beta x} e^{cx/2} g$. The L^2-projection associated with the eigenvalue $\tilde{\lambda} = 1$ and eigenfunction $\tilde{w}(x) = \mathrm{sech}(x)$ is $\tilde{E}_c(\tilde{w}) = |\mathrm{sech}(\cdot)|_{L^2}^{-2} (\tilde{w}, \mathrm{sech}(\cdot))_{L^2} \mathrm{sech}(\cdot) = \frac{1}{2} (\tilde{w}, \mathrm{sech}(\cdot))_{L^2} \mathrm{sech}(\cdot)$, and transforming back, we arrive at (17). With the projections $E_c : C_b^0 \to \mathrm{range}(E_c) = \mathbb{R}\varphi_{\beta,c} =: X_c$ and $E_s = \mathrm{Id} - E_c : C_b^0 \to \mathrm{range}(E_s) =: X_s$, the phase space C_b^0 is split as $C_b^0 = X_c \oplus X_s$. By construction we have the invariance $E_c \Lambda = \Lambda E_c$ as well as $E_s \Lambda = \Lambda E_s$, see e.g. [14, Thm. 5.7], and we moreover note that the projections are bounded independently of small ε.

Returning now to (15), (16), we make the ansatz

$$\begin{aligned} u &= -\varepsilon w_c (B^{-1} M \varphi_{\beta,c}) + \varepsilon^2 \tilde{u}, \\ w &= \varepsilon w_c \varphi_{\beta,c} + \varepsilon^{5/2} w_s, \end{aligned}$$

for the solution of (15), (16), where $w_c \in \mathbb{R}$, $\tilde{u} \in C_b^0$, and $w_s \in X_s$. It is explained below in Lemma 3.2 why $B^{-1}M$ is a bounded operator. A short calculation reveals that the equations for the new variables $(\tilde{u}, w_c, w_s)$ to be satisfied are

$$0 = B\tilde{u} + \varepsilon^{1/2} M w_s + \varepsilon \partial_x \Big(\Big[- w_c (B^{-1} M \varphi_{\beta,c}) + \varepsilon \tilde{u} \Big]^3 \Big), \tag{18}$$

$$\begin{aligned} 0 = {} & w_c + w_c^3 \, e_c \Big((\partial_x - \beta) \Big[(B^{-1} M \varphi_{\beta,c})^2 \varphi_{\beta,c} \Big] \Big) \\ & -2\varepsilon w_c^2 \, e_c \Big((\partial_x - \beta) \Big[(B^{-1} M \varphi_{\beta,c}) \, \tilde{u} \, \varphi_{\beta,c} \Big] \Big) \\ & +\varepsilon^2 w_c \, e_c \Big((\partial_x - \beta) \Big[\tilde{u}^2 \varphi_{\beta,c} \Big] \Big) \\ & +\varepsilon^{3/2} e_c \Big((\partial_x - \beta) \Big(\Big[- w_c (B^{-1} M \varphi_{\beta,c}) + \varepsilon \tilde{u} \Big]^2 w_s \Big) \Big), \end{aligned} \tag{19}$$

$$0 = \Lambda_s w_s + \varepsilon^{1/2} E_s (\partial_x - \beta) \Big(\Big[- w_c (B^{-1} M \varphi_{\beta,c}) + \varepsilon \tilde{u} \Big]^2 [w_c \varphi_{\beta,c} + \varepsilon^{3/2} w_s] \Big), \tag{20}$$

where $\Lambda_s = \Lambda|_{X_s} = \Lambda E_s$. To motivate the next steps, note that the system is of the principal form

$$\begin{aligned} 0 &= B\tilde{u} + \mathcal{O}(\varepsilon^{1/2}), \\ 0 &= w_c + w_c^3\, e_c\Big((\partial_x - \beta)\Big[\left(B^{-1}M\varphi_{\beta,c}\right)^2 \varphi_{\beta,c}\Big]\Big) + \mathcal{O}(\varepsilon), \\ 0 &= \Lambda_s w_s + \mathcal{O}(\varepsilon^{1/2}). \end{aligned}$$

In the second equation we have

$$0 = w_c - \gamma_c w_c^3 + \mathcal{O}(\varepsilon),$$

where γ_c are the constants defined by

$$\gamma_c = -e_c\Big((\partial_x - \beta)\Big[\left(B^{-1}M\varphi_{\beta,c}\right)^2 \varphi_{\beta,c}\Big]\Big). \tag{21}$$

The properties of γ_c are summarized in the following

Lemma 3.1 *For $\varepsilon > 0$ small enough we have $\gamma_c > 0$. Moreover, if for fixed β we let*

$$\gamma = \gamma_{c=2} = -e_{c=2}\Big((\partial_x - \beta)\Big[\left(B^{-1}M\varphi_{\beta,c=2}\right)^2 \varphi_{\beta,c=2}\Big]\Big),$$

then

$$|\gamma_c - \gamma| \leq C\varepsilon^2. \tag{22}$$

Proof. By definition of γ_c in (21), and in view of (17), we have

$$\begin{aligned} \gamma_c &= -\frac{1}{2}\int_{\mathbb{R}} e^{-\beta x} e^{cx/2} \operatorname{sech}(x)\,(\partial_x - \beta)\,\Big[\left(B^{-1}M\varphi_{\beta,c}\right)^2 \varphi_{\beta,c}\Big]\,dx \\ &= \frac{1}{2}\int_{\mathbb{R}} e^{-\beta x} e^{cx/2} \operatorname{sech}(x)\,\Big[\frac{c}{2} - \operatorname{sech}(x)\sinh(x)\Big]\left(B^{-1}M\varphi_{\beta,c}\right)^2 \varphi_{\beta,c}\,dx. \end{aligned} \tag{23}$$

Since this tends to

$$\gamma = \gamma_{c=2} = \frac{1}{2}\int_{\mathbb{R}} e^{-\beta x} \operatorname{sech}^2(x)\left(B^{-1}M\varphi_{\beta,c}\right)^2 \varphi_{\beta,c}\,dx > 0$$

as $c \to 2^-$, the continuity of the right-hand side of (23) in c implies that also $\gamma_c > 0$ for $|c-2|$ small, i.e., $\varepsilon > 0$ small. Expansion with respect to ε shows estimate (22). □

Hence for $\varepsilon = 0$ the equations are

$$0 = B\tilde{u}, \quad 0 = w_c - \gamma w_c^3, \quad 0 = \Lambda_s w_s,$$

having the non-trivial solutions $(0, \pm\gamma^{-1/2}, 0)$. It remains to prove persistence of these solutions under the higher order terms.

First we take $(0, \gamma^{-1/2}, 0)$ as new origin and introduce $\tilde{a}$ by $w_c = \gamma^{-1/2} + \tilde{a}$. Then (18)-(20), written in new variables $(\tilde{u}, \tilde{a}, w_s)$, read as

$$\tilde{u} = -\varepsilon^{1/2}(B^{-1}Mw_s) - \varepsilon B^{-1}\partial_x\Big(g_1(\tilde{u}, \tilde{a}, w_s)\Big), \tag{24}$$

$$\tilde{a} = g_2(\tilde{u}, \tilde{a}, w_s), \tag{25}$$

$$w_s = -\varepsilon^{1/2}\Lambda_s E_s(\partial_x - \beta)\Big(g_3(\tilde{u}, \tilde{a}, w_s)\Big), \tag{26}$$

where

$$\begin{aligned}
g_1(\tilde{u}, \tilde{a}, w_s) &= \Big(- [\gamma^{-1/2} + \tilde{a}](B^{-1}M\varphi_{\beta,c}) + \varepsilon\tilde{u}\Big)^3, \\
g_2(\tilde{u}, \tilde{a}, w_s) &= \frac{1}{2}\Big\{3\gamma^{-1}(\gamma - \gamma_c)\tilde{a} + \gamma^{-3/2}(\gamma - \gamma_c) - \Big(3\gamma^{-1/2}\tilde{a}^2 + \tilde{a}^3\Big)\gamma_c \\
&\quad -2\varepsilon[\gamma^{-1/2} + \tilde{a}]^2\, e_c\Big((\partial_x - \beta)\Big[(B^{-1}M\varphi_{\beta,c})\, \tilde{u}\, \varphi_{\beta,c}\Big]\Big) \\
&\quad +\varepsilon^2[\gamma^{-1/2} + \tilde{a}]\, e_c\Big((\partial_x - \beta)\Big[\tilde{u}^2\varphi_{\beta,c}\Big]\Big) \\
&\quad +\varepsilon^{3/2}e_c\Big((\partial_x - \beta)\Big(\Big[- [\gamma^{-1/2} + \tilde{a}](B^{-1}M\varphi_{\beta,c}) + \varepsilon\tilde{u}\Big]^2 w_s\Big)\Big)\Big\}, \\
g_3(\tilde{u}, \tilde{a}, w_s) &= \Big(- [\gamma^{-1/2} + \tilde{a}](B^{-1}M\varphi_{\beta,c}) + \varepsilon\tilde{u}\Big)^2 \Big([\gamma^{-1/2} + \tilde{a}]\varphi_{\beta,c} + \varepsilon^{3/2}w_s\Big).
\end{aligned}$$

We define the operator

$$F = F(\tilde{u}, \tilde{a}, w_s) : \quad B_{\delta_1}^{C_b^0}(0) \times \Big(] - C_1\varepsilon, C_1\varepsilon[\Big) \times B_{\delta_1}^{X_s}(0) \to C_b^0 \times \mathbb{R} \times X_s$$

through the right-hand sides in (24)-(26), the superscripts denoting the space in which the respective balls are taken; the constants $\delta_1 > 0$ and $C_1 > 0$ are to be chosen later.
To estimate F, we utilize the following technical lemma.

Lemma 3.2 *We have*

$$\begin{aligned}
&\|B^{-1}M\|_{C_b^0 \to C_b^0} \leq C, \quad \|B^{-1}\partial_x\|_{C_b^0 \to C_b^0} \leq C, \\
&\|e_c(\partial_x \cdot)\|_{C_b^0 \to \mathbb{R}} \leq C, \quad \|\Lambda_s^{-1}E_s(\partial_x - \beta)\|_{C_b^0 \to C_b^0} \leq C,
\end{aligned}$$

for a constant $C > 0$ independent of ε.

Proof. Observe that the first bound follows from the second, since

$$B^{-1}Mw = B^{-1}\partial_x\Big(\int_{-\infty}^{x} 2\,\mathrm{sech}^2(y)e^{-\beta y}w(y)\,dy\Big).$$

Concerning the second, the unique bounded solution of $\partial_x^2 u + c\partial_x u = Bu = \partial_x w$ which depends linearly (not affine) on w is

$$u(x) = e^{-cx}\int_{-\infty}^{x} e^{cy}w(y)\,dy,$$

hence $\|B^{-1}\partial_x w\|_{L^\infty} = \|u\|_{L^\infty} \leq C\|w\|_{L^\infty}$. Next, $\|e_c(\partial_x \cdot)\|_{C_b^0 \to \mathbb{R}} \leq C$ follows from (17) through integration by parts.

In order to show that the operator $\Lambda_s^{-1} E_s(\partial_x - \beta)$ can be extended to a bounded operator in C_b^0 we first note that $\Lambda_s^{-1} = \Lambda_s^{-1} E_s$ is a bounded operator from C_b^0 to C_b^2. If we denote by $[M_1, M_2] = M_1 M_2 - M_2 M_1$ the commutator of the two operators M_1 and M_2, we calculate

$$
\begin{aligned}
\Lambda_s^{-1}\partial_x &= \Lambda_s^{-1}\partial_x \Lambda_s \Lambda_s^{-1} = \partial_x \Lambda_s^{-1} + \Lambda_s^{-1}[\partial_x, \Lambda_s]\Lambda_s^{-1}, \\
[\partial_x, \Lambda_s] &= \partial_x \Lambda E_s - \Lambda E_s \partial_x = [\partial_x, \Lambda] E_s - \Lambda [E_s, \partial_x], \\
[E_s, \partial_x] &= (\mathrm{Id} - E_c)\partial_x - \partial_x(\mathrm{Id} - E_c) = -[E_c, \partial_x].
\end{aligned}
$$

Since E_c is finite-dimensional, $[E_s, \partial_x] : C_b^0 \to C_b^n$ is a bounded operator for every $n \in \mathbb{N}_0$, in particular $[E_s, \partial_x] : C_b^2 \to C_b^2$ is bounded. Due to

$$
\begin{aligned}
[\partial_x, \Lambda]w &= \partial_x(\Lambda w) - \Lambda(\partial_x w) \\
&= \partial_x\Big(\partial_x^2 w + (c - 2\beta)\partial_x w + (\beta^2 - c\beta + \alpha r(x))w\Big) \\
&\quad - \partial_x^3 w - (c - 2\beta)\partial_x^2 w - (\beta^2 - c\beta + \alpha r(x))\partial_x w = \alpha r'(x) w,
\end{aligned}
$$

the operator $[\partial_x, \Lambda] : C_b^0 \to C_b^0$ loses no derivatives and is bounded. Putting all these observations together, we hence find that $\Lambda_s^{-1}\partial_x$ can be extended to a bounded operator in C_b^0; note that here $\Lambda_s^{-1}\Lambda[E_c, \partial_x]\Lambda_s^{-1} w = [E_c, \partial_x]\Lambda_s^{-1} w$, since $\Lambda_s^{-1} w \in C_b^2$, thus also $[E_c, \partial_x]\Lambda_s^{-1} w \in C_b^2$. $\square$

Writing $F = (F_1, F_2, F_3)$ it hence follows that

$$
\begin{aligned}
\|F_1(\tilde{u}, \tilde{a}, w_s)\|_{L^\infty} &\leq C\varepsilon^{1/2}\|w_s\|_{L^\infty} + C\varepsilon\Big(1 + |\tilde{a}|^3 + \|\tilde{u}\|_{L^\infty}^3\Big), \\
|F_2(\tilde{u}, \tilde{a}, w_s)| &\leq C\Big[\varepsilon^2|\tilde{a}| + \varepsilon^2 + |\tilde{a}|^2 + |\tilde{a}|^3 + \varepsilon(1 + |\tilde{a}|^2)\|\tilde{u}\|_{L^\infty} + \varepsilon^2(1 + |\tilde{a}|)\|\tilde{u}\|_{L^\infty}^2 \\
&\qquad + \varepsilon^{3/2}(1 + |\tilde{a}|^2)(1 + \|\tilde{u}\|_{L^\infty}^2)\|w_s\|_{L^\infty}\Big], \\
\|F_3(\tilde{u}, \tilde{a}, w_s)\|_{L^\infty} &\leq C\varepsilon^{1/2}\Big(1 + |\tilde{a}|^2 + \|\tilde{u}\|_{L^\infty}^2\Big)(1 + |\tilde{a}| + \|w_s\|_{L^\infty}).
\end{aligned}
$$

Thus we may select $\delta_1 > 0$ and $C_1 > 0$ such that F is a self-map and a contraction for $\varepsilon > 0$ sufficiently small. Therefore F has a unique fixed point, and working the above steps backwards, we have found a solution of (15) and (16), i.e., of (13) and (14). Since we can use the same argument with $-\gamma^{-1/2}$ instead of $+\gamma^{-1/2}$, we have shown:

Theorem 3.3 *For $\varepsilon > 0$ small, there exist two nontrivial equilibria $(u_\pm, w_\pm)$ of (13) and (14), satisfying*

$$
\begin{aligned}
u_\pm(x) &= \pm\varepsilon\gamma^{-1/2}(B^{-1}M\varphi_{\beta,c})(x) + \mathcal{O}(\varepsilon^2), \\
w_\pm(x) &= \pm\varepsilon\gamma^{-1/2}\varphi_{\beta,c}(x) + \mathcal{O}(\varepsilon^2),
\end{aligned}
$$

in $C_b^0 \times C_b^0$. In a C_b^0-neighbourhood of the origin, these functions are the only nontrivial equilibria for system (13) and (14).

In particular, we have found stationary solutions $u_\pm$ of (1). Note also that in order to get the formulas for $u_\pm$ and $w_\pm$, we have used that

$$|\tilde{a}| = \mathcal{O}(\varepsilon)$$

for the fixed point $(\tilde{u}, \tilde{a}, w_s)$, as follows from the definition of F.
Moreover, we remark that this theorem is not contradicting the fact that the unstable manifold at $U = 0$ for (1) is tangential to the eigenfunction $\varphi_{0,c}$. We have here a singular situation, and the coefficients for the flow on the unstable manifold diverge as $\varepsilon \to 0^+$.

We fix a further property of u_+ that will play a role in the next section.

Lemma 3.4 *There is a constant $a \in \mathbb{R}$ such that $\lim_{x\to-\infty} u_+(x) = a = \mathcal{O}(\varepsilon)$. More precisely, for all $\beta \in]0,2[$ there exists $C > 0$ such that*

$$|u_+(x) - a|e^{-\beta x} \leq C \quad \textit{and} \quad |u_+'(x)|e^{-\beta x} \leq C, \quad x \in \mathbb{R}. \tag{27}$$

Proof. Stationary solutions $U = U(x) = u_+(x)$ of (1) satisfy the ODE

$$\begin{aligned} U' &= V, \\ V' &= -cV - \alpha r(x)U - 3U^2V. \end{aligned} \tag{28}$$

For $x \to \pm\infty$ this non-autonomous system becomes the autonomous system

$$\begin{aligned} U' &= V, \\ V' &= -cV - 3U^2V. \end{aligned} \tag{29}$$

The solutions of (29) which are uniformly bounded for all $x \in \mathbb{R}$ are exactly the equilibria $(U, V) = (a, 0)$ with $a \in \mathbb{R}$. Therefore every uniformly bounded solution of (28) has to converge towards an equilibrium point $(U, V) = (a, 0)$ for $x \to -\infty$ (and also for $x \to \infty$).

The linearization about $(U, V) = (a, 0)$ in (29) possesses the eigenvalues 0 and $-c - 3a^2$. Hence there exists a one-dimensional stable manifold and a one-dimensional centre manifold which coincides with the family of equilibria $\{(U, V) \mid V = 0\}$. A solution of (28) converging to $(a, 0)$ for $x \to -\infty$ has to be in the centre manifolds of $(0, 0)$. The centre manifolds of the non-autonomous system (28) converge with rate $|\alpha r(x)u_+(x)| \sim \varepsilon e^{-2|x|}$ towards the centre manifold of the autonomous system (29). But since the flow on the centre manifold of the autonomous system (29) is trivial, (27) follows. We note in passing that a similar reasoning yields $\lim_{x\to+\infty} u_+(x) = 0$. $\square$

4 Nonlinear stability of the bifurcating equilibria

In this section we prove the nonlinear stability of the bifurcating equilibria $u_\pm$, again with respect to sufficiently small spatially localized perturbations. Once more the difficulty results from the linearizations about $u_\pm$ possessing continuous spectrum up the imaginary axis. The following theorem can be interpreted as an *exchange of stability* from $U = 0$ to $U = u_\pm$, similar to the case of mere discrete spectrum within the framework of classical bifurcation theory.

The proof is related to the one given in [8] for the stability of modulated fronts in the Swift-Hohenberg equation.

We return to (1) for $0 < 1 - \frac{c^2}{4} = \varepsilon^2 \ll 1$ and introduce the deviation u from the equilibrium $U_0 = u_\pm$ by $U(x,t) = U_0(x) + u(x,t)$; we henceforth consider $U_0 = u_+$. Then u satisfies

$$\partial_t u = \partial_x^2 u + c\partial_x u + \alpha r(x)u + 3\partial_x(U_0^2 u) + 3\partial_x(U_0 u^2) + \partial_x(u^3).$$

We know that U_0 converges to a constant $a = \mathcal{O}(\varepsilon)$ as $x \to -\infty$, recall Lemma 3.4. Making the ansatz $U_0(x) = a + g(x)$, it follows that

$$\begin{aligned}\partial_t u &= \partial_x^2 u + c\partial_x u + \alpha r(x)u + 3a^2\partial_x u + 6a\partial_x(gu) + 3\partial_x(g^2 u)\\ &\quad +3a\partial_x(u^2) + 3\partial_x(gu^2) + \partial_x(u^3).\end{aligned}$$

Next we re-introduce the function $w(x,t) = u(x,t)e^{\beta x}$, with $\beta \in]0,2[$ fixed, and since we will always have $|c-2|$ small, also $\beta < c < 2$. It is necessary to augment the PDE for u by an additional one for w, in order to control the influence of the bifurcating equilibrium U_0 and of the spatial amplification induced by $\alpha r(x)$.

Then u and w obey

$$\begin{aligned}\partial_t u &= \partial_x^2 u + c\partial_x u + \alpha r(x)e^{-\beta x}w\\ &\quad +3a^2\partial_x u + 6a\partial_x(ge^{-\beta x}w) + 3\partial_x(g^2 e^{-\beta x}w)\\ &\quad +3a\partial_x(u^2) + 3\partial_x(gue^{-\beta x}w) + \partial_x(u^3), \end{aligned}\tag{30}$$

$$\begin{aligned}\partial_t w &= \partial_x^2 w + (c-2\beta)\partial_x w + (\beta^2 - c\beta + \alpha r(x))w + 3(\partial_x - \beta)(U_0^2 w)\\ &\quad +3(\partial_x - \beta)(U_0 uw) + (\partial_x - \beta)(u^2 w).\end{aligned}\tag{31}$$

Here we once more expressed $\alpha r(x)u$, and now as well the spatial amplification through U_0, in terms of w. The latter is possible, as $\sup_{x\in\mathbb{R}} |g(x)e^{-\beta x}| \le C$ due to Lemma 3.4. It will turn out that w is exponentially damped, whereas u remains bounded.

We are going to prove the following result on the nonlinear stability of U_0.

Theorem 4.1 *For $\varepsilon > 0$ sufficiently small, there exist $\mu > 0$ (depending on ε and β) such that the following holds. For all $\delta_1 > 0$ there exists $\delta_2 > 0$ with*

$$\sup_{t\in[0,\infty[} \Big(\|u(\cdot,t)\|_{L^\infty} + e^{\mu t}\|w(\cdot,t)\|_{L^\infty} \Big) \le \delta_1$$

for the solutions (u,w) of (30) and (31), provided the data satisfy

$$\|u(\cdot,0)\|_{L^\infty} + \|w(\cdot,0)\|_{L^\infty} \le \delta_2.$$

Proof. In order to prove the exponential decay of w we need the following lemma.

Lemma 4.2 *For $\varepsilon > 0$ sufficiently small the spectrum of the linear operator*

$$\Lambda_2 w = \partial_x^2 w + (c - 2\beta)\partial_x w + (\beta^2 - c\beta + \alpha r(x))w + 3(\partial_x - \beta)(U_0^2 w)$$

lies strictly in the left half-plane. More precisely, there exist $\mu > 0$ (depending on ε and β) and $C > 0$ such that the analytic semigroup $e^{\Lambda_2 t}$ generated by Λ_2 satisfies

$$\|e^{\Lambda_2 t}\|_{L^\infty \to L^\infty} \le Ce^{-2\mu t}, \quad \|e^{\Lambda_2 t}\partial_x\|_{L^\infty \to L^\infty} \le Ce^{-2\mu t}t^{-1/2}.$$

Proof. Since Λ_2 is a sectorial operator, the estimates on the semigroup $e^{\Lambda_2 t}$ will follow from the assertion about the spectrum. If we again write $w = \varepsilon w_c \varphi_{\beta,c} + \varepsilon^{5/2} w_s$, with $w_c \in \mathbb{R}$ and $w_s = E_s w_s \in X_s$, then exactly the same calculations as in Section 3 show that $\partial_t w = \Lambda_2 w$ is split in

$$\begin{aligned}
\partial_t w_c &= -2\varepsilon^2 w_c + \mathcal{O}(\varepsilon^3),\\
\partial_t w_s &= \Lambda_s w_s + \mathcal{O}(\varepsilon^{1/2}),
\end{aligned}$$

where $\Lambda_s = \Lambda E_s$. This immediately implies the assertion about the spectrum. □

Then we define

$$\begin{aligned}
a(t) &= \sup_{s\in[0,t]} \|u(s)\|_{L^\infty},\\
b(t) &= \sup_{s\in[0,t]} \Big(e^{\mu s}\|w(s)\|_{L^\infty}\Big).
\end{aligned}$$

By means of the variation of constants formula, (31) for w can be rewritten as

$$w(t) = e^{\Lambda_2 t} w_0 + \int_0^t e^{\Lambda_2 (t-s)}(\partial_x - \beta)N_0(s)\, ds,$$

where

$$N_0 = 3U_0 u w + u^2 w.$$

Hence it follows that

$$\begin{aligned}
e^{\mu t}\|w(t)\|_{L^\infty} \le\ & e^{\mu t}\|e^{\Lambda_2 t} w_0\|_{L^\infty}\\
& + e^{\mu t}\int_0^t \Big(\beta\|e^{\Lambda_2(t-s)}\|_{L^\infty\to L^\infty} + \|e^{\Lambda_2(t-s)}\partial_x\|_{L^\infty\to L^\infty}\Big)\|N_0(s)\|_{L^\infty}\, ds\\
\le\ & C\|w_0\|_{L^\infty} + C\Big(a(t)b(t) + a(t)^2 b(t)\Big)\int_0^t e^{-\mu(t-s)}(1 + (t-s)^{-1/2})\, ds,
\end{aligned}$$

and thus by definition of $b(t)$

$$b(t) \le C\Big(b(0) + a(t)b(t) + a(t)^2 b(t)\Big). \tag{32}$$

We next note that, by standard theory of semilinear parabolic equations, for $t > 0$ the solutions $u(t)$ and $w(t)$ are analytic in space. In particular, for $t \geq 1$ we have the estimate

$$\begin{aligned} \|\partial_x u(t)\|_{L^\infty} &\leq C\Big(\sup_{s\in[t-1,t]} \|u(s)\|_{L^\infty} + \sup_{s\in[t-1,t]} \|w(s)\|_{L^\infty}\Big) \leq C\Big(a(t) + e^{-\mu t} b(t)\Big), \\ \|\partial_x w(t)\|_{L^\infty} &\leq C \sup_{s\in[t-1,t]} \|w(s)\|_{L^\infty} \leq C e^{-\mu t} b(t), \end{aligned}$$

for a constant $C > 0$ independent of t, assuming $a(t) \leq 1$ and $b(t) \leq 1$; see e.g. [13].

To bound u we use the maximum principle. We write (30) in the form

$$\partial_t u = \partial_x^2 u + c\partial_x u + 3a^2 \partial_x u + 6au\partial_x u + 3u^2 \partial_x u + N_1 + \partial_x N_2, \tag{33}$$

where

$$\begin{aligned} N_1 &= \alpha r(x) e^{-\beta x} w, \\ N_2 &= 6age^{-\beta x} w + 3g^2 e^{-\beta x} w + 3ge^{-\beta x} uw. \end{aligned}$$

Using the above smoothing properties and $\sup_{x\in\mathbb{R}} |r(x)e^{-\beta x}| \leq C$ shows that for $t \geq 1$

$$\|N_1(t)\|_{L^\infty} \leq Ce^{-\mu t} b(t), \qquad \|\partial_x N_2(t)\|_{L^\infty} \leq Ce^{-\mu t} b(t)\Big(1 + a(t)\Big),$$

the latter since $\sup_{x\in\mathbb{R}} |g(x)e^{-\beta x}| \leq C$ and $\sup_{x\in\mathbb{R}} |g'(x)e^{-\beta x}| = \sup_{x\in\mathbb{R}} |u_+'(x)e^{-\beta x}| \leq C$ according to Lemma 3.4, as well as $\|g\|_{L^\infty} \leq |a| + \|U_0\|_{L^\infty} \leq C$.
Therefore the maximum principle yields

$$a(t) \leq a(1) + C\int_1^t e^{-\mu s} b(s)\Big(1 + a(s)\Big)\, ds \leq a(1) + Cb(t) + C\int_1^t e^{-\mu s} a(s)\, ds$$

for $t \geq 1$, the latter as long as $b(t) \leq 1$. Hence

$$a(t) \leq C\Big(a(1) + b(t)\Big), \quad t \geq 1,$$

by Gronwall's inequality. Since $a(t) \leq a(1)$ for $t \in [0, 1]$, and $a(0) = \|u_0\|_{L^\infty} \to 0$ enforces $a(1) \to 0$, utilizing (32) we may then proceed as in the proof of Theorem 2.1 to see that the assertion follows. □

Remark 4.3 We used the maximum principle only for simplicity. There are more general methods which can be successfully applied also in more complicated situations; see [3, 4].

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