Mathematical results related to dispersion management in nonlinear optical fibers

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1. Introduction

Traditionally the transmission of optical pulses in nonlinear fiber optics was intimately connected to the classical soliton solution of the NLS equation that arises as a ground state of the equation, after averaging out the rapid oscillations of the power. This standard optical soliton decays like $\sim e^{-|x|}$ and preserves its shape during propagation by compensating the constant dispersion in the fiber through the nonlinearity. Starting at about 1995, however the concept of dispersion-managed optical solitons (DM solitons) was introduced. The basic set-up for these devices consists in two optical fibers of opposite dispersions that are concatenated into a line. Furthermore, a periodic chain of amplifiers is used to compensate for the fiber losses. It turned out that in real-world applications DM solitons could be used for highly efficient data transmission, in particular leading to an excellent performance in systems that are designed with a large variation of the dispersions in two adjacent pieces of the line, along with a low average.

To motivate the equation that will be of interest to us in this short survey, consider an optical fiber extended in the z-direction of $\mathbb{R}^3$. It is assumed that the fiber has a constant circular cross-section in the transversal x, y-directions. A z-segment of length L^+ and dispersion $\beta^+ > 0$ is followed by a segment of length L^- and dispersion $\beta^- < 0$. Then the piece $[0, L^+] \cup [L^+, L^+ + L^-]$ is periodically repeated along the z-axis. In order to make some simplifying assumptions it is supposed that the fiber is unimodal and supports a monochromatic wave. Denoting ω_1 a fixed frequency and k_1 the associated wave number, the ansatz $E(x, y, z, t) = \kappa A(z, t)\Phi(x, y)e^{i(k_1 z - \omega_1 t)}$ is made for the electromagnetic field E. Here Φ is a (transversal) eigenfunction. To lowest order in κ the equation

$$iA_z + \beta_2(z)A_{tt} + i\beta_3(z)A_{ttt} + |A|^2 A = 0$$

formally arises [10] from the Maxwell equation for E, where $\beta_2(z) = \beta^+$ in $[0, L^+]$ and $\beta_2(z) = \beta^-$ in $]L^+, L^+ + L^-[$, and also a non-constant third order dispersion function $\beta_3(z)$ has been included. Due to the periodic change of the dispersion and the periodic amplification, the system will exhibit rapid oscillations of the pulse width and power. This fast dynamics is averaged out by replacing $\beta_j(z)$ with $\varepsilon^{-1}\beta_j(\varepsilon^{-1}z)$ and performing a formal averaging over ε on one segment. Renaming $z \to t$, $t \to x$, $A \to u$, the resulting propagation equation for the slow dynamics is found to be

$$iu_t + \beta_2 u_{xx} + i\beta_3 u_{xxx} + \langle Q\rangle(u) = 0, \tag{1}$$

where $\langle Q\rangle(u) = \int_0^1 T(-t)(|T(t)u|^2 T(t)u)\, dt$ is the averaged nonlinearity for the function $u(t, x) = (T(t)u)(x)$ solving $iu_t + \beta_2 u_{xx} + i\beta_3 u_{xxx} = 0$ and $T(0)u = u$. The constants $\beta_j \geq 0$ denote the residual dispersions; for instance, $\beta_2 = L^+\beta^+ + L^-\beta^-$ for $\beta_2(z)$ as described above.

2. Summary of Results

First we consider the case where $\beta_3 = 0$. In [18] it was shown that the averaging outlined above is mathematically justified. Hence it is reasonable to look for ground state solutions, i.e., minimizers of the functional $H(u) = \alpha \int_{\mathbb{R}} |u_x|^2\,dx - \int_0^1 \int_{\mathbb{R}} |T(t)|^4\,dx\,dt$ under the constraint $\int_{\mathbb{R}} |u|^2\,dx = 1$. Note that a minimizer u_* leads to the periodic solution $u(t,x) = e^{i\omega t} u_*(x)$ of (1) for some ω, and hence to a nearly stable pulse for the non-averaged equation. If $\alpha > 0$, then H has a minimizer; see [21, 4]. The regularity of such minimizers and further properties are investigated in [19]. If $\alpha = 0$, then H still has a minimizer [7], and such minimizers also arise as the singular limit $\alpha \to 0^+$ of minimizers u_α for $\alpha > 0$ [6]. From a technical viewpoint, the difficulty of such a result is due to the invariances of the functional, and furthermore it is owed to the fact that there a no bounds on minimal sequences $(u_j)_{j\in\mathbb{N}}$ in spaces different from $L^2(\mathbb{R})$. In [7] a new and general method was devised that relies on applying the concentration compactness principle to both unit-mass sequences $(u_j)_{j\in\mathbb{N}}$ and $(\hat{u}_j)_{j\in\mathbb{N}}$ ('two-level concentration compactness'). The paper [16] reproved the existence of a minimizer u_* with different methods (using $X^{s,b}$-spaces) that also allowed to show that $u_* \in C^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ is smooth. For $x \in \mathbb{R}^2$, there is no minimizer, and there is also no minimizer for $x \in \mathbb{R}$, if $|T(t)|^4$ is replaced by $|T(t)|^6$ in H; see [16].

Closely related to H at $\alpha = 0$ is the functional $H_s(u) = -\int_{\mathbb{R}} \int_{\mathbb{R}} |T(t)|^6\,dx\,dt$. Refining the method from [7], it was proved in [8] that the constraint variational problem for H_s admits a minimum, i.e., the best constant $S > 0$ in the Strichartz inequality $\|u\|_{L^6_{tx}(\mathbb{R}\times\mathbb{R})} \leq S\|u_0\|_{L^2(\mathbb{R})}$ is attained. In [1], this result was reproved by a more elementary method that relies on interpreting the space-time Fourier transform $\widetilde{u^3}(\tau,\xi)$ in a clever way as a (τ,ξ)-dependent inner product and the Strichartz estimate as an application of the Cauchy-Schwarz inequality to this inner product. In particular, as the cases of equality in the Cauchy-Schwarz inequality are known, the best constant could be evaluated to be $S = 12^{-1/12}$ with corresponding minimizer $u_*(x) = e^{-|x|^2}$ (and all orbits thereof under the symmetry groups). In two dimensions, $x \in \mathbb{R}^2$, and $|T(t)|^6$ in H_s replaced by $|T(t)|^4$, the best constant is $2^{-1/2}$ and obtained from the same minimizer. Furthermore, [1] contains similar results for some Strichartz inequalities for wave equations. Yet by another method, without making use of the Fourier transform at all, similar results are obtained in [2].

For the case $\beta_3 > 0$ in (1), i.e., higher-order dispersion, the existence of a minimizer is due to [11, 12] in the case of non-zero average dispersion. For zero average dispersion see [9], where also certain dispersion relations of order higher than three could be included; once again, this paper relies on the method of two-level concentration compactness.

Further references related to the subject of dispersion management include [3, 5, 14, 15, 20, 22, 23].

Quite recently, also so-called diffraction-managed optical fibers attracted some interest [13, 17]. Mathematically, here the continuous problem for $x \in \mathbb{R}$ has to be replaced by a discrete version.

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