On the absence of radiationless motion for a rotating classical charge

Markus Kunze

Universität Duisburg-Essen, Fachbereich Mathematik,
D - 45117 Essen, Germany

Key words: particle-field model, global asymptotic stablility, Wiener condition, almost periodic functions

Abstract

For the Abraham-Lorentz model of a spinning charge a new approach is used to prove that all solutions converge to the set of stationary solutions in the limit $t \to \pm\infty$. This new method allows one to get rid of the additional assumptions that have been imposed before (e.g., the Wiener condition).

1 Introduction and main results

In the words of [10, p. 213], 'the state of the classical electron theory reminds one of a house under construction that was abandoned by its workmen upon receiving news of an approaching plague'. A particularly interesting and left open problem related to classical electron models is the question whether or not radiationless motion is possible, i.e., whether or not a particle could move in such a way such that it continuously catches up its own radiation. This issue has been discussed controversially in a large number of publications, one of the pioneering works being [4]. The article [10] is a good summary of related physics papers up to 1982 (making it also clear that in many cases unjustified linearization methods have been applied). The recent book [11] reviews newer references also.

Several mathematically rigorous results of global asymptotic stability type (i.e., absence of radiationless motion) were obtained in the last decade [11]. Denoting a solution schematically by $Y(t)$, these theorems typically assert that

$$Y(t) \to \mathcal{S} \quad \text{as} \quad t \to \pm\infty \tag{1.1}$$

in some kind of local energy norm (see below) and for *all* initial data $Y(0) = Y_0$ satisfying mild regularity or decay hypotheses. Depending on the choice of the particular model, $\mathcal{S}$ either denotes the set of all stationary states or a manifold of soliton-type solutions; therefore this problem is closely linked to the questions and results discussed in [12]. However, the rigorous results on radiationless motion so far have been requiring at least one of the two following additional conditions:

(i) A smallness condition on the nonlinearity which is usually formulated as a smallness condition on the charge-to-mass ratio e/m. This is helpful, since schematically the equation of motion is $m\ddot{q} = eF(q, \dot{q})$ and e/m small allows for a contraction type argument.

(ii) The Wiener condition.

To explain the Wiener condition it is useful to remark that a basic quantitative estimate for this kind of problems is obtained by keeping track of the amount of local energy that is radiated off to infinity. Very roughly speaking, this estimate implies that

$$\lim_{t\to\infty} (\ddot{q} * g)(t) = 0, \tag{1.2}$$

where g is an explicitly known scalar function that is related to the charge distribution. If it is assumed that its Fourier transform $\hat{g}$ has no zeroes, then Wiener's tauberian theorem asserts that (1.2) implies the acceleration relaxation $\lim_{t\to\infty} \ddot{q}(t) = 0$ also, which is the key step for proving (1.1). Accordingly, the requirement that $\hat{g}(\tau) \neq 0$ for all $\tau \in \mathbb{R}$ (or the corresponding assertion for the charge distribution) was termed the Wiener condition.

It is the purpose of the present paper to investigate the case where neither (i) nor (ii) is assumed. Therefore we consider the simplest classical particle-field model for which the Wiener condition is violated. It consists of a spinning charged particle at rest at the origin in $\mathbb{R}^3$ coupled to its self-generated Maxwell field. As it will be seen below, if for instance the charge distribution is taken to be a uniformly charged sphere or a uniformly charged ball, then the associated function $\hat{g}$ will have countably many zeroes. Nevertheless we will be able to prove that, under mild assumptions on the initial data, all solutions are attracted to the set of stationary solutions in a suitable sense. This holds without any further assumption for the charged sphere. For the charged ball (and also in the general case, if a natural nonresonance condition on the zeroes of $\hat{g}$ is included) this global asymptotic stability result remains true provided that countably many masses are excluded. That is, if we consider the particle's mass to be a parameter of the system, then outside a countable set of 'exceptional masses' all solutions of the system converge to the set of stationary solutions. In particular, this latter property is generic. The appearance of such exceptional or resonant masses was already observed in [4] on a linearized level; see Remark 1.5(b) below for more information.

The novelty of the approach taken in this paper consists of considering the equation for the dynamical quantity (here: the angular velocity) as a dynamical system and to study its limit points. Due to the estimate obtained from the energy dissipation it turns out that all possible limit points are almost periodic functions. Since such functions can be well approximated by means of trigonometric polynomials useful conclusions can be drawn about those limit points which are solutions of the associated limiting equation. It is conceivable that the method of proof will lead to improved results for other classical particle-field models as well.

According to the Abraham-Lorentz model for a rotating charge with positive bare inertia [11, p. 125], the governing field equations for the system described above are the Maxwell-Lorentz equations

$$\begin{aligned} \partial_t E(t,x) &= \mathrm{rot} B(t,x) - 4\pi(\omega(t)\wedge x) f_{\mathrm{e}}(x), \qquad \partial_t B(t,x) = -\mathrm{rot} E(t,x), && (1.3)\\ \mathrm{div} E(t,x) &= 4\pi f_{\mathrm{e}}(x), \qquad \mathrm{div} B(t,x) = 0, && (1.4) \end{aligned}$$

for $t \in \mathbb{R}$ and $x \in \mathbb{R}^3$, where f_{e} is the charge distribution. The angular velocity $\omega(t) \in \mathbb{R}^3$ is to be determined from

$$I_{\mathrm{b}}\, \dot{\omega}(t) = \int_{\mathbb{R}^3} x \wedge \Big[E(t,x) + (\omega(t)\wedge x)\wedge B(t,x)\Big]\, f_{\mathrm{e}}(x)\, dx, \tag{1.5}$$

where

$$I_{\rm b}=\frac{2}{3}\,m_{\rm b}\int_{\mathbb{R}^3}|x|^2 f_{\rm e}(x)\,dx \tag{1.6}$$

is the bare moment of inertia associated to the bare mass $m_{\rm b}$; all other constants are set equal to unity. The right-hand side of (1.5) is called the torque vector. For simplicity the distributions that model the charge distribution and the mass distributions, respectively, are chosen to be proportional, but this does not really matter. They are both given by $f_{\rm e}$ which we assume to be a radially symmetric measure or function of compact support. More precisely, the required properties of $f_{\rm e}$ are as follows.

$$f_{\rm e}(x)=f_{\rm e}(|x|)\text{ is radial},\quad f_{\rm e}(x)=0\quad\text{for}\quad |x|>R_0,\quad\text{and}\quad\int_{\mathbb{R}^3}|f_{\rm e}(x)|\,dx<\infty. \tag{1.7}$$

At many places $f_{\rm e}(x)$ will be identified with its radial version $f_{\rm e}(r)$.

We will investigate the asymptotic ($t\to\pm\infty$) behavior of solutions to (1.3)–(1.5) for suitable initial data

$$\omega(0)=\omega_0,\quad E(0,x)=E_0(x),\quad B(0,x)=B_0(x). \tag{1.8}$$

Here $\omega_0\in\mathbb{R}^3$ and for the initial fields E_0, B_0 we assume that

$$E_0(-x)=-E_0(x),\quad B_0(-x)=B_0(x)\quad(x\in\mathbb{R}^3)\quad\text{and}\quad\operatorname{div}E_0=4\pi f_{\rm e},\quad\operatorname{div}B_0=0, \tag{1.9}$$

are verified. The symmetry assumptions on E_0 and B_0 propagate in time and they are needed to insure that $(q(t)=0,\omega(t),E(t,x),B(t,x))$ gives rise to a consistent particular solution of the full Abraham-Lorentz model of a spinning charge in motion [11, Section 10.2].

It should be noted that the Abraham-Lorentz model is the classical counterpart of the Pauli-Fierz model of non-relativistic quantum electrodynamics, the latter being kind of a quantized version of the former; see [11].

For the initial fields E_0 and B_0 we suppose that there is $\gamma>1/2$ such that for every $R>0$ large enough

$$|x|(|E_0(x)|+|B_0(x)|)+|x|^2(|\nabla E_0(x)|+|\nabla B_0(x)|)\le C(R)\,|x|^{-\gamma},\quad |x|>R, \tag{1.10}$$

and

$$|x|(|\nabla\nabla E_0(x)|+|\nabla\nabla B_0(x)|)\le C(R),\quad |x|>R, \tag{1.11}$$

are verified. Most likely these hypotheses imposed on the initial data could be improved, but this is not the main aspect of this work.

For all $\omega\in\mathbb{R}^3$ the system (1.3)–(1.5) admits a stationary state $(\omega,E_\omega(x),B_\omega(x))$; see Lemma 7.3 below. As mentioned above, our main results are of global asymptotical stability type and they concern the long-time behavior of all solutions $(\omega(t),E(t,x),B(t,x))$ whose initial data satisfy (1.9) and (1.10). Among other things, it will be shown that such solutions converge to the set of stationary solutions

$$\mathcal{S}=\{(\omega,E_\omega,B_\omega):\omega\in\mathbb{R}^3\}$$

in the local energy norm,

$$\operatorname{dist}_R\Big((\omega(t),E(t),B(t)),\mathcal{S}\Big)\to 0\quad\text{as}\quad t\to\infty\quad\text{for every}\quad R>0, \tag{1.12}$$

where

$$\mathrm{dist}_R\Big((\omega(t),E(t),B(t)),\mathcal{S}\Big)=\inf_{\bar\omega\in\mathbb{R}^3}\Big(|\omega(t)-\bar\omega|+\|E(t)-E_{\bar\omega}\|_{L^2(B_R(0))}+\|B(t)-B_{\bar\omega}\|_{L^2(B_R(0))}\Big)$$

is the (local in space) distance of the solution to $\mathcal{S}$. Note that due to the Hamiltonian nature of the system a global in space convergence cannot be expected.

First we consider the uniformly charged (unit) sphere.

Theorem 1.1 *Take*

$$f_{\mathrm{e}}(x)=\delta(|x|-1)$$

and let the initial data (ω_0,E_0,B_0) be such that (1.9) and (1.10) hold. Then the corresponding solution $(\omega(t),E(t,x),B(t,x))$ of (1.3)–(1.5) and (1.8) satisfies

$$\dot\omega(t)\to 0\quad\text{and}\quad\ddot\omega(t)\to 0\quad\text{as}\quad t\to\infty,$$

and ω is asymptotically slowly varying. Furthermore, (1.12) holds.

This result will be proved in Section 5 below.

Remarks 1.2 (a) The method of proof will show that it is also possible to include models with more general bare spin/angular velocity relation, like those considered in [3, Section 5.3]; also see [2]. These kinds of systems are fully relativistic, whereas the Abraham-Lorentz model from above is only semi-relativistic.

(b) We will neither obtain the pointwise convergence of $\omega(t)$ nor a decay rate for $\dot\omega(t)$ or $\ddot\omega(t)$.

(c) A function ω is said to be asymptotically slowly varying, if $|\omega(t+T)-\omega(t)|\to 0$ as $t\to\infty$ uniformly for T in compact subsets of $\mathbb{R}$.

(d) We do not deal with the case where $m_{\mathrm{b}}=0$. See [4] for some remarks, and also [2, Section A.3.3] for a discussion of the singular limit $m_{\mathrm{b}}\to 0$ and $I_{\mathrm{b}}\to 0$. ♢

Next we turn to general charge distributions satisfying (1.7). In this case we need to impose a further hypothesis on f_{e}. To introduce it, consider the function

$$g(t)=t\int_{|t|}^{\infty} r f_{\mathrm{e}}(r)\,dr,\quad t\in\mathbb{R},\tag{1.13}$$

along with its Fourier transform

$$\hat g(\tau)=\sqrt{\frac{2}{\pi}}\,i\int_0^{\infty} r^3 f_{\mathrm{e}}(r)\phi_1(\tau r)\,dr,\quad \tau\in\mathbb{R},\tag{1.14}$$

where

$$\phi_1(s)=\frac{s\cos(s)-\sin(s)}{s^2},\quad s\in\mathbb{R}.\tag{1.15}$$

The function $\hat g$ is defined on $\mathbb{R}$, odd, and does not vanish identically, since $f_{\mathrm{e}}\neq\delta_0$ is assumed. Writing out the series for cos and sin, we see that $\hat g$ has the analytic continuation

$$\hat g(z)=-2\sqrt{\frac{2}{\pi}}\,i\sum_{j=0}^{\infty}\frac{(j+1)}{(2j+3)!}\Big[\int_0^{\infty} dr\, r^{2j+4}\, f_{\mathrm{e}}(r)\Big]\,z^{2j+1},\quad z\in\mathbb{C}.$$

Thus its set of zeroes

$$\{\hat{g} = 0\} = \{\tau \in \mathbb{R} : \hat{g}(\tau) = 0\} =: \{\mu_j : j \in \mathbb{Z}\} \tag{1.16}$$

is (at most) countable and no $\tau \in \{\hat{g} = 0\}$ can be an accumulation point of $\{\hat{g} = 0\} \setminus \{\tau\}$; here we let $\mu_0 = 0$ and $\mu_{-j} = -\mu_j$ and note that $\hat{g}(0) = 0$ due to $\phi_1(0) = 0$.

Definition 1.3 *We say that the nonresonance condition* (NRC) *is satisfied for $f_{\rm e}$, if for $l \neq 0$ the relation $\mu_j + \mu_k = \mu_l$ has no solutions, except for the trivial ones where $j = 0$, $k = l$ or $j = l$, $k = 0$.*

For instance, it will be argued in Lemma 7.5 below that (NRC) holds for the uniformly charged (unit) ball $f_{\rm e} = \mathbf{1}_{\{|x|<1\}}$.

Our second main result is as follows.

Theorem 1.4 *Suppose that $f_{\rm e}$ satisfies (1.7) and* (NRC). *Let the initial data (ω_0, E_0, B_0) be such that (1.9) and (1.10) are verified. Then there is an at most countable set $\mathcal{M}_{\rm exc} \subset]0, \infty[$ of exceptional masses such that the following holds. If $m_{\rm b} \notin \mathcal{M}_{\rm exc}$, then the solution $(\omega(t), E(t, x), B(t, x))$ of (1.3)–(1.5) and (1.8) satisfies*

$$\dot{\omega}(t) \to 0 \quad and \quad \ddot{\omega}(t) \to 0 \quad as \quad t \to \infty,$$

and ω is asymptotically slowly varying. Furthermore, (1.12) holds.

The proof is given in Section 6.

Remarks 1.5 (a) Remarks 1.2(b), (c) also apply in this general case.
(b) The set $\mathcal{M}_{\rm exc}$ is explicit and can be calculated from $f_{\rm e}$; see (6.1), and furthermore (7.18) for the example of the uniformly charged ball. It is however unclear whether such exceptional masses do really occur, i.e., whether the set $\mathcal{S}$ of stationary states could be non-attracting for some $m_{\rm b} \in \mathcal{M}_{\rm exc}$. For instance, a periodic or more complicated solution cannot a priori be excluded for $m_{\rm b} \in \mathcal{M}_{\rm exc}$. If in general there were exceptional masses, this would give rise to a kind of 'mass spectrum' for excited charge states in this classical model. $\diamondsuit$

2 Energy dissipation

From (1.13) recall that

$$g(t) = \int_0^\infty dr\, r f_{\rm e}(r)\, \mathbf{1}_{[-r,r]}(t) t = t \int_{|t|}^\infty dr\, r f_{\rm e}(r), \quad t \in \mathbb{R}.$$

Then g is odd and (1.7) implies that

$$g(t) = 0 \quad \text{for} \quad |t| > R_0, \tag{2.1}$$

and in particular $g \in L^1(\mathbb{R})$.

Lemma 2.1 *For every solution to (1.3)–(1.5) and (1.8) as in Theorems 1.1 or 1.4,*

$$\dot{\omega} * g \in L^2(\mathbb{R}),$$

where as usual $(u * v)(t) = \int_{\mathbb{R}} u(t-s)v(s)\,ds$ *denotes the convolution of the functions* u *and* v.

Proof: For simplicity we assume that E_0 and B_0 are smooth and compactly supported in a ball of radius $R_1 > 0$ in $\mathbb{R}^3$. A similar argument works if only (1.10) holds, as can be seen along the lines of [9]. For $R > 0$ the local energy in $B_R(0) \subset \mathbb{R}^3$ is

$$\mathcal{E}_R(t) = \frac{1}{2}\,I_{\rm b}\,|\omega(t)|^2 + \frac{1}{8\pi}\int_{|x|<R} (|E(t,x)|^2 + |B(t,x)|^2)\,dx.$$

From (1.3)–(1.5) it follows that

$$\dot{\mathcal{E}}_R(t) = \int_{|x|>R} E\cdot j\,dx + \frac{1}{4\pi}\int_{|x|=R} \bar{x}\cdot(B\wedge E)\,dS(x),$$

where $j(t,x) = (\omega(t)\wedge x)f_e(x)$ and $\bar{x} = |x|^{-1}x$ is the unit normal. Thus $j(t,x) = 0$ for $|x| > R_0$ implies that

$$\dot{\mathcal{E}}_R(t) = \frac{1}{4\pi}\int_{|x|=R} \bar{x}\cdot(B\wedge E)\,dS(x) \tag{2.2}$$

for $R > R_0$. Defining $\rho(x) = f_{\rm e}(x)$, the Maxwell equations (1.3), (1.4) are rewritten as wave equations for E and B whose solutions are

$$\begin{aligned} E(t,x) &= E_{\rm data}(t,x) - \int_{|y-x|<t} \frac{1}{4\pi|y-x|}\,\partial_t j(t-|x-y|,y)\,dy \\ &\quad - \int_{|y-x|=t} \frac{y-x}{4\pi|y-x|^2}\,\rho(y)\,dS(y) - \int_{|y-x|<t} \frac{y-x}{4\pi|y-x|^3}\,\rho(y)\,dy, \\ B(t,x) &= B_{\rm data}(t,x) + \int_{|y-x|=t} \frac{1}{4\pi|y-x|^2}\,(y-x)\wedge j(0,y)\,dS(y) \\ &\quad + \int_{|y-x|<t} \frac{1}{4\pi|y-x|^3}\,(y-x)\wedge j(t-|x-y|,y)\,dy \\ &\quad + \int_{|y-x|<t} \frac{1}{4\pi|y-x|^2}\,(y-x)\wedge \partial_t j(t-|x-y|,y)\,dy, \end{aligned}$$

for $t \in [0,\infty[$. Note that ρ and j could be measures in x, so the usual terms $\nabla\rho$ in the integrand of E and ${\rm rot} j$ in the integrand of B had to be re-expressed. Concerning the data terms, for instance

$$\begin{aligned} E_{\rm data}(t,x) &= \frac{1}{4\pi t^2}\int_{|y-x|=t} \Big((y-x)\cdot\nabla E_0(y) + E_0(y)\Big)\,dS(y) \\ &\quad + \frac{1}{4\pi t}\int_{|y-x|=t} \Big({\rm rot}B_0(y) - 4\pi j(0,y)\Big)\,dS(y) \end{aligned}$$

holds. Thus if $t \geq R + \max\{R_0, R_1\}$ and $|x| = R$, then $E_{\rm data}(t,x) = 0$, and similarly $B_{\rm data}(t,x) = 0$. Next we expand the inhomogeneous parts of E and B in R^{-1}. To begin with, recall that $j(t,x) =$

$0=\rho(x)$ for $|x|>R_0$. Hence defining $Q=B_{R_0+1}(0)\subset\mathbb{R}^3$ we obtain

$$\begin{aligned}
E(t,x) &= -\int_Q \frac{1}{4\pi|y-x|}\,\partial_t j(t-|x-y|,y)\,dy-\int_Q \frac{y-x}{4\pi|y-x|^3}\,\rho(y)\,dy,\\
B(t,x) &= \int_Q \frac{1}{4\pi|y-x|^3}\,(y-x)\wedge j(t-|x-y|,y)\,dy\\
&\quad+\int_Q \frac{1}{4\pi|y-x|^2}\,(y-x)\wedge \partial_t j(t-|x-y|,y)\,dy,
\end{aligned}$$

for $t\geq R+\max\{R_0,R_1\}+1=R+t_0$ and $|x|=R$, since then $B_t(x)\cap Q=Q$ and $\partial B_t(x)\cap Q=\emptyset$. It follows that

$$E(t,x) = E_{\rm rad}(t,x)+E_{\rm err}(t,x), \tag{2.3}$$
$$B(t,x) = B_{\rm rad}(t,x)+B_{\rm err}(t,x), \tag{2.4}$$

for $t\geq R+t_0$, $|x|=R$, and $R\geq 2(R_0+1)$, where

$$\begin{aligned}
E_{\rm rad}(t,x) &= -\frac{1}{4\pi|x|}\int_Q \partial_t j(t-|x|+\bar{x}\cdot y,y)\,dy,\\
B_{\rm rad}(t,x) &= -\frac{x}{4\pi|x|^2}\wedge\int_Q \partial_t j(t-|x|+\bar{x}\cdot y,y)\,dy,
\end{aligned}$$

are the radiation parts and $|E_{\rm err}(t,x)|+|B_{\rm err}(t,x)|\leq CR^{-2}$. Let us for example check the formula for E. If $y\in Q$, then $|y-x|\geq|x|-|y|\geq|x|/2=R/2$. Hence

$$\Big|\int_Q \frac{y-x}{4\pi|y-x|^3}\,\rho(y)\,dy\Big|\leq CR^{-2}\int_Q |f_{\rm e}(y)|\,dy\leq CR^{-2}$$

contributes to the error term. Also

$$\Big|\frac{1}{|y-x|}-\frac{1}{|x|}\Big|=\frac{||x|-|y-x||}{|x||y-x|}\leq 2R^{-2}|y|\leq CR^{-2}$$

and $\partial_t j(t,x)=(\dot{\omega}(t)\wedge x)f_{\rm e}(x)$. Therefore

$$\begin{aligned}
&\Big|-\int_Q \frac{1}{4\pi|y-x|}\,\partial_t j(t-|x-y|,y)\,dy+\frac{1}{4\pi R}\int_Q \partial_t j(t-|x-y|,y)\,dy\Big|\\
&\leq\ CR^{-2}\|\dot{\omega}\|_{L^\infty}\int_Q |y||f_{\rm e}(y)|\,dy\leq CR^{-2}
\end{aligned}$$

by Lemma 7.2. Next, $||x|-|x-y|-\bar{x}\cdot y|\leq CR^{-1}$ implies that

$$\begin{aligned}
&\Big|-\frac{1}{4\pi R}\int_Q \partial_t j(t-|x-y|,y)\,dy-E_{\rm rad}(t,x)\Big|\\
&\leq\ CR^{-2}\|\ddot{\omega}\|_{L^\infty}\int_Q |y||f_{\rm e}(y)|\,dy\leq CR^{-2},
\end{aligned}$$

once again by Lemma 7.2. Therefore (2.3) is verified, and the proof of (2.4) is similar. Now observe that $|E_{\rm rad}(t,x)|+|B_{\rm rad}(t,x)|\leq CR^{-1}$ and $B=B_{\rm rad}+B_{\rm err}=\bar{x}\wedge E_{\rm rad}+B_{\rm err}$. Hence if $t\geq R+t_0$, $|x|=R$, and $R\geq 2(R_0+1)$, then

$$\begin{aligned}
\bar{x}\cdot(B\wedge E) &= \bar{x}\cdot\Big([\bar{x}\wedge E_{\rm rad}+B_{\rm err}]\wedge[E_{\rm rad}+E_{\rm err}]\Big)=-\bar{x}\cdot\Big(E_{\rm rad}\wedge(\bar{x}\wedge E_{\rm rad})\Big)+F_{\rm err}\\
&= -|\bar{x}\wedge E_{\rm rad}|^2+F_{\rm err},
\end{aligned}$$

where $|F_{\rm err}(t,x)| \le CR^{-3}$. Returning to (2.2), we have shown that

$$\Big|\dot{\mathcal{E}}_R(t) + \frac{1}{4\pi}\int_{|x|=R} |\bar{x}\wedge E_{\rm rad}(t,x)|^2\, dS(x)\Big| \le CR^{-1}$$

for all $R \ge 2(R_0+1)$ and $t \ge R+t_0$. Fix $T \ge t_0 = \max\{R_0, R_1\}+1$ and $R \ge 2(R_0+1)$. Integration from $T_1 = R + t_0$ to $T_2 = R + T$ yields

$$\int_{R+t_0}^{R+T} dt \int_{|x|=R} dS(x)\, |\bar{x}\wedge E_{\rm rad}(t,x)|^2 \le C\Big(E_R(R+t_0) + E_R(R+T)\Big) + C(T-t_0)R^{-1}.$$

Since $0 \le \mathcal{E}_R(t) \le \mathcal{E}(t) = \mathcal{E}(0)$ by Lemma 7.1, we may insert the definition of $E_{\rm rad}$, shift the t-integration by R, and put $x = R\sigma$ for $|\sigma| = 1$ to find

$$\int_{t_0}^{T} dt \int_{|\sigma|=1} dS(\sigma)\, \Big|\sigma \wedge \int_Q \partial_t j(t+\sigma\cdot y, y)\, dy\Big|^2 \le C + C(T-t_0)R^{-1}.$$

Passing to the limit $R \to \infty$ first and then taking the limit $T \to \infty$, we obtain

$$\int_{t_0}^{\infty} dt \int_{|\sigma|=1} dS(\sigma)\, \Big|\sigma \wedge \int_{\mathbb{R}^3} (\dot{\omega}(t+\sigma\cdot y)\wedge y) f_{\rm e}(y)\, dy\Big|^2 \le C. \tag{2.5}$$

Since the system is time reversible and $[-t_0, t_0]$ is a finite time interval, we may as well replace $\int_{t_0}^{\infty} dt$ by $\int_{\mathbb{R}}$ dt in (2.5). Recalling that $f_{\rm e}$ is radial, explicit integration then yields

$$\int_{\mathbb{R}} dt \Big| \int_{\mathbb{R}} ds\, \dot{\omega}(t-s) \int_0^\infty dr\, r f_{\rm e}(r)\, \mathbf{1}_{[-r,r]}(s) s\Big|^2 \le C.$$

This completes the proof of the lemma. $\square$

3 The torque equation

In this section we rewrite the right-hand side of (1.5) in a different way. If $(\omega(t), E(t,x), B(t,x))$ is a solution of (1.3)–(1.5) and (1.8) as in Theorems 1.1 or 1.4, define

$$F(t,x) = \begin{pmatrix} E(t,x) \\ B(t,x) \end{pmatrix} \quad \text{and} \quad F_\omega(x) = \begin{pmatrix} E_\omega(x) \\ B_\omega(x) \end{pmatrix},$$

and moreover introduce

$$Z(t,x) = F(t,x) - F_{\omega(t)}(x) = \begin{pmatrix} E(t,x) - E_{\omega(t)}(x) \\ B(t,x) - B_{\omega(t)}(x) \end{pmatrix}. \tag{3.1}$$

Using the Maxwell operator $\mathcal{M}(E,B) = ({\rm rot}B, -{\rm rot}E)$ for the fields E, B satisfying the constraints ${\rm div}E = {\rm div}B = 0$, it hence follows from (1.3), (1.4), and $\mathcal{M}(F_\omega) = (4\pi(\omega\wedge x)f_{\rm e}, 0)$ [see Lemma 7.3] that

$$\dot{Z} = \mathcal{M}Z - G,$$

where

$$G(t,x) = \begin{pmatrix} G_1(t,x) \\ G_2(t,x) \end{pmatrix} = \begin{pmatrix} 0 \\ (\dot{\omega}(t)\cdot\nabla_\omega) B_{\omega(t)}(x) \end{pmatrix} = \begin{pmatrix} 0 \\ B_{\dot{\omega}(t)}(x) \end{pmatrix};$$

note that $\omega \mapsto B_\omega$ is linear, cf. (7.4), and the first component G_1 of G is zero in view of $\nabla_\omega E_\omega = 0$. Depending on the regularity of E_0 and B_0, the relation $\dot{Z} = \mathcal{M}Z - G$ is to be understood in the mild solution form

$$Z(t,x) = [\mathcal{U}(t)Z(0,\cdot)](x) - \int_0^t ds\, [\mathcal{U}(t-s)G(s,\cdot)](x), \tag{3.2}$$

where $(\mathcal{U}(t))_{t\in\mathbb{R}}$ denotes the group of isometries in $L^2(\mathbb{R}^3)^3 \oplus L^2(\mathbb{R}^3)^3$ generated by the Maxwell operator $\mathcal{M}$.

In the next lemma $W(t,s,x) = [\mathcal{U}(t)G(s,\cdot)](x)$ is determined.

Lemma 3.1 *Let g be defined by (1.13). Then under the above hypotheses,*

$$\begin{aligned} W_1(t,s,x) &= -\frac{2\pi}{|x|}\Big(g(t-|x|) + g(t+|x|) - \frac{1}{|x|}\int_{-|x|}^{|x|} g(t-\tau)\,d\tau\Big)\,\dot{\omega}(s)\wedge\bar{x}, \qquad (3.3)\\ W_2(t,s,x) &= \frac{2\pi}{|x|}\Big(g(t+|x|) - g(t-|x|) + \frac{1}{|x|^2}\int_{-|x|}^{|x|} g(t-\tau)\tau\,d\tau\Big)\,\dot{\omega}(s)\\ &\quad - \frac{2\pi}{|x|}\Big(g(t+|x|) - g(t-|x|) + \frac{3}{|x|^2}\int_{-|x|}^{|x|} g(t-\tau)\tau\,d\tau\Big)\,(\bar{x}\cdot\dot{\omega}(s))\,\bar{x}, \end{aligned}$$

where $\bar{x} = |x|^{-1}x$.

Proof: First we follow [9] to solve $\dot{\Phi} = \mathcal{M}\Phi$, $\Phi(0) = \Phi^{(0)}$, for $\Phi = (\Phi_1, \Phi_2)$ under the constraints $\mathrm{div}\Phi_1 = \mathrm{div}\,\Phi_2 = 0$. For the complex field $\Psi = \Phi_1 + i\Phi_2$ this means that $\partial_t\Phi = -i\,\nabla\wedge\Phi$, and thus $\partial_t\hat{\Phi} = k\wedge\hat{\Phi} =: m(k)\hat{\Phi}$ for the matrix $m(k)$ representing $k\wedge$. Therefore we obtain $\hat{\Psi}(t) = \exp(tm(k))\hat{\Psi}(0)$. Since $m(k)^2 = k\otimes k - |k|^2\mathrm{Id}$ and $m(k)^3 = -|k|^2 m(k)$, etc., the exponential can be evaluated explicitly to be

$$\exp(tm(k)) = \cos(|k|t)\,\mathrm{Id} + |k|^{-2}(1-\cos(|k|t))(k\otimes k) + |k|^{-1}\sin(|k|t)\,m(k).$$

As $\Psi(0) = \Phi_1^{(0)} + i\,\Phi_2^{(0)}$ has $\mathrm{div}\Psi(0) = 0$, we get

$$\hat{\Psi}(t) = \exp(tm(k))\hat{\Psi}(0) = \cos(|k|t)\,\hat{\Psi}(0) + |k|^{-1}\sin(|k|t)\,k\wedge\hat{\Psi}(0),$$

and the corresponding relations

$$\begin{aligned} \hat{\Phi}_1(t) &= \cos(|k|t)\,\hat{\Phi}_1^{(0)} + i|k|^{-1}\sin(|k|t)\,k\wedge\hat{\Phi}_2^{(0)}, \qquad (3.4)\\ \hat{\Phi}_2(t) &= \cos(|k|t)\,\hat{\Phi}_2^{(0)} - i|k|^{-1}\sin(|k|t)\,k\wedge\hat{\Phi}_1^{(0)}, \qquad (3.5) \end{aligned}$$

for the components Φ_1 and Φ_2. Application to $\Phi^{(0)}(x) = G(s,x) = (0, G_2(s,x))$ for fixed s yields

$$\begin{aligned} \widehat{W}_1(t,s,k) &= |k|^{-1}\sin(|k|t)\,(\mathrm{rot}\,G_2(s,\cdot))^\wedge(k),\\ \widehat{W}_2(t,s,k) &= \cos(|k|t)\,\hat{G}_2(s,k). \end{aligned}$$

Next we recall that $\mathrm{rot}\,G_2(s,x) = \mathrm{rot}B_{\dot{\omega}(s)}(s,x) = 4\pi(\dot{\omega}(s)\wedge x)f_{\mathrm{e}}(x)$ by Lemma 7.3. Let ϕ_1 be defined by (1.15). Since f_{e} is radial, the Fourier transform is evaluated as

$$(\mathrm{rot}\,G_2(s,\cdot))^\wedge(k) = 4\sqrt{2\pi}i\,(\dot{\omega}(s)\wedge\bar{k})\int_0^\infty dr\, r^3 f_{\mathrm{e}}(r)\,\phi_1(r|k|) \tag{3.6}$$

for $\bar{k} = |k|^{-1}k$. By taking the inverse Fourier transform, this yields

$$\begin{aligned} W_1(t,s,x) &= 8(\dot{\omega}(s)\wedge\bar{x})\int_0^\infty d\tau\,\tau\sin(\tau t)\phi_1(\tau|x|)\int_0^\infty dr\,r^3 f_{\mathrm{e}}(r)\phi_1(\tau r) \\ &= 4(\dot{\omega}(s)\wedge\bar{x})\int_{\mathbb{R}} d\tau\,\tau\sin(\tau t)\phi_1(\tau|x|)\int_0^\infty dr\,r^3 f_{\mathrm{e}}(r)\phi_1(\tau r). \end{aligned}$$

Now observe that g from (1.13) has

$$\hat{g}(\tau) = \sqrt{\frac{2}{\pi}}\,i\int_0^\infty dr\,r^3 f_{\mathrm{e}}(r)\phi_1(\tau r),$$

cf. (1.14). Furthermore, $\phi_0(s) = \frac{\sin(s)}{s}$ satisfies $\phi_0'(s) = \phi_1(s)$ and $\widehat{h_{|x|}}(\tau) = \phi_0(\tau|x|)$ for

$$h_{|x|}(s) = \sqrt{\frac{\pi}{2}}\,\frac{1}{|x|}\,\mathbf{1}_{[-|x|,\,|x|]}(s), \quad s\in\mathbb{R}.$$

It follows that

$$\begin{aligned} W_1(t,s,x) &= 4(\dot{\omega}(s)\wedge\bar{x})\,\mathrm{Im}\int_{\mathbb{R}} d\tau\,\tau\,e^{i\tau t}\,\phi_1(\tau|x|)\Big(\int_0^\infty dr\,r^3 f_{\mathrm{e}}(r)\phi_1(\tau r)\Big) \\ &= -\,2\sqrt{2\pi}\,(\dot{\omega}(s)\wedge\bar{x})\,\frac{d}{d|x|}\,\mathrm{Im}\Big(i\int_{\mathbb{R}} d\tau\,e^{i\tau t}\,\widehat{h_{|x|}}(\tau)\,\hat{g}(\tau)\Big) \\ &= -\,2\sqrt{2\pi}\,(\dot{\omega}(s)\wedge\bar{x})\,\frac{d}{d|x|}\,(h_{|x|}*g)(t) \\ &= -\,2\pi(\dot{\omega}(s)\wedge\bar{x})\Big(\frac{1}{|x|}\,[g(t-|x|)+g(t+|x|)] - \frac{1}{|x|^2}\int_{-|x|}^{|x|} g(t-\tau)\,d\tau\Big), \end{aligned}$$

proving (3.3). Concerning the second component W_2, the argument is similar. First, $\mathrm{rot}B_{\dot{\omega}} = 4\pi(\dot{\omega}\wedge x)f_{\mathrm{e}}$ and $\mathrm{div}B_{\dot{\omega}} = 0$ implies that

$$\begin{aligned} \widehat{G_2}(s,k) &= \widehat{B_{\dot{\omega}(s)}}(s,k) = \frac{4\pi i}{|k|^2}\,k\wedge((\dot{\omega}(s)\wedge x)f_{\mathrm{e}})^\wedge(k) \\ &= -4\sqrt{2\pi}\,\frac{1}{|k|^3}\,k\wedge(\dot{\omega}(s)\wedge k)\int_0^\infty dr\,r^3 f_{\mathrm{e}}(r)\phi_1(r|k|), \end{aligned}$$

in accordance with (3.6). The inverse Fourier transform of

$$\widehat{W_2}(t,s,k) = \cos(|k|t)\,\widehat{G_2}(s,k) = -4\sqrt{2\pi}\,\frac{\cos(|k|t)}{|k|^3}\,k\wedge(\dot{\omega}(s)\wedge k)\int_0^\infty dr\,r^3 f_{\mathrm{e}}(r)\phi_1(r|k|)$$

is calculated to be

$$\begin{aligned} W_2(t,s,x) &= -4\Big[\int_{\mathbb{R}} d\tau\,\tau\cos(\tau t)\Big(\phi_0(\tau|x|) + \frac{1}{\tau|x|}\,\phi_1(\tau|x|)\Big)\int_0^\infty dr\,r^3 f_{\mathrm{e}}(r)\phi_1(\tau r)\Big]\,\dot{\omega}(s) \\ &\quad + 4\Big[\int_{\mathbb{R}} d\tau\,\tau\cos(\tau t)\Big(\phi_0(\tau|x|) + \frac{3}{\tau|x|}\,\phi_1(\tau|x|)\Big)\int_0^\infty dr\,r^3 f_{\mathrm{e}}(r)\phi_1(\tau r)\Big]\,(\bar{x}\cdot\dot{\omega}(s))\,\bar{x}. \end{aligned}$$

In addition,

$$\begin{aligned}\int_{\mathbb{R}} d\tau\,\tau\cos(\tau t)\phi_0(\tau|x|)\int_0^\infty dr\,r^3 f_{\rm e}(r)\phi_1(\tau r) &= -\sqrt{\frac{\pi}{2}}\,{\rm Re}\Big(\frac{d}{dt}\int_{\mathbb{R}} d\tau\,e^{i\tau t}\widehat{h_{|x|}}(\tau)\,\hat{g}(\tau)\Big)\\ &= -\sqrt{\frac{\pi}{2}}\,\frac{d}{dt}\,(h_{|x|}*g)(t)\\ &= -\frac{\pi}{2|x|}\,[g(t+|x|)-g(t-|x|)],\end{aligned}$$

and in view of

$$-i|x|^{-1}(h_{|x|}(s)s)^\wedge(\tau)=\phi_1(\tau|x|)$$

also

$$\begin{aligned}\frac{1}{|x|}\int_{\mathbb{R}} d\tau\cos(\tau t)\phi_1(\tau|x|)\int_0^\infty dr\,r^3 f_{\rm e}(r)\phi_1(\tau r) &= -\sqrt{\frac{\pi}{2}}\,\frac{1}{|x|^2}\,{\rm Re}\Big(\int_{\mathbb{R}} d\tau\,e^{i\tau t}(h_{|x|}(s)s)^\wedge(\tau)\,\hat{g}(\tau)\Big)\\ &= -\sqrt{\frac{\pi}{2}}\,\frac{1}{|x|^2}\,((h_{|x|}(s)s)*g)(t)\\ &= -\frac{\pi}{2|x|^3}\int_{-|x|}^{|x|} g(t-\tau)\tau\,d\tau.\end{aligned}$$

Using these relations above shows that W_2 is as claimed. □

Corollary 3.2 *For $|x|\le R$,*

$$|W_1(t,s,x)|+|W_2(t,s,x)|\le\frac{C}{|x|}\,\mathbf{1}_{\{|t|\le R+R_0\}}\,|\dot\omega(s)|.$$

In particular, if $t\ge R+R_0$, then

$$\begin{aligned}&\Big\|\int_0^t W_1(t-s,s,\cdot)\,ds\Big\|_{L^2(B_R(0))}+\Big\|\int_0^t W_2(t-s,s,\cdot)\,ds\Big\|_{L^2(B_R(0))}\\ &\quad\le\; C(R)\,\max\{|\dot\omega(\tau)|:\tau\in[t-(R+R_0),t]\}.\end{aligned}$$

Proof: This is a direct consequence of Lemma 3.1 and the support properties (2.1) of g. □

Now we turn to rewriting (1.5). By (7.8), (7.9), (3.1), and (3.2),

$$\begin{aligned}I_{\rm b}\,\dot\omega(t) &= \int_{\mathbb{R}^3} x\wedge\Big[E(t,x)+(\omega(t)\wedge x)\wedge B(t,x)\Big]\,f_{\rm e}(x)\,dx\\ &= \int_{\mathbb{R}^3} x\wedge\Big[Z_1(t,x)+(\omega(t)\wedge x)\wedge Z_2(t,x)\Big]\,f_{\rm e}(x)\,dx\\ &= \int_{\mathbb{R}^3} x\wedge\Big[[\mathcal{U}(t)Z(0,\cdot)]_1(x)+(\omega(t)\wedge x)\wedge[\mathcal{U}(t)Z(0,\cdot)]_2(x)\Big]\,f_{\rm e}(x)\,dx\\ &\quad-\int_0^t ds\int_{\mathbb{R}^3} dx\,f_{\rm e}(x)\,x\wedge\Big[W_1(t-s,s,x)+(\omega(t)\wedge x)\wedge W_2(t-s,s,x)\Big]\\ &= \mathcal{T}_{\rm hom}(t)+\mathcal{T}_{\rm inh}(t)\end{aligned}\tag{3.7}$$

for $t\in\mathbb{R}$. The next estimate concerns $\mathcal{T}_{\rm hom}(t)$.

Lemma 3.3 *Under the above hypotheses (1.10) and (1.11) there is a constant $C > 0$ such that*

$$|\mathcal{T}_{\rm hom}(t)| \le C\,(1 + |\omega(t)|)\,t^{-(1+\gamma)} \quad \textit{and} \quad |\dot{\mathcal{T}}_{\rm hom}(t)| \le C(1 + |\omega(t)| + |\dot\omega(t)|)$$

holds for $t \ge 2R_0$.

Proof: Taking the inverse Fourier transform it follows from (3.4) and (3.5) that

$$\begin{aligned}
\Phi_1(t,x) &= \frac{1}{4\pi t^2}\int_{|y-x|=t} dS(y)\,\Big[t\,\mathrm{rot}\,Z_{0,2}(y) + Z_{0,1}(y) + ((y-x)\cdot\nabla)Z_{0,1}(y)\Big],\\
\Phi_2(t,x) &= \frac{1}{4\pi t^2}\int_{|y-x|=t} dS(y)\,\Big[-t\,\mathrm{rot}\,Z_{0,1}(y) + Z_{0,2}(y) + ((y-x)\cdot\nabla)Z_{0,2}(y)\Big],
\end{aligned}$$

for

$$\Phi(t,x) = [\mathcal{U}(t)Z(0,\cdot)](x) = (\Phi_1(t,x), \Phi_2(t,x)), \tag{3.8}$$

where

$$(\Phi_1^{(0)}, \Phi_2^{(0)}) = \Phi(0) = Z(0,\cdot) = (E_0 - E_{\omega_0}, B_0 - B_{\omega_0});$$

see [9]. From (1.10) and (7.5) we have

$$|\Phi_j^{(0)}(y)| + |y||\nabla\Phi_j^{(0)}(y)| \le C(R)\,|y|^{-(1+\gamma)}, \quad |y| \ge R,$$

for $j = 1,2$. If $t \ge 2R$ and $|x| \le R$, then $|y-x| = t$ implies that $|y| \ge |y-x| - |x| \ge t - R \ge t/2 \ge R$. Hence

$$|\Phi(t,x)| \le C(R)\,t^{-(1+\gamma)}, \quad t \ge 2R, \quad |x| \le R. \tag{3.9}$$

Using (1.7) and this estimate for $R = R_0$, we get $|\mathcal{T}_{\rm hom}(t)| \le C\,(1 + |\omega(t)|)\,t^{-(1+\gamma)}$ for $t \ge 2R_0$. Next observe that $\partial_t\Phi_1 = \mathrm{rot}\,\Phi_2$ and $\partial_t\Phi_2 = -\mathrm{rot}\,\Phi_1$ by construction. Since

$$|\nabla\Phi_j^{(0)}(y)| + |y||\nabla\nabla\Phi_j^{(0)}(y)| \le C(R), \quad |y| \ge R,$$

for $j = 1,2$ by (1.11) and (7.6), it follows as above that

$$|\dot\Phi(t,x)| \le C(R), \quad t \ge 2R, \quad |x| \le R,$$

which in turn yields the bound on $\dot{\mathcal{T}}_{\rm hom}(t)$. □

By means of Lemma 3.1 the inhomogeneous part $\mathcal{T}_{\rm inh}(t)$ can be expressed in a more convenient way. Let

$$\kappa_1(t) = \int_0^\infty r^3 f_{\rm e}(r)\varphi_1(t,r)\,dr, \tag{3.10}$$

$$\kappa_2(t) = \int_0^\infty r^4 f_{\rm e}(r)\varphi_{23}(t,r)\,dr, \tag{3.11}$$

for $t \in \mathbb{R}$, where

$$\varphi_1(t,r) = -\frac{2\pi}{r}\Big(g(t-r) + g(t+r) - \frac{1}{r}\int_{-r}^{r} g(t-\tau)\,d\tau\Big), \tag{3.12}$$

$$\varphi_{23}(t,r) = -\frac{4\pi}{r^3}\int_{-r}^{r} g(t-\tau)\tau\,d\tau = -\frac{4\pi}{r^3}\int_{t-r}^{t+r}(t-\tau)g(\tau)\,ds, \tag{3.13}$$

for $t \in \mathbb{R}$ and $r \in [0, \infty[$. Then

$$\dot{\kappa}_2(t) = -2\kappa_1(t) \quad \text{for} \quad t \in \mathbb{R}. \tag{3.14}$$

Furthermore, $\varphi_1(t, r) = \varphi_{23}(t, r) = 0$ for $|t| > 2R_0$ and $r \in [0, R_0]$ by (2.1). It follows that

$$\kappa_1(t) = \kappa_2(t) = 0 \quad \text{for} \quad |t| > 2R_0. \tag{3.15}$$

Also note that κ_1 is odd and κ_2 is even.

Lemma 3.4 *Under the above hypotheses,*

$$\mathcal{T}_{\rm inh}(t) = -\frac{8\pi}{3} \int_0^t \dot{\omega}(t-s)\kappa_1(s)\, ds - \frac{4\pi}{3}\, \omega(t) \wedge \int_0^t \dot{\omega}(t-s)\kappa_2(s)\, ds$$

for $t \in \mathbb{R}$.

Proof: Using Lemma 3.1,

$$W_1(t, s, x) = \varphi_1(t, |x|)\, \dot{\omega}(s) \wedge \bar{x} \quad \text{and} \quad W_2(t, s, x) = \varphi_{21}(t, |x|)\, \dot{\omega}(s) + \varphi_{22}(t, |x|)\, (\bar{x} \cdot \dot{\omega}(s))\, \bar{x},$$

where φ_1 is given by (3.12) and

$$\begin{aligned}
\varphi_{21}(t, r) &= \frac{2\pi}{r} \Big(g(t+r) - g(t-r) + \frac{1}{r^2} \int_{-r}^{r} g(t-\tau)\tau\, d\tau \Big), \\
\varphi_{22}(t, r) &= -\frac{2\pi}{r} \Big(g(t+r) - g(t-r) + \frac{3}{r^2} \int_{-r}^{r} g(t-\tau)\tau\, d\tau \Big).
\end{aligned}$$

Then

$$\begin{aligned}
&x \wedge \Big[W_1(t-s, s, x) + (\omega(t) \wedge x) \wedge W_2(t-s, s, x) \Big] \\
&= \varphi_1(t-s, |x|)(|x|\dot{\omega}(s) - x \cdot \dot{\omega}(s)\, \bar{x}) + x \cdot W_2(t-s, s, x)\, \omega(t) \wedge x \\
&= \frac{1}{|x|}\, \varphi_1(t-s, |x|)(|x|^2 \mathrm{Id} - x \otimes x)\, \dot{\omega}(s) + \varphi_{23}(t-s, |x|)\, \omega(t) \wedge ((x \otimes x)\, \dot{\omega}(s))
\end{aligned}$$

for

$$\varphi_{23}(t, r) = \varphi_{21}(t, r) + \varphi_{22}(t, r)$$

as in (3.13). From the symmetry of $f_{\rm e}$ it follows that

$$\begin{aligned}
\mathcal{T}_{\rm inh}(t) &= -\int_0^t ds \int_{\mathbb{R}^3} dx\, f_{\rm e}(x)\, \frac{1}{|x|}\, \varphi_1(t-s, |x|)(|x|^2 \mathrm{Id} - x \otimes x)\, \dot{\omega}(s) \\
&\quad - \omega(t) \wedge \int_0^t ds \int_{\mathbb{R}^3} dx\, f_{\rm e}(x)\, \varphi_{23}(t-s, |x|)\, (x \otimes x)\, \dot{\omega}(s) \\
&= -\frac{8\pi}{3} \int_0^t ds\, \dot{\omega}(t-s) \Big(\int_0^\infty dr\, r^3 f_{\rm e}(r) \varphi_1(s, r) \Big) \\
&\quad - \frac{4\pi}{3}\, \omega(t) \wedge \int_0^t ds\, \dot{\omega}(t-s) \Big(\int_0^\infty dr\, r^4 f_{\rm e}(r) \varphi_{23}(s, r) \Big),
\end{aligned}$$

as was to be shown. □

4 The limiting equation

In what follows we will frequently refer to the notation and results summarized in Section 7.5 below. By Lemma 7.2 we have the bounds

$$\|\omega\|_{L^\infty} + \|\dot{\omega}\|_{L^\infty} + \|\ddot{\omega}\|_{L^\infty} < \infty. \tag{4.1}$$

For the function $\Omega : \mathbb{R} \to \mathbb{R}^6$ given by $\Omega(t) = (\omega(t), \dot{\omega}(t))$ this means that $\|\Omega\|_{L^\infty} + \|\dot{\Omega}\|_{L^\infty} < \infty$, and thus $\Omega \in C_b^1(\mathbb{R})$ and $\Gamma^+(\Omega) \neq \emptyset$ for the limit set. Let $Y = (Y_1, Y_2) \in \Gamma^+(\Omega)$. Then Y_1 and Y_2 are continuous and furthermore

$$\omega(t + h_k) \longrightarrow Y_1(t) \quad \text{and} \quad \dot{\omega}(t + h_k) \longrightarrow Y_2(t) \tag{4.2}$$

uniformly on every compact t-interval as $k \to \infty$ for some fixed sequence $h_k \to \infty$. Since

$$\int_{t_1}^{t_2} \dot{\omega}(s + h_k)\, ds = \omega(t_2 + h_k) - \omega(t_1 + h_k), \quad t_2 > t_1,$$

we can pass to the limit $k \to \infty$ to conclude that Y_1 is differentiable and $\dot{Y}_1 = Y_2$. In addition, (4.1) implies that Y_1 and Y_2 are bounded. By (3.7) and Lemma 3.4,

$$\begin{aligned} I_{\mathrm{b}}\, \dot{\omega}(t + h_k) \; &= \; \mathcal{T}_{\mathrm{hom}}(t + h_k) - \frac{8\pi}{3} \int_0^{t+h_k} \dot{\omega}(t + h_k - s)\kappa_1(s)\, ds \\ &\quad - \frac{4\pi}{3}\, \omega(t + h_k) \wedge \int_0^{t+h_k} \dot{\omega}(t + h_k - s)\kappa_2(s)\, ds \end{aligned}$$

for all $t \in \mathbb{R}$ and $k \in \mathbb{N}$. If k is sufficiently large (more precisely: $t + h_k \geq 2R_0$), then $\int_0^{t+h_k} (\ldots)\, ds = \int_0^{2R_0} (\ldots)\, ds = \int_0^{\infty} (\ldots)\, ds$ by (3.15). Thus passing to the limit $k \to \infty$ we obtain from (4.2) and Lemma 3.3 the limiting equation

$$I_{\mathrm{b}}\, Y_2(t) = -\frac{8\pi}{3} \int_0^\infty Y_2(t - s)\kappa_1(s)\, ds - \frac{4\pi}{3}\, Y_1(t) \wedge \int_0^\infty Y_2(t - s)\kappa_2(s)\, ds \tag{4.3}$$

for all $t \in \mathbb{R}$. Since $\dot{\omega}$ is Lipschitz continuous, (2.1) implies that also $\dot{\omega} * g$ is Lipschitz continuous. Hence $(\dot{\omega} * g)(t) \to 0$ as $t \to \infty$ by Lemma 2.1. For fixed $t \in \mathbb{R}$ therefore by (2.1),

$$\begin{aligned} (Y_2 * g)(t) \; &= \; \int_{-R_0}^{R_0} Y_2(t - s) g(s)\, ds \longleftarrow \int_{-R_0}^{R_0} \dot{\omega}(t + h_k - s) g(s)\, ds \\ &= \; (\dot{\omega} * g)(t + h_k) \to 0 \end{aligned}$$

as $k \to \infty$. Hence we arrive at the relation $Y_2 * g = 0$. Thus $\sigma(Y_2 * g) = \emptyset$ for the spectrum, so that

$$\sigma(Y_2) \subset \{\hat{g} = 0\}$$

by (7.23) and since $g \in L^1(\mathbb{R})$. In view of $\phi_1(0) = 0$ also $\hat{g}(0) = 0$; recall (1.15) and (1.14). Hence $\dot{Y}_1 = Y_2$ in conjunction with (7.24) and (1.16) implies that

$$\sigma(Y_2) \subset \sigma(Y_1) \subset \sigma(Y_2) \cup \{0\} \subset \{\hat{g} = 0\} = \{\mu_j : j \in \mathbb{Z}\}. \tag{4.4}$$

In the introduction we noted that $\{\hat{g} = 0\}$ is at most countable, and hence so are $\sigma(Y_1)$ and $\sigma(Y_2)$. Now observe that Y_2 is Lipschitz continuous by (4.2), since $|\dot{\omega}(t + h_k) - \dot{\omega}(s + h_k)| \leq C|t - s|$ in view of (4.1). Thus Y_1 and Y_2 are bounded, uniformly continuous, and they have at most countable spectra. As a consequence, both Y_1 and Y_2 are almost periodic. Next we note that as a consequence of (3.14),

$$\int_0^\infty Y_2(t-s)\kappa_2(s)\,ds = -2\int_0^\infty Y_1(t-s)\kappa_1(s)\,ds + \kappa_2(0)\,Y_1(t) \quad \text{for} \quad t\in\mathbb{R}, \tag{4.5}$$

where explicitly

$$\kappa_2(0) = \frac{16\pi}{3}\int_0^\infty da\,a^4 f_{\rm e}(a)\int_a^\infty dr\,r\,f_{\rm e}(r) \tag{4.6}$$

is calculated. Returning to (4.3), (4.5) yields the limiting equation

$$I_{\rm b}\,Y_2(t) = -\frac{8\pi}{3}\int_0^\infty Y_2(t-s)\kappa_1(s)\,ds + \frac{8\pi}{3}\,Y_1(t)\wedge\int_0^\infty Y_1(t-s)\kappa_1(s)\,ds \tag{4.7}$$

for $t\in\mathbb{R}$.

5 Proof of Theorem 1.1

Here we have $f_{\rm e}(x) = \delta(|x|-1)$, so that $f_e(r) = \delta(r-1)$. Then

$$\hat{g}(\tau) = \sqrt{\frac{2}{\pi}}\,i\,\phi_1(\tau), \tag{5.1}$$

$$\tilde{\kappa}_1(\tau) = -4\sqrt{2\pi}\,\phi_1(\tau)\Big(i+\frac{1}{\tau}\Big)\,e^{-i\tau}, \tag{5.2}$$

for $\tau\in\mathbb{R}$ by (1.14) and Lemma 7.6. By (7.29), due to $\dot{Y}_1 = Y_2$, and using (7.31), we have for $\tau\in\mathbb{R}$

$$\begin{aligned}(Y_2 *_0 \kappa_1)^\flat(\tau) &= \sqrt{2\pi}\,Y_2^\flat(\tau)\tilde{\kappa}_1(\tau) = \sqrt{2\pi}i\tau\,Y_1^\flat(\tau)\tilde{\kappa}_1(\tau),\\ (Y_1 *_0 \kappa_1)^\flat(\tau) &= \sqrt{2\pi}\,Y_1^\flat(\tau)\tilde{\kappa}_1(\tau).\end{aligned}$$

If $\tau\notin\sigma(Y_1)$, then $Y_1^b(\tau) = 0$ by (7.27). If $\tau\in\sigma(Y_1)$, then $\hat{g}(\tau)=0$ by (4.4), whence $\tilde{\kappa}_1(\tau)=0$ in view of (5.1) and (5.2). Therefore $Y_2 *_0 \kappa_1 = Y_1 *_0 \kappa_1 = 0$ by the uniqueness theorem for the Bohr transform. Accordingly, the limiting equation (4.7) yields $Y_2 = 0$, so that $\sigma(Y_2)=\emptyset$ and $\sigma(Y_1)\subset\{0\}$ by (4.4). The latter relation implies that Y_1 equals a constant vector.

If we summarize the argument that was started in Section 4, then so far we have proved that for $\Omega(t) = (\omega(t),\dot{\omega}(t))$ every function $Y=(Y_1,Y_2)\in\Gamma^+(\Omega)$ satisfies $Y=(C,0)$ for some constant vector $C\in\mathbb{R}^3$. From Lemma 7.7 we deduce that $\dot{\Omega}(t)\to 0$ as $t\to\infty$ and Ω is asymptotically slowly varying. Hence $|\omega(t+T)-\omega(t)|\to 0$ as $t\to\infty$ uniformly for T in compact subsets of $\mathbb{R}$, $\dot{\omega}(t)\to 0$, and $\ddot{\omega}(t)\to 0$ as $t\to\infty$ are obtained.

Next we consider the asymptotic behavior of the fields and prove (1.12). To begin with,

$$\begin{aligned}&{\rm dist}_R\Big((\omega(t),E(t),B(t)),\mathcal{S}\Big)\\ &\le \|E(t)-E_{\omega(t)}\|_{L^2(B_R(0))} + \|B(t)-B_{\omega(t)}\|_{L^2(B_R(0))}\\ &= \|Z_1(t)\|_{L^2(B_R(0))} + \|Z_2(t)\|_{L^2(B_R(0))}\\ &= \|\Phi_1(t)\|_{L^2(B_R(0))} + \|\Phi_2(t)\|_{L^2(B_R(0))} + \Big\|\int_0^t W_1(t-s,s,\cdot)\,ds\Big\|_{L^2(B_R(0))}\\ &\quad + \Big\|\int_0^t W_2(t-s,s,\cdot)\,ds\Big\|_{L^2(B_R(0))}\end{aligned}$$

by (3.1), (3.2), and (3.8). Hence (3.9) and Corollary 3.2 yield

$$\mathrm{dist}_R\Big((\omega(t), E(t), B(t)), \mathcal{S}\Big) \le C(R)\,\Big[t^{-(1+\gamma)} + \max\{|\dot\omega(\tau)| : \tau \in [t-(R+R_0), t]\}\Big]$$

for $t \ge \max\{2R, R+R_0\}$, which gives (1.12) in view of $\dot\omega(t) \to 0$ as $t \to \infty$. This completes the proof of Theorem 1.1. □

6 Proof of Theorem 1.4

Define

$$\mathcal{M}_{\rm exc} = \Big\{-\sqrt{2\pi}\,\Big(\int_0^\infty r^4 f_{\rm e}(r)\,dr\Big)^{-1}\tilde\kappa_1(\mu_j) : j \in \mathbb{Z}\Big\}, \tag{6.1}$$

where $\{\mu_j : j \in \mathbb{Z}\}$ are the zeroes of $\hat g$, see (1.16), and

$$\tilde\kappa_1(\tau) = (\kappa_1 \mathbf{1}_{[0,\infty[})^\wedge(\tau) = -4\sqrt{2\pi}\int_0^\infty da\, a^3 f_{\rm e}(a)\phi_1(\tau a)\int_a^\infty dr\, r^2 f_{\rm e}(r)\Big(i + \frac{1}{r\tau}\Big)\,e^{-i\tau r}, \quad \tau \in \mathbb{R},$$

for the function κ_1 as introduced in (3.10). Thus $\mathcal{M}_{\rm exc}$ is countable and can be determined from the charge density $f_{\rm e}$. Consider the system (1.3)–(1.5) for $m_{\rm b} \notin \mathcal{M}_{\rm exc}$. Let again $Y = (Y_1, Y_2) \in \Gamma^+(\Omega)$ be a limit point of $\Omega(t) = (\omega(t), \dot\omega(t))$. Once more it is the aim to conclude from (4.7) that Y_1 is constant and $Y_2 = 0$. For this suppose that $\sigma(Y_2) \ne \emptyset$ and write $\sigma(Y_2) = \{\lambda_j : j \in \mathbb{N}\}$; if $\sigma(Y_2)$ is finite, then the proof is easier. By (4.4),

$$\sigma(Y_2) \subset \sigma(Y_1) \subset \sigma(Y_2) \cup \{0\} = \{\lambda_j : j \in \mathbb{N}_0\} \subset \{\hat g = 0\},$$

where we let $\lambda_0 = 0$. We may assume that $\sigma(Y_1) = \{\lambda_j : j \in \mathbb{N}_0\}$, since the proof is again easier (and similar) in the case where $\sigma(Y_1) = \{\lambda_j : j \in \mathbb{N}\}$. Choose trigonometric polynomials

$$P_m(t) = \sum_{j=0}^{r_m} \nu_{mj}\, Y_1^\flat(\lambda_j)\, e^{i\lambda_j t} \quad \text{and} \quad Q_m(t) = i\sum_{j=1}^{s_m} \sigma_{mj}\,\lambda_j\, Y_1^\flat(\lambda_j)\, e^{i\lambda_j t}$$

such that

$$\lim_{m\to\infty} \|P_m - Y_1\|_{L^\infty(\mathbb{R})} = 0 \quad \text{and} \quad \lim_{m\to\infty} \|Q_m - Y_2\|_{L^\infty(\mathbb{R})} = 0, \tag{6.2}$$

where $\nu_{mj} \in]0,1]$ and $\sigma_{mj} \in]0,1]$ are suitable coefficients such that $\lim_{m\to\infty}\nu_{mj} = 1$ as well as $\lim_{m\to\infty}\sigma_{mj} = 1$ for every j; see (7.32) below. Fix $\varepsilon > 0$. If $m_0 \in \mathbb{N}$ is sufficiently large and $m \ge m_0$, then $|\Delta_m(t)| \le C\varepsilon$ for all $t \in \mathbb{R}$, where

$$\Delta_m(t) = I_{\rm b}\, Q_m(t) + \frac{8\pi}{3}\int_0^\infty Q_m(t-s)\kappa_1(s)\,ds - \frac{8\pi}{3}\, P_m(t) \wedge \int_0^\infty P_m(t-s)\kappa_1(s)\,ds.$$

This follows from (4.7) and (6.2), since in particular $\kappa_1 \in L^1(\mathbb{R})$. Thus if $m \ge m_0$ and $\lambda \in \mathbb{R}$, then also $|\langle \Delta_m, e^{i\lambda t}\rangle_{\mathbb{M}}| \le C\varepsilon$. Noting that by (7.30)

$$\begin{aligned}\Delta_m(t) \;&=\; i\sum_{j=1}^{s_m} \sigma_{mj}\,\lambda_j\, Y_1^\flat(\lambda_j) e^{i\lambda_j t}\Big(I_{\rm b} + \frac{8\pi}{3}\sqrt{2\pi}\,\tilde\kappa_1(\lambda_j)\Big) \\ &\quad - \frac{8\pi}{3}\sqrt{2\pi}\sum_{j,\,k=0}^{r_m} \nu_{mj}\,\nu_{mk}\,(Y_1^\flat(\lambda_j) \wedge Y_1^\flat(\lambda_k))\, e^{i(\lambda_j+\lambda_k)t}\,\tilde\kappa_1(\lambda_k),\end{aligned}$$

we obtain from (7.26) that

$$\Big| i \sum_{j=1}^{s_m} \sigma_{mj}\, \lambda_j\, Y_1^\flat(\lambda_j)\Big(I_\mathrm{b} + \frac{8\pi}{3}\sqrt{2\pi}\,\tilde\kappa_1(\lambda_j)\Big)\delta(\lambda_j-\lambda)$$
$$-\frac{8\pi}{3}\sqrt{2\pi}\sum_{j,\,k=0}^{r_m}\nu_{mj}\,\nu_{mk}\,(Y_1^\flat(\lambda_j)\wedge Y_1^\flat(\lambda_k))\,\tilde\kappa_1(\lambda_k)\,\delta(\lambda_j+\lambda_k-\lambda)\Big| \le C\varepsilon.$$

Then take $\lambda=\lambda_l\in\sigma(Y_2)$ for some fixed $l\in\mathbb{N}$. We have $\lambda_l\neq 0$ by (7.27) and (7.31), since $\sigma(Y_2)\subset\{\hat g=0\}$ and $\{\hat g=0\}\setminus\{0\}$ does not have the accumulation point 0. Recalling (4.4), the nonresonance condition (NRC) yields that the only solutions to $\lambda_j+\lambda_k-\lambda_l=0$ are given by the trivial ones $j=0$, $k=l$ and $j=l$, $k=0$. To summarize, if $l\in\mathbb{N}$ and $\varepsilon>0$ are fixed, then for all m sufficiently large,

$$\Big| i\sigma_{ml}\,\lambda_l\,Y_1^\flat(\lambda_l)\Big(I_\mathrm{b}+\frac{8\pi}{3}\sqrt{2\pi}\,\tilde\kappa_1(\lambda_l)\Big)$$
$$-\frac{8\pi}{3}\sqrt{2\pi}\,\nu_{m0}\,\nu_{ml}\,(\tilde\kappa_1(\lambda_l)-\tilde\kappa_1(0))\,[Y_1^\flat(0)\wedge Y_1^\flat(\lambda_l)]\Big|\le C\varepsilon.$$

Passing to the limits $m\to\infty$ first and then $\varepsilon\to 0$, we obtain the relation

$$i\,\lambda_l\,Y_1^\flat(\lambda_l)\Big(I_\mathrm{b}+\frac{8\pi}{3}\sqrt{2\pi}\,\tilde\kappa_1(\lambda_l)\Big)=\frac{8\pi}{3}\sqrt{2\pi}\,(\tilde\kappa_1(\lambda_l)-\tilde\kappa_1(0))\,[Y_1^\flat(0)\wedge Y_1^\flat(\lambda_l)]$$

for all $l\in\mathbb{N}$. Upon taking the inner product with $Y_1^\flat(\lambda_l)\in\mathbb{R}^3$, it follows that

$$i\,\lambda_l\,|Y_1^\flat(\lambda_l)|^2\Big(I_\mathrm{b}+\frac{8\pi}{3}\sqrt{2\pi}\,\tilde\kappa_1(\lambda_l)\Big)=0.$$

Since $\lambda_l\in\sigma(Y_1)$ implies $Y_1^\flat(\lambda_l)\neq 0$ we get

$$I_\mathrm{b}+\frac{8\pi}{3}\sqrt{2\pi}\,\tilde\kappa_1(\lambda_l)=0.$$

By (1.6) this relation is equivalent to

$$m_\mathrm{b}=-\sqrt{2\pi}\,\Big(\int_0^\infty r^4 f_\mathrm{e}(r)\,dr\Big)^{-1}\tilde\kappa_1(\lambda_l),$$

which however is excluded since $\lambda_l\in\{\hat g=0\}$ and $m_\mathrm{b}\notin\mathcal{M}_\mathrm{exc}$. Therefore we have shown that $\sigma(Y_2)=\emptyset$ and $\sigma(Y_1)=\{0\}$. As a consequence, $Y_2=0$, and Y_1 equals a (non-zero) constant vector. [Note that in the case where $\sigma(Y_1)=\{\lambda_j : j\in\mathbb{N}\}=\sigma(Y_2)$ we would have $Y_1=0$.] Thus we proof can be completed in the same way as was the proof of Theorem 1.1. □

7 Appendix: Some technicalities

7.1 Existence of the dynamics and a priori bounds

Lemma 7.1 *Suppose that $\omega_0\in\mathbb{R}^3$, and $E_0,B_0\in L^2(\mathbb{R}^3)$ are such that (1.9) holds. Then the system (1.3)–(1.5) with initial data (1.8) has a unique (weak) solution. It conserves the energy*

$$\mathcal{E}(t)=\frac{1}{2}\,I_\mathrm{b}\,|\omega(t)|^2+\frac{1}{8\pi}\int_{\mathbb{R}^3}(|E(t,x)|^2+|B(t,x)|^2)\,dx.$$

Proof: See [6, Prop. 2.2]. □

Lemma 7.2 *Under the hypotheses (1.10) and (1.11) on the initial data we have*

$$\|\omega\|_{L^\infty} + \|\dot{\omega}\|_{L^\infty} + \|\ddot{\omega}\|_{L^\infty} < \infty.$$

Proof: The bound on ω is obtained from the conservation of energy. What concerns $\dot{\omega}$, recalling (3.7) we have

$$I_{\mathrm{b}}\, \dot{\omega}(t) = \mathcal{T}_{\mathrm{hom}}(t) + \mathcal{T}_{\mathrm{inh}}(t).$$

If $t \geq 2R_0$, then

$$|\mathcal{T}_{\mathrm{hom}}(t)| \leq C\,(1 + |\omega(t)|)\, t^{-(1+\gamma)} \leq C\, t^{-(1+\gamma)}$$

by Lemma 3.3. An analogous estimate can be derived for $t \leq -2R_0$, and $\mathcal{T}_{\mathrm{hom}}$ is bounded for $|t| \leq 2R_0$. Furthermore, Lemma 3.4 and (3.14) imply that

$$\mathcal{T}_{\mathrm{inh}}(t) = -\frac{8\pi}{3}\int_0^t \dot{\omega}(t-s)\kappa_1(s)\,ds - \frac{8\pi}{3}\,\omega(t) \wedge \int_0^t \omega(t-s)\kappa_1(s)\,ds - \frac{4\pi}{3}\,(\omega(t)\wedge\omega_0)\,\kappa_2(t) \quad (7.1)$$

for $t \in \mathbb{R}$. Due to the compact support of κ_1 and κ_2 the last two terms are bounded. For the first term, writing out the definitions we get

$$\begin{aligned}
&\int_0^t \dot{\omega}(t-s)\kappa_1(s)\,ds \\
&\quad = \; 2\pi \int_0^\infty dr\, r^2 f_{\mathrm{e}}(r) \int_0^t ds\, \dot{\omega}(t-s)\Big[g(s-r) + g(s+r) - \frac{1}{r}\int_{-r}^{r} g(s-\tau)\,d\tau\Big].
\end{aligned}$$

Consider for instance the contribution of $g(s-r)$ for $r \in [0, R_0]$. We distinguish the cases $s \in [0, r]$ and $s \in [r, t]$ to integrate by parts in s using (1.13). For $t > R_0$ it follows that

$$\begin{aligned}
\int_0^t ds\, \dot{\omega}(t-s)\, g(s-r) \;&=\; -\,\omega(t)g(r) - \omega_0\, g(t-r) + \int_0^r ds\, \omega(t-s)(s-r)^2 f_{\mathrm{e}}(|s-r|) \\
&\quad - \int_r^t ds\, \omega(t-s)(s-r)^2 f_{\mathrm{e}}(|s-r|) + \int_0^t ds\, \omega(t-s)\int_{|s-r|}^\infty da\, a f_{\mathrm{e}}(a).
\end{aligned}$$

Since $|s-r| \leq R_0$ is required, we have $s \leq 2R_0$. Hence it $t > 2R_0$, then

$$\begin{aligned}
\int_0^t ds\, \dot{\omega}(t-s)\, g(s-r) \;&=\; -\,\omega(t)g(r) + \int_0^r ds\, \omega(t-s)(s-r)^2 f_{\mathrm{e}}(|s-r|) \\
&\quad - \int_r^{2R_0} ds\, \omega(t-s)(s-r)^2 f_{\mathrm{e}}(|s-r|) \\
&\quad + \int_0^{2R_0} ds\, \omega(t-s)\int_{|s-r|}^\infty da\, a f_{\mathrm{e}}(a). \qquad (7.2)
\end{aligned}$$

This function of $t > 2R_0$ is bounded. As the other cases and contributions can be handled similarly, we obtain the boundedness $\dot{\omega}$. To bound the second derivative $\ddot{\omega}$, we use

$$I_{\mathrm{b}}\, \ddot{\omega}(t) = \dot{\mathcal{T}}_{\mathrm{hom}}(t) + \dot{\mathcal{T}}_{\mathrm{inh}}(t).$$

According to Lemma 3.3 and the previous steps we have

$$|\dot{\mathcal{T}}_{\rm hom}(t)| \le C(1+|\omega(t)|+|\dot\omega(t)|) \le C.$$

The derivative $\dot{\mathcal{T}}_{\rm inh}$ is calculated from (7.1). The last two terms yield a bounded contribution, since $\|\omega\|_{L^\infty} + \|\dot\omega\|_{L^\infty} < \infty$ and $\dot\kappa_2 = -2\kappa_1$ by (3.14). For the first term we consider for instance the contribution of $g(s-r)$ which led to (7.2). Differentiating the right-hand side of (7.2) yields a bounded function of $t > 2R_0$. Thus we may argue as before to conclude that $\|\ddot\omega\|_{L^\infty} < \infty$ also. □

7.2 Stationary states

The stationary states $(\omega, E_\omega(x), B_\omega(x))$ of (1.3)–(1.5) are described in the next lemma; cf. [11, Section 10.2]. We use $\hat\varphi(k) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ik\cdot x}\varphi(x)\,dx$ as the Fourier transform of a function $\varphi : \mathbb{R}^n \to \mathbb{R}$.

Lemma 7.3 *The stationary states (1.3)–(1.5) are*

$$\begin{aligned}
\omega(t) &\equiv \omega, \\
E_\omega(x) &= 4\pi\,\frac{x}{|x|^3}\int_0^{|x|} dr\, r^2 f_{\rm e}(r), && (7.3)\\
B_\omega(x) &= \frac{8\pi}{3}\Big(\int_{|x|}^\infty dr\, r f_{\rm e}(r)\Big)\omega - \frac{4\pi}{3}\Big(\frac{1}{|x|^3}\int_0^{|x|} dr\, r^4 f_{\rm e}(r)\Big)[\omega - 3(\bar x\cdot\omega)\bar x]. && (7.4)
\end{aligned}$$

[Observe that in fact $E_\omega = E$ is independent of ω. Nevertheless this notation is used throughout to emphasize that this E is part of the stationary state.] For every $R > 0$ large enough there is a constant $C(R) > 0$ such that

$$|x|^2|E_\omega(x)| + |x|^3|B_\omega(x)| + |x|^3|\nabla E_\omega(x)| + |x|^4|\nabla B_\omega(x)| \le C(R)\,(1+|\omega|) \qquad (7.5)$$

and

$$|x|(|\nabla\nabla E_\omega(x)| + |\nabla\nabla B_\omega(x)|) \le C(R)\,(1+|\omega|) \qquad (7.6)$$

are verified for $|x| > R$.

Proof: The relations to be satisfied are

$$\dot\omega = 0, \quad \operatorname{rot} B_\omega(x) = 4\pi(\omega\wedge x) f_{\rm e}(x), \quad \operatorname{div} B_\omega(x) = 0, \quad \operatorname{rot} E_\omega(x) = 0, \quad \operatorname{div} E_\omega(x) = 4\pi f_{\rm e}(x).$$

From this both E and B are obtained using the observation that, in general, the solution F of the equations $\operatorname{rot}F = G_1$ and $\operatorname{div}F = g_2$ is given by $\hat F(k) = i|k|^{-2}(k\wedge \hat G_1(k) - k\hat g_2(k))$ in Fourier space. Using $(xf_{\rm e}(x))\hat{}\,(k) = \bar k\,\hat g(|k|)$ it is found that

$$\hat E_\omega(k) = -\frac{4\pi i}{|k|^2}\,k\hat f_{\rm e}(k) \quad \text{and} \quad \hat B_\omega(k) = \frac{4\pi i}{|k|^3}\,\hat g(|k|)\,(|k|^2\omega - (k\cdot\omega)k). \qquad (7.7)$$

Furthermore, it can be seen that these functions E_ω and B_ω already give rise to a solution of (1.5). Indeed, for a fixed $\omega\in\mathbb{R}^3$

$$4\pi\int (x\wedge E_\omega) f_{\rm e}(x)\,dx = \int (x\wedge E_\omega)\operatorname{div}E_\omega\,dx = 0 \qquad (7.8)$$

is verified through integration by parts and observing $\mathrm{div}(x \wedge E_\omega) = E_\omega \cdot \mathrm{rot}\, x - x \cdot \mathrm{rot} E_\omega = 0$. In addition,

$$4\pi \int x \wedge \Big[(\omega \wedge x) \wedge B_\omega\Big]\, f_{\mathrm{e}}(x)\, dx = \int x \wedge (\mathrm{rot} B_\omega \wedge B_\omega)\, dx = -\int \mathrm{div} B_\omega (x \wedge B_\omega)\, dx = 0, \quad (7.9)$$

and thus

$$\int x \wedge \Big[E_\omega(x) + (\omega \wedge x) \wedge B_\omega(x)\Big] f_{\mathrm{e}}(x)\, dx = 0$$

for every $\omega \in \mathbb{R}^3$. Taking the inverse Fourier transform of (7.7) then leads to (7.3) and (7.4), which in turn yield (7.5). □

Note that $E_\omega(-x) = -E_\omega(x)$ and $B_\omega(-x) = B_\omega(x)$ by the symmetry of f_{e}. We also remark that (7.6) is certainly not optimal, but sufficient for our purposes.

We consider two important special cases.

Example 7.4 (a) For the uniformly charged sphere, $f_{\mathrm{e}}(x) = \delta(|x| - 1)$,

$$\begin{aligned} E_\omega(x) &= 4\pi \frac{x}{|x|^3} \mathbf{1}_{\{|x|>1\}}(x), \\ B_\omega(x) &= \frac{8\pi}{3} \mathbf{1}_{\{|x|<1\}}\, \omega - \frac{4\pi}{3|x|^3} \mathbf{1}_{\{|x|>1\}} \left[\omega - 3(\bar{x} \cdot \omega)\bar{x}\right]. \end{aligned}$$

(b) For the uniformly charged ball, $f_{\mathrm{e}} = \mathbf{1}_{\{|x|<1\}}$,

$$E_\omega(x) = \frac{4\pi}{3} \frac{x}{|x|^3} \min\{1, |x|^3\}, \quad (7.10)$$

$$\begin{aligned} B_\omega(x) = \frac{4\pi}{15} \Big[&\Big([5 - 6|x|^2]\, \mathbf{1}_{\{|x|<1\}} - \frac{1}{|x|^3} \mathbf{1}_{\{|x|>1\}} \Big) \omega \\ &+ 3\Big(|x|^2\, \mathbf{1}_{\{|x|<1\}} + \frac{1}{|x|^3} \mathbf{1}_{\{|x|>1\}} \Big) (\bar{x} \cdot \omega)\bar{x} \Big]. \end{aligned} \quad (7.11)$$

For other particular charge distributions (7.3) and (7.4) can be evaluated in a similar way. ♢

7.3 The uniformly charged ball

In this section we include some additional remarks concerning the uniformly charged ball, where $f_{\mathrm{e}} = \mathbf{1}_{\{|x|<1\}}$. The stationary fields are given in (7.10) and (7.11). Furthermore, a straightforward calculation using (7.19) below then shows that

$$\begin{aligned} g(t) &= \mathbf{1}_{\{|t|\le 1\}}(t)\, \frac{t}{2}(1 - t^2), \\ \hat{g}(\tau) &= \sqrt{\frac{2}{\pi}} \frac{i}{\tau^4} \Big[(\tau^2 - 3)\sin(\tau) + 3\tau\cos(\tau) \Big], \\ \tilde{\kappa}_1(\tau) &= -4\sqrt{2\pi}\, \frac{1}{\tau^7} \Big[\sqrt{\frac{\pi}{2}} (\tau^2 - 3)\tau^4 \Big(\frac{i}{2}\cos(\tau) + \sin(\tau) \Big) \hat{g}(\tau) + 3\sqrt{\frac{\pi}{2}} \tau^5 \Big(\cos(\tau) - \frac{i}{2}\sin(\tau) \Big) \hat{g}(\tau) \\ &\qquad + \frac{i\pi}{4} \tau^8 \hat{g}(\tau)^2 + \frac{1}{10} \tau^3 (\tau^2 - 30) \Big], \end{aligned} \quad (7.12)$$

for $t, \tau \in \mathbb{R}$.

Concerning the nonresonance condition, we have the following result.

Lemma 7.5 *The nonresonance condition* (NRC) *from Definition 1.3 is satisfied for the uniformly charged ball.*

Proof: We have $\{\hat{g} = 0\} = \{\tau \in \mathbb{R} : (\tau^2 - 3)\sin(\tau) + 3\tau\cos(\tau) = 0\}$; note that always $\hat{g}(0) = 0$. If $\tau \neq 0$ is a solution to $\hat{g}(\tau) = 0$, then $\tau \neq k\pi$ for all $k \in \mathbb{Z}$, so that $\cot(\tau) = 1/\tau - \tau/3$. First we claim that if τ is a solution and $\tau \geq 6$, then $|\tau - l\pi| \leq \pi/4$ for some $l \in \mathbb{N}$. For, if $|\tau - l\pi| > \pi/4$ for all $l \in \mathbb{N}$, we would have $|\sin(\tau)| \geq 1/\sqrt{2}$ and accordingly

$$\frac{11}{6} \leq \Big|\frac{1}{\tau} - \frac{\tau}{3}\Big| = |\cot(\tau)| = \Big|\frac{\cos(\tau)}{\sin(\tau)}\Big| \leq \sqrt{2},$$

which is a contradiction. Next we refine the preceding estimate and prove that if $\tau \geq 15$ is a solution and $\tau \in](k-1)\pi, k\pi[$ for some $k \in \mathbb{N}$, then

$$|\tau - k\pi| \leq 10k^{-1} \tag{7.13}$$

is verified. To check this claim, suppose that $|\tau - k\pi| > 10k^{-1}$ holds. By the first step there is $l \in \mathbb{N}$ such that $\tau \in](l-1/4)\pi, l\pi[\cup]l\pi, (l+1/4)\pi[$. However, if $\tau \in]l\pi, (l+1/4)\pi[$, then $\cot(\tau) > 0$ but $1/\tau - \tau/3 < 0$, which is impossible. Thus $l = k$ and $|\tau - k\pi| \leq \pi/4$ by the first step. Next we Taylor expand sin about $k\pi$ to obtain $|\sin(\tau) - \sigma(\tau - k\pi)| \leq (\tau - k\pi)^2/2$, where $\sigma \in \{-1, 1\}$. Thus we get

$$|\sin(\tau)| \geq |\tau - k\pi| - (\tau - k\pi)^2/2 \geq (1 - \pi/8)|\tau - k\pi| \geq |\tau - k\pi|/2 \geq 5k^{-1}.$$

Hence $\tau \in](k-1)\pi, k\pi[$, and accordingly $k > 15/\pi$, yields the contradiction

$$\frac{2}{5}\,k \leq \frac{(k-1)\pi}{6} \leq \frac{(k-1)\pi}{3} - \frac{1}{(k-1)\pi} \leq \Big|\frac{1}{\tau} - \frac{\tau}{3}\Big| = |\cot(\tau)| = \Big|\frac{\cos(\tau)}{\sin(\tau)}\Big| \leq \frac{1}{5}\,k.$$

As a further step we show that if $\tau \geq 15$ is a solution and $\tau \in](k-1)\pi, k\pi[$ for some $k \in \mathbb{N}$, then

$$\Big|\tau - k\pi + \frac{3}{k\pi}\Big| \leq C_1 k^{-3} \tag{7.14}$$

holds, where $C_1 = 4027$. For, we already know that $|\tau - k\pi| \leq \pi/4$. If $|\zeta - k\pi| \leq \pi/4$, then $|\cos(\zeta)| \geq 1/\sqrt{2}$. By Taylor expansion of tan about $k\pi$, $\tan(\tau) = \tau - k\pi + (1/3)(\tau - k\pi)^3(1 + 2\sin^2\zeta)/\cos^4\zeta$ for some ζ satisfying $|\zeta - k\pi| \leq \pi/4$. From the second step it follows that

$$|\tan(\tau) - \tau + k\pi| \leq 4000\,k^{-3}. \tag{7.15}$$

Furthermore,

$$\Big|\frac{3\tau}{3-\tau^2} + \frac{3}{k\pi}\Big| = \frac{3}{k\pi}\Big|\frac{\tau(k\pi - \tau) + 3}{3 - \tau^2}\Big| \leq \frac{6}{k\pi}\,\frac{(10\tau/k + 3)}{\tau^2} \leq \frac{24}{k\pi}\,\frac{(10\pi + 3)}{k^2\pi^2} \leq 27\,k^{-3}. \tag{7.16}$$

Since $\cot(\tau) = 1/\tau - \tau/3$ means that $\tan(\tau) = 3\tau/(3 - \tau^2)$, (7.14) is a consequence of (7.15) and (7.16).

Now we can prove that $\mu_j+\mu_k=\mu_l$ cannot have a nontrivial solution. Case (i): $\mu_j,\mu_k\geq 4C_1\pi^2$. Select $J,K,L\in\mathbb{N}$ such that $\mu_j\in](J-1)\pi,J\pi[$, $\mu_k\in](K-1)\pi,K\pi[$, and $\mu_l\in](L-1)\pi,L\pi[$. Then by (7.14),

$$\Big|(J+K-L)\pi-\frac{3}{\pi}\Big(\frac{1}{J}+\frac{1}{K}-\frac{1}{L}\Big)\Big|\leq C_1(J^{-3}+K^{-3}+L^{-3}).$$

Since in particular $\mu_j,\mu_k,\mu_l\geq 500$, we get $J,K,L>500/\pi$, and accordingly

$$|(J+K-L)\pi|\leq\frac{3}{\pi}\,\frac{3\pi}{500}+3C_1\frac{\pi^3}{500^3}<\pi.$$

As a consequence, $L=J+K$, and hence

$$\frac{JK+K^2+J^2}{JK(J+K)}=\Big|\frac{1}{J}+\frac{1}{K}-\frac{1}{J+K}\Big|\leq\frac{\pi}{3}\,C_1(J^{-3}+K^{-3}+(J+K)^{-3}).$$

If we assume w.l.o.g. that $K\geq J$, then we obtain

$$\frac{1}{2J}\leq\pi\,C_1J^{-3},$$

which however contradicts the fact that $J>500/\pi$. Case (ii): $\mu_j,\mu_k\in]0,4C_1\pi^2[$. For this case an inspection of the finitely many possibilities (e.g. by sufficiently precise numerical approximation as was done by the author) shows that the relation $\mu_j+\mu_k=\mu_l$ does not admit a nontrivial solution. Case (iii): $\mu_j\in]0,4C_1\pi^2[$ and $\mu_k\geq 4C_1\pi^2$. Fix $J,K,L\in\mathbb{N}$ as in (i). If $\mu_j+\mu_k=\mu_l$, then also $\mu_l\geq 4C_1\pi^2$. Since $\tau=0$ is the only zero of $(\tau^2-3)\sin(\tau)+3\tau\cos(\tau)$ in $[-5,5]$, in particular $\cot(\mu_j)=1/\mu_j-\mu_j/3<0$, which yields $\mu_j\in](J-1/2)\pi,J\pi[$. Thus by (7.13) and due to $K,L>500/\pi$,

$$\begin{aligned}|(J+K-L)\pi| &\leq |\mu_j+K\pi-L\pi|+\frac{\pi}{2}\leq|\mu_j+K\pi-L\pi|+\frac{\pi}{2}\\ &\leq 10(K^{-1}+L^{-1})+\frac{\pi}{2}\leq 20\,\frac{\pi}{500}+\frac{\pi}{2}<\pi.\end{aligned}$$

It follows that $L=J+K$. Thus by (7.14) for error terms R_l and R_k such that $|R_l|\leq C_1L^{-3}$ and $|R_k|\leq C_1K^{-3}$,

$$\mu_j=\mu_l-\mu_k=L\pi-\frac{3}{L\pi}-K\pi+\frac{3}{K\pi}+R_l+R_k=J\pi+\frac{3}{\pi}\Big(\frac{1}{K}-\frac{1}{L}\Big)+R_l+R_k. \tag{7.17}$$

Since $K>4C_1\pi$ we get

$$L\pi=J\pi+K\pi<\mu_j+\frac{\pi}{2}+K\pi<4C_1\pi^2+\frac{\pi}{2}+K\pi<3K\pi,$$

and therefore due to $L\geq K+1$,

$$\begin{aligned}|R_l|+|R_k| &\leq C_1(L^{-3}+K^{-3})\leq 2C_1K^{-3}\leq 6C_1L^{-1}K^{-2}\leq\frac{6C_1\pi}{4C_1\pi}L^{-1}K^{-1}\\ &\leq \frac{3}{2\pi}\Big(\frac{1}{K}-\frac{1}{L}\Big).\end{aligned}$$

From (7.17) we obtain the contradiction

$$\mu_j=J\pi+\frac{3}{\pi}\Big(\frac{1}{K}-\frac{1}{L}\Big)+R_l+R_k\geq J\pi+\frac{3}{2\pi}\Big(\frac{1}{K}-\frac{1}{L}\Big)>J\pi.$$

Case (iv): $\mu_j \in]-4C_1\pi^2, 0[$ and $\mu_k \geq 4C_1\pi^2$. Since $\mu_{-j} = -\mu_j$, we have $\mu_k = \mu_l + \mu_{-j}$ and $\mu_{-j} \in]0, 4C_1\pi^2[$. If $\mu_l \in]0, 4C_1\pi^2[$, then we are back to case (ii), whereas if $\mu_l \geq 4C_1\pi^2$, then case (iii) applies. The remaining cases can be handled using (i)-(iv) and the symmetry $\mu_{-j} = -\mu_j$. $\Box$

Next we consider the set $\mathcal{M}_{\rm exc}$ of exceptional masses from (6.1) for the uniformly charged ball. Recall from (1.16) that $\{\hat{g} = 0\} = \{\mu_j : j \in \mathbb{Z}\}$. If $\hat{g}(\mu_j) = 0$, then

$$\tilde{\kappa}_1(\mu_j) = -\frac{2}{5}\sqrt{2\pi}\,\Big(\frac{\mu_j^2 - 30}{\mu_j^4}\Big)$$

by (7.12). Therefore

$$\mathcal{M}_{\rm exc} = \Big\{4\pi\,\Big(\frac{\mu_j^2 - 30}{\mu_j^4}\Big) : j \in \mathbb{Z}\Big\} \tag{7.18}$$

is obtained.

7.4 κ_1 and κ_2

The kernels κ_1 and κ_2 are defined in (3.10) and (3.11). Here we outline the calculation of

$$\tilde{\kappa}_j(\tau) = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-i\tau s}\kappa_j(s)\,ds$$

for $j = 1, 2$ and $\tau \in \mathbb{R}$.

Lemma 7.6 *Explicitly,*

$$\begin{aligned}
\tilde{\kappa}_1(\tau) &= -4\sqrt{2\pi}\int_0^\infty da\, a^3 f_{\rm e}(a)\phi_1(\tau a)\int_a^\infty dr\, r^2 f_{\rm e}(r)\Big(i + \frac{1}{r\tau}\Big)\, e^{-i\tau r}, && (7.19)\\
\tilde{\kappa}_2(\tau) &= \frac{2i}{\tau}\,\tilde{\kappa}_1(\tau) - \frac{8i}{3\tau}\sqrt{2\pi}\int_0^\infty da\, a^4 f_{\rm e}(a)\int_a^\infty dr\, r f_{\rm e}(r), && (7.20)
\end{aligned}$$

where ϕ_1 is given by (1.15).

Proof: To begin with

$$\begin{aligned}
\tilde{\kappa}_1(\tau) &= \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-i\tau s}\kappa_1(s)\,ds = \frac{1}{\sqrt{2\pi}}\int_0^\infty dr\, r^3 f_{\rm e}(r)\int_0^\infty ds\, e^{-i\tau s}\varphi_1(s, r)\\
&= -\sqrt{2\pi}\int_0^\infty dr\, r^2 f_{\rm e}(r)\int_0^\infty ds\, e^{-i\tau s}\Big(g(s-r) + g(s+r) - \frac{1}{r}\int_{-r}^r g(s-\sigma)\,d\sigma\Big)\\
&= -\sqrt{2\pi}\int_0^\infty dr\, r^2 f_{\rm e}(r)\int_0^\infty da\, a f_{\rm e}(a)\int_0^\infty ds\, e^{-i\tau s}\Big(\mathbf{1}_{[-a,\,a]}(s-r)(s-r)\\
&\qquad + \mathbf{1}_{[-a,\,a]}(s+r)(s+r) - \frac{1}{r}\int_{-r}^r d\sigma\,\mathbf{1}_{[-a,\,a]}(s-\sigma)(s-\sigma)\Big),
\end{aligned}$$

recall (3.12) and (1.13). Now

$$\begin{aligned}
T_1 &= \int_0^\infty ds\, e^{-i\tau s}\,\mathbf{1}_{[-a,\,a]}(s-r)(s-r)\\
&= \mathbf{1}_{\{r<a\}}\int_0^{r+a} ds\, e^{-i\tau s}(s-r) + \mathbf{1}_{\{r\geq a\}}\int_{r-a}^{r+a} ds\, e^{-i\tau s}(s-r)\\
&= \mathbf{1}_{\{r<a\}}\,\frac{1}{\tau^2}\Big(e^{-i\tau(r+a)}(1 + ia\tau) + ir\tau - 1\Big) + \mathbf{1}_{\{r\geq a\}}\, 2ia^2 e^{-i\tau r}\phi_1(\tau a) && (7.21)
\end{aligned}$$

and

$$
\begin{aligned}
T_2 &= \int_0^\infty ds\, e^{-i\tau s}\, \mathbf{1}_{[-a,\,a]}(s+r)(s+r) = \mathbf{1}_{\{r<a\}} \int_0^{a-r} ds\, e^{-i\tau s}(s+r) \\
&= \mathbf{1}_{\{r<a\}}\, \frac{1}{\tau^2}\Big(e^{-i\tau(a-r)}(1+ia\tau) - ir\tau - 1\Big).
\end{aligned} \tag{7.22}
$$

Hence

$$
T_1 + T_2 = \mathbf{1}_{\{r<a\}}\, \frac{2}{\tau^2}\Big((1+ia\tau)e^{-i\tau a}\cos(\tau r) - 1\Big) + \mathbf{1}_{\{r\ge a\}}\, 2ia^2 e^{-i\tau r}\phi_1(\tau a).
$$

Furthermore, this also yields

$$
\begin{aligned}
T_3 &= -\frac{1}{r}\int_0^\infty ds\, e^{-i\tau s}\int_{-r}^{r} d\sigma\, \mathbf{1}_{[-a,\,a]}(s-\sigma)(s-\sigma) \\
&= -\frac{1}{r}\int_0^r d\sigma \int_0^\infty ds\, e^{-i\tau s}\, \Big(\mathbf{1}_{[-a,\,a]}(s-\sigma)(s-\sigma) + \mathbf{1}_{[-a,\,a]}(s+\sigma)(s+\sigma)\Big) \\
&= -\frac{1}{r}\int_0^r d\sigma \Big(\mathbf{1}_{\{\sigma<a\}}\, \frac{2}{\tau^2}\Big((1+ia\tau)e^{-i\tau a}\cos(\tau\sigma) - 1\Big) + \mathbf{1}_{\{\sigma\ge a\}}\, 2ia^2 e^{-i\tau\sigma}\phi_1(\tau a)\Big) \\
&= -\,\mathbf{1}_{\{r<a\}}\, \frac{2}{\tau^2}\Big((1+ia\tau)e^{-i\tau a}\,\phi_0(\tau r) - 1\Big) \\
&\quad -\,\mathbf{1}_{\{r\ge a\}}\, \frac{2a}{r\tau^2}\Big[(1+ia\tau)e^{-i\tau a}\,\phi_0(\tau a) - 1 - \tau(e^{-i\tau r} - e^{-i\tau a})\,\phi_1(\tau a)\Big],
\end{aligned}
$$

and thus after some simplification

$$
T_1 + T_2 + T_3 = \mathbf{1}_{\{r<a\}}\, \frac{2r}{\tau}(1+ia\tau)e^{-i\tau a}\phi_1(\tau r) + \mathbf{1}_{\{r\ge a\}}\, \frac{2a^2}{r\tau}(1+ir\tau)e^{-i\tau r}\phi_1(\tau a).
$$

Therefore

$$
\begin{aligned}
\tilde\kappa_1(\tau) &= -\sqrt{2\pi}\int_0^\infty dr\, r^2 f_{\mathrm{e}}(r)\int_0^\infty da\, a f_{\mathrm{e}}(a)\,(T_1+T_2+T_3) \\
&= -\frac{2\sqrt{2\pi}}{\tau}\int_0^\infty dr\, r^2 f_{\mathrm{e}}(r)\Big[\Big(\frac{1}{r}+i\tau\Big)\, e^{-i\tau r}\int_0^r da\, a^3 f_{\mathrm{e}}(a)\phi_1(\tau a) \\
&\qquad\qquad + r\phi_1(\tau r)\int_r^\infty da\, a f_{\mathrm{e}}(a)\,(1+ia\tau)e^{-i\tau a}\Big]
\end{aligned}
$$

yields (7.19), since $\int_0^\infty dr \int_0^r da = \int_0^\infty da \int_a^\infty dr$ in the first integral, and r and a can be interchanged in the second integral. Finally, the relation (7.20) is a consequence of (7.19) and (3.14), if we use (4.6). □

7.5 Limit points, spectra, and almost periodic functions

In this section we review the definition and some properties of limit points, spectra of functions, and almost periodic functions. All these results and more information can be found in e.g. [1, 5, 7]. Generally speaking, it seems that the class of almost periodic functions (or distributions) will play an important role for the understanding of global asymptotic properties [8, 12].

The space of bounded and uniformly continuous vector-valued functions $u : \mathbb{R} \to \mathbb{R}^n$ is denoted by $\mathrm{BUC}(\mathbb{R};\mathbb{R}^n)$ or simply by $\mathrm{BUC}(\mathbb{R})$, whereas $\mathrm{BC}(\mathbb{R})$ stands for the bounded and continuous functions. In particular, $C_b^1(\mathbb{R}) \subset \mathrm{BUC}(\mathbb{R})$. If $u : \mathbb{R} \to \mathbb{R}^n$ is a function, then $(\tau_h u)(t) = u(t+h)$ is its translate by $h \in \mathbb{R}$. The ω-limit set of $u \in \mathrm{BUC}(\mathbb{R})$ is

$$\Gamma^+(u) = \Big\{ v : \exists\, h_k \to \infty \text{ such that } \tau_{h_k} u \to v \text{ uniformly on every compact interval in } \mathbb{R} \Big\}.$$

Then $\Gamma^+(u) \neq \emptyset$.

Lemma 7.7 *Suppose that $u, \dot{u} \in \mathrm{BUC}(\mathbb{R})$. Then the following assertions are equivalent.*

(a) $\Gamma^+(u)$ contains only constant functions.

(b) u is (uniformly) asymptotically slowly varying, i.e., $|u(t+T) - u(t)| \to 0$ as $t \to \infty$ uniformly for T in compact subsets of $\mathbb{R}$.

(c) $\dot{u}(t) \to 0$ as $t \to \infty$.

The (norm) spectrum $\sigma(u)$ of $u \in \mathrm{BC}(\mathbb{R})$ is defined as

$$\sigma(u) = \Big\{ \tau \in \mathbb{R} : \hat{\varphi}(\tau) = 0 \text{ holds for all } \varphi \in L^1(\mathbb{R}) \text{ such that } \varphi * u = 0 \Big\}.$$

Then $\sigma(u) = \emptyset$ iff $u = 0$ and $\sigma(u) = \{0\}$ iff u equals a non-zero constant. If $u \in \mathrm{BC}(\mathbb{R})$ and $g \in L^1(\mathbb{R})$, then

$$\sigma(u) \subset \sigma(u * g) \cup \{\tau \in \mathbb{R} : \hat{g}(\tau) = 0\}. \tag{7.23}$$

If $u \in \mathrm{BC}(\mathbb{R}) \cap C^1(\mathbb{R})$ and $\dot{u} \in \mathrm{BC}(\mathbb{R})$, then

$$\sigma(\dot{u}) \subset \sigma(u) \subset \sigma(\dot{u}) \cup \{0\}. \tag{7.24}$$

Now we turn to almost periodic functions.

Definition 7.8 *Let $u : \mathbb{R} \to \mathbb{R}^n$ be a function. If $\varepsilon > 0$, then $h \in \mathbb{R}$ is said to be an ε-almost period of u, if $\sup_{t\in\mathbb{R}} |u(t-h) - u(t)| < \varepsilon$. The function u is almost periodic, if it is continuous and for every $\varepsilon > 0$ there is $T > 0$ such that each interval $[t_0, t_0 + T]$ in $\mathbb{R}$ contains an ε-almost period of u.*

The space of almost periodic functions will be denoted by $\mathrm{AP}(\mathbb{R})$. Then $\mathrm{AP}(\mathbb{R}) \subset \mathrm{BUC}(\mathbb{R})$ holds. Conversely, if $u \in \mathrm{BUC}(\mathbb{R})$ and $\sigma(u)$ is at most countable, then $u \in \mathrm{AP}(\mathbb{R})$. If $u \in \mathrm{AP}(\mathbb{R})$, then the mean value

$$\mathbb{M}(u) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u(t)\, dt$$

does exist. It even holds that

$$\sup_{s\in\mathbb{R}} \Big| \frac{1}{2T} \int_{-T+s}^{T+s} u(t)\, dt - \mathbb{M}(u) \Big| \to 0, \quad T \to \infty.$$

Since

$$\Big| \frac{1}{T} \int_0^T u(t)\, dt - \mathbb{M}(u) \Big| = \Big| \frac{1}{2(T/2)} \int_{-T/2+T/2}^{T/2+T/2} u(t)\, dt - \mathbb{M}(u) \Big| \le \sup_{s\in\mathbb{R}} |\ldots| \to 0$$

as $t \to \infty$, also

$$\mathbb{M}(u) = \lim_{T\to\infty} \frac{1}{T} \int_0^T u(t)\, dt \tag{7.25}$$

is satisfied. If $\tau \in \mathbb{R}$, then $e^{-i\tau t}u \in \mathrm{AP}(\mathbb{R})$, and hence the Bohr transform

$$u^\flat(\tau) = \mathbb{M}(e^{-i\tau t}u), \quad \tau \in \mathbb{R},$$

is well-defined. The uniqueness theorem asserts that is $u, v \in \mathrm{AP}(\mathbb{R})$ and $u^\flat = v^\flat$, then $u = v$ holds. An inner product on $\mathrm{AP}(\mathbb{R})$ can be defined by

$$\langle u, v\rangle_{\mathbb{M}} = \mathbb{M}(u\bar{v}),$$

and

$$\langle e^{i\lambda t}, e^{i\mu t}\rangle_{\mathbb{M}} = \delta(\lambda - \mu). \tag{7.26}$$

The relation of the spectrum to the Bohr transform is

$$\sigma(u) = \overline{\{\tau : u^\flat(\tau) \neq 0\}}, \tag{7.27}$$

and $\sigma(u)$ is at most countable. If $u \in \mathrm{AP}(\mathbb{R})$ and $g \in L^1([0, \infty[)$, then $u *_0 g \in \mathrm{AP}(\mathbb{R})$ for

$$(u *_0 g)(t) = \int_0^\infty u(t-s)g(s)\, ds, \quad t \in \mathbb{R}. \tag{7.28}$$

Furthermore,

$$(u *_0 g)^\flat(\tau) = \sqrt{2\pi}\, u^\flat(\tau)\tilde{g}(\tau), \quad \tau \in \mathbb{R}, \tag{7.29}$$

where

$$\tilde{g}(\tau) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-i\tau t} g(t)\, dt, \quad \tau \in \mathbb{R}. \tag{7.30}$$

To check (7.29), note that by (7.25)

$$\begin{aligned} (u *_0 g)^\flat(\tau) &= \mathbb{M}(e^{-i\tau t}\,(u *_0 g)) = \lim_{T\to\infty} \frac{1}{T} \int_0^T dt\, e^{-i\tau t} \int_0^\infty ds\, u(t-s)g(s) \\ &= \lim_{T\to\infty} \frac{1}{T} \int_0^\infty ds\, e^{-i\tau s}\, g(s) \int_{-s}^{T-s} dt\, e^{-i\tau t}\, u(t) = \sqrt{2\pi}\, \tilde{g}(\tau)u^\flat(\tau), \end{aligned}$$

since $g \in L^1([0, \infty[)$, u is bounded, and $\frac{1}{T}\int_{-s}^{T-s} dt\, e^{-i\tau t}u(t) \to \mathbb{M}(e^{-i\tau t}u) = u^b(\tau)$ as $T \to \infty$ pointwise for $s \in [0, \infty[$. Thus (7.29) is verified. Next, if $u \in C^1(\mathbb{R})$ is such that $u, \dot{u} \in \mathrm{AP}(\mathbb{R})$, then

$$\dot{u}^\flat(\tau) = i\tau u^\flat(\tau), \quad \tau \in \mathbb{R}. \tag{7.31}$$

This follows from integration by parts and the boundedness of u. An almost periodic function u can be uniformly approximated by trigonometric polynomials. For this, write $\sigma(u) = \{\lambda_j : j \in \mathbb{N}\}$. Then

$$\lim_{m\to\infty} \|Q_m - u\|_{L^\infty(\mathbb{R})} = 0$$

for the functions

$$Q_m(t) = \sum_{j=1}^{r_m} \nu_{mj} u^\flat(\lambda_j)\, e^{i\lambda_j t},$$

where $\nu_{mj} \in]0, 1]$ are suitable coefficients. If j is fixed, then

$$\lim_{m\to\infty} \nu_{mj} = 1. \tag{7.32}$$

For, (7.26) yields

$$\langle Q_m, e^{i\lambda_j t}\rangle_{\mathbb{M}} = \sum_{k=1}^{r_m} \nu_{mk} u^\flat(\lambda_k)\, \langle e^{i\lambda_k t}, e^{i\lambda_j t}\rangle_{\mathbb{M}} = \sum_{k=1}^{r_m} \nu_{mk} u^\flat(\lambda_k)\, \delta(\lambda_k - \lambda_j) = \nu_{mj} u^\flat(\lambda_j).$$

Thus

$$u^\flat(\lambda_j) = \mathbb{M}(e^{-i\lambda_j t} u) = \langle u, e^{i\lambda_j t}\rangle_{\mathbb{M}} = \lim_{m\to\infty} \langle Q_m, e^{i\lambda_j t}\rangle_{\mathbb{M}} = u^\flat(\lambda_j) \lim_{m\to\infty} \nu_{mj}$$

leads to (7.32).

Acknowledgement: The author is grateful to Michael Stoll for his suggestions concerning the proof of Lemma 7.5. Furthermore thanks are due to an anonymous referee whose comments helped to the improve the presentation.

References

[1] Amerio L. & Prouse G.: *Almost-Periodic Functions and Functional Equations*, van Nostrand, New York 1971

[2] Appel W. & Kiessling M. K.-H.: Mass and spin renormalization in Lorentz electrodynamics, *Annals of Phys. (N.Y.)* **289**, 24-83 (2001)

[3] Appel W. & Kiessling M. K.-H.: Scattering and radiation damping in gyroscopic Lorentz electrodynamics, *Letters Math. Phys.* **59**, 31-46 (2002)

[4] Bohm D. & Weinstein M.: The self-oscillations of a charged particle, *Phys. Rev.* 74, 1789-1798 (1948)

[5] Gripenberg G., Londen S.-O. & Staffans O.: *Volterra Integral and Functional Equations*, Cambridge University Press, Cambridge-New York 1990

[6] Imaikin V., Komech A.I. & Spohn H.: Rotating charge coupled to the Maxwell field: scattering theory and adiabatic limit, *Monatsh. Math.* **142**, 143-156 (2004)

[7] Katznelson Y.: *An Introduction to Harmonic Analysis*, Dover Publications, New York 1976

[8] Komech A.I. & Komech A.A.: Global attractor for a nonlinear oscillator coupled to the Klein-Gordon field, *Arch. Ration. Mech. Anal.* **185**, 105-142 (2007)

[9] Komech A.I. & Spohn H.: Long-time asymptotics for the coupled Maxwell-Lorentz equation, *Comm. Partial Differential Equations* **25**, 559-584 (2000)

[10] Pearle P.: Classical electron models, pp. 211-295 in *Electromagnetism: Paths to Research*, Ed. D. Teplitz, Plenum Press, New York 1982

[11] Spohn H.: *Dynamics of Charged Particles and Their Radiation Field*, Cambridge University Press, Cambridge-New York 2004

[12] Tao T.: Why are solitons stable?, *Bull. Amer. Math. Soc.* **46**, 1-33 (2009)