The formation of black holes in spherically symmetric gravitational collapse

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Abstract

We consider the spherically symmetric, asymptotically flat Einstein-Vlasov system. We find explicit conditions on the initial data, with ADM mass M, such that the resulting spacetime has the following properties: there is a family of radially outgoing null geodesics where the area radius r along each geodesic is bounded by $2M$, the timelike lines $r = c \in [0, 2M]$ are incomplete, and for $r > 2M$ the metric converges asymptotically to the Schwarzschild metric with mass M. The initial data that we construct guarantee the formation of a black hole in the evolution. We give examples of such initial data with the additional property that the solutions exist for all $r \geq 0$ and all Schwarzschild time, i.e., we obtain global existence in Schwarzschild coordinates in situations where the initial data are not small. Some of our results are also established for the Einstein equations coupled to a general matter model characterized by conditions on the matter quantities.

1 Introduction

The spherically symmetric, asymptotically flat Einstein-Vlasov system describes in the context of General Relativity the time evolution of a large ensemble of particles which interact only through the gravitational field which they create collectively. Such a self-gravitating, collisionless gas is used in astrophysics to model galaxies or globular clusters [6]. To be specific, we consider a spacetime manifold S with Lorentz metric

$$ds^2 = -e^{2\mu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

in Schwarzschild coordinates. Here $t \in \mathbb{R}$ is the time coordinate, $r \in [0, \infty[$ is the area radius, i.e., $4\pi r^2$ is the area of the orbit of the symmetry group SO(3) labeled by r, and the angles $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$ parameterize these orbits. Asymptotic flatness means that the metric quantities λ and μ satisfy the boundary conditions

$$\lim_{r\to\infty} \lambda(t, r) = \lim_{r\to\infty} \mu(t, r) = 0. \tag{1.1}$$

For a metric of this form the 00, 11, and 01 components of the Einstein equations read

$$e^{-2\lambda}(2r\lambda_r - 1) + 1 = 8\pi r^2\rho, \tag{1.2}$$

$$e^{-2\lambda}(2r\mu_r + 1) - 1 = 8\pi r^2 p, \tag{1.3}$$

$$\lambda_t = -4\pi r e^{\mu+\lambda} j, \tag{1.4}$$

where the subscripts r and t indicate the partial derivative with respect to r or t, respectively, and the right hand sides are related to the energy-momentum tensor $T_{\alpha\beta}$ via

$$\rho = e^{-2\mu}T_{00},\ p = e^{-2\lambda}T_{11},\ j = -e^{-(\lambda+\mu)}T_{01}. \tag{1.5}$$

All the particles in the ensemble are assumed to have the same rest mass, normalized to unity, and to move forward in time. Hence, their number density f is a non-negative function supported on the mass shell $PS = \{g_{\alpha\beta}p^\alpha p^\beta = -1,\ p^0 > 0\}$, a submanifold of the tangent bundle TS of the spacetime manifold S; p^α are the canonical momenta corresponding to the coordinates on S. In order to exploit the symmetry it is useful to introduce non-canonical variables on momentum space in which the Vlasov equation for $f = f(t, r, w, L)$ takes the form

$$\partial_t f + e^{\mu-\lambda} \frac{w}{E} \partial_r f - \left(\lambda_t w + e^{\mu-\lambda} \mu_r E - e^{\mu-\lambda} \frac{L}{r^3 E} \right) \partial_w f = 0. \tag{1.6}$$

Here $E = E(r, w, L) := \sqrt{1 + w^2 + L/r^2} = e^\mu p^0$, and $w \in]-\infty, \infty[$ and $L \in [0, \infty[$ can be thought of as the radial component of the momentum and the square of the angular momentum respectively. The latter is conserved along characteristics of the Vlasov equation. The quantities ρ, p, and j read

$$\rho(t, r) = \frac{\pi}{r^2} \int_{-\infty}^{\infty} \int_0^{\infty} E f(t, r, w, L) \, dL \, dw, \tag{1.7}$$

$$p(t, r) = \frac{\pi}{r^2} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{w^2}{E} f(t, r, w, L) \, dL \, dw, \tag{1.8}$$

$$j(t, r) = \frac{\pi}{r^2} \int_{-\infty}^{\infty} \int_0^{\infty} w \, f(t, r, w, L) \, dL \, dw. \tag{1.9}$$

For a detailed derivation of the system (1.2)–(1.9) we refer to [18]. As initial data we prescribe a distribution function $\mathring{f} = \mathring{f}(r, w, L) \geq 0$ which is C^1, compactly supported in $]0, \infty[\times] - \infty, \infty[\times]0, \infty[$, and such that

$$4\pi \int_0^r \eta^2 \mathring{\rho}(\eta) \, d\eta = 4\pi^2 \int_0^r \int_{-\infty}^{\infty} \int_0^{\infty} E \mathring{f}(\eta, w, L) \, dL \, dw \, d\eta < \frac{r}{2}. \tag{1.10}$$

Such initial data we call *regular*. The origin $r = 0$ is excluded from the support for technical reasons, but this can be avoided by using Cartesian coordinates. Regular initial data launch a unique local solution for which all the derivatives which appear in the system exist classically [18, 19]. The solution of (1.2) is given by

$$e^{-2\lambda(t,r)} = 1 - \frac{2m(t, r)}{r} \text{ where } m(t, r) := 4\pi \int_0^r \eta^2 \rho(t, \eta) \, d\eta, \tag{1.11}$$

so (1.10) is necessary in order that this relation makes sense at least initially; geometrically speaking (1.10) says that the initial data do not contain a trapped surface. Since $\mathring{f}$ has compact support and this property is inherited by $f(t)$, the integrals in (1.7)–(1.9) exist, and they are given in terms of f alone, which is the main reason for using the non-canonical variables w and L. As stated, the system is overdetermined, but for a solution of the subsystem (1.2), (1.3), (1.6), (1.7), (1.8) all other components of the field equations hold as well [18, 19]. Besides the 01 component (1.4) also the 22 and 33 components are nontrivial, but they are not needed for our analysis. The remaining components vanish identically due to the symmetry assumption. By (1.4), (1.11), and the compact support of $j(t)$, the quantity $M = m(t, \infty)$ is conserved, and is the ADM mass of the solution.

Our aim is to find explicit conditions on the initial data such that the corresponding solutions have the following property: There is an outgoing radial null geodesic γ^+ originating from $r = r_0 > 0$, i.e.,

$$\frac{d\gamma^+}{ds}(s) = e^{(\mu-\lambda)(s,\gamma^+(s))},\ \gamma^+(0) = r_0, \tag{1.12}$$

such that the solution exists on the outer region

$$D := \{(t, r) \in [0, \infty[^2 \mid r \geq \gamma^+(t)\}, \tag{1.13}$$

and γ^+ has the property that

$$\lim_{s\to\infty} \gamma^+(s) < \infty. \tag{1.14}$$

This indicates that the matter distribution undergoes a gravitational collapse, and a black hole forms. In fact we obtain a more detailed picture which supports this interpretation: There exists an extremal, radially outgoing null geodesic γ^* in the outer domain D such that $\lim_{s\to\infty} \gamma^*(s) = 2M$, and as $t \to \infty$ the metric converges for $r > 2M$ to the Schwarzschild metric representing a black hole of mass M; recall that M is the ADM mass of the solution. The established behavior of the solutions is stable in the sense that, except for "boundary cases", properly restricted small perturbations of the corresponding initial data lead to solutions with the same properties.

For the Einstein equations coupled to a general matter model, i.e., if the field equations (1.2)–(1.4) are supplemented by an evolution equation for the matter replacing the Vlasov equation and by the definitions of the corresponding components of the energy-momentum tensor, some of our results remain true, provided the matter model satisfies specific assumptions. In order to give a broader impact to our analysis we include this general-matter case, but we emphasize that only the Vlasov matter model is presently known to satisfy all the required assumptions. As a corollary to our main result we obtain initial data for the Einstein-Vlasov system which lead to the formation of black holes and for which the solutions exist for all Schwarzschild time and all $r \geq 0$. To the best of our knowledge this is the first global existence result in Schwarzschild coordinates for initial data which lead to gravitational collapse and the formation of black holes and which in particular are not small. Small data are known to result in dispersing solutions and singularity-free, geodesically complete spacetimes [19].

We now put our results into the larger context of General Relativity. One of the many striking predictions of this theory is that under appropriate conditions astrophysical objects like stars or galaxies undergo a gravitational collapse resulting in a spacetime singularity. This was first proven by Oppenheimer and Snyder [15] who constructed a semi-explicit example of a homogeneous spherically symmetric ball of dust, i.e., of a pressure-less fluid, which collapses under its self-consistent, general relativistic gravitational interaction. During the collapse the scalar curvature blows up at the centre of symmetry, and the geometry of spacetime breaks down there, i.e., a spacetime singularity forms. In the 1960s Penrose [16] proved that the formation of spacetime singularities from regular initial data is not restricted to spherically symmetric, especially constructed or isolated examples but is a genuine, stable feature of spacetimes. However, this result gives little information about the geometric structure of a spacetime containing such a singularity. In particular, it is not known in general if every spacetime singularity arising from the gravitational collapse of regular data is covered by an event horizon. The (weak) cosmic censorship conjecture asserts that generically this is the case, and the validity of this conjecture is one of the major open problems in classical mathematical relativity; see [23] for more information. To deal with this conjecture in full generality is out of reach of the present level of mathematics, but under the assumption of spherical symmetry progress has been made in recent years. One important insight is that the answer is sensitive to which model is chosen to describe the matter. Christodoulou [7] showed that for dust, i.e., the matter model used by Oppenheimer and Snyder, cosmic censorship is violated. On the other hand, in a series of papers Christodoulou investigated a massless scalar field as matter model and showed in 1999 that weak and strong cosmic censorship hold true for this matter model; see [11] and the references therein.

One aspect of our result is that there is a set of initial data which leads to gravitational collapse such that weak cosmic censorship holds. This point should be related to an earlier result by Rendall [22], who showed that there exist initial data for the spherically symmetric Einstein-Vlasov system such that a trapped surface forms in the evolution. The occurrence of a trapped surface signals the formation of an event horizon. Indeed, Dafermos [12] proved that weak cosmic censorship holds if a spherically symmetric spacetime contains a trapped surface and the matter model satisfies certain hypotheses which were then verified for Vlasov matter in [13]. By combining these results it follows that initial data exist which lead to gravitational collapse and for which weak cosmic censorship holds. However, the proof in [22] rests on a continuity argument, and it is not possible to decide whether or not a given initial data set will give rise to a black hole. This is in contrast to the explicit conditions on the initial data together with the detailed asymptotic structure that we obtain in the present paper. In this regard it is natural to relate our results to those of Christodoulou on the spherically symmetric Einstein-scalar field system [8, 9, 10]. In [8] it is shown that if the final Bondi mass M is different from zero, the region exterior to the sphere $r = 2M$ tends to the Schwarzschild metric with mass M. Theorem 2.4 below shows that solutions of the spherically Einstein-Vlasov system, under certain conditions on the initial data, also converge to the Schwarzschild metric asymptotically. Furthermore, in [9] explicit conditions on the initial data are specified which guarantee the formation of trapped surfaces. This paper played a crucial role in Christodoulou's proof of the weak and strong cosmic censorship conjectures for the Einstein-scalar field system, cf. [10, 11]. The conditions on the initial data in [9] allow the ratio of the Hawking mass and the area radius to cover the full range, i.e., $2m/r \in]0, 1[$, whereas our conditions require $2m/r$ to be close to one. However, we believe that to understand gravitational collapse in the case of Vlasov matter the essential situation is when $2m/r$ is large. We thus hope that our results will lead to progress in the general understanding of gravitational collapse and the weak cosmic censorship conjecture in the case of Vlasov matter.

The paper proceeds as follows. In the next section we state our main results for the Einstein-Vlasov system, where we specify classes of spherically symmetric initial data which lead to solutions showing the above behavior. In Section 3 we formulate one of our results, Theorem 2.2, for the general Einstein-matter system and on the level of the macroscopic matter quantities single out precisely the conditions needed for our arguments to go through. After stating some general auxiliary results in Section 4 we prove, in Section 5, Theorem 3.1 which is the general-matter version of Theorem 2.2. The latter result is then established in Section 6 by showing that Vlasov matter satisfies the required general conditions on the matter for a suitable class of initial data. Theorem 2.1 is established in Section 7 together with Corollary 2.3 on global existence in Schwarzschild coordinates. In Section 8 we prove the convergence of our solutions to a Schwarzschild black hole of the corresponding ADM mass in the case of Vlasov matter.

To conclude the introduction we refer to [1] and the references there for more background on the Einstein-Vlasov system, and we mention [5] where in particular the present results are related to a formulation of weak cosmic censorship proposed in [11].

2 Main results for Vlasov matter

To state our main results let $0 < r_0 < r_1$ be given, put $M = r_1/2$ (this is going to be the ADM mass of the solution), and fix $0 < M_{\text{out}} < M$ such that

$$\frac{2(M - M_{\text{out}})}{r_0} < \frac{8}{9}. \tag{2.1}$$

The value 8/9 is chosen for definiteness only. Two different theorems will be stated below, corresponding to the following two situations.

(i) Let $R_1 > r_1$ be such that

$$R_1 - r_1 < \frac{r_1 - r_0}{6}, \tag{2.2}$$

or

(ii) let $R_1 > r_1$ be such that

$$\sqrt{\frac{R_1 - r_1}{R_1}} < \min\left\{\frac{1}{6}, \frac{r_0^2}{12\kappa R_1 M}, \frac{r_1 - r_0}{8\kappa R_1}\right\}, \tag{2.3}$$

where the (explicit) constant $\kappa > 0$ will be specified in Theorems 2.2 and 3.1 below.

Finally, we define

$$R_0 := \frac{1}{2}(r_1 + R_1).$$

Denote by $\mathring{\rho}$ the energy density induced by the initial distribution function $\mathring{f}$. We require that all the matter in the outer region $[r_0, \infty[$ is initially located in the strip $[R_0, R_1]$, with M_{out} being the corresponding fraction of the ADM mass M, i.e.,

$$\int_{r_0}^{\infty} 4\pi r^2 \mathring{\rho}(r)dr = \int_{R_0}^{R_1} 4\pi r^2 \mathring{\rho}(r)dr = M_{\text{out}}. \tag{2.4}$$

Furthermore, the remaining fraction $M - M_{\text{out}}$ should be initially located within the ball of area radius r_0, i.e.,

$$\int_0^{r_0} 4\pi r^2 \mathring{\rho}(r)dr = M - M_{\text{out}}. \tag{2.5}$$

We are now in the position to formulate our main results for Vlasov matter. Corresponding to Case (i) above, we prove

Theorem 2.1 *Let r_0, r_1, M, and $M_{\rm out}$ be given as above, and let R_1 satisfy (2.2). Then there exists a set $\mathcal{I}_1$ of regular initial data for the spherically symmetric Einstein-Vlasov system such that if $\overset{\circ}{f} \in \mathcal{I}_1$, then (2.4) and (2.5) hold, the corresponding solution exists on D, and*

$$\lim_{s\to\infty} \gamma^+(s) < \infty, \quad \lim_{s\to\infty} \int_{\gamma^+(s)}^{\infty} 4\pi r^2 \rho(s,r)\, dr > 0,$$

where γ^+ satisfies (1.12).

By abuse of notation we denote by D both the outer region in spacetime defined by (1.13) and the part of the mass shell with $(t, r) \in D$.

The next theorem addresses Case (ii) above, assuming the stronger condition (2.3). This allows for a more straightforward proof, and the constraints on the momentum variables of the initial distribution function $\overset{\circ}{f}$ which are used to specify the set $\mathcal{I}_1$ will be slightly relaxed. Hence, the initial data set $\mathcal{I}_1$ does not contain $\mathcal{I}_2$ in Theorem 2.2 below, but it is larger in the sense that data in $\mathcal{I}_2$ are quite close to containing a trapped surface, which is not necessarily the case for data in $\mathcal{I}_1$. The precise form of $\mathcal{I}_1$ and $\mathcal{I}_2$ will be specified in the proofs.

Theorem 2.2 *Let r_0, r_1, M, and $M_{\rm out}$ be given as above and let R_1 satisfy (2.3) with $\kappa = 6$. Then there exists a set $\mathcal{I}_2$ of regular initial data for the spherically symmetric Einstein-Vlasov system such that if $\overset{\circ}{f} \in \mathcal{I}_2$, then (2.4) and (2.5) hold, the corresponding solution exists on D, and*

$$\lim_{s\to\infty} \gamma^+(s) < \infty, \quad \lim_{s\to\infty} \int_{\gamma^+(s)}^{\infty} 4\pi r^2 \rho(s,r)\, dr > 0,$$

where γ^+ satisfies (1.12).

The Einstein-Vlasov system has a wide variety of static, spherically symmetric solutions with finite ADM mass and finite radius, i.e., compact support of the matter, cf. [20] and the references therein. Particularly interesting examples of initial data for which our results apply are obtained if the matter for $r \le r_0$ is represented by such a static solution.

Corollary 2.3 *Let f_s be a static solution of the spherically symmetric Einstein-Vlasov system with finite ADM mass $M_s > 0$ and finite radius $r_s > 0$. Define $r_0 = r_s$, let $r_1 > r_0$ be arbitrary, $M = r_1/2$, and $M_{\rm out} = M - M_s > 0$. Then the initial data sets $\mathcal{I}_1$ and $\mathcal{I}_2$ both contain* *data $\mathring{f}$ which coincide with the given static solution for $0 \leq r \leq r_0$. The corresponding solution f of the Einstein-Vlasov system exists for all $r \geq 0$ and $t \geq 0$, and it coincides with the static solution f_s for all $r \leq \gamma^+(t)$ and $t \geq 0$.*

We prove this result at the end of Section 7. It represents a global existence result for the Einstein-Vlasov system in Schwarzschild time for initial data that are not small.

In the next section we formulate a version of Theorem 2.2 for quite general matter models. One reason for this is that the main mechanism behind our method becomes very transparent by posing sufficient conditions on the macroscopic matter terms rather than conditions on the initial distribution function $\mathring{f}$ as we did in the theorems above. In the proofs it will turn out that for the classes of initial data that we specify we can establish the following additional result which shows that the solution evolves towards a Schwarzschild black hole of mass M.

Theorem 2.4 *In the situation of Theorem 2.1 or Theorem 2.2 the following holds:*

(a) *There exist constants $\alpha, \beta > 0$ depending only on the initial data set $\mathcal{I}_1$ or $\mathcal{I}_2$ respectively such that if $t \geq 0$ and $r \geq 2M + \alpha e^{-\beta t}$, then $f(t, r, \cdot, \cdot) = 0$, i.e., we have vacuum, and the metric equals the Schwarzschild metric*

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

representing a black hole of mass M.

(b) *For all $t \geq 0$ and $\gamma^+(t) \leq r \leq 2M + \alpha e^{-\beta t}$ we have $\lim_{t\to\infty} \mu(t, r) = -\infty$ for all $r \leq 2M$. Furthermore, for $c \in [0, 2M]$ the timelike lines $r = c$ are incomplete and their proper lengths are uniformly bounded by a constant depending on α, β, and M.*

(c) *Let*

$$r^* := \sup\{r \geq r_0 \mid \text{the radially outgoing null geodesic } \gamma \text{ with } \gamma(0) = r \text{ satisfies } \lim_{s\to\infty} \gamma(s) < \infty\},$$

and let γ^* be the radially outgoing null geodesic with $\gamma^*(0) = r^*$. Then $\lim_{s\to\infty} \gamma^*(s) = 2M$, and every radially outgoing null geodesic γ with $\gamma(0) > r^*$ is future complete with $\lim_{s\to\infty} \gamma(s) = \infty$.

3 The result for general matter models

In this section we specify the general assumptions on a matter model sufficient for our method to be applied. In order to keep the discussion consistent with the Vlasov part of our arguments we use the notation introduced in (1.5). Firstly, we assume that the following two conditions are satisfied.

- The dominant energy condition holds. (DEC)
- The radial pressure p is non-negative. (NNP)

The dominant energy condition (DEC) plays a central role in general relativity and is the main criterion that a matter model should satisfy to be considered realistic. We refer to [14] for its definition. The non-negative pressure condition (NNP) is restrictive in the sense that it rules out, for example, a Maxwell field as matter model. However, for most astrophysical models it is a standard assumption, with e.g. fluid models satisfying this condition. For the purpose of this paper we only need to focus on two consequences of these two criteria, cf. [14] and [17]. (DEC) implies, together with (NNP), that

$$0 \leq p \leq \rho \text{ and } |j| \leq \rho. \tag{3.1}$$

Furthermore, by (DEC) any geodesic $(s, R(s))$ of a material particle or a light ray satisfies

$$\left| \frac{dR(s)}{ds} \right| \leq e^{(\mu-\lambda)(s,R(s))}. \tag{3.2}$$

The meaning of the latter condition is that locally the speed of energy flow is less than or equal to the speed of light.

Let λ, μ, ρ, p, j correspond to a solution of the general spherically symmetric Einstein-matter system in Schwarzschild coordinates, i.e., (1.1)–(1.4) supplied with suitable evolution equations for the matter and an energy-momentum tensor being an appropriate function of the matter and the metric. In order to investigate the global structure of the solutions it is necessary that they exist globally in an appropriate sense. In the situation at hand they need to be defined on the outer region D from (1.13). In the spherically symmetric case the main obstruction for obtaining global solutions arises from the difficulties related to the centre of symmetry $r = 0$. For example, for a massless scalar field or a collisionless gas as matter model it has been shown that solutions remain regular away from $r = 0$ for general initial data, cf. [11, 2, 21]. On the other hand, for dust a singularity of shell crossing type can also occur at some $r > 0$. Although in that case there are no true geometric spacetime singularities, such behavior has to be ruled out in order not to interfere with the analysis of the solution on D. This can be achieved by proper assumptions on the initial data, cf. [7]. In view of (3.2) a possible break down of solutions at $r = 0$ will have no influence on the outer domain D. Hence we formulate a third condition, concerning global existence of solutions in the outer domain:

- For solutions launched by data from the set $\mathcal{I}$, γ^+ defined by (1.12) exists on $[0, \infty[$, and λ, μ, ρ, p, $j \in C^1(D)$. (GLO)

The three conditions above are of a quite general nature. The fourth and final condition however, is tightly connected to our method of proof.

- There exists a constant $c_1 > 0$ such that $\rho \leq -c_1 j$ in D. (GCC)

The acronym (GCC) stands for "gravitational collapse condition". We emphasize that for Vlasov matter there are by our main results initial data sets such that (GCC) holds. As a first consequence of (GCC) and (3.1), note that $j \leq 0$ in D, i.e., the matter is ingoing for all times. In this respect our present results complement [4], where purely outgoing matter was considered.

Let us now assume that our matter model satisfies (DEC) and (NNP), and that there exists an initial data set $\mathcal{I}$ such that (GLO) and (GCC) hold as well. Then we have the following result, which should be viewed as a version of Theorem 2.2 for general matter.

Theorem 3.1 *Let r_0, r_1, M, and M_{out} be given as above and let R_1 satisfy (2.3) with $\kappa = 2c_1$. Assume that there exists an initial data set $\mathcal{I}_3 \subset \mathcal{I}$ such that (2.4) and (2.5) hold for all initial data in $\mathcal{I}_3$. Then for any solution launched by initial data in $\mathcal{I}_3$,*

$$\lim_{s\to\infty} \gamma^+(s) < \infty, \quad \lim_{s\to\infty} \int_{\gamma^+(s)}^{\infty} 4\pi r^2 \rho(s, r)\, dr > 0,$$

where γ^+ satisfies (1.12).

The detailed information on the gravitational collapse which for Vlasov matter is provided in Theorem 2.4 is not available in the present situation.

4 Preliminaries

In this section we collect some general facts concerning the spherically symmetric Einstein-matter equations under the assumptions (DEC) and (NNP) that have been specified in the previous section. A quantity which plays an important role is the quasi-local mass $m(t,r)$. We assume $M > 0$ and define

$$m(t,r) := M - \int_r^\infty 4\pi\eta^2 \rho(t,\eta)\, d\eta. \tag{4.1}$$

Then $\lim_{r\to\infty} m(t,r) = M$, $0 \leq m \leq M$, and $m_r = 4\pi r^2 \rho$. Defining λ by $e^{-2\lambda} = 1 - 2m/r$, (1.2) and the boundary condition in (1.1) are satisfied. We require that

$$\mathring{m}(r) < \frac{r}{2}, \quad r \in]0,\infty[, \tag{4.2}$$

a condition that again will be included in the notion of regular initial data. By (1.2) and (1.3),

$$\lambda_r = \Big(4\pi r\rho - \frac{m}{r^2}\Big) e^{2\lambda}, \quad \mu_r = \Big(\frac{m}{r^2} + 4\pi r p\Big) e^{2\lambda}. \tag{4.3}$$

In view of (1.1), $\mu = \hat{\mu} + \check{\mu}$, where we define

$$\hat{\mu}(t,r) := -\int_r^\infty \frac{m(t,\eta)}{\eta^2}\, e^{2\lambda(t,\eta)}\, d\eta, \tag{4.4}$$

$$\check{\mu}(t,r) := -\int_r^\infty 4\pi\eta\, p(t,\eta)\, e^{2\lambda(t,\eta)}\, d\eta. \tag{4.5}$$

Lemma 4.1 *The following assertions hold.*

(a) $2\hat{\mu} \leq \mu - \lambda \leq \hat{\mu} \leq \hat{\mu} + \lambda$ *and* $\mu + \lambda \leq \hat{\mu} + \lambda$.

(b) $(\mu - \lambda)(t,r) = 2\hat{\mu}(t,r) + \int_r^\infty 4\pi\eta\,(\rho - p)(t,\eta)\, e^{2\lambda(t,\eta)}\, d\eta$.

(c) $\hat{\mu}_t(t,r) = \int_r^\infty 4\pi j(t,\eta)\, e^{(\mu+\lambda)(t,\eta)} e^{2\lambda(t,\eta)}\, d\eta$. *In particular, if* $j \leq 0$, *then also* $\hat{\mu}_t \leq 0$.

Proof: The claims follow straightforwardly in view of the boundary conditions (1.1) and the formulas for μ, $\hat{\mu}$, $\check{\mu}$, and λ from above. $\square$

The first part of the following lemma is due to [2], and the second part can be proved similarly.

Lemma 4.2 *For* $r \in [0,\infty[$,

$$\int_r^\infty 4\pi\eta\,(\rho + p)(t,\eta)\, e^{(\mu+\lambda)(t,\eta)} e^{2\lambda(t,\eta)}\, d\eta = 1 - e^{(\mu+\lambda)(t,r)} \leq 1,$$

$$\int_r^\infty 4\pi\eta\,\rho(t,\eta)\, e^{(\hat{\mu}+\lambda)(t,\eta)} e^{2\lambda(t,\eta)}\, d\eta = 1 - e^{(\hat{\mu}+\lambda)(t,r)} \leq 1.$$

5 Proof of Theorem 3.1

In this section we use the hypotheses stated in Section 3 to prove Theorem 3.1. The proof is short and emphasizes that the crucial mechanism is captured in (GCC). Consider the out- and ingoing null geodesics γ^+ and γ^- defined by

$$\frac{d\gamma^\pm}{ds}(s) = \pm\, e^{(\mu-\lambda)(s,\gamma^\pm(s))}, \quad \gamma^+(0) = r_0 < r_1 = \gamma^-(0). \tag{5.1}$$

The claims follow if we can show that these geodesics never intersect. By continuity and monotonicity there exists $T \in]0,\infty]$ such that

$$r_0 \le \gamma^+(t) < \gamma^-(t) \le r_1, \quad t \in [0,T[; \tag{5.2}$$

it will be shown that actually $T=\infty$. In view of (2.4) we have initially that $\rho = p = j = 0$ for $r \ge R_1$. (GCC) implies that $j \le 0$ in D, i.e., the flow of matter is ingoing. Therefore

$$\rho = p = j = 0 \quad \text{and} \quad m = M \quad \text{for} \quad (t,r) \in [0,T[\times[R_1,\infty[. \tag{5.3}$$

By Lemma 4.2, (3.1), (GCC), and Lemma 4.1(c) for $s \in [0,T[$ and $r \in [\gamma^+(s),\infty[$,

$$\begin{aligned}
1 - e^{(\mu+\lambda)(s,r)} &= \int_r^\infty 4\pi\eta\,(\rho+p)(s,\eta)\, e^{(\mu+\lambda)(s,\eta)} e^{2\lambda(s,\eta)}\, d\eta \\
&\le -2c_1R_1 \int_r^\infty 4\pi j(s,\eta)\, e^{(\mu+\lambda)(s,\eta)} e^{2\lambda(s,\eta)}\, d\eta = -2c_1R_1\hat\mu_t(s,r),
\end{aligned}$$

since $j(s,\eta) \ne 0$ implies $\eta \le R_1$. Thus

$$\hat\mu_t(s,r) \le -\frac{1}{2c_1R_1}\Big(1 - e^{(\mu+\lambda)(s,r)}\Big). \tag{5.4}$$

This in turn implies that

$$\begin{aligned}
&\hat\mu(t,\gamma^\pm(t)) - \hat\mu(0,\gamma^\pm(0)) = \int_0^t \frac{d}{ds}\,\hat\mu(s,\gamma^\pm(s))\, ds \\
&= \int_0^t \Big(\hat\mu_t(s,\gamma^\pm(s)) \pm \hat\mu_r(s,\gamma^\pm(s)) e^{(\mu-\lambda)(s,\gamma^\pm(s))}\Big)\, ds \\
&\le \int_0^t \Big(-\frac{1}{2c_1R_1}\Big(1 - e^{(\mu+\lambda)(s,\gamma^\pm(s))}\Big) \pm \frac{m(s,\gamma^\pm(s))}{\gamma^\pm(s)^2}\, e^{(\mu+\lambda)(s,\gamma^\pm(s))}\Big)\, ds \\
&\le -\frac{t}{2c_1R_1} + \int_0^t \Big(\frac{1}{2c_1R_1} + \frac{m(s,\gamma^\pm(s))}{\gamma^\pm(s)^2}\Big)\, e^{(\mu+\lambda)(s,\gamma^\pm(s))}\, ds.
\end{aligned} \tag{5.5}$$

Now for any $r \in [r_0, r_1]$ and $t \in [0, T[$ it follows from $\hat{\mu}_r \geq 0$ and $e^{-2\lambda} = 1 - 2m/r$ that

$$\hat{\mu}(t, r) \leq \hat{\mu}(t, R_1) = -\int_{R_1}^{\infty} \frac{M\, d\eta}{\eta^2(1 - 2M/\eta)}. \tag{5.6}$$

Using $M = r_1/2$ we get $\hat{\mu}(t, R_1) = \frac{1}{2}\log(\frac{R_1 - r_1}{R_1})$, so that for $r \in [r_0, r_1]$,

$$e^{\hat{\mu}(t,r)} \leq e^{\hat{\mu}(t,R_1)} = \sqrt{(R_1 - r_1)/R_1}. \tag{5.7}$$

By (3.2) and the properties of the initial matter distribution there is vacuum in the region $\gamma^+(t) \leq r \leq \gamma^-(t)$. Hence $m(t, r) = M - M_{\text{out}}$ and (2.1) imply that

$$e^{\lambda(t,r)} \leq \frac{1}{\sqrt{1 - 2(M - M_{\text{out}})/r_0}} < 3 \tag{5.8}$$

for $\gamma^+(t) \leq r \leq \gamma^-(t)$. From Lemma 4.1(a) and (2.3), recalling $\kappa = 2c_1$, we obtain in particular that

$$e^{(\mu+\lambda)(s,\gamma^\pm(s))} \leq e^{(\hat{\mu}+\lambda)(s,\gamma^\pm(s))} < \min\left\{\frac{1}{2}, \frac{r_0^2}{8c_1 R_1 M}\right\} =: d.$$

Thus (5.5) yields

$$\begin{aligned}\hat{\mu}(t, \gamma^\pm(t)) - \hat{\mu}(0, \gamma^\pm(0)) &\leq -\frac{t}{2c_1 R_1} + d\int_0^t \left(\frac{1}{2c_1 R_1} + \frac{M}{r_0^2}\right) ds \\ &= -\left(\frac{1-d}{2c_1 R_1} - d\,\frac{M}{r_0^2}\right)t \leq -\left(\frac{1}{4c_1 R_1} - d\,\frac{M}{r_0^2}\right)t \leq -\frac{t}{8c_1 R_1}\end{aligned} \tag{5.9}$$

for $t \in [0, T[$. Hence Lemma 4.1(a) leads to the estimate

$$\begin{aligned}|\gamma^\pm(t) - \gamma^\pm(0)| &= \left|\int_0^t e^{(\mu-\lambda)(s,\,\gamma^\pm(s))}\, ds\right| \leq \int_0^t e^{\hat{\mu}(s,\,\gamma^\pm(s))}\, ds \\ &\leq e^{\hat{\mu}(0,\,\gamma^\pm(0))} \int_0^t e^{-\frac{s}{8c_1 R_1}}\, ds \leq 8c_1 R_1 \sqrt{\frac{R_1 - r_1}{R_1}},\end{aligned}$$

where we used (5.7) in the last inequality. By the third condition in (2.3), $\sqrt{(R_1 - r_1)/R_1} < (r_1 - r_0)/(16c_1 R_1)$, so that $|\gamma^\pm(t) - \gamma^\pm(0)| < (r_1 - r_0)/2$ for $t \in [0, T[$. Since $\gamma^-(0) - \gamma^+(0) = r_1 - r_0$, this implies that $\gamma^-(T) - \gamma^+(T) > 0$. Hence, if we choose T in (5.2) to be maximal, then $T = \infty$, i.e., γ^+ and γ^- never intersect. This completes the proof of Theorem 3.1. $\square$

6 Proof of Theorem 2.2

We first check that the conditions (DEC), (NNP), and (GLO) hold for Vlasov matter. Then we show that there exists a class of initial data such that the corresponding solutions satisfy (GCC) with $c_1 = 3$. Theorem 2.2 then follows from Theorem 3.1.

The characteristic system associated to the Vlasov equation (1.6) is

$$\frac{dR}{ds} = e^{(\mu-\lambda)(s,R)}\,\frac{W}{E}, \tag{6.1}$$

$$\frac{dW}{ds} = -\lambda_t(s,R)W - e^{(\mu-\lambda)(s,R)}\mu_r(s,R)E + e^{(\mu-\lambda)(s,R)}\frac{L}{R^3E}, \tag{6.2}$$

$$\frac{dL}{ds} = 0. \tag{6.3}$$

If $s \mapsto (R,W,L)(s)$ is a solution with data $(R,W,L)(0) = (r,w,L)$, then $f(s,R(s),W(s),L) = \mathring{f}(r,w,L)$ is constant in s. Hence $(R(s),W(s),L) \in \operatorname{supp} f(s)$ iff $(r,w,L) \in \operatorname{supp} \mathring{f}$. Such characteristics will be addressed as characteristics in $\operatorname{supp} f$.

Direct inspection of the definition in (1.8) shows that (NNP) holds for Vlasov matter. It is moreover well-known that (DEC) is satisfied for Vlasov matter; see [1, Sec. 1.4]. Alternatively, we can check (3.1) and (3.2) directly. The latter follows from (6.1) above, whereas the former is a consequence of the expressions for the matter terms (1.7), (1.8), and (1.9).

To see that (GLO) holds we consider the spherically symmetric Einstein-Vlasov system on D, with $e^{-2\lambda} = 1 - 2m/r$ and (4.1) replacing the usual boundary condition $\lambda(t,0) = 0$ of a regular centre and with (1.12) included. We need to show firstly that regular initial data supported on $\{r > r_0\}$ launch a local solution on D which can be extended as long as $P(t) := \sup\{|w| \mid (r,w,L) \in \operatorname{supp} f(t),\ r > \gamma^+(t)\}$ remains bounded, and secondly that $P(t)$ cannot blow up in finite time. The latter follows by the estimates in [21] where is is shown that the w-support of a solution on the whole space cannot blow up in finite time, provided matter is bounded away from the centre or is controlled in a neighborhood of the centre. A local existence and continuation result of the required type is usually shown by an iterative scheme, cf. [18, 19]. The difficulty with such a scheme in the present situation is that γ^+ and hence D would change with the iteration. Rather than dealing with this difficulty here we by-pass it by observing that data from our initial data set launch solutions where the support of the matter on D stays strictly to the right of γ^+. For such data $\mathring{f}_{\text{out}}$ we take an arbitrary initial data $\mathring{f}_{\text{in}}$

supported in $\{r < r_0\}$ such that $\mathring{f}_{\rm out} + \mathring{f}_{\rm in}$ has mass M. This launches a local solution on the whole space which we restrict to D. Assuming that the solution on D is maximally extended with finite existence time T and that P is bounded on $[0,T[$ we pick $t_0 \in]0,T[$ and solve the system on the whole space with the data $f_{\rm out}(t_0) + \mathring{f}_{\rm in}$ prescribed at $t = t_0$. The corresponding local solution exists on a time interval the length of which is bounded from below uniformly in t_0. This follows from the bound on P, cf. Step 7 in the proof of Thm. 3.1 in [18]. If we choose t_0 close enough to T we have extended the solution, and the local existence and continuation result is established. Notice that the matter inside $\{r < \gamma^+(t)\}$ can influence the solution on D only through its mass which remains constant in the situation we consider.

It remains to show that (GCC) holds. To this end we let $0 < r_0 < r_1 < R_1$, $R_0 = (r_1 + R_1)/2$, and $M = r_1/2$. For a parameter $W_- < 0$ to be specified below and regular data $\mathring{f}$ with ADM mass M we formulate the

General support condition: For all $(r, w, L) \in \mathrm{supp}\, \mathring{f}$ the following holds:

$$r \in]0, r_0] \cup [R_0, R_1],$$

and if $r \in [R_0, R_1]$ then $w \le W_-$ and also

$$0 < L < \frac{3L}{\eta}\, \mathring{m}(\eta) + \eta\, \mathring{m}(\eta),\ \ \eta \in [r_0, R_1]. \tag{6.4}$$

We use the notation $\mathring{m}$ when $\rho = \mathring{\rho}$ in (4.1). Furthermore, we abbreviate

$$\Gamma = \Gamma(r_1, R_1) := \sqrt{\frac{R_1 - r_1}{R_1 + r_1}}. \tag{6.5}$$

The following lemma shows that if the support condition holds, then the particles in the outer domain D keep moving inward in a controlled way.

Lemma 6.1 *Let $\mathring{f}$ be regular and satisfy the general support condition for some $W_- < 0$. Then for all $(r, w, L) \in \mathrm{supp}\, f(t)$ such that $(t, r) \in D$,*

$$w \le \Gamma(r_1, R_1) W_-.$$

In particular, $j \le 0$ on D.

Proof: Let $[0, T[$ denote the maximal time interval such that for $t < T$

$$w < 0 \text{ for } (r, w, L) \in \mathrm{supp}\, f(t) \text{ with } (t, r) \in D. \tag{6.6}$$

Since $W_- < 0$, $T > 0$ by continuity. By the definition of j,

$$j(t, r) \le 0 \text{ for } (t, r) \in D_T := D \cap ([0, T[\times [0, \infty[). \tag{6.7}$$

Let $(R, W, L)(s)$ be a characteristic in $\operatorname{supp} f$. Then

$$
\begin{aligned}
\frac{d}{ds}(e^{-\lambda}W) &= -\, e^{-\lambda}\Big(W\lambda_t + W\lambda_r \frac{dR}{ds} - \frac{dW}{ds}\Big) \\
&= \frac{4\pi R}{E}\, e^{\mu}(2WEj - W^2\rho - E^2 p) + e^{\mu}\Big(1 - \frac{2m}{R}\Big)\frac{L}{R^3 E} \\
&\quad + e^{\mu}\frac{m}{R^2}\Big(\frac{w^2}{E} - E\Big) \\
&= -\,\frac{4\pi^2}{R}\, e^{\mu} \int_{-\infty}^{\infty}\int_0^{\infty} \Bigg[\sqrt{\frac{\tilde{E}}{E}}\, w - \sqrt{\frac{E}{\tilde{E}}}\, \tilde{w}\Bigg]^2 f\, d\tilde{L}\, d\tilde{w} \\
&\quad - e^{\mu}\frac{m}{R^2}\Big(\frac{1 + L/R^2}{E} + \frac{2L}{R^2 E}\Big) + e^{\mu}\frac{L}{R^3 E}\,,
\end{aligned}
$$

where $E = E(R, W, L)$ and $\tilde{E} = \tilde{E}(R, \tilde{w}, \tilde{L})$. Therefore

$$
\frac{d}{ds}(e^{-\lambda}W) \le -e^{\mu}\frac{m}{R^2}\Big(\frac{1 + L/R^2}{E} + \frac{2L}{R^2 E}\Big) + e^{\mu}\frac{L}{R^3 E}.
$$

Differentiating $e^{-2\lambda} = 1 - 2m/r$ w.r. to t and using (1.4) leads to $m_t = -4\pi r^2 e^{\mu-\lambda} j$, which by (6.7) is non-negative on D_T. It follows that $m(s, r) \ge m(0, r) = \mathring{m}(r)$. Thus as long as the characteristic remains in D_T,

$$
\begin{aligned}
\frac{d}{ds}(e^{-\lambda}W) &\le -e^{\mu}\frac{\mathring{m}(R)}{R^2}\Big(\frac{1 + L/R^2}{E} + \frac{2L}{R^2 E}\Big) + e^{\mu}\frac{L}{R^3 E} \\
&= e^{\mu}\,\frac{1}{R^3 E}\Big(L - \frac{3L}{R}\,\mathring{m}(R) - R\,\mathring{m}(R)\Big).
\end{aligned}
$$

Now $R(0) \in [R_0, R_1]$ and $\dot{R}(s) \le 0$ by (6.1) and (6.6) yields $R_1 \ge R(0) \ge R(s) \ge \gamma^+(s) \ge r_0$. Hence condition (6.4) implies that, as long as the characteristic remains in D_T, $\frac{d}{ds}(e^{-\lambda}W) < 0$, so that

$$
W(s) \le e^{\lambda(s,\, R(s)) - \lambda(0,\, R(0))}\, W_- \le \Big(\min_{r \in [R_0, R_1]} e^{-\lambda(0,\, r)}\Big)\, W_-.
$$

Furthermore, $e^{-\lambda(0,\,r)} \ge (1 - 2M/R_0)^{1/2} = ((R_1 - r_1)/(R_1 + r_1))^{1/2}$ holds for $r \in [R_0, R_1]$, and recalling (6.5) it follows that $W(s) \le \Gamma(r_1, R_1) W_- < 0$, as long as the characteristic remains in D_T. By the maximality of T in (6.6), $T = \infty$, and the proof is complete. □

In order to specify the initial data set $\mathcal{I}_2$, let r_0, r_1, M, and M_{out} be given as in Section 2 and let R_1 be such that (2.3) holds for $\kappa = 6$. We require that $W_- < 0$ satisfies the estimate

$$
\Gamma(r_1, R_1)\, |W_-| \ge 1. \tag{6.8}
$$

Then

$$\mathcal{I}_2 := \Big\{ \mathring{f} \mid \mathring{f} \text{ is regular, satisfies (2.4), (2.5), the general support condition,} \\ \text{and for } (r,w,L) \in \operatorname{supp} \mathring{f} \text{ with } r \in [R_0, R_1], \sqrt{L}/r_0 \le \Gamma\, |W_-| \Big\}. \tag{6.9}$$

Consider now a solution f launched by initial data from this set. Condition (6.8) and Lemma 6.1 imply that

$$|w| \ge \Gamma(r_1, R_1)\, |W_-| \ge 1 \quad \text{on} \quad \operatorname{supp} f \cap D, \tag{6.10}$$

and since L is conserved along characteristics, (6.9) leads to $\sqrt{L}/r \le \sqrt{L}/r_0 \le |w|$ for all particles in $\operatorname{supp} f \cap D$. Hence the definition (1.7) of ρ implies that on D,

$$\begin{aligned} \rho(t,r) &\le \frac{\pi}{r^2} \int_{-\infty}^{\infty} \int_0^{\infty} f\, dL\, dw + \frac{\pi}{r^2} \int_{-\infty}^{\infty} \int_0^{\infty} |w| f\, dL\, dw \\ &\quad + \frac{\pi}{r^2} \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{L}/r\, f\, dL\, dw \\ &\le 3\, \frac{\pi}{r^2} \int_{-\infty}^{\infty} \int_0^{\infty} |w| f\, dL\, dw = 3\, |j(t,r)|. \end{aligned} \tag{6.11}$$

Accordingly, $\mathcal{I}_2$ satisfies (GCC) with $c_1 = 3$, and Theorem 2.2 follows from Theorem 3.1. □

We briefly show that the set $\mathcal{I}_2$ is far from being empty. Therefore fix $0 < r_0 < r_1 < R_0 < R_1$, $M = r_1/2$, and $0 < M_{\rm out} < M$ such that $R_0 = (r_1 + R_1)/2$, (2.1), and (2.3) are satisfied. Let $0 \le f_1 \in C^1$ have r-support in $[r_0 - \delta, r_0]$ for some $0 < \delta < r_0/9$, and let $0 \le f_2 \in C^1$ have r-support in $[R_0, R_1]$. Fix the compact w-support of f_2 in $]-\infty, W_-]$ with $W_- < 0$ such that (6.8) holds, and fix its L-support in $[0, L_2]$ so that $\sqrt{L_2}/r_0 \le \Gamma(r_1, R_1)\, |W_-|$ and

$$L < (M - M_{\rm out}) \Big(\frac{3L}{\eta} + \eta \Big), \quad L \in [0, L_2], \quad \eta \in [r_0, R_1].$$

Now take $\mathring{f} = A f_1 + B f_2$, where $A > 0$ and $B > 0$ are chosen such that (2.4) and (2.5) are satisfied. Note that $\mathring{m}(\eta) \ge M - M_{\rm out}$ for $\eta \in [r_0, R_1]$, whence (6.4) holds as well; thus the general support condition if verified. It remains to check (4.2). If $r \in]0, r_0 - \delta]$, then $\mathring{m}(r) = 0$. If $r \in [r_0 - \delta, R_0]$, then $\mathring{m}(r) \le M - M_{\rm out}$ yields in view of (2.1),

$$\frac{2\mathring{m}}{r} \le \frac{2(M - M_{\rm out})}{r_0 - \delta} < 1.$$

If $r \in [R_0, \infty[$, then $2\mathring{m}/r \leq 2M/R_0 < 1$, since $2M = r_1 < R_0$. Hence $\mathring{f}$ is regular and has all the properties that are required in the definition of $\mathcal{I}_2$.

Remark. The set $\mathcal{I}_2$ has "non-empty interior", in the sense that sufficiently small perturbations of initial data in the "interior" of this set belong to $\mathcal{I}_2$ as well, provided that the support is changed very little and M is left invariant. This is due to the fact that the various parameters entering into the definition of $\mathcal{I}_2$ are defined in terms of inequalities and hence can be varied.

7 Proof of Theorem 2.1

The set up is closely related to the set up in the proof of Theorem 2.2. As we saw above, (DEC), (NNP), and (GLO) are satisfied for Vlasov matter, and we will again construct an initial data set such that (GCC) holds with $c_1 = 3$. However, since this result relies on condition (2.2) instead of (2.3), we cannot simply invoke Theorem 3.1 after (GCC) has been verified; instead an additional step needs to be added to the proof. For this new argument a slightly stronger condition on the momentum variable w needs to be imposed on $\operatorname{supp} \mathring{f}$. We now require that $W_- < 0$ satisfies

$$\Gamma(r_1, R_1)^2 |W_-|^2 \geq \frac{10}{d}, \tag{7.1}$$

where

$$d := \min\left\{\frac{1}{2}, \frac{r_0}{12R_1}, \frac{r_1 - r_0}{300R_1}\right\}.$$

Then

$$\mathcal{I}_1 := \Big\{\mathring{f} \mid \mathring{f} \text{ is regular, satisfies (2.4), (2.5), the general support condition,} \\ \text{and for } (r, w, L) \in \operatorname{supp} \mathring{f} \text{ with } r \in [R_0, R_1], \sqrt{L}/r_0 \leq 1.\Big\} \tag{7.2}$$

The same construction as at the end of the previous section shows that this set is not empty, and the same remark as at the end of the previous section applies. Let f be a solution launched by initial data from $\mathcal{I}_1$. It is clear from these conditions that Lemma 6.1 applies, and since $10/d \geq 1$, it follows that (6.10) holds as well. Thus the argument leading to $\rho \leq 3|j|$ on D in the proof of Theorem 2.2 applies again. Hence, (GCC) is satisfied with $c_1 = 3$. Next consider the expression

$$\rho(s, r) - p(s, r) = \frac{\pi}{r^2} \int_{-\infty}^{\infty} \int_0^{\infty} \left(E - \frac{w^2}{E}\right) f(s, r, w, L)\, dL\, dw.$$

Since $E^2 \geq w^2 \geq \Gamma^2(r_1, R_1)\, W_-^2$ by Lemma 6.1, we get for $r \in [\gamma^+(s), R_1]$ from $\sqrt{L}/r_0 \leq 1$,

$$E - \frac{w^2}{E} = \frac{1}{E}\,(E^2 - w^2) = \frac{1}{E}\left(1 + \frac{L}{r^2}\right) \leq \frac{2}{E} \leq \frac{2}{\Gamma^2\, W_-^2}\, E =: c_0 E, \qquad (7.3)$$

so that

$$\rho(s, r) - p(s, r) \leq c_0 \rho(s, r). \qquad (7.4)$$

After this preparation, we again show that the out- and ingoing null geodesics γ^+ and γ^- do not intersect. We choose $T \in]0, \infty[$ such that (5.2) holds. In this case we cannot rely on the smallness of $e^{\hat{\mu}}$ as in the proof of Theorem 3.1, so we need to control the evolution also when $e^{\hat{\mu}}$ is not small. For this part the estimate (7.4) is essential. We fix $t_*^\pm \in [0, T[$ by requiring that

$$e^{(\hat{\mu}+\lambda)(s,\,\gamma^\pm(s))} > d \text{ for } s \in [0, t_*^\pm[, \quad e^{(\hat{\mu}+\lambda)(s,\,\gamma^\pm(s))} \leq d \text{ for } s \in [t_*^\pm, T[.$$

First we note that $t_*^\pm$ is well-defined, since

$$\frac{d}{ds}\,(\hat{\mu} + \lambda)(s,\, \gamma^\pm(s)) = \Big(\hat{\mu}_t - 4\pi r\, e^{\mu+\lambda}(j \mp \rho)\Big) \leq 0. \qquad (7.5)$$

Step 1: Consider $s \in [0, t_*^\pm]$; if $t_*^\pm = 0$, then this step is omitted. For $\eta \geq \gamma^\pm(s)$ we have $d \leq e^{(\hat{\mu}+\lambda)(s,\gamma^\pm(s))} \leq e^{(\hat{\mu}+\lambda)(s,\,\eta)}$, since $(\hat{\mu}+\lambda)_r = 4\pi r \rho\, e^{2\lambda} \geq 0$. Hence Lemma 4.1(b) and (7.4) yield

$$\begin{aligned}
(\mu - \lambda)(s, \gamma^\pm(s)) &= 2\hat{\mu}(s, \gamma^\pm(s)) + \int_{\gamma^\pm(s)}^\infty 4\pi\eta\, (\rho - p)(s, \eta)\, e^{2\lambda(s,\,\eta)}\, d\eta \\
&\leq 2\hat{\mu}(s, \gamma^\pm(s)) + \frac{c_0}{d} \int_{\gamma^\pm(s)}^\infty 4\pi\eta\, \rho(s, \eta)\, e^{(\hat{\mu}+\lambda)(s,\,\eta)} e^{2\lambda(s,\,\eta)}\, d\eta \\
&\leq 2\hat{\mu}(s, \gamma^\pm(s)) + \frac{c_0}{d},
\end{aligned}$$

where for the last estimate Lemma 4.2 has been used.

Now we make the following observation: There is at least one characteristic $(\bar{R}, \bar{W}, \bar{L})(s)$ with $\bar{R}(0) \in [R_0, R_1]$, which does not leave the strip $[r_1, R_1]$ during the finite time interval $[0, T]$. In fact, if at time $t = T$ all characteristics had left the strip $[r_1, R_1]$ (and thus had entered the region $r < r_1$), then $m(T, r_1) = M$. From $e^{-2\lambda} = 1 - 2m/r$ and $2M = r_1$ it would follow that $\lambda(T, r_1) = \infty$. However, this contradicts (GLO) which holds for Vlasov matter.

Since $\gamma^{\pm}(s) \leq r_1 \leq \bar{R}(s)$ and $\hat{\mu}_r \geq 0$, we thus obtain in view of Lemma 4.1(a) that

$$\begin{aligned}(\mu-\lambda)(s,\gamma^{\pm}(s)) &\leq 2\hat{\mu}(s,\gamma^{\pm}(s)) + \frac{c_0}{d} \leq 2\hat{\mu}(s,\bar{R}(s)) + \frac{c_0}{d}\\ &\leq (\mu-\lambda)(s,\bar{R}(s)) + \frac{c_0}{d}, \quad s\in[0,t_*^{\pm}].\end{aligned}$$

Next note that $|W| \geq 1$ by (6.10), and hence due to (6.1) and observing $\bar{R}^2 \geq r_0^2 \geq L$,

$$|\dot{\bar{R}}| = \frac{|W|}{E}\, e^{\mu-\lambda} \geq \frac{|W|}{\sqrt{2+W^2}}\, e^{\mu-\lambda} \geq \frac{1}{2}\, e^{\mu-\lambda}.$$

Therefore we obtain for all $t\in[0,t_*^{\pm}]$ the estimate

$$\begin{aligned}|\gamma^{\pm}(t)-\gamma^{\pm}(0)| &= \left|\int_0^t \pm\, e^{(\mu-\lambda)(s,\gamma^{\pm}(s))}\, ds\right| \leq e^{\frac{c_0}{d}} \int_0^t e^{(\mu-\lambda)(s,\bar{R}(s))}\, ds\\ &\leq -2e^{\frac{c_0}{d}} \int_0^t \dot{\bar{R}}(s)\, ds = 2e^{\frac{c_0}{d}}(\bar{R}(0)-\bar{R}(t))\\ &\leq 2e^{\frac{c_0}{d}}(R_1-r_1). \end{aligned}\tag{7.6}$$

Step 2: Let $t\in[t_*^{\pm},T[$; if $t_*^{\pm}=T$, then this step is omitted. The arguments here are basically the ones presented in Section 5. The computation leading to (5.5) is almost identical, and

$$\begin{aligned}&\hat{\mu}(t,\gamma^{\pm}(t)) - \hat{\mu}(t_*^{\pm},\gamma^{\pm}(t_*^{\pm}))\\ &\qquad \leq -\frac{t-t_*^{\pm}}{2c_1R_1} + \int_{t_*^{\pm}}^t \left(\frac{1}{2c_1R_1} + \frac{m(s,\gamma^{\pm}(s))}{\gamma^{\pm}(s)^2}\right) e^{(\mu+\lambda)(s,\gamma^{\pm}(s))}\, ds\end{aligned}\tag{7.7}$$

for $c_1=3$. By Lemma 4.1(a), $e^{(\mu+\lambda)(s,\gamma^{\pm}(s))} \leq e^{(\hat{\mu}+\lambda)(s,\gamma^{\pm}(s))} \leq d$. Using the definition of d we obtain by a similar chain of estimates as in (5.9)

$$\hat{\mu}(t,\gamma^{\pm}(t)) - \hat{\mu}(t_*^{\pm},\gamma^{\pm}(t_*^{\pm}) \leq -\frac{1}{8c_1R_1}(t-t_*^{\pm}), \quad t\in[t_*^{\pm},T[.$$

Hence by Lemma 4.1(a),

$$\begin{aligned}|\gamma^{\pm}(t)-\gamma^{\pm}(t_*^{\pm})| &= \left|\int_{t_*^{\pm}}^t e^{(\mu-\lambda)(s,\gamma^{\pm}(s))}\, ds\right| \leq \int_{t_*^{\pm}}^t e^{\hat{\mu}(s,\gamma^{\pm}(s))}\, ds\\ &\leq e^{(\hat{\mu}+\lambda)(t_*^{\pm},\gamma^{\pm}(t_*^{\pm}))} \int_{t_*^{\pm}}^{\infty} e^{-\frac{(s-t_*^{\pm})}{8c_1R_1}}\, ds \leq 8c_1R_1d.\end{aligned}\tag{7.8}$$

Adding the contributions (7.6) from Step 1 and (7.8) from Step 2, the final estimate $|\gamma^\pm(t) - \gamma^\pm(0)| \leq 2e^{c_0/d}(R_1 - r_1) + 8c_1R_1d$ is obtained for all $t \in [0, T[$. From (7.3) and (7.1) we have $c_0/d \leq 1/5$. The third condition on d together with (2.2) thus imply that $|\gamma^\pm(t) - \gamma^\pm(0)| < (r_1 - r_0)/2$. As in the proof of Theorem 3.1 we conclude that γ^+ and γ^- do not intersect, completing the proof of Theorem 2.1. $\Box$

It remains to prove Corollary 2.3.

Proof of Corollary 2.3: Let f_s be a static solution. By [3], $2m_s(r)/r < 8/9$ for $r > 0$ where m_s is the local ADM mass induced by f_s. In particular, $M_s < r_s/2 < r_1/2 = M$, and (2.1) holds. As described above we can now specify the matter distribution for $r \geq r_0$, and we obtain initial data $\mathring{f}$ in $\mathcal{I}_1$ or in $\mathcal{I}_2$ which coincide with the given static solution for $0 \leq r \leq r_0$.

Since no matter travels from the outer domain D to the inner one where $r \leq \gamma^+(t)$, the only way the matter in the outer domain can affect the static solution is through the metric. Consider the time-independent version of the Vlasov equation (1.6). Dropping all the time derivatives we see that in the remaining equation the factor $e^{\lambda-\mu}$ can be canceled. Therefore, the static Einstein-Vlasov system is formulated in terms the quantities f, λ, and μ_r, but not μ itself. Recalling $e^{-2\lambda} = 1 - 2m/r$ and (4.3), we see that λ and μ_r are, on $r \leq \gamma^+(t)$, not affected by the matter in the outer domain D. Therefore $f = f_s$, λ, and μ_r remain time-independent for $r \leq \gamma^+(t)$. $\Box$

Note that the metric coefficient μ of course does change on the interior region; cf. Theorem 2.4(b).

8 Proof of Theorem 2.4

As a first step we estimate $\mu-\lambda$ from below for $r > 2M$, using Lemma 4.1(a).

$$
\begin{aligned}
(\mu - \lambda)(t, r) &\geq 2\hat{\mu}(t, r) = -2 \int_r^\infty \frac{m(t, \eta)}{\eta^2} e^{2\lambda(t,\eta)} d\eta \\
&= -2 \int_r^\infty \frac{m(t, \eta)}{\eta\,(\eta - 2m(t, \eta))} d\eta \geq -2 \int_r^\infty \frac{M}{\eta\,(\eta - 2M)} d\eta \\
&= \ln \frac{r - 2M}{r}, \quad r > 2M.
\end{aligned}
$$

Now consider any characteristic in the matter support and let $R(t)$ denote its radial coordinate. Then by Lemma 6.1 and as long as $R(t) > 2M$,

$$
\frac{dR}{ds} = e^{(\mu-\lambda)(s,R)} \frac{W}{E} \leq -Ce^{(\mu-\lambda)(s,R)} \leq -C\,\frac{R - 2M}{R};
$$

for initial data from the set $\mathcal{I}_1$ respectively $\mathcal{I}_2$ one can take $C := \Gamma|W_-|/\sqrt{2+\Gamma^2 W_-^2}$ respectively $C := 1/\sqrt{3}$. Integrating this differential inequality we find that as long as $R(t) > 2M$ the estimate

$$\begin{aligned}-Ct &\geq \int_{R(0)}^{R(t)} \frac{r}{r-2M}\,dr = R(t) - R(0) + 2M \ln \frac{R(t)-2M}{R(0)-2M} \\ &\geq 2M - R_1 + 2M \ln \frac{R(t)-2M}{R(0)-2M}\end{aligned}$$

holds, and hence $R(t) \leq 2M + (R_1 - 2M)\exp(R_1 - 2M - Ct/2M)$. This proves the support estimate in part (a). Since all the matter, which has ADM mass M, is contained in the region where $r \leq 2M + \alpha e^{-\beta t} =: \sigma(t)$, the assertion on the metric follows. Moreover, for any $r \leq \sigma(t)$ the monotonicity of μ with respect to r implies that

$$\mu(t,r) \leq \ \mu(t,\sigma(t)) = \hat{\mu}(t,\sigma(t)) = \ln \left(\frac{\sigma(t)-2M}{\sigma(t)}\right)^{1/2},$$

which is the first assertion of part (b). The second follows immediately since the integral $\int_0^\infty e^{\mu(t,r)}dt$ is the proper length of a coordinate line of constant r, θ, and φ in the outer region D. This completes the proof of part (b).

As to (c) we first observe that any radially outgoing null geodesic which enters the region $r > 2M$ escapes to $r = \infty$ and is future complete, since by part (a) the metric on $r > 2M + \epsilon$ where $\epsilon > 0$ is arbitrary eventually equals the Schwarzschild one for which the asserted properties of the geodesics hold. Now consider the extremal geodesic γ^*. If there existed some time $t > 0$ such that $\gamma^*(t) > 2M$, then by continuous dependence on the initial data the same would be true for all radially outgoing null geodesics with $\gamma(0)$ sufficiently close to but less than r^*. Hence such geodesics would escape to $r = \infty$ in contradiction to the definition of r^*. This shows that the extremal, radially outgoing null geodesic γ^* has the property that $\lim_{t\to\infty}\gamma^*(t) \leq 2M$.

It remains to show that the limit above cannot be strictly less than $2M$. To this end we consider a radially outgoing null geodesic as long as $\gamma(t) < \sigma(t) = 2M + \alpha e^{-\beta t}$. Then

$$\frac{d\gamma}{ds} = e^{(\mu-\lambda)(s,\gamma(s))} \leq e^{\mu(s,\sigma(s))} = \left(\frac{\sigma(s)-2M}{\sigma(s)}\right)^{1/2} \leq Ce^{-\beta s/2},$$

and hence for any $0 \leq t_0 \leq t$ and as long as $\gamma(t) < \sigma(t)$ we have $\gamma(t) \leq \gamma(t_0) + Ce^{-\beta t_0/2}$, where the constant $C > 0$ again depends only on the initial data set. Assume that $R^* := \lim_{t\to\infty} \gamma^*(t) < 2M$, choose $t_0 > 0$ such that $R^* + Ce^{-\beta t_0/2} < 2M$, and consider the radially outgoing null geodesic γ^{**} with $\gamma^{**}(t_0) = R^*$. By construction, $\gamma^{**}(t) < 2M < \sigma(t)$ for all $t \geq t_0$, and since $\gamma^{**}(t_0) = R^* > \gamma^*(t_0)$ it follows that $\gamma^{**}(0) > \gamma^*(0) = r^*$. Hence γ^{**} is a radially outgoing null geodesic which at time $t = 0$ starts to the right of r^* and does not escape to $r = \infty$. This contradicts the definition of r^*. $\square$

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