Twist mappings with non-periodic angles

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1 Introduction

Consider the map

$$f: \quad \theta_1 = F(\theta, r), \quad r_1 = G(\theta, r).$$

The functions F and G are defined for $\theta \in \mathbb{R}$, $r \in]a, b[$, and satisfy the periodicity conditions

$$F(\theta + 2\pi, r) = F(\theta, r) + 2\pi, \quad G(\theta + 2\pi, r) = G(\theta, r). \tag{1}$$

After the identification $\theta + 2\pi \equiv \theta$, the domain of f can be interpreted as an annulus or a cylinder. Let us think that it is a cylinder with vertical coordinate r. We say that the map f has *twist* if

$$\frac{\partial F}{\partial r} > 0,$$

and it is *exact symplectic* if the differential form $r_1 d\theta_1 - r d\theta$ is exact. This means that there exists a smooth function $H = H(\theta, r)$ that is 2π-periodic in θ and such that

$$r_1 d\theta_1 - r d\theta = dH.$$

The above definitions have simple geometrical interpretations which will be discussed later. The reversed inequality $\frac{\partial F}{\partial r} < 0$ is also admissible as a twist condition.

Exact symplectic twist maps play an important role in the qualitative theory of Hamiltonian systems of low dimension. See [4, 7, 11, 26] for the general theory and [3, 9, 17, 27, 34, 36, 37, 38] for applications. Typically these maps appear in the study of systems of the type

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p}, \quad \dot{p} = -\frac{\partial \mathcal{H}}{\partial q}, \quad (q, p) \in \mathbb{R}^d,$$

in the cases

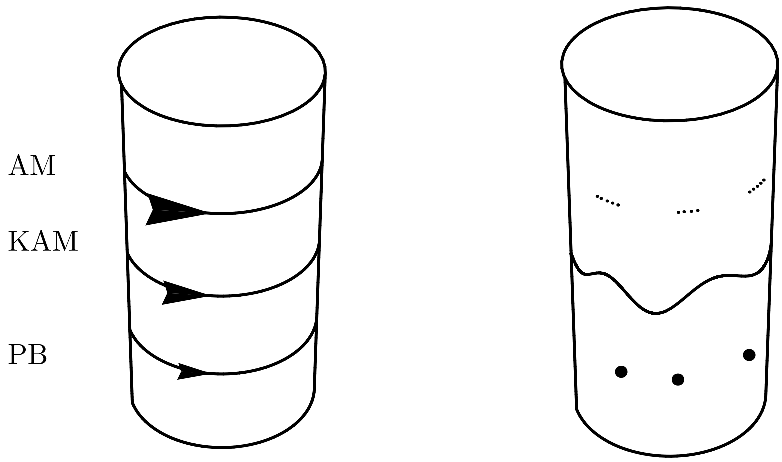
- 2 degrees of freedom and autonomous, $d = 2$ and $\mathcal{H} = \mathcal{H}(q, p)$
- 1 degree of freedom and time periodic, $d = 1$ and $\mathcal{H} = \mathcal{H}(t, q, p)$ with $\mathcal{H}(t + 2\pi, q, p) = \mathcal{H}(t, q, p)$.

The second case is sometimes referred to as the case of 1.5 degrees of freedom. The periodicity in time is usually employed to guarantee the periodicity of the angle θ in the associated twist map. In this course we will show that twist maps are also useful in the study of Hamiltonian systems with one degree of freedom but with general dependence on time. The key point will be to change the domain of the map f: Instead of a cylinder we will work on the horizontal strip $-\infty < \theta < \infty$, $a < r < b$, without any periodicity assumption on the angle θ. Before entering into the details we will discuss some results at an intuitive (non-rigorous) level. This will be useful to describe the contents of the course.

Let us start with the *integrable twist map*

$$T: \quad \theta_1 = \theta + \varphi(r), \;\; r_1 = r,$$

where $\varphi : [a, b] \to \mathbb{R}$ is a smooth function such that $\varphi' > 0$. This map has twist ($\frac{\partial F}{\partial r} = \varphi'$) and it is exact symplectic on the cylinder. To check this last property we notice that $r_1 d\theta_1 - r d\theta = d\Phi$ where $\Phi = \Phi(r)$ is a primitive of the function $r\varphi'(r)$. The function $I(\theta, r) = r$ is a first integral, that is $I(\theta_1, r_1) = I(\theta, r)$, and each circle $r = r_*$ is invariant under T. The twist condition implies that the rotation number $\omega = \varphi(r_*)$ associated to each of these circles increases with r_*. When the rotation number ω is commensurable with 2π, say $\frac{\omega}{2\pi} = \frac{p}{q}$ in reduced form, then all orbits in the invariant circle are periodic and satisfy $\theta_{n+q} = \theta_n + 2\pi p$, $r_{n+q} = r_n$. On the contrary, when $\frac{\omega}{2\pi}$ is irrational, orbits are quasi-periodic with frequencies 2π and ω. A key property of the map T is that many of its invariant sets persist under small perturbations in the class of exact symplectic twist mappings. This is a delicate theory because there are different cases depending on the arithmetic properties of ω. Given a compact interval $[\varphi_-, \varphi_+]$ with $\varphi(a+) < \varphi_- < \varphi_+ < \varphi(b-)$ and a small perturbation T_ϵ of T in the class of exact symplectic twist maps, then for each $\omega \in [\varphi_-, \varphi_+]$, ω commensurable with 2π, there are at least two periodic orbits with rotation number ω. This is a consequence of the Poincaré-Birkhoff theorem (see [2, 29]). In the case where ω is not commensurable with 2π there are two possibilities: Either the invariant curve associated to ω persists and all motions on this curve are quasi-periodic with frequencies 2π and ω, or the invariant circle breaks down and an invariant Cantor set appears. The dynamics of the Cantor set is of Denjoy type and has rotation number ω. These are consequences of KAM and Aubry-Mather theories (see [3, 21, 34]).



In the above discussions it is essential that the perturbation T_ϵ is exact symplectic. In the cylinder $r \in]-1, 1[$ and for $\epsilon > 0$ the map

$$T_{\epsilon,1} : \quad \theta_1 = \theta + \varphi(r), \;\; r_1 = (1-\epsilon)r,$$

has no invariant set with rotation number $\omega \neq 0$, and the map

$$T_{\epsilon,2} : \quad \theta_1 = \theta + \varphi(r), \;\; r_1 = r + \epsilon,$$

has no invariant sets at all.

All the previously mentioned results can be derived from by now classical theorems in the theory of twist maps. Let us now go to a less standard situation and consider maps f on the strip $-\infty < \theta < +\infty$, $r \in]a, b[$. In particular the periodicity conditions on F and G as imposed before will in general be dropped. The twist condition still makes sense and we replace the concept of exact symplectic map in the cylinder with the following definition. The map f on the strip is *E-symplectic* if the differential form $r_1 d\theta_1 - r d\theta$ is exact and its primitive $H = H(\theta, r)$ is bounded on each region $\mathbb{R} \times [A, B]$ with $a < A < B < b$. Notice that now the function H is not periodic in θ and so this boundedness condition is not automatic. The integrable twist map T is E-symplectic because the function Φ is bounded on compact intervals. The sets $r = r_*$, invariant under T, are now straight lines where the orbits move with increasing velocities as r_* goes from a to b. The rotation number is recovered from the limit

$$\lim_{|n|\to\infty} \frac{\theta_n}{n}, \tag{2}$$

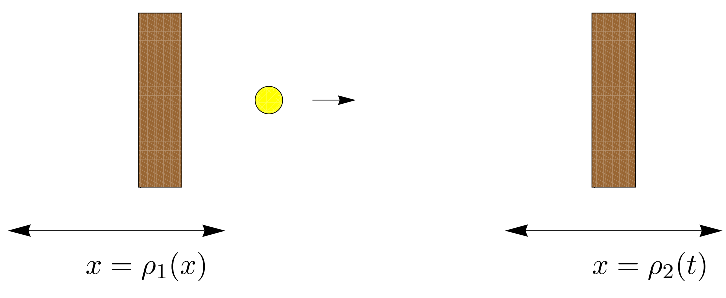
which exists for each orbit $(\theta_n, r_n)_{n\in\mathbb{Z}}$ and coincides with $\omega = \varphi(r_*)$. Assuming that the strip is wide enough, we will prove that there is still some persistence of invariant sets for small perturbations of T in the class of E-symplectic maps. In particular we will prove the existence of complete orbits which are bounded in the variable r. However, it does not seem possible to associate a rotation number to these sets. As an example consider for $\epsilon > 0$ the map

$$T_{\epsilon,3}: \quad \theta_1 = \theta + r, \quad r_1 = r + \frac{\epsilon}{1+\theta_1^2},$$

where $r \in]a,b[$ for $0 < a < b$. All orbits of this map are strictly increasing in θ and so rotation numbers cannot exist, at least if they are understood in the sense of (2). On the other hand this map is an E-symplectic perturbation of T. Actually, $r_1 d\theta_1 - r d\theta = dH$ with

$$H(\theta, r) = \frac{1}{2} r^2 + \epsilon \arctan \theta_1.$$

These results on perturbations of the map T have many consequences. As an application to mechanical problems we can consider the following ping-pong game. Two players move their rackets ($\equiv$ parallel moving walls) according to known protocols, say $x = \rho_1(t)$ and $x = \rho_2(t)$ with $\rho_1(t) < \rho_2(t)$. The ball is hit alternatively by the players and all impacts are assumed to be perfectly elastic.



In the absence of gravity the motion of the ball is described by a Hamiltonian system with one degree of freedom defined by

$$\mathcal{H}(t,q,p) = \frac{1}{2}p^2 + V(t,q) \text{ with } V(t,q) = \begin{cases} 0 & : \quad \rho_1(t) \le q \le \rho_2(t) \\ +\infty & : \quad \text{otherwise} \end{cases}.$$

The question is to decide whether the velocity of the ball remains bounded or can become arbitrarily large. For simplicity we assume that one of the walls is fixed, say $\rho_1 \equiv 0$. Then we can consider the successor map $(t_0, v_0) \mapsto (t_1, v_1)$, associating consecutive impacts against the fixed wall. Times of impact are t_0, t_1 and velocities after impacts are $v_0 \ge 0$, $v_1 \ge 0$. Ideally we would like to determine if $\sup v_n < +\infty$ or $\sup v_n = +\infty$, for each sequence of iterates (t_n, v_n). When the function $\rho_2(t)$ is 2π-periodic, the successor map satisfies the periodicity conditions (1) and it seems reasonable to interpret t_0 as an angle. However, the successor map is not E-symplectic. A computation shows that the form $v_1^2 dt_1 - v_0^2 dt_0$ is closed and this fact suggests the use of new variables: time t_0 and kinetic energy $E_0 = \frac{1}{2}v_0^2$. With the identification $\theta = t_0$ and $r = E_0$, the map $(t_0, E_0) \mapsto (t_1, E_1)$ becomes exact symplectic when ρ_2 is periodic and E-symplectic in the non-periodic case. These properties hold for large energies, as well as the twist condition

$$\frac{\partial t_1}{\partial E_0} < 0 \quad \text{if } E_0 \gg 0.$$

The intuition behind this formula is that the time employed by the ball to go back to the fixed wall will decrease as the energy increases. We have reformulated our problem in terms of an exact symplectic twist map but in general this map is not a perturbation of the integrable twist map. The symplectic change of variables

$$\tau(t) = \int_0^t \frac{ds}{\rho_2(s)}, \quad W = \rho_2(t)^2 E,$$

leads to a new map $(\tau_0, W_0) \mapsto (\tau_1, W_1)$ which is close to T for $\varphi(W) = \sqrt{\frac{2}{W}}$ and W_0 large enough. The results on twist maps described previously are applicable and many consequences for the ping-pong model can be deduced. When $\rho_2(t)$ is a 2π-periodic and smooth function, say of class C^5, KAM theory implies that the map has invariant curves in $W_0 \gg 0$. These curves act as barriers for the orbits (τ_n, W_n) so that all motions have bounded velocity. A complete proof of this result can be found in [17] and some extensions to the quasi-periodic case with diophantine conditions can be found in [38]. The use of KAM theory forces the assumption on the smoothness of ρ_2. An ingenious example in [37] shows that motions with unbounded velocity can exist for certain functions $\rho_2(t)$ which are still periodic but only continuous. Without the periodicity assumption, motions with unbounded velocity can exist even if $\rho_2(t)$ is very smooth. Examples are constructed in [16]. Also in that paper it is proved that there exist infinitely many motions with bounded velocity when $\rho_2 \in C^3(\mathbb{R})$ satisfies $\|\rho_2^{(k)}\|_\infty < \infty$ for $k = 0, 1, 2, 3$. Moreover, unbounded motions remain close to some of these bounded motions for arbitrarily long periods of time.

The plan of these notes is as follows. First we will discuss the theory of twist maps in the plane and present a result on the persistence of bounded orbits. The notion of *generating function* will be crucial. This is a classical tool in mechanics which allows to represent E-symplectic maps in terms of one single function. Some connections between generating functions and the Calculus of Variations will be discussed. We will restrict ourselves to the variational framework associated to Newtonian equations. Finally the general theory will be applied to the study of a ping-pong model. The one mentioned before is technically difficult, so we will analyze a simpler variant which only involves one single racket and gravity. The notes are concluded with some bibliographical comments.

2 Symplectic maps in the plane and in the cylinder

We will work on the plane $\mathbb{R}^2$ with cartesian coordinates (θ, r). Sometimes we will also work on the cylinder $\mathbb{T} \times \mathbb{R}$ with $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. A generic point in the cylinder will be denoted by $(\overline{\theta}, r)$ with $\overline{\theta} = \theta + 2\pi\mathbb{Z}$. The covering map $p : \mathbb{R}^2 \to \mathbb{T} \times \mathbb{R}$, $(\theta, r) \mapsto (\overline{\theta}, r)$, is useful to lift maps from the cylinder to the universal covering $\mathbb{R}^2$.

Let us start with the plane. We work with C^k embeddings, $k \geq 1$, defined on a strip $\Sigma = \mathbb{R}\times]a, b[$, $-\infty \leq a < b \leq +\infty$. More precisely, consider a C^k map

$$f : \Sigma \subset \mathbb{R}^2 \to \mathbb{R}^2, \ \ (\theta, r) \mapsto (\theta_1, r_1),$$

satisfying

(i) f is one-to-one

(ii) $\det f'(\theta, r) \neq 0 \quad \forall (\theta, r) \in \Sigma.$

The class of these maps will be denoted by $\mathcal{E}^1(\Sigma)$. A map $f \in \mathcal{E}^1(\Sigma)$ is called *symplectic* if it preserves the differential form $\omega = d\theta \wedge dr$. This means that, in the set Σ,

$$d\theta_1 \wedge dr_1 = d\theta \wedge dr. \tag{3}$$

If we express the map f in coordinates

$$\theta_1 = F(\theta, r), \quad r_1 = G(\theta, r),$$

then the condition (3) can be reformulated as

$$\det f' = \frac{\partial F}{\partial \theta}\frac{\partial G}{\partial r} - \frac{\partial F}{\partial r}\frac{\partial G}{\partial \theta} = 1 \text{ on } \Sigma.$$

This is the classical definition of area-preserving map.

Exercise 1 *Prove that $f \in \mathcal{E}^1(\Sigma)$ is symplectic if and only if the two conditions below hold:*

(a) f is orientation-preserving

(b) for each (Lebesgue) measurable set $\Omega \subset \Sigma$, the image $\Omega_1 = f(\Omega)$ is also measurable and $\mu(\Omega) = \mu(\Omega_1)$. Here μ is the Lebesgue measure in the plane.

Given $f \in \mathcal{E}^2(\Sigma)$ we consider the 1-form

$$\alpha = r_1 d\theta_1 - r d\theta.$$

Then $d\alpha = -d\theta_1 \wedge dr_1 + d\theta \wedge dr$ and so α is closed if and only if f is symplectic. The strip Σ is contractible and therefore closed and exact forms coincide. In particular, if f is symplectic there must exist a function H with $\alpha = dH$. After taking differentials in this identity we conclude that the converse is also true. Summing up, a map $f \in \mathcal{E}^2(\Sigma)$ is symplectic if and only if

$$dH = r_1 d\theta_1 - r d\theta \quad \text{for some } H \in C^2(\Sigma). \tag{4}$$

This identity can be expressed as

$$H_\theta = GF_\theta - r, \;\; H_r = GF_r. \tag{5}$$

When f is only in $\mathcal{E}^1(\Sigma)$, the components of α given by (5) are only continuous and the operation of taking the differential of α becomes more delicate.

Exercise 2 *Assume that $f \in \mathcal{E}^1(\Sigma)$. Prove that f is symplectic if and only if there exists a function $H \in C^1(\Sigma)$ such that $dH = \alpha$. Hint: If f is symplectic define the potential H by the standard integral of α and check* *directly that (5) holds. For the converse construct sequences $F^\epsilon, G^\epsilon \in C^2(\Sigma)$ with $\|F - F^\epsilon\|_{C^1(\Sigma)} + \|G - G^\epsilon\|_{C^1(\Sigma)} \to 0$ as $\epsilon \to 0$ and consider the integral*

$$\int_\Sigma \{[G^\epsilon (F^\epsilon)_\theta - r]\phi_r - G^\epsilon (F^\epsilon)_r \phi_\theta\}\, d\theta\, dr$$

for each test function $\phi \in \mathcal{D}(\Sigma)$. Integrate by parts and pass to the limit.

The equivalence between closed and exact 1-forms is no longer true in the cylinder. Consider the strip immersed in the cylinder $\overline{\Sigma} = \mathbb{T}\times]a, b[$. Since Σ is its universal covering, all 1-forms on $\overline{\Sigma}$ of class C^1 can be expressed as

$$\beta = A(\theta, r)d\theta + B(\theta, r)dr$$

with $A, B \in C^1(\Sigma)$ and 2π-periodic in θ. When β is closed it is possible to find a function $H = H(\theta, r)$ with $dH = \beta$. The problem is that sometimes H is not periodic in θ and so it becomes a multi-valued function when regarded in the cylinder.

Exercise 3 *Prove that exact 1-forms in the cylinder can be characterized as closed 1-forms satisfying*

$$\int_0^{2\pi} A(\theta, r_*)d\theta = 0$$

for some $r_* \in]a, b[$.

This difference between the plane and the cylinder plays a role when one tries to extend the notion of symplectic map to $\mathbb{T} \times \mathbb{R}$. Let us start with a map

$$\overline{f} : \overline{\Sigma} \subset \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}, \ \ (\overline{\theta}, r) \mapsto (\overline{\theta}_1, r_1),$$

satisfying the same conditions as in the case of the plane. The class of these maps is $\mathcal{E}^1(\overline{\Sigma})$. Every $f \in \mathcal{E}^1(\overline{\Sigma})$ has a lift $f = (F, G)$ in $\mathcal{E}^1(\Sigma)$. The coordinates satisfy

$$F(\theta + 2\pi, r) = F(\theta, r) + 2n\pi, \ \ G(\theta + 2\pi, r) = G(\theta, r).$$

In principle n could be any integer but since our map is an orientation-preserving embedding it can only take the values $n = -1$ or $n = 1$. We say that $\overline{f}$ is symplectic if its lift is symplectic as a map of the plane. Notice that, up to an additive constant $2N\pi$, the lift is unique and so this definition is all right.

Exercise 4 *Extend exercise 1 to the cylinder using the measure transported from the plane via the covering map,*

$$\mu_{\mathbb{T}\times\mathbb{R}}(\overline{A}) = \mu_{\mathbb{R}^2}(p^{-1}(\overline{A}) \cap ([0, 2\pi] \times \mathbb{R})).$$

We say that $\overline{f} \in \mathcal{E}^1(\overline{\Sigma})$ is *exact symplectic* if there is a function $H \in C^1(\overline{\Sigma})$, and hence 2π-periodic in θ, such that

$$dH = r_1 d\theta_1 - r d\theta.$$

Using exercise 2 we observe that exact symplectic maps are always symplectic. Since not all closed forms are exact in the cylinder, we can expect that there are symplectic maps which are not exact.

Exercise 5 *Prove that $\overline{f} \in \mathcal{E}^1(\overline{\Sigma})$ is exact symplectic if and only if it is symplectic and*

$$\int_0^{2\pi} G(\theta, r_*) \frac{\partial F}{\partial \theta}(\theta, r_*) d\theta = 2\pi r_*$$

for some $r_* \in]a, b[$.

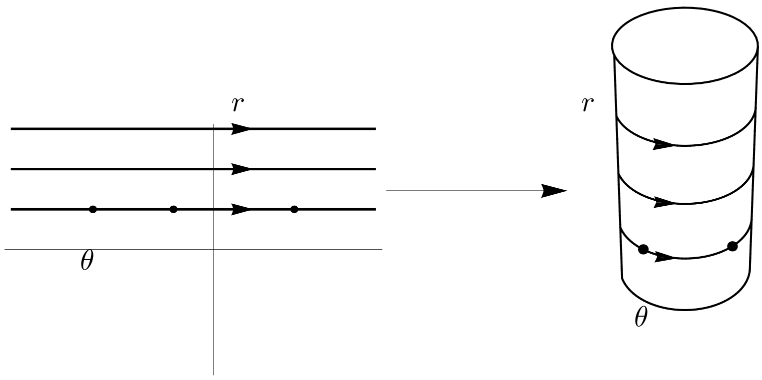
The notion of exact symplectic map can also be characterized in terms of measure theory. Given an arbitrary Jordan curve $\Gamma \subset \overline{\Sigma}$ which is C^1, regular and non-contractible, the image $\Gamma_1 = \overline{f}(\Gamma) \subset \mathbb{T} \times \mathbb{R}$ is another Jordan curve enjoying the same properties. Let us fix some $r_0 \in \mathbb{R}$ such that $\Gamma \cup \Gamma_1 \subset \{r > r_0\}$ and let A and A_1 denote the bounded components of $\{r > r_0\} \setminus \Gamma$ and $\{r > r_0\} \setminus \Gamma_1$, respectively. Then, if $\overline{f}$ is exact symplectic, $\mu(A) = \mu(A_1)$.

Exercise 6 *Prove that the previous property is a characterization of exact symplectic maps.*

To illustrate the above definitions we consider some simple maps in the cylinder and the corresponding lifts in the plane.

Example 1: $f(\theta, r) = (\theta + \omega, r)$ for fixed $\omega \in]0, 2\pi[$.

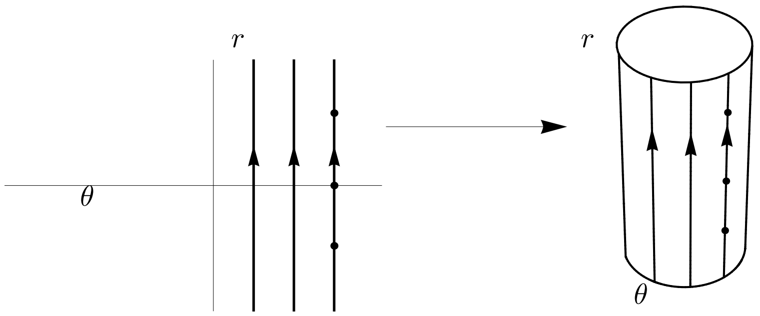
In the plane this map is a translation in the horizontal direction. It can be seen as the lift of a rotation.



From $r_1 d\theta_1 - r d\theta = r d(\theta + \omega) - r d\theta = 0$ we deduce that the condition (4) holds with $H \equiv 0$. Hence, rotations are exact symplectic maps.

Example 2: $f(\theta, r) = (\theta, r + \lambda)$ for fixed $\lambda \in \mathbb{R} \setminus \{0\}$.

This map can be interpreted as a vertical translation.



Now, $r_1 d\theta_1 - r d\theta = (r + \lambda) d\theta - r d\theta = \lambda d\theta$. In the plane the condition (4) holds with $H(\theta, r) = \lambda\theta$. The differential form $\lambda d\theta$ is not exact in the cylinder and so $\overline{f}$ is symplectic but not exact.

We finish this section with another characterization of exact symplectic maps. It is less standard but it is useful to suggest how to introduce a related notion in the plane.

Exercise 7 *Assume that* $\overline{f} \in \mathcal{E}^1(\overline{\Sigma})$ *is symplectic and* $H \in C^1(\Sigma)$ *is such that* $dH = r_1 d\theta_1 - r d\theta$. *Prove that the three conditions below are equivalent:*

(i) $\overline{f}$ *is exact symplectic*

(ii) H *is* 2π*-periodic in* θ

(iii) H *is bounded on each strip* $\mathbb{R} \times [A, B]$ *with* $a < A < B < b$.

Let us consider now a general map $f \in \mathcal{E}^1(\Sigma)$, possibly not 2π-periodic in θ. We say that f is E*-symplectic* if there exists a function $H \in C^1(\Sigma)$ satisfying

$$dH = r_1 d\theta_1 - r d\theta$$

and

$$\sup \{|H(\theta, r)| : \theta \in \mathbb{R}, \ A \le r \le B\} < \infty$$

for each A, B with $a < A < B < b$.

In the introduction we already presented some examples of E-symplectic maps. As a further example consider the map

$$\theta_1 = \theta + r, \quad r_1 = r + a + b \sin(\theta + r) + c \sin \sqrt{2}(\theta + r),$$

with $a, b, c \in \mathbb{R}$. In this case $G(\theta, r)$ is not 2π-periodic, if b and c do not vanish, but in the plane it satisfies (4) taking

$$H(\theta, r) = \frac{1}{2} r^2 + a(\theta + r) - b \cos(\theta + r) - \frac{c}{\sqrt{2}} \cos \sqrt{2}(\theta + r).$$

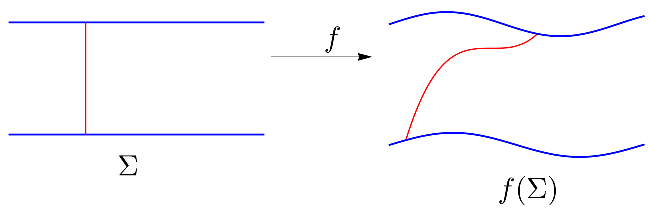
Therefore f is E-symplectic when $a = 0$. Recall that θ is an unbounded variable.

3 The twist condition and the generating function

A map $f = (F, G) \in \mathcal{E}^1(\Sigma)$ has *twist* if

$$\frac{\partial F}{\partial r}(\theta, r) > 0 \quad \forall (\theta, r) \in \Sigma. \tag{6}$$

Geometrically this means that vertical segments are twisted to the right.



From an analytic point of view the condition (6) is employed to solve the implicit function problem

$$\theta_1 = F(\theta, r). \tag{7}$$

In this way a function $r = R(\theta, \theta_1)$ is obtained. It is defined on the region

$$\Omega = \{(\theta, \theta_1) \in \mathbb{R}^2 : \; F(\theta, a) < \theta_1 < F(\theta, b)\}$$

where

$$F(\theta, a) = \lim_{r \downarrow a} F(\theta, r), \quad F(\theta, b) = \lim_{r \uparrow b} F(\theta, r).$$

Notice that

$$-\infty \leq F(\theta, a) < F(\theta, b) \leq +\infty \;\text{ for each } \theta \in \mathbb{R}.$$

Exercise 8 *Prove that Ω is open and connected. Hint:* $\Omega = \bigcup \Omega_\epsilon$, $\Omega_\epsilon = \{(\theta, \theta_1) \in \mathbb{R}^2 : \; F(\theta, a + \epsilon) < \theta_1 < F(\theta, b - \epsilon)\}$.

The function R is in $C^1(\Omega)$ and, by implicit differentiation,

$$F_\theta \circ \mathcal{R} + (F_r \circ \mathcal{R}) R_\theta = 0, \quad (F_r \circ \mathcal{R}) R_{\theta_1} = 1, \tag{8}$$

where $\mathcal{R}(\theta, \theta_1) = (\theta, R(\theta, \theta_1))$.

Assuming that there exists a function $H \in C^1(\Sigma)$ such that $dH = r_1 d\theta_1 - r d\theta$, the *generating function* of f is defined as

$$h(\theta, \theta_1) = -H(\theta, R(\theta, \theta_1)), \quad (\theta, \theta_1) \in \Omega.$$

Combining the identities (5), (8), and differentiating $h = -H \circ \mathcal{R}$ we obtain

$$\frac{\partial h}{\partial \theta}(\theta, \theta_1) = R(\theta, \theta_1), \;\; \frac{\partial h}{\partial \theta_1}(\theta, \theta_1) = -G(\theta, R(\theta, \theta_1)). \tag{9}$$

In a less formal language we can say that the map f given by $\theta_1 = F(\theta, r)$, $r_1 = G(\theta, r)$, is now expressed as

$$\frac{\partial h}{\partial \theta}(\theta, \theta_1) = r, \quad \frac{\partial h}{\partial \theta_1}(\theta, \theta_1) = -r_1.$$

This formula says that the map, originally defined in terms of two functions F and G, can be given in terms of a single function only, the generating function. This is reminiscent of the role played by the Hamiltonian function in the theory of Hamiltonian systems. The above formulas also have a consequence for the regularity of the generating function, because (9) implies that h is in $C^2(\Omega)$. Moreover,

$$\frac{\partial^2 h}{\partial\theta\partial\theta_1} > 0 \quad \text{in } \Omega.$$

This follows from the twist condition together with (9) and (8), since

$$h_{\theta\theta_1} = R_{\theta_1} = 1/(F_r \circ \mathcal{R}) > 0.$$

Assume now that $\overline{f} \in \mathcal{E}^1(\overline{\Sigma})$ is a map in the cylinder whose coordinates satisfy

$$F(\theta + 2\pi, r) = F(\theta, r) + 2\pi, \;\; G(\theta + 2\pi, r) = G(\theta, r).$$

When the lift is symplectic and has twist, the domain Ω and the function R are invariant under the translation

$$T(\theta, \theta_1) = (\theta + 2\pi, \theta_1 + 2\pi).$$

This means that $T(\Omega) = \Omega$ and $R \circ T = R$. The second identity is a consequence of the uniqueness of solution for the implicit function problem (7) and the generalized periodicity of F. The generating function is not always invariant under T. Indeed the identity $h \circ T = h$ is equivalent to

$$H(\theta + 2\pi, R(\theta, \theta_1)) = H(\theta, R(\theta, \theta_1)),$$

as can be deduced from the definition of h and the periodicity of R. For fixed θ the function $\theta_1 \mapsto R(\theta, \theta_1)$ maps the interval $]F(\theta, a), F(\theta, b)[$ onto $]a, b[$. Hence the above identity, valid for all θ and θ_1, is equivalent to

$$H(\theta + 2\pi, r) = H(\theta, r) \;\; \forall (\theta, r) \in \mathbb{R}\times]a, b[.$$

This is just the periodicity of H with respect to θ and, from the definition of h, we deduce that

$$h(\theta + 2\pi, \theta_1 + 2\pi) = h(\theta, \theta_1), \;\; (\theta, \theta_1) \in \Omega, \tag{10}$$

holds whenever $\overline{f}$ is exact symplectic.

To illustrate the previous notions let us go back to the example at the end of the last section. We consider the symplectic map in $\Sigma = \mathbb{R}^2$ given by

$$\theta_1 = \theta + r, \;\; r_1 = r + a + b\sin(\theta + r) + c\sin\sqrt{2}(\theta + r).$$

Since $\frac{\partial\theta_1}{\partial r} = 1$ the map has twist. Moreover $\Omega = \mathbb{R}^2$ and $R(\theta, \theta_1) = \theta_1 - \theta$. The generating function is

$$h(\theta, \theta_1) = -\frac{1}{2}(\theta_1 - \theta)^2 - a\theta_1 + b\cos\theta_1 + \frac{c}{\sqrt{2}}\cos\sqrt{2}\theta_1.$$

When $c = 0$ this map can be defined on the cylinder, and h satisfies the periodicity condition (10) in the case where $c = 0$ and $a = 0$.

Exercise 9 *Assume that $h \in C^2(\mathbb{R}^2)$, $h = h(\theta, \theta_1)$, is a function satisfying $\frac{\partial^2 h}{\partial\theta\partial\theta_1} > 0$ and, for some numbers a, b,*

$$\inf_{\theta_1\in\mathbb{R}} \frac{\partial h}{\partial\theta}(\theta, \theta_1) \leq a < b \leq \sup_{\theta_1\in\mathbb{R}} \frac{\partial h}{\partial\theta}(\theta, \theta_1)$$

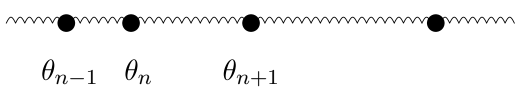
for each $\theta \in \mathbb{R}$. Then there exists a twist symplectic map $f \in \mathcal{E}^1(\Sigma)$, $\Sigma = \mathbb{R}\times]a, b[$, such that h is its generating function. Here it is understood that h is restricted to an appropriate domain.

4 The variational principle

We will construct a functional such that its critical points are in correspondence with the orbits generated by symplectic twist maps. First we present a concrete example, arising in solid state physics.

4.1 The Frenkel-Kontorowa model

Let us imagine an infinite chain of atoms placed on a line, the positions of the atoms being described by bi-infinite sequences $(\theta_n)_{n\in\mathbb{Z}}$. It is assumed that every atom n is attracted by its neighbors $n-1$ and $n+1$, according to Hooke's law (with constant C). In addition there is a force derived from a potential $V = V(\theta)$ acting on the real line.



To find the equilibrium positions of the chain it is enough to impose that the sum of forces acting on each atom vanishes. That is,

$$C(\theta_{n-1} - \theta_n) + C(\theta_{n+1} - \theta_n) - V'(\theta_n) = 0.$$

We arrive at the second order difference equation

$$\theta_{n+1} + \theta_{n-1} - 2\theta_n = \frac{1}{C} V'(\theta_n), \quad n \in \mathbb{Z}, \tag{11}$$

which can be seen as a discrete counterpart of the Newtonian equation $\ddot{\theta} = \frac{1}{C} V'(\theta)$.

Alternatively we can look for critical points of the potential energy

$$\Phi((\theta_n)_{n\in\mathbb{Z}}) = \sum_{n\in\mathbb{Z}} \Big[\frac{C}{2}(\theta_{n+1} - \theta_n)^2 + V(\theta_n)\Big].$$

It is straightforward to check that the system of conditions $\frac{\partial\Phi}{\partial\theta_n} = 0$ leads to (11). Of course this computation is purely formal since typically the series defining Φ will be divergent. One way to proceed rigorously is to consider finite strings $(\theta_n)_{|n|\leq N}$ and to assume that the end points are fixed and known, say $\theta_{-N} = A_{-N}$ and $\theta_N = A_N$. Then we can consider the truncated potential energy

$$\Phi_N((\theta_n)_{n<|N|}) = \sum_{-N\leq n<N} \Big[\frac{C}{2}(\theta_{n+1} - \theta_n)^2 + V(\theta_n)\Big].$$

The critical points of Φ_N satisfy (11) for $|n| < N$.

Let us now assume that the potential V is in $C^2(\mathbb{R})$ and let us interpret the function

$$h(\theta, \theta_1) = -\frac{C}{2}(\theta_1 - \theta)^2 - V(\theta)$$

as the generating function of a symplectic twist map. This makes sense since $h_{\theta\theta_1} = C > 0$ and so Exercise 9 is applicable. The associated map is defined by

$$r = \frac{\partial h}{\partial\theta}(\theta, \theta_1), \quad r_1 = -\frac{\partial h}{\partial\theta_1}(\theta, \theta_1),$$

or

$$f: \quad \theta_1 = \theta + \frac{1}{C} r + \frac{1}{C} V'(\theta), \quad r_1 = r + V'(\theta).$$

The previous discussion leads to an interesting conclusion: given a "critical point" $(\theta^*_n)_{n\in\mathbb{Z}}$ of Φ, the sequence $(\theta^*_n, r^*_n)_{n\in\mathbb{Z}}$ with $r^*_n = C(\theta^*_{n+1} - \theta^*_n) + V'(\theta^*_n)$ is an f-orbit. The process can be also reversed.

Exercise 10 *Prove that the map f defined above is E-symplectic in $\mathbb{R}^2$ when $\|V\|_\infty + \|V'\|_\infty < \infty$. Under what conditions is there an induced exact symplectic map $\overline{f}$ in the cylinder?*

Exercise 11 *Prove that f is conjugate to the "standard map" $\theta_1 = \theta + \frac{1}{C} r$, $r_1 = r + V'(\theta_1)$. Hint: $r \mapsto r + V'(\theta)$.*

4.2 A general framework

Assume now that $f \in \mathcal{E}^1(\Sigma)$ is a twist symplectic map and let $h = h(\theta, \theta_1)$ denote its generating function. Given $N \geq 1$ consider the function

$$\Phi_N((\theta_n)_{|n|\leq N}) = \sum_{-N\leq n<N} h(\theta_n, \theta_{n+1}),$$

where $\theta_{\pm N} = A_{\pm N}$ are fixed numbers. This function is of class C^3 on the domain

$$\Omega_N = \{(\theta_n)_{|n|<N} :\ (\theta_n, \theta_{n+1}) \in \Omega,\ -N \leq n < N\}.$$

Exercise 12 *Prove that Ω_N is an open and connected subset of $\mathbb{R}^{2N-1}$.*

Critical points of Φ_N are solutions of

$$\partial_1 h(\theta_n, \theta_{n+1}) + \partial_2 h(\theta_{n-1}, \theta_n) = 0,\ |n| < N,\ \theta_{\pm N} = A_{\pm N}. \tag{12}$$

The sequence $(\theta_n)_{|n|\leq N}$ obtained in this way leads to a segment of f-orbit with the definitions $r_N = \partial_2 h(\theta_{N-1}, \theta_N)$ and $r_n = \partial_1 h(\theta_n, \theta_{n+1})$ if $-N \leq n < N$. Actually, from the definition of the function $R(\theta, \theta_1)$ and (9) we deduce that $a < r_n < b$ if $-N \leq n < N$. This inequality also holds for $n = N$, as follows from (12). Putting together (9), (12) and the definition of R it is easy to deduce that $f(\theta_n, r_n) = (\theta_{n+1}, r_{n+1})$ for $-N \leq n < N$.

The previous process can be reversed in order to obtain critical points of Φ_N from f-orbits. Our goal is to construct complete f-orbits with certain additional properties. To this end we will prove the existence of critical points of Φ_N and let $N \to \infty$. This is clarified by the following result, valid for the general second order difference equation

$$E(\theta_{n-1}, \theta_n, \theta_{n+1}) = 0 \tag{13}$$

where $E : S \to \mathbb{R}$ is a continuous function defined on

$$S = \{(\theta_{-1}, \theta_0, \theta_1) \in \mathbb{R}^3 :\ \delta \leq \theta_0 - \theta_{-1} \leq \Delta \text{ and } \delta \leq \theta_1 - \theta_0 \leq \Delta\}$$

for some $\Delta > \delta \geq 0$.

Lemma 13 *Assume that for $N \geq N^*$ there exists a finite sequence $(\theta_n^{[N]})_{|n|\leq N}$ satisfying (13) for $|n| < N$. Moreover, assume that*

$$\lim_{N\to+\infty} \theta_{\pm N}^{[N]} = \pm\infty.$$

Then there exists a complete solution of (13). This means that there is a sequence $(\theta_n)_{n\in\mathbb{Z}}$ satisfying (13) for all $n \in \mathbb{Z}$.

Proof. Let us consider the space of sequences $\mathbb{R}^{\mathbb{Z}}$ endowed with the product topology. We recall that this space is metrizable and the associated convergence is just the convergence of each coordinate. Inside $\mathbb{R}^{\mathbb{Z}}$ we consider the space

$$K_\infty = \{\Theta = (\theta_n)_{n\in\mathbb{Z}} : \ \delta \le \theta_{n+1} - \theta_n \le \Delta \ \forall n \in \mathbb{Z}, \ |\theta_0| \le \Delta\}.$$

This space is compact because it can be viewed as a closed subset of

$$\hat{K}_\infty = \{\Theta = (\theta_n)_{n\in\mathbb{Z}} : \ \theta_n \in [n\delta - \Delta, (n+1)\Delta] \ \forall n \in \mathbb{Z}\}$$

and $\hat{K}_\infty$ is compact by Tichonoff's Theorem on the product of compact spaces. We will look for a solution of (13) lying in K_∞.

For large N we can assume $\theta_N^{[N]} > 0 > \theta_{-N}^{[N]}$ and so there exists an integer $\nu = \nu(N)$, $-N \le \nu < N$, such that $\theta_\nu^{[N]} < 0 \le \theta_{\nu+1}^{[N]}$. Since $0 < \theta_{\nu+1}^{[N]} - \theta_\nu^{[N]} \le \Delta$ we conclude that

$$|\theta_\nu^{[N]}| \le \Delta. \tag{14}$$

Also we notice that

$$\lim_{N\to+\infty} [\pm N - \nu(N)] = \pm\infty. \tag{15}$$

For the sign $+$ this limit is justified using the estimates

$$\theta_N^{[N]} - \Delta \le \theta_N^{[N]} - \theta_\nu^{[N]} = \sum_{n=\nu}^{N-1} (\theta_{n+1}^{[N]} - \theta_n^{[N]}) \le (N - \nu)\Delta.$$

The case of the sign $-$ is treated similarly.

The equation (13) is autonomous and so the shifted sequence $\tilde{\theta}_n^{[N]} = \theta_{n-\nu}^{[N]}$, satisfies (13) for $|n - \nu| < N$. Next we complete the finite sequence $(\tilde{\theta}_n^{[N]})$ so that it becomes a point $\tilde{\Theta}^{[N]}$ of K_∞. A simple way to achieve this is to define

$$\tilde{\theta}_n^{[N]} = \omega(n - N - \nu) + \tilde{\theta}_N^{[N]}, \ \text{ if } n > N + \nu$$

and

$$\tilde{\theta}_n^{[N]} = \omega(n + N - \nu) + \tilde{\theta}_{-N}^{[N]}, \ \text{ if } n < -N + \nu,$$

where $\omega = \frac{1}{2}(\delta + \Delta)$. Using (14) it is easy to check that $\tilde{\Theta}^{[N]}$ is contained in K_∞. By compactness we can extract a convergent subsequence $(\tilde{\Theta}^{[\sigma(N)]})$, where $\sigma : \mathbb{N} \to \mathbb{N}$ is increasing. We claim that the limit $\Theta = (\theta_n)_{n\in\mathbb{Z}}$ is a complete solution (13). To check this assertion we observe that for fixed n and large N, $\tilde{\theta}_k^{[N]} = \theta_{k-\nu}^{[N]}$ if $|n-k| \leq 1$. This is a consequence of (15). Then

$$E(\tilde{\theta}_{n-1}^{[\sigma(N)]}, \tilde{\theta}_n^{[\sigma(N)]}, \tilde{\theta}_{n+1}^{[\sigma(N)]}) = 0, \quad N \text{ large},$$

and we can pass to the limit $(N \to +\infty)$ using the continuity of E.■

We will apply the previous lemma to the difference equation

$$\partial_1 h(\theta_n, \theta_{n+1}) + \partial_2 h(\theta_{n-1}, \theta_n) = 0, \quad \delta \leq \theta_{n+1} - \theta_n \leq \Delta,$$

where δ and Δ are positive numbers such that

$$F(\theta, a) < \delta < \Delta < F(\theta, b) \quad \forall \theta \in \mathbb{R}.$$

After finding solutions for $|n| < N$ as critical points of Φ_N, we will pass to the limit.

This section will be finished with a discussion of the nature of the critical points of Φ_N in the simplest instance. Consider the map $\theta_1 = \theta + r$, $r_1 = r$. The generating function is

$$h(\theta, \theta_1) = -\frac{1}{2}(\theta_1 - \theta)^2.$$

After fixing $A_{\pm N} \in \mathbb{R}$ we observe that the function $-\Phi_N$ is coercive on $\mathbb{R}^{2N-1}$. To establish this claim, if $\Theta_N = (\theta_n)_{|n|<N}$ is a generic point in $\mathbb{R}^{2N-1}$ and n_0 is an integer such that $|n_0| < N$ and $|\theta_{n_0}| = \max_{|n|<N} |\theta_n| = \|\Theta_N\|_\infty$, then

$$-4\Phi_N(\Theta_N) \geq \ 2\textstyle\sum_{n_0 \leq n < N} (\theta_{n+1} - \theta_n)^2 \ \geq \left(\ \sum_{n_0 \leq n < N} |\theta_{n+1} - \theta_n| \ \right)^2.$$

The last term dominates $|\theta_{n_0} - A_N|^2$ and, since $|\theta_{n_0} - A_N| \geq \|\Theta_N\|_\infty - |A_N|$, we deduce that $\Phi_N(\Theta_N) \to -\infty$ as $\|\Theta_N\|_\infty \to \infty$. The conditions $\frac{\partial \Phi_N}{\partial \theta_n} = 0$ lead to the discrete Dirichlet problem

$$\begin{cases} \theta_{n+1} + \theta_{n-1} - 2\theta_n = 0, \\ \theta_{-N} = A_{-N}, \ \theta_N = A_N. \end{cases}$$

This problem has the unique solution $\theta_n^{[N]} = n\omega + c$ for $\omega = \frac{A_N - A_{-N}}{2N}$ and $c = \frac{1}{2}(A_N + A_{-N})$. As a consequence $\Theta_N^* = (\theta_n^{[N]})_{|n|<N}$ is the unique critical point of Φ_N and

$$\max \Phi_N = \Phi_N(\Theta_N^*).$$

Let us fix $\delta < \Delta$ and assume that $\delta < \frac{A_N - A_{-N}}{2N} < \Delta$. Then Θ_N^* is in the interior of the compact set

$$S_N = \{\Theta_N = (\theta_n)_{|n|<N} : \ \delta \leq \theta_{n+1} - \theta_n \leq \Delta, \text{ if } -N \leq n < N\}.$$

This observation will be relevant later.

5 Existence of complete orbits

In this section we fix two positive numbers $\Delta > \delta > 0$ and consider the strip

$$S = \{(\theta, \theta_1) \in \mathbb{R}^2 : \ \delta \leq \theta_1 - \theta \leq \Delta\}$$

and a given function $h = h(\theta, \theta_1)$ in $C^1(S)$. Our goal is to prove the existence of a complete orbit of the difference equation

$$\partial_1 h(\theta_n, \theta_{n+1}) + \partial_2 h(\theta_{n-1}, \theta_n) = 0, \ \ n \in \mathbb{Z}. \tag{16}$$

Notice that this setting implicitly implies that the complete solution satisfies $(\theta_n, \theta_{n+1}) \in S$ for each n. The prototype of function h will be $h_*(\theta, \theta_1) = -\alpha(\theta - \theta_1)^2$ with α a positive constant. We will impose two conditions that roughly say that h is close to h_* and the strip S is sufficiently wide, "width" being measured by the quotient Δ/δ.

Theorem 14 *Assume that $h \in C^1(S)$ and there are two positive numbers $\overline{\alpha}$, $\underline{\alpha}$ with $\overline{\alpha} < 2\underline{\alpha}$ and*

$$-\overline{\alpha}(\theta_1 - \theta)^2 \leq h(\theta_1, \theta) \leq -\underline{\alpha}(\theta_1 - \theta)^2 \quad \forall (\theta, \theta_1) \in S. \tag{17}$$

Then there exists a number $\sigma \geq 1$, depending only on the quotient $\overline{\alpha}/\underline{\alpha}$, such that if $\sigma^2 \delta \leq \Delta$ then the equation (16) has a complete solution.

As will be seen from the proof, the number σ can be computed explicitly. To obtain results of qualitative nature it is enough to interpret $\sigma = \sigma(q)$ as an increasing function depending on $q = \overline{\alpha}/\underline{\alpha} \in [1, 2[$. This is illustrated by the following consequence on the existence of equilibria for the Frenkel-Kontorowa model.

Corollary 15 *Assume that the potential V is bounded and of class C^1. Then the equation*

$$\theta_{n+1} + \theta_{n-1} - 2\theta_n = V'(\theta_n), \ \ n \in \mathbb{Z},$$

has infinitely many complete solutions $(\theta_{n,N})_{n \in \mathbb{Z}}$ with $N = 1, 2, \ldots$. Moreover, the upper and lower rotation numbers

$$\underline{\omega}_N := \liminf_{|n| \to \infty} \frac{\theta_{n,N}}{n} \leq \limsup_{|n| \to \infty} \frac{\theta_{n,N}}{n} =: \overline{\omega}_N$$

satisfy $\overline{\omega}_N < \infty$ for each N and $\underline{\omega}_N \to +\infty$ as $N \to +\infty$.

Proof. To prove the corollary we select the number $\sigma_0 = \sigma(3/2)$ corresponding to $\overline{\alpha}/\underline{\alpha} = 3/2$ and work on the region $S: \ \delta \leq \theta_1 - \theta \leq \Delta$, where $\delta > 0$ is a parameter to be adjusted and $\Delta = \sigma_0^2 \delta$. Our equation is just (16) for

$$h(\theta, \theta_1) = -\frac{1}{2}(\theta - \theta_1)^2 - V(\theta).$$

Moreover, if $(\theta, \theta_1) \in S$,

$$h(\theta, \theta_1) \leq -\frac{1}{2}(\theta - \theta_1)^2 + \|V\|_\infty \leq -\frac{1}{2}(\theta - \theta_1)^2 + \frac{\|V\|_\infty}{\delta^2}(\theta - \theta_1)^2.$$

A similar lower estimate can be obtained to see that the condition (17) holds for

$$\underline{\alpha} = \frac{1}{2} - \frac{\|V\|_\infty}{\delta^2}, \quad \overline{\alpha} = \frac{1}{2} + \frac{\|V\|_\infty}{\sigma_0^4 \delta^2}.$$

For large δ the inequality $\overline{\alpha}/\underline{\alpha} \leq 3/2$ holds and so the constant $\sigma = \sigma(\overline{\alpha}/\underline{\alpha})$ given by the theorem satisfies $\sigma \leq \sigma_0$. Then $\sigma^2\delta \leq \sigma_0^2 \delta = \Delta$ and the theorem is applicable. This shows the existence of an equilibrium for the Frenkel-Kontorowa model $(\theta_n^\delta)_{n\in\mathbb{Z}}$ with $\delta \leq \theta_{n+1} - \theta_n \leq \sigma^2\delta$, $n \in \mathbb{Z}$. Letting $\delta = N$ we obtain infinitely many equilibria, and the assertions on the rotation numbers are easily seen to be verified.$\blacksquare$

Later we will present other applications of the theorem or of some variant of it. In all cases h will be the generating function of a twist symplectic map f. Indeed the condition (17) automatically implies that f is E-symplectic.

Proof of theorem 14. For each $N \geq 3$ we select two numbers $A_{\pm N}$ satisfying

$$A_{-N} = -A_N, \quad N\delta \leq A_N \leq N\Delta, \tag{18}$$

and consider the following subset of $\mathbb{R}^{2N-1}$:

$$S_N = \{\Theta_N = (\theta_n)_{|n|\leq N} : \ \delta \leq \theta_{n+1} - \theta_n \leq \Delta \text{ for each } n = -N, \ldots, N-1\},$$

with the convention $\theta_{\pm N} = A_{\pm N}$. This set is non-empty since it contains at least the point $(\frac{n}{N} A_N)$. It is easily proved that S_N is closed and contained in the ball $\|\Theta_N\|_\infty \leq A_N$ and, since we are in finite dimension, we can deduce that this set is compact. The continuous function

$$\Phi_N : S_N \to \mathbb{R}, \quad \Phi_N(\Theta_N) = \sum_{-N\leq n<N} h(\theta_n, \theta_{n+1}),$$

attains its maximum at some point $\Theta_N^* \in S_N$,

$$\Phi_N(\Theta_N^*) = \max_{S_N} \Phi_N.$$

We will prove that, for an appropriate choice of the sequence A_N, the point Θ_N^* is in the interior of S_N. Hence this is a critical point of Φ_N that can be also interpreted as a solution of (12). Finally we can apply lemma 13 to complete the proof. From now on we will concentrate on the claim

$$\Theta_N^* \in \mathrm{int}(S_N). \tag{19}$$

To this end we make a couple of observations on the configuration of the atoms of Θ_N^*.

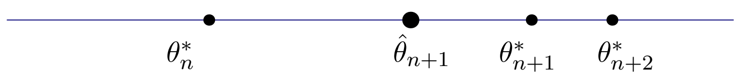
(i) *There exists $L > 1$ such that*

$$\frac{1}{L}(\theta_{n+1}^* - \theta_n^*) \leq \theta_{n+2}^* - \theta_{n+1}^* \leq L(\theta_{n+1}^* - \theta_n^*)$$

for each $n = -N, \ldots, N-2$. Moreover L only depends on the quotient $\overline{\alpha}/\underline{\alpha}$.

To prove this assertion we modify Θ_N^* by replacing θ_{n+1}^* with the mid-point between θ_n^* and θ_{n+2}^*; that is,

$$\hat{\Theta}_N = (\hat{\theta}_n)_{|n|\leq N}, \ \ \hat{\theta}_m = \theta_m^* \text{ if } m \neq n+1, \ \ \hat{\theta}_{n+1} = \frac{1}{2}(\theta_n^* + \theta_{n+2}^*).$$



The new point $\hat{\Theta}_N$ also belongs to S_N. Indeed

$$\hat{\theta}_{n+2} - \hat{\theta}_{n+1} = \hat{\theta}_{n+1} - \hat{\theta}_n = \frac{1}{2}(\theta_{n+2}^* - \theta_n^*) = \frac{1}{2}(\theta_{n+2}^* - \theta_{n+1}^*) + \frac{1}{2}(\theta_{n+1}^* - \theta_n^*)$$

and these differences remain between δ and Δ. The maximizing property of Θ_N^* implies that $\Phi_N(\Theta_N^*) \geq \Phi_N(\hat{\Theta}_N)$, leading to

$$h(\theta_n^*, \theta_{n+1}^*) + h(\theta_{n+1}^*, \theta_{n+2}^*) \geq h(\theta_n^*, \hat{\theta}_{n+1}) + h(\hat{\theta}_{n+1}, \theta_{n+2}^*).$$

As a consequence

$$-\underline{\alpha}[(\theta_{n+1}^* - \theta_n^*)^2 + (\theta_{n+2}^* - \theta_{n+1}^*)^2] \geq -\overline{\alpha}\,\frac{(\theta_{n+2}^* - \theta_n^*)^2}{2}.$$

Assume that $\ell := \frac{\theta_{n+2}^* - \theta_{n+1}^*}{\theta_{n+1}^* - \theta_n^*} \geq 1$, otherwise we would define ℓ as the inverse fraction. Using that $\theta_{n+2}^* - \theta_n^* = (1+\ell)(\theta_{n+1}^* - \theta_n^*)$ we are led to the inequality

$$\varphi(\ell) := \frac{2(1+\ell^2)}{(1+\ell)^2} \leq \overline{\alpha}/\underline{\alpha}.$$

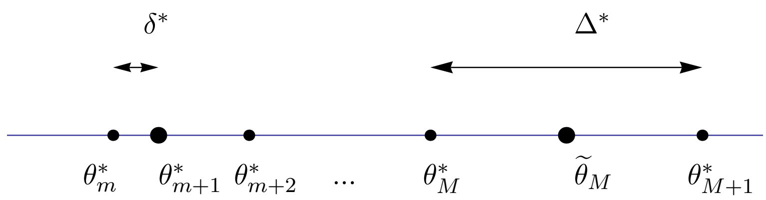
The function $\varphi : [1, +\infty[\to [1, 2[$ is an increasing homeomorphism and so $\ell \le \varphi^{-1}(\overline{\alpha}/\underline{\alpha})$. This implies that (i) holds with $L = \varphi^{-1}(\overline{\alpha}/\underline{\alpha})$. Notice that at this point we are using $\overline{\alpha} < 2\underline{\alpha}$.

(ii) *There exists $\sigma > 1$ such that*

$$\frac{\Delta^*}{\delta^*} \le \sigma,$$

where $\Delta^* = \max_{-N \le n < N}(\theta^*_{n+1} - \theta^*_n)$ and $\delta^* = \min_{-N \le n < N}(\theta^*_{n+1} - \theta^*_n)$. Moreover σ only depends on the quotient $\overline{\alpha}/\underline{\alpha}$.

Let us assume that $\Delta^* = \theta^*_{M+1} - \theta^*_M$ and $\delta^* = \theta^*_{m+1} - \theta^*_m$ with $m, M \in \{-N, \ldots, N-1\}$. If $|m-M| \le 1$ we can apply the previous step and deduce that $\Delta^*/\delta^* \le L$. From now on we assume that $|m-M| \ge 2$, say $M \ge m+2$.



We modify Θ^*_N in a new way: After eliminating θ^*_{m+1} a new atom is inserted between θ^*_M and θ^*_{M+1}. Let $\tilde{\Theta}_N = (\tilde{\theta}_n)_{|n|<N}$ be defined as $\tilde{\theta}_n = \theta^*_n$ if $n \le m$ or $n > M$, $\tilde{\theta}_n = \theta^*_{n+1}$ if $m < n < M$ and $\tilde{\theta}_M = \frac{1}{2}(\theta^*_M + \theta^*_{M+1})$. We prove that $\tilde{\Theta}_N \in S_N$ as soon as $\Delta^*/\delta^* \ge L+1$, where L is given by step (i). Actually,

$$\tilde{\theta}_{m+1} - \tilde{\theta}_m = \theta^*_{m+2} - \theta^*_m \le (L+1)\delta^* \le \Delta^* \le \Delta$$

and

$$\tilde{\theta}_{M+1} - \tilde{\theta}_M = \tilde{\theta}_M - \tilde{\theta}_{M-1} = \frac{\Delta^*}{2} \ge \frac{L+1}{2}\delta^* \ge \delta^* \ge \delta,$$

so that $\tilde{\Theta}_N \in S_N$. Then from $\Phi_N(\Theta^*_N) \ge \Phi_N(\tilde{\Theta}_N)$ we deduce that

$$h(\theta^*_m, \theta^*_{m+1}) + h(\theta^*_{m+1}, \theta^*_{m+2}) + h(\theta^*_M, \theta^*_{M+1})$$

$$\ge h(\theta^*_m, \theta^*_{m+2}) + h(\theta^*_M, \tilde{\theta}_M) + h(\tilde{\theta}_M, \theta^*_{M+1}).$$

Hence

$$-\underline{\alpha}[(\theta^*_m - \theta^*_{m+1})^2 + (\theta^*_{m+1} - \theta^*_{m+2})^2 + (\theta^*_M - \theta^*_{M+1})^2]$$
$$\ge -\overline{\alpha}[(\theta^*_m - \theta^*_{m+2})^2 + 2(\tilde{\theta}_M - \theta^*_M)^2]$$

and, using again (i), we are led to $\psi_L(\Delta^*/\delta^*) \leq \overline{\alpha}/\underline{\alpha}$, where the function ψ_L is defined as

$$\psi_L(q) = \frac{1 + L^{-2} + q^2}{(1+L)^2 + \frac{1}{2}q^2}.$$

This function is strictly increasing on the interval $[1 + L, \infty[$ and satisfies $\psi_L(1+L) < 1$ as well as $\psi_L(\infty) = 2$. Hence the inequality $\psi_L(\Delta^*/\delta^*) \leq \overline{\alpha}/\underline{\alpha}$ is equivalent to $\Delta^*/\delta^* \leq \psi_L^{-1}(\overline{\alpha}/\underline{\alpha})$, i.e., we have proved (ii) taking

$$\sigma = \max\{1 + L, \psi_L^{-1}(\overline{\alpha}/\underline{\alpha})\}.$$

Now that we have shown (ii) we can complete the proof of the theorem. Define

$$A_N = \frac{1}{2}(\sigma^{-1}\Delta + \sigma\delta)N.$$

From the assumption $\sigma^2\delta < \Delta$ we observe that $\delta < \sigma\delta < \sigma^{-1}\Delta < \Delta$, and A_N is the mid point of the interval $[\sigma\delta, \sigma^{-1}\Delta]$. This implies that (18) holds. We are going to prove that for this choice of the sequence $\{A_N\}$ the claim (19) holds. By contradiction assume that $\Delta^* = \Delta$ or $\delta^* = \delta$. Then either $\delta^* \geq \frac{1}{\sigma}\Delta$ or $\Delta^* \leq \sigma\delta$. To fix ideas let us consider the first case $\Delta^* = \Delta$, $\delta^* \geq \frac{1}{\sigma}\Delta$. Then

$$2A_N = \sum_{n=-N}^{N-1}(\theta^*_{n+1} - \theta^*_n) \geq \frac{2N\Delta}{\sigma},$$

and this contradicts the definition of A_N. The case $\delta^* = \delta$ is treated similarly.■

Exercise 16 *Show that the previous proof allows to compute* $\sigma = \sigma(\overline{\alpha}/\underline{\alpha})$ *explicitly. Hint:* $\sigma = \frac{4}{3}\sqrt{\frac{85}{3}}$ *for* $\overline{\alpha}/\underline{\alpha} = 5/4$.

Exercise 17 *Compute two numbers* δ *and* Δ *such that the equation*

$$\theta_{n+1} - 2\theta_n + \theta_{n-1} = \sin\theta_n + \cos(\sqrt{2}\theta_n), \quad n \in \mathbb{Z}$$

has a solution lying in $\delta \leq \theta_{n+1} - \theta_n \leq \Delta$.

Exercise 18 *Prove that the conclusion of theorem 14 still holds when the condition (17) is replaced by*

$$-\overline{\alpha}(\theta_1 - \theta)^k \leq h(\theta_1, \theta) \leq -\underline{\alpha}(\theta_1 - \theta)^k \quad \forall(\theta, \theta_1) \in S \tag{20}$$

with $k > 1$ *and* $\overline{\alpha} < 2^{k-1}\underline{\alpha}$. *Hint:* $\varphi(\ell) = \frac{2^{k-1}(1+\ell^k)}{(1+\ell)^k}$, $\psi_L(q) = \frac{1+L^{-k}+q^k}{(1+L)^k+2^{1-k}q^k}$.

Exercise 19 *Prove that the conclusion of theorem 14 also holds when the condition (17) is replaced by*

$$-\overline{\alpha}(\theta_1-\theta)^{-k} \leq h(\theta_1,\theta) \leq -\underline{\alpha}(\theta_1-\theta)^{-k} \quad \forall(\theta,\theta_1)\in S \tag{21}$$

with $k>0$ *and* $\overline{\alpha}<2^k\underline{\alpha}$. *Hint:* $\varphi(\ell)=\frac{(1+\ell^{-k})(1+\ell)^k}{2^{k+1}}$, $\psi_L(q)=\frac{1+L^{-k}+q^{-k}}{2^{-k}+2^{1+k}q^{-k}}$, $\theta^*_{m+2}-\theta^*_m\geq 2\delta^*$.

In the applications of theorem 14 or the variants given by the previous exercises we must know how to compute h or at least how to estimate it in order to verify (17), (20) or (21). The next two sections are devoted to the computation of generating functions in two interesting mechanical situations.

6 The action functional of a Newtonian equation

Consider the differential equation

$$\ddot{x}=-V_x(t,x),\quad t\in[0,1],\ x\in\mathbb{R}, \tag{22}$$

where the potential $V:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous and has two partial derivatives with respect to x, V_x and V_{xx}, which are also continuous functions of (t,x). It will be assumed that the Cauchy problem is globally well posed. This can be guaranteed if V_x has linear growth, that is

$$|V_x(t,x)|\leq A|x|+B,\quad (t,x)\in[0,1]\times\mathbb{R},$$

for some $A,B>0$. Given $x_0,v_0\in\mathbb{R}$, the solution satisfying $x(0)=x_0$, $\dot{x}(0)=v_0$, will be denoted by $x(t;x_0,v_0)$. If we interpret these initial conditions as coordinates, say $\theta=x_0$ and $r=v_0$, then we can define the Poincaré map

$$f:\mathbb{R}^2\to\mathbb{R}^2,\quad \theta_1=x(1;\theta,r),\quad r_1=\dot{x}(1;\theta,r).$$

The classical theorems on the Cauchy problem[1] imply that f is a C^1-diffeomorphism. Moreover f is symplectic. This can be justified using Liouville's theorem on Hamiltonian flows but we will prove it in a different way. Consider the Lagrangian function

$$L(t,\theta,r)=\frac{1}{2}\,\dot{x}(t;\theta,r)^2-V(t,x(t;\theta,r))$$

[1] Notice that no smoothness in t has been assumed.

and its time average

$$H(\theta, r) = \int_0^1 L(t, \theta, r)\, dt.$$

This function is of class C^1 with partial derivatives

$$H_\theta = \int_0^1 L_\theta\, dt = \int_0^1 \{\dot{x}\frac{\partial \dot{x}}{\partial \theta} - V_x \frac{\partial x}{\partial \theta}\}\, dt,$$

$$H_r = \int_0^1 L_r\, dt = \int_0^1 \{\dot{x}\frac{\partial \dot{x}}{\partial r} - V_x \frac{\partial x}{\partial r}\}\, dt.$$

Commuting ∂_t with ∂_θ and ∂_r and integrating by parts,

$$\int_0^1 \dot{x}\frac{\partial \dot{x}}{\partial \theta} dt = [\dot{x}\frac{\partial x}{\partial \theta}]_{t=0}^{t=1} - \int_0^1 \ddot{x}\frac{\partial x}{\partial \theta} dt, \quad \int_0^1 \dot{x}\frac{\partial \dot{x}}{\partial r} dt = [\dot{x}\frac{\partial x}{\partial r}]_{t=0}^{t=1} - \int_0^1 \ddot{x}\frac{\partial x}{\partial r} dt.$$

From (22) we conclude that $dH = r_1 d\theta_1 - r d\theta$ and so f is symplectic.

The map f will induce a map $\overline{f}$ on the cylinder if the potential satisfies

$$V(t, x + 2\pi) = V(t, x) + p(t), \quad (t, x) \in [0, 1] \times \mathbb{R}, \tag{23}$$

for some function $p : [0, 1] \to \mathbb{R}$. This condition of generalized periodicity implies that

$$x(t; \theta + 2\pi, r) = x(t; \theta, r) + 2\pi, \quad \dot{x}(t; \theta + 2\pi, r) = \dot{x}(t; \theta, r),$$

and letting $t = 1$, $f(\theta + 2\pi, r) = f(\theta, r) + (2\pi, 0)$. Hence $\overline{f}$ is symplectic.

Exercise 20 *Prove that the Poincaré map $\overline{f}$ associated to*

$$\ddot{x} + a \sin x = p(t)$$

is exact symplectic if and only if $\int_0^1 p(t)dt = 0$. Here $a > 0$ is a parameter and $p : [0, 1] \to \mathbb{R}$ is a given continuous function.

Exercise 21 *Assume that, instead of (23), the potential satisfies*

$$V(t, x) = B(t, x) + p(t)x$$

where $p : [0, 1] \to \mathbb{R}$ is continuous and B, B_x are bounded. Prove that the Poincaré map is E-symplectic if $\int_0^1 p(t)dt = 0$. Hint: The kinetic energy $T(t) = \frac{1}{2}\dot{x}(t)^2$ satisfies $|\dot{T}| \leq CT^{1/2}$ and $\int_0^1 p(t)x(t)dt = -\int_0^1 P(t)\dot{x}(t)dt$ for $P(t) = \int_0^t p(s)ds$.

Next we are going to discuss under what conditions the Poincaré map will satisfy the twist condition. The partial derivative $\frac{\partial F}{\partial r} = \frac{\partial \theta_1}{\partial r}$ can be expressed as

$$\frac{\partial \theta_1}{\partial r} = y(1)$$

where $y(t)$ is the solution of the variational equation

$$\ddot{y} + V_{xx}(t, x(t; \theta, r))y = 0$$

such that $y(0) = 0$, $\dot{y}(0) = 1$. The twist condition becomes $y(1) > 0$ and can be proved using Sturm comparison theory. Actually it holds when the potential satisfies

$$V_{xx}(t, x) < \pi^2, \quad (t, x) \in [0, 1] \times \mathbb{R}. \tag{24}$$

In this case our solution $y(t)$ must oscillate less than the solution of the comparison equation $\ddot{z} + \pi^2 z = 0$, which is $z(t) = \sin \pi t$. This implies that $y(t) > 0$ if $t \in]0, 1]$ and therefore the twist holds.

Another hypothesis implying the twist condition is

$$(2n\pi)^2 < V_{xx}(t, x) < ((2n + 1)\pi)^2, \quad (t, x) \in [0, 1] \times \mathbb{R}, \tag{25}$$

for some $n = 1, 2, \ldots$. Now the oscillations of $y(t)$ are between those of $z_-(t) = \sin(2n\pi t)$ and $z_+(t) = \sin((2n + 1)\pi t)$. Hence $y(t)$ has exactly $2n$ zeros on $]0, 1[$ and $y(1)$ is positive.

Exercise 22 *Find conditions on the parameters a and ω to guarantee that the Poincaré map associated to $\ddot{x} + \omega^2 x + a \sin x = p(t)$ is a twist symplectic map.*

Once we know that the twist condition holds, to compute the generating function we solve the equation

$$x(1; \theta, r) = \theta_1$$

and find r for given θ and θ_1. This is equivalent to finding $r = \dot{x}(0)$, where $x(t)$ is the solution of the Dirichlet problem

$$\ddot{x} = -V_x(t, x), \;\; x(0) = \theta, \; x(1) = \theta_1. \tag{26}$$

The conditions (24) or (25) are sufficient to guarantee that this problem has at most one solution. This is obviously a consequence of the twist condition. However these conditions are not sufficient for the existence of solution.

Exercise 23 *Find θ and θ_1 such that*

$$\ddot{x} + \pi^2 x - \arctan x = 0, \quad x(0) = \theta, \; x(1) = \theta_1,$$

has no solution. Hint: Multiply the equation by $\sin \pi t$ and integrate between $t = 0$ and $t = 1$.

A classical result in the theory of nonlinear boundary value problems says that (26) has a unique solution, if (24) or (25) are replaced by the corresponding stronger conditions

$$V_{xx}(t, x) \leq \Gamma < \pi^2, \quad (t, x) \in [0, 1] \times \mathbb{R}, \tag{27}$$

$$(2n\pi)^2 < \gamma \leq V_{xx}(t, x) \leq \Gamma < ((2n + 1)\pi)^2, \quad (t, x) \in [0, 1] \times \mathbb{R}, \tag{28}$$

where $n = 1, 2, \dots$ and γ and Γ are given constants. From now on we assume that (27) or (28) are satisfied and so the set Ω defined in section 3 is $\mathbb{R}^2$. The solution of (26) will be denoted by $\xi(t; \theta, \theta_1)$ and the discussions of section 3 on the regularity of the function $R = R(\theta, \theta_1)$ together with the standard theorems on differentiability with respect to initial conditions imply that ξ and $\dot{\xi}$ are of class C^1 in $[0, 1] \times \mathbb{R}^2$. Notice that

$$\xi(t; \theta, \theta_1) = x(t; \theta, R(\theta, \theta_1)) \;\text{ and }\; R(\theta, \theta_1) = \dot{\xi}(0; \theta, \theta_1).$$

The generating function $h(\theta, \theta_1) = -H(\theta, R(\theta, \theta_1))$ is well defined on the whole plane by

$$h(\theta, \theta_1) = -\int_0^1 \left[\frac{1}{2}\,\dot{\xi}(t; \theta, \theta_1)^2 - V(t, \xi(t; \theta, \theta_1))\right] dt.$$

The reader who is familiar with the classical theory of the Calculus of Variations will recognize this expression. Up to a sign, the generating function is the restriction of the action functional to fields of extremals defined by $\xi = \xi(t; \theta, \theta_1)$. More precisely, if we consider the Sobolev space $H^1(0, 1)$ and the functional

$$\mathcal{A} : H^1(0, 1) \to \mathbb{R}, \;\; \mathcal{A}[x] = \int_0^1 \left[\frac{1}{2}\,\dot{x}(t)^2 - V(t, x(t))\right] dt,$$

then

$$h(\theta, \theta_1) = -\mathcal{A}[\xi(\cdot; \theta, \theta_1)].$$

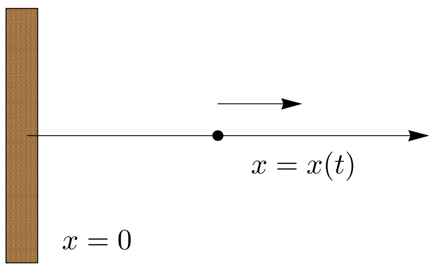
Exercise 24 *Compute the generating function associated to $\ddot{x} + \omega^2 x = 0$ with $2n\pi < \omega < 2(n + 1)\pi$ for some $n = 1, 2, \dots$.*

Finally we propose a more difficult exercise dealing with an application of theorem 14 to the framework of this section.

Exercise 25 *Prove that the equation* $\ddot{x} + a \sin x = p(t)$ *with* $0 < a < \pi^2$ *and* $p(t+1) = p(t)$, $\int_0^1 p(t)dt = 0$, *has a solution satisfying* $\delta \leq x(t+1) - x(t) \leq \Delta$ *for some* $\Delta > \delta > 0$.

7 Impact problems and generating functions

Let us consider a particle moving on the half-line $x = x(t) \geq 0$. It satisfies a Newtonian law for $x > 0$ but at the end point $x = 0$ there is an obstacle and the particle bounces elastically.



The function $x(t)$ is a solution of the impact problem

$$\begin{cases} \ddot{x} = -V_x(t, x), \quad t \in \mathbb{R}, \\ x(t) \geq 0, \\ x(\tau) = 0 \Rightarrow \dot{x}(\tau^+) = -\dot{x}(\tau^-), \end{cases} \tag{29}$$

where $V : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and has two partial derivatives in x, V_x and V_{xx}. Moreover it is assumed that both derivatives are continuous with respect to both variables (t, x). In short, $V \in C^{0,2}(\mathbb{R} \times \mathbb{R})$.

The exact meaning of the above impact problem is clarified by the following definition. A *bouncing solution* of (29) is a continuous function $x : \mathbb{R} \to [0, \infty[$ and a sequence of times $(t_n)_{n \in \mathbb{Z}}$ satisfying

(i) $\inf_{n\in\mathbb{Z}}(t_{n+1} - t_n) > 0$,

(ii) $x(t_n) = 0$ and $x(t) > 0$ for $t \in]t_n, t_{n+1}[$ and $n \in \mathbb{Z}$,

(iii) the restriction of $x(t)$ to $[t_n, t_{n+1}]$ is of class C^2 and satisfies the differential equation,

(iv) $\dot{x}(t_n^+) = -\dot{x}(t_n^-)$ for $n \in \mathbb{Z}$.

This solution will be *bounded* if furthermore

(v) $\sup_{t\in\mathbb{R}} |x(t)| + \operatorname{ess\,sup}_{t\in\mathbb{R}} |\dot{x}(t)| < \infty$,

(vi) $\sup_{n\in\mathbb{Z}}(t_{n+1} - t_n) < \infty$.

Notice that $\dot{x}(t)$ is well defined for $t \neq t_n$ and so the essential supremum makes sense.

Exercise 26 *Compute the bouncing solutions for a linear spring with obstacle, $V(t,x) = \frac{1}{2}x^2$. Prove that all of them are bounded.*

We will present a method for the construction of bouncing solutions. The first step will be the study of a boundary value problem.

7.1 The Dirichlet problem

Let us consider the problem

$$\ddot{x} = -V_x(t,x), \quad x(t_0) = x(t_1) = 0. \tag{30}$$

From now on we will assume that the potential satisfies two additional conditions:

(**C1**) $V_{xx}(t,x) \leq 0$ for each $(t,x) \in \mathbb{R}^2$.

(**C2**) There exist two numbers $c_1, c_2 \in \mathbb{R}$ and two functions $\psi, \phi \in C^2(\mathbb{R})$ such that

$$\ddot{\psi}(t) + c_1 \leq V_x(t,x) \leq \ddot{\phi}(t) + c_2 \text{ for each } (t,x) \in \mathbb{R}^2.$$

Moreover, $\sup_{t\in\mathbb{R}} |\dot{\psi}(t)| < \infty$.

These assumptions have strong consequences for the problem (30). First of all we present a result showing that there is a unique solution.

Lemma 27 *Assume that* (**C1**) *and* (**C2**) *hold. Then problem (30) has a unique solution on the interval $[t_0, t_1]$ for each $t_1 - t_0 > 0$.*

Proof. The existence will be obtained via the method of upper and lower solutions. Let $\alpha(t)$ and $\beta(t)$ be the solutions of the linear problems

$$\ddot{\alpha} = -\ddot{\phi}(t) - c_2,\ \alpha(t_0) = \alpha(t_1) = 0 \ \text{ and } \ \ddot{\beta} = -c_1 - \ddot{\psi}(t),\ \beta(t_0) = \beta(t_1) = 0.$$

From (**C2**) we deduce that $-\ddot{\beta} \leq -\ddot{\alpha}$ and so, by the Maximum Principle, $\alpha(t) \geq \beta(t)$ everywhere. Moreover, using (**C2**),

$$-\ddot{\alpha}(t) = c_2 + \ddot{\phi}(t) \geq V_x(t, \alpha(t)),\ -\ddot{\beta}(t) = c_1 + \ddot{\psi}(t) \leq V_x(t, \beta(t)).$$

This shows that $\alpha(t)$ and $\beta(t)$ is a couple of ordered upper and lower solutions. Therefore the problem (30) has a solution lying in $\alpha \geq x \geq \beta$.

For the uniqueness we assume that $x_1(t)$ and $x_2(t)$ are two solutions of (30). We notice that the difference $y(t) = x_1(t) - x_2(t)$ satisfies the linear problem

$$\ddot{y} + \alpha(t)y = 0,\ \ y(t_0) = y(t_1) = 0, \tag{31}$$

where $\alpha(t) = \int_0^1 V_{xx}(t, \lambda x_1(t) + (1-\lambda)x_2(t))\, d\lambda$. The condition (**C1**) implies that $\alpha \leq 0$ everywhere. By Sturm comparison theory we deduce that (31) is disconjugate and so we arrive at a contradiction unless y vanishes identically and $x_1 = x_2$. Instead of using comparison techniques we can also prove the uniqueness just by multiplying the equation with y and using integration by parts. ■

Exercise 28 *Prove the Maximum Principle used above: Let $y(t)$ be the solution of $-\ddot{y} = p(t)$, $y(t_0) = y(t_1) = 0$ with $p \in C[t_0, t_1]$. If $p(t) \geq 0$ and $\int_{t_0}^{t_1} p(t)dt > 0$ then $y(t) > 0$ for $t \in]t_0, t_1[$.*

To guarantee the positivity of the solution of (30) it is enough to know that the lower solution $\beta(t)$ is positive. This will be the case provided that $t_1 - t_0$ is large enough. To check this fact it is convenient to employ the explicit formula for β given by

$$\beta(t) = \frac{c_1}{2}(t_1 - t)(t - t_0) + \frac{\psi(t_1) - \psi(t_0)}{t_1 - t_0}(t - t_0) + \psi(t_0) - \psi(t).$$

Exercise 29 *Prove that $\beta(t) > 0$ for each $t \in]t_0, t_1[$ if $t_1 - t_0 > \frac{8}{c}\|\dot{\psi}\|_\infty$. Hint: First study the interval $]t_0, \frac{t_0+t_1}{2}[$.*

To complete our study of the Dirichlet problem we present a result on differentiability with respect to the end points. The unique solution of (30) will be denoted by $x_D(t; t_0, t_1)$.

Lemma 30 *The map* $(t;t_0,t_1)\in D\mapsto(x_D(t;t_0,t_1),\dot{x}_D(t;t_0,t_1))\in\mathbb{R}^2$ *is of class* C^1, *where* $D=\{(t;t_0,t_1)\in\mathbb{R}^3:\ t_1-t_0>0,\ t_0\le t\le t_1\}$.

Proof. Let $x(t;t_0,x_0,v_0)$ be the solution of

$$\ddot{x}=-V_x(t,x),\ \ x(t_0)=x_0,\ \dot{x}(t_0)=v_0.$$

Since $\ddot{\psi}(t)+c_1\le V_x(t,x)\le\ddot{\phi}(t)+c_2$, this solution is well defined and smooth for $t\in]-\infty,+\infty[$ and $(t_0,x_0,v_0)\in\mathbb{R}^3$. Let us consider the equation in v_0,

$$x(t_1;t_0,0,v_0)=0.$$

It is equivalent to solving (30) and so we know that it has a unique solution $v_0=v_0(t_0,t_1)$. The Implicit Function Theorem will imply that $v_0(t_0,t_1)$ is of class C^1 if we prove that

$$\frac{\partial x}{\partial v_0}(t_1;t_0,0,v_0)>0\ \text{ for } t_1>t_0 \text{ and } v_0\in\mathbb{R}.$$

The function $y(t)=\frac{\partial x}{\partial v_0}(t;t_0,0,v_0)$ is a solution of the initial value problem

$$\ddot{y}+V_{xx}(t,x(t;t_0,0,v_0))y=0,\ \ y(t_0)=0,\ \dot{y}(t_0)=1.$$

From (**C1**) we deduce that this linear equation is disconjugate and so $y(t_1)$ has to be positive.■

7.2 The condition of elastic bouncing

A naive approach for the construction of bouncing solutions could consist in juxtaposing solutions of Dirichlet problems for prescribed sequences of impact times. Given a sequence $(t_n)_{n\in\mathbb{Z}}$, the function

$$x(t):=x_D(t;t_n,t_{n+1})\ \text{ for } t\in[t_n,t_{n+1}],\ n\in\mathbb{Z},\tag{32}$$

would be the candidate for a bouncing solution. Indeed, if we assume that the sequence satisfies

$$t_{n+1}-t_n>\frac{8}{c_1}\|\dot{\psi}\|_\infty,\ \ n\in\mathbb{Z},\tag{33}$$

then the conditions $(i),(ii)$, and (iii) of the definition are satisfied. Here we are using the previous discussions, in particular exercise 29. In most cases this procedure does not lead to a bouncing solution because the elasticity condition given by (iv) does not necessarily hold. Next we present a method for the construction of a judicious sequence of impacts.

Consider the function

$$h(t_0,t_1)=\int_{t_0}^{t_1} L(t,x_D(t;t_0,t_1),\dot{x}_D(t;t_0,t_1))\,dt, \tag{34}$$

where L is the Lagrangian function associated to $\ddot{x}=-V_x$. More preceisely, here

$$L(t,x,\dot{x})=\frac{1}{2}\,\dot{x}^2-V(t,x)+V(t,0).$$

We recall that the Newtonian equation can be expressed in the Lagrangian framework as

$$\partial_x L-\frac{d}{dt}(\partial_{\dot{x}}L)=0. \tag{35}$$

The function h is of class C^1 in the region $\{(t_0,t_1)\in\mathbb{R}^2:\ t_1-t_0>0\}$. This is a consequence of lemma 30. An integration by parts leads to

$$\begin{aligned}
&\partial_{t_0}h(t_0,t_1)\\
&= -L(t_0,x_D(t_0;t_0,t_1),\dot{x}_D(t_0;t_0,t_1))+\int_{t_0}^{t_1}\{(\partial_x L)\frac{\partial x_D}{\partial t_0}+(\partial_{\dot{x}}L)\frac{\partial \dot{x}_D}{\partial t_0}\}\,dt\\
&= -\frac{1}{2}\dot{x}_D(t_0;t_0,t_1)^2+[(\partial_{\dot{x}}L)\frac{\partial x_D}{\partial t_0}]_{t=t_0}^{t=t_1}+\int_{t_0}^{t_1}[(\partial_x L)-\frac{d}{dt}(\partial_{\dot{x}}L)]\frac{\partial x_D}{\partial t_0}\,dt.
\end{aligned}$$

From $x_D(t_0;t_0,t_1)=x_D(t_1;t_0,t_1)=0$ we deduce that

$$\dot{x}_D(t_0;t_0,t_1)+\frac{\partial x_D}{\partial t_0}(t_0;t_0,t_1)=\frac{\partial x_D}{\partial t_0}(t_1;t_0,t_1)=0.$$

These identities together with (35) imply that

$$\partial_{t_0}h(t_0,t_1)=\frac{1}{2}\,\dot{x}_D(t_0;t_0,t_1)^2.$$

After differentiating with respect to t_1 we arrive at

$$\partial_{t_1}h(t_0,t_1)=-\frac{1}{2}\,\dot{x}_D(t_1;t_0,t_1)^2.$$

Assume now that (t_n) is a sequence solving

$$\partial_{t_0}h(t_n,t_{n+1})+\partial_{t_1}h(t_{n-1},t_n)=0,\ \ n\in\mathbb{Z}. \tag{36}$$

If the condition (33) holds, then the function defined by (32) is non-negative and it satisfies $\dot{x}(t_n^+)^2=\dot{x}(t_n^-)^2$. Then $\dot{x}(t_n^-)\le 0\le \dot{x}(t_n^+)$ and so the condition (iv) holds and $x(t)$ becomes a bouncing solution.

Exercise 31 *Assume that $(t_n)_{n\in\mathbb{Z}}$ satisfies (33), (36) and $\sup(t_{n+1}-t_n)<\infty$. Moreover, let $\sup_{t\in\mathbb{R}}|\dot{\phi}(t)|<\infty$. Prove that $x(t)$ is bounded.*

7.3 A bouncing ball

Let us apply the previous discussions to a concrete model. Assume that a horizontal plate (the racket) is moving according to some prescribed protocol and a particle (the ball) is in free fall until hitting the plate, when it bounces elastically. In more analytic terms assume that the unknown $z = z(t)$ is the vertical position of the particle and the given function $w(t)$ is the position of the plate. For $z > w(t)$ the free fall is modelled by $\ddot{z} = -g$, where $g > 0$ is the gravitational constant. The elastic impact is easily modelled through the relative position $x(t) = z(t) - w(t)$,

$$x(\tau) = 0 \Rightarrow \dot{x}(\tau^+) = -\dot{x}(\tau^-).$$

Assuming that $w(t)$ is of class C^2 we find that $x(t)$ is a solution of the impact problem (29) with

$$V(t,x) = (g + \ddot{w}(t))x. \tag{37}$$

From now on we assume that the position and velocity of the plate are bounded; that is,

$$w \in C^2(\mathbb{R}) \;\text{ and }\; \|w\|_\infty + \|\dot{w}\|_\infty < \infty. \tag{38}$$

This is sufficient to guarantee that (**C1**) and (**C2**) are satisfied for $c_1 = c_2 = g$ and $\phi = \psi = w$.

In this case the simplicity of the potential allows for explicit computations.

Exercise 32 *Determine $x_D(t; t_0, t_1)$ in terms of $w(t)$. Hint: $\beta(t)$.*

Exercise 33 *Use the previous exercise together with (34) to prove that the generating function is*

$$\begin{aligned} h(t_0,t_1) &= -\frac{g^2}{24}(t_1-t_0)^3 - \frac{g}{2}(w(t_1)+w(t_0))(t_1-t_0) \\ &\quad + \frac{(w(t_1)-w(t_0))^2}{2(t_1-t_0)} + g\int_{t_0}^{t_1} w(t)\,dt - \frac{1}{2}\int_{t_0}^{t_1} \dot{w}(t)^2\,dt. \end{aligned}$$

Hint: $\int_{t_0}^{t_1} \dot{x}_D^2(t)\,dt = -\int_{t_0}^{t_1} x_D(t)\ddot{x}_D(t)\,dt$.

We do not need this exact formula for h, since for our purposes it is sufficient to determine the dominant term as $t_1 - t_0 \to \infty$. From the above exercise,

$$h(t_0,t_1) = -\frac{g^2}{24}(t_1-t_0)^3 + R(t_0,t_1)$$

where

$$|R(t_0, t_1)| \leq C(t_1 - t_0) \text{ for } t_1 > t_0.$$

Here C is a constant depending only on $\|w\|_\infty + \|\dot{w}\|_\infty$.

We are going to apply exercise 18 with $k = 3$ and fixed numbers $\underline{\alpha}$, $\overline{\alpha}$ satisfying $\underline{\alpha} < \frac{g}{24} < \overline{\alpha}$ and $\overline{\alpha} < 4\underline{\alpha}$. Then there exists $d > 0$ such that

$$-\overline{\alpha}(t_1 - t_0)^3 \leq h(t_1 - t_0) \leq -\underline{\alpha}(t_1 - t_0)^3 \quad \text{for } t_1 - t_0 \geq d.$$

The number σ associated to $\underline{\alpha}$ and $\overline{\alpha}$ can be computed in order to find complete orbits of the difference equation (36) lying in $\delta \leq t_{n+1} - t_n \leq \sigma^2\delta$ if $\delta \geq d$. These sequences of impact times lead to bouncing solutions. Actually, the conditions (v) and (vi) are also satisfied and so these solutions are bounded. The condition (vi) is automatic from the construction of the sequence (t_n). To verify (v) we notice that $x(t)$ is a solution of the Dirichlet problem

$$\ddot{x} = -(g + \ddot{w}(t)), \quad x(t_n) = x(t_{n+1}) = 0$$

and, going back to exercise 32, we obtain a bound for $\|x\|_\infty + \|\dot{x}\|_\infty$ in terms of g, σ, δ and $\|w\|_\infty + \|\dot{w}\|_\infty$. Alternatively we could apply exercise 31.

We sum up the previous discussions.

Theorem 34 *Assume that $w(t)$ satisfies (38) and consider the impact problem (29) with potential given by (37). Then there exist positive constants $\sigma > 1$ and d such that for each $\delta \geq d$ there exists a bounded solution with impact times $(t_n^\delta)_{n\in\mathbb{Z}}$ satisfying*

$$\delta \leq t_{n+1}^\delta - t_n^\delta \leq \sigma^2\delta, \quad n \in \mathbb{Z}.$$

8 Comments and bibliographical remarks

1. Introduction. In [38] Zharnitsky replaced standard angles on $\mathbb{S}^1$ by angles on a torus $\mathbb{S}^1 \times \ldots \times \mathbb{S}^1$. These generalized angles were then employed to reformulate KAM theory for quasi-periodic maps. The paper [38] motivated us to consider non-periodic angles.

Standard versions of the Poincaré-Birkhoff and Aubry-Mather theorems concern diffeomorphisms mapping the cylinder onto itself. In our setting the image of the map T_ϵ is not necessarily contained in the region $a < r < b$. This is not a serious problem since one can first apply KAM theory in order to find invariant curves. Then the region between two of these invariant curves is mapped onto itself. This region can be symplectically deformed into a compact cylinder of the type $A \leq r \leq B$ where the standard theory applies. An alternative is the use of more sophisticated versions of P-B and A-M theorems. See for instance [10, 19, 31].

The map $T_{\epsilon,3}$ was presented as an example in [15]. In that paper there also some results on the existence of orbits with rotation number for certain maps with non-periodic angle.

The mechanical model described in the introduction is sometimes called Fermi-Ulam ping-pong. The problem of deciding whether the velocity can become unbounded is of physical significance in connection with the so-called Fermi acceleration. This is explained in a paper by Dolgopyat [9]. The relativistic version of the model is probably more significant in physics and has been considered, in the periodic case, by Pustyl'nikov, see [32].

2. Symplectic maps in the plane and in the cylinder. Given a homeomorphism $f : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$, the theory of covering spaces allows to construct a lifting $\tilde{f} : \mathbb{T} \times \mathbb{R} \to \mathbb{R}^2$. Then $p \circ \tilde{f}$ is a homeomorphism of the plane and this is what we mean by a lift from the cylinder to the plane. More details on this point can be found in [4]. The condition (3) can be formulated in a slightly more abstract language as $f^*\omega = \omega$, where $f^*\omega$ is the pull-back of the two form $\omega = d\theta \wedge dr$. Symplectic manifolds are the natural setting to define symplectic maps, the plane and the cylinder endowed with the form ω are just two simple examples of this class of manifolds. We refer to [2, 13, 25, 29] for the general theory. The notion of E-symplectic map was introduced in [14].

3. The twist condition and the generating function. Twist maps are also studied in more degrees of freedom. See the papers by Herman [12] and by Bialy-MacKay [6]. The presentation of the notion of generating function on the cylinder is inspired by the frameworks defined by Mather in [22] and by Moser in [28]. Generating functions can be defined on general symplectic manifolds. We refer to the textbooks [2, 26, 29] on Hamiltonian Dynamics. Usually generating functions are associated to symplectic maps by local constructions based on the implicit function theorem.

4. The variational principle. The presentation of the Frenkel-Kontorowa model follows Aubry's paper [5]. This paper contains interesting results on the dynamics of the standard map when the potential V is a trigonometric function. The variational principle is just an adaptation of classical ideas in Aubry-Mather theory. The only difference is that A-M theory is concerned with generating functions satisfying the periodicity condition

$$h(\theta + 2\pi, \theta_1 + 2\pi) = h(\theta, \theta_1).$$

Mather arrived at a different variational principle in [21]. It was concerned with functions of the continuous variable θ instead of discrete sequences (θ_n). Mather's original motivation was to understand what remains of invariant curves after perturbations when KAM theory does no longer apply. The book by Moser [28] presents a general view of the different variational principles connected with Aubry-Mather theory. The use of results about compactness in spaces of sequences in lemma 13 is influenced by the paper [1]. There Angenent extended Aubry-Mather theory to more general classes of difference equations using a method of upper and lower solutions. Finally we mention the paper by MacKay, Slijepčević, and Stark [24], where the variational principle is employed for a non-standard application.

5. Existence of complete orbits. Theorem 14 first appeared in [14]. In the original version the condition $\overline{\alpha} < 2\underline{\alpha}$ was replaced by the more restrictive condition $\overline{\alpha} < (3/2)\underline{\alpha}$. The two variants of the theorem presented in exercises 18 and 19 are taken from [15] and [16]. Again there is some improvement in the conditions on the quotient $\overline{\alpha}/\underline{\alpha}$. The steps (i) and (ii) of the proof of theorem 14 are inspired by the techniques employed by Terracini and Verzini in [35]. See also [30].

6. The action functional of a Newtonian equation. The standard versions of the theorem on differentiability with respect to initial conditions assume the differentiability of the vector field with respect to both variables t and x. However, the differentiability in t is not required to differentiate the solution at a fixed time $t = t_1$. See for instance the book by Lefschetz [20].

Alternative proofs of the result in exercise 20 can be found in the papers by Franks [10] and You [36].

If (25) is replaced by

$$((2n+1)\pi)^2 < V_{xx}(t,x) < (2n\pi)^2,$$

then there is backward twist, which means that the derivative $\frac{\partial\theta_1}{\partial r}$ is negative. A classical result in the theory of boundary value problems, see [18], says that the problem

$$\ddot{z} = -\tilde{V}_z(t,z), \quad z(0) = z(1) = 0,$$

has a unique solution if $\tilde{V}_{zz} \leq \Gamma < \pi^2$ or $(n\pi)^2 < \gamma \leq \tilde{V}_{zz} \leq \Gamma < ((n+1)\pi)^2$. This problem has homogeneous boundary conditions but the previous result is applicable to (26) after the change of variables $x(t) = z(t) + (\theta_1 - \theta)t + \theta$.

Very clear discussions on the connection between the generating function and the action functional can be found in Moser's course [28]. They apply to general Lagrangian systems but we have preferred to restrict ourselves to Newtonian equations to make the discussion simpler and also to present a non-local formulation. In [33] Radmazé considered the problem of minimization of the action functional in a space of periodic functions, say $H^1(\mathbb{R}/\mathbb{Z})$. He found that there is a minimizer if and only if the function $\mathcal{R}(\theta) := h(\theta, \theta)$ reaches a minimum at some θ^*. Moreover, the minimizer is precisely $\xi(t; \theta^*, \theta^*)$. This is also described in Caratheodory's book [8].

Exercise 25 is a particular case of some results in [15] for equations of the type

$$\ddot{x} + V'(x) = p(t), \;\; p(t+1) = p(t), \;\; \int_0^1 p(t)dt = 0,$$

with $V(x)$ bounded. An alternative way to solve it is to find a generalized periodic solution of the type $x(t+1) = x(t) + 2\pi$ by minimization of the action functional. More information on this approach can be found in the paper by Mawhin on the pendulum equation [23].

7. Impact problems and generating functions. The connections between the generating function and the action functional for impact problems is discussed in [15]. The key is the formula (34), which also explains the connection between the so-called Nehari method and our method of construction of bouncing solutions via the equation (36). The basic idea of Nehari's method (see [35]) is to consider the function

$$h(t_0, t_1) = \inf \left\{ \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))\, dt : \; x \in H_0^1(t_0, t_1), \; x > 0 \text{ on }]t_0, t_1[\right\},$$

and then to obtain a solution satisfying the boundary conditions $x(t_0) = x(t_{n+1}) = 0$ and having exactly n zeros in $]t_0, t_{n+1}[$ by finding a maximum of the function

$$\Phi(t_1, t_2, \ldots, t_n) = \sum_{k=0}^{n} h(t_k, t_{k+1}).$$

From formula (34) and the uniqueness of solution to the Dirichlet problem we observe that h is precisely the generating function and the critical points of Φ are the solutions of (36) on a finite interval of indexes.

The problem of the bouncing ball with gravity was considered in [32] by Pustyl'nikov in the case where the motion of the racket $w(t)$ is periodic. In particular he obtained an interesting result on the existence of motions with unbounded velocity for certain functions $w(t)$ which are periodic, smooth and have large norm.

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